SIAM J. MATH. ANAL.
Vol. 23, No. 5, pp. 1141-1166, September 1992

EXISTENCE AND NONEXISTENCE OF SOLITARY WAVE SOLUTIONS TO HIGHER-ORDER MODEL EVOLUTION EQUATIONS*

SATYANAD KICHENASSAMY† AND PETER J. OLVER†‡

Abstract. The problem of existence of solitary wave solutions to some higher-order model evolution equations arising from water wave theory is discussed. A simple direct method for finding monotone solitary wave solutions is introduced, and by exhibiting explicit necessary and sufficient conditions, it is illustrated that a model admit exact sech^2 solitary wave solutions. Moreover, it is proven that the only fifth-order perturbations of the Korteweg-deVries equation that admit solitary wave solutions reducing to the usual one-soliton solutions in the limit are those admitting families of explicit sech^2 solutions.

Key words. solitary wave, nonlinear evolution equation, water waves, singular perturbation

AMS(MOS) subject classifications. 76B25, 35Q51, 35Q53, 35B25, 76B15

1. Introduction. In the study of equations modeling wave phenomena, one of the fundamental objects of study is the traveling wave solution, meaning a solution of constant form moving with a fixed velocity. The determination of such solutions is accomplished by solving a reduced differential equation in fewer independent variables by one. In particular, the traveling wave solutions for a one-dimensional wave equation are found by solving a connection problem for an associated ordinary differential equation. Of particular interest are three types of traveling waves: the *solitary waves*, which are localized traveling waves, asymptotically zero at large distances, the *periodic waves*, and the *kink waves*, which rise or descend from one asymptotic state to another. All of these are, in the completely integrable case, solitons, coming from the inverse scattering solution to an eigenvalue problem, and dependent on a free parameter. On the other hand, the existence of these types of solutions is not dependent on integrability of the model, or the connection with an inverse scattering transform method of solution, as evidenced by the φ^4 theory; cf. [37], [38], and the examples described here. Except in the simplest instances, it is by no means obvious that such types of traveling wave solutions even exist. In addition, once existence is known, the delicacy of the connection problem to be solved makes their numerical computation rather difficult to effect in an easy, practical manner.

In this paper, we concentrate on the determination of solitary waves, whose importance for fluids came to the forefront with Scott Russell's experimental observation of solitary water waves in the Edinburgh canal [33]. Airy's premature dismissal of these solutions based on a linearized analysis of the free boundary problem necessitated the construction of suitable models exhibiting such solutions. This was accomplished, in the case of long waves over shallow water, through Boussinesq's bidirectional models and, subsequently, the celebrated Korteweg-deVries model, whose solitary wave solutions are explicit sech^2 solutions, which, moreover, have the remarkable soliton property of interacting without change of form. More recently, Amick and Toland, [4], and others, [1], [2], [19], have proved the existence of such waves for the full water wave problem. For small amplitude waves, the Korteweg-deVries solitons do a good job of modeling solitary water waves, [13]. However, the model fails to replicate such important physical phenomena as having a wave of maximal height,

* Received by the editors April 16, 1991; accepted for publication (in revised form) January 17, 1992.
† School of Mathematics, University of Minnesota, Minneapolis, Minnesota, 55455.
‡ The research of this author was supported in part by National Science Foundation grant DMS-89-01600.

originally conjectured by Stokes (cf. [1]) and the breaking of large amplitude waves. Owing to the difficulty of analyzing the water wave problem directly, the construction of suitable models is of great importance. One possible approach is to retain higher-order terms in the Boussinesq perturbation expansion, leading to fifth-order model evolution equations. One of the principal purposes of this paper is to show that there are definite difficulties with this procedure, in that for most of these higher-order models, solitary wave solutions of the appropriate form do not even exist! Indeed, this holds for almost all versions of the models derived from the water wave problem. (An alternative approach would be to employ the two-timing approach advocated by Segur, [42], and others, in which the higher-order terms in the expansion are forced evolution equations governed by the leading order Korteweg-deVries equation. However, it is hard to see how the requisite phenomena of maximal height and breaking would manifest themselves in this approach.)

The present paper is devoted to the analysis of solitary wave solutions to a general class of scalar fifth-order evolution equations; see (2.1) below. We begin by discussing the various models that are included in this class, such as the fifth-order Korteweg-deVries equations, other integrable equations, water wave models, and models from elastic media with microstructure. The third section discusses known results on explicit solitary wave solutions for certain models, numerical results, and a nonexistence result of Amick and McLeod for the critical surface tension water wave model. Next we present a simplified approach to the determination of explicit monotone traveling wave solutions, which reduces the fifth-order evolution equation to a third-order ordinary differential equation. This leads to explicit criteria for the existence of exact sech^2 solitary wave solutions, which imply that these models admit either 0, 1, 2, ∞, or $\infty+1$ exact sech^2 solitary wave solutions. Here ∞ indicates a one-parameter family of solutions valid for a range of wave speeds, and these particular models are explicitly characterized by a pair of simple algebraic relations on the coefficients. Interestingly, even for fifth-order Korteweg-deVries models, there is the possibility of having more than one solitary wave solution for a given wave speed, leading to unusual "bound state solutions." Finally, we present a nonexistence result that says, in essence, that the only models which are perturbations of the usual Korteweg-deVries equation and that possess solitary wave solutions reducing, in the limit, to Korteweg-deVries solitons are those that have a one-parameter family of explicit sech^2 solitary waves. See Theorem 13 for a precise formulation. Our proof relies on a general method introduced by the first author [24] in a similar study of breather solutions of Klein-Gordon equations, which we outline at the end of § 3. Our result does not completely rule out all solitary wave solutions, but only those which reduce to Korteweg-deVries solitary waves in an appropriate scaling limit; nevertheless, it does demonstrate that "physically relevant" solitary wave solutions do not, in general, exist. This has some interesting implications for perturbation theories, which we discuss in the final section.

2. Higher-order model equations. We will consider a class of fifth-order model evolution equations of the general form

$$\begin{aligned} &u_t+\mu u_{xxx}+\alpha u_{xxxxx}+\beta u u_{xxx}+\delta u_x u_{xx}+P'(u)u_x \\ &\quad = u_t+[\mu u_{xx}+\alpha u_{xxxx}+\beta u u_{xx}+\gamma u_x^2+P(u)]_x=0. \end{aligned} \tag{2.1}$$

Here α, β, γ, $\delta=2\gamma+\beta$, and μ are assumed to be constants, and $P(u)$ is an analytic function of the dependent variable. Many of these models require that P be a cubic polynomial

$$P(u)=pu+qu^2+ru^3, \tag{2.2}$$

where p, q, r are constants, although this will not be necessary for most of our analysis. (However, *only* these models will admit explicit sech^2 solitary waves.) Note that we can assume without loss of generality that $p = 0$ by going to a suitable moving coordinate frame. In the models derived by perturbation expansion, the coefficients in (2.1) will depend on a small parameter, ε, in terms of which p is of order 1, q, μ are of order ε, and α, β, δ, and r of order ε^2.

The general class of equations (2.1) includes many well-known equations that have been studied at length in the literature. If the ε^2 terms are absent, the model (2.1) reduces to the well-known Korteweg-deVries equation

$$u_t + pu_x + \mu u_{xxx} + 2quu_x = 0, \tag{2.3}$$

which serves to model many different wave phenomena requiring a balance between dispersion and nonlinearity, [33], [46]. Also of note is the modified Korteweg-deVries equation

$$u_t + pu_x + \mu u_{xxx} + 3ru^2u_x = 0. \tag{2.4}$$

Both the Korteweg-deVries and modified Korteweg-deVries equations are known to be integrable via inverse scattering techniques, [33], [42], [46], the scattering operator for the Korteweg-deVries equation being the well-studied Schrödinger operator $L = D^2 + v$, where the potential $v(x, t)$ is a suitable multiple of $u(x, t)$, and $D = d/dx$. In particular, their solitary wave solutions are solitons, and interact without change of form. Their speed is related to the value of the associated spectral parameter (eigenvalue). There are additional integrable models included in the class (2.1). The particular parameter values

$$\beta = \tfrac{5}{3}\kappa\alpha, \quad \delta = \tfrac{10}{3}\kappa\alpha, \quad r = \tfrac{5}{18}\kappa^2\alpha, \quad q = \tfrac{1}{2}\kappa\mu, \tag{2.5}$$

where $\kappa \neq 0$, describe a four-parameter family of integrable fifth-order Korteweg-deVries equations [33], which are soluble by the scattering problem associated with the same Schrödinger operator. (More accurately, the models given by (2.5) are linear combinations of purely fifth-order (corresponding to the parameter α) and third-order (corresponding to the parameter μ) Korteweg-deVries equations.) The Sawada-Kotera equation [41],

$$u_t + u_{xxxxx} + 30uu_{xxx} + 30u_xu_{xx} + 180u^2u_x = 0, \tag{2.6}$$

and the Kaup equation [21],

$$u_t + u_{xxxxx} + 30uu_{xxx} + 75u_xu_{xx} + 180u^2u_x = 0, \tag{2.7}$$

are also known to be integrable, being associated with the scattering problem for the third-order operator $M = D^3 + vD + w$; cf. [21]. For the Sawada-Kotera equation, $v = 6u$ and $w = 0$, whereas for the Kaup equation $v = 6u$ and $w = 3u_x$. However, in contrast to the higher-order Korteweg-deVries equations, we cannot add in third-order terms to these equations without destroying their integrability.

Other models of the general form (2.1) that are (almost certainly) not integrable also arise in applications. In [34], [35] the second author proposed certain special cases of the general fifth-order model (2.3) as models for the unidirectional propagation of shallow water waves over a flat surface. (See [29] for extensions which include bottom topography.) These arose from two sources: first as the second-degree correction to the standard Korteweg-deVries model for the undirectional propagation of long waves in shallow water arising in the Boussinesq expansion for the full water wave problem. Second, using a general theory of noncanonical perturbation expansions of

Hamiltonian systems, these types of models appear as "Hamiltonian versions" of the Korteweg-deVries model, incorporating the correct first degree expansions of both the water wave Hamiltonian functional (energy) and the Hamiltonian operator. Indeed, whereas the full water wave problem admits a Hamiltonian structure due to Zakharov [50] and the Korteweg-deVries equation admits two distinct Hamiltonian structures [36], neither of these matches the perturbation expansion of Zakharov's structure. Alternatively, we can verify that the first-order perturbation expansion of the water wave energy functional is *not* conserved under the Korteweg-deVries flow. The Hamiltonian models attempt to rectify these unexpected difficulties. In the water wave models, $u(x, t)$ represents either the surface elevation or the horizontal velocity measured at a fraction $0 \leqq \theta \leqq 1$ of the undisturbed fluid depth. There are two small parameters called α, β in [34], [35], but, to avoid confusion, we denote them here by ε, which measures the ratio of wave amplitude to undisturbed fluid depth, and κ, which measures the square of the ratio of fluid depth to wave length. In the shallow water regime, ε and κ are assumed to have the same order of magnitude. The Bond number, which represents a dimensionless magnitude of surface tension, is denoted by τ. In all models, the leading order (Korteweg-deVries) terms are all the same:

$$p=1, \quad \mu=\kappa\frac{1-3\tau}{6}, \quad q=\frac{3}{4}\varepsilon, \tag{2.8}$$

representing a Korteweg-deVries equation except when the Bond number has the critical value $\tau=\frac{1}{3}$. (See below.) The models differ only in the higher-order terms, which take the following forms:

u = horizontal velocity at depth θ; second-order model

$$\alpha=\kappa^2\frac{19-30\tau-45\tau^2}{360}, \quad \beta=\kappa\varepsilon\frac{5-3\tau}{12}, \quad \delta=\kappa\varepsilon\frac{53-36\theta^2-39\tau}{24}, \quad r=0, \tag{2.9}$$

u = horizontal velocity at depth θ; Hamiltonian model

$$\begin{aligned}\alpha&=-\kappa^2\frac{(5-6\theta^2-3\tau)(2-3\theta^2)}{18}, \quad \beta=\kappa\varepsilon\frac{53-66\theta^2-27\tau}{24},\\ \delta&=\kappa\varepsilon\frac{139-168\theta^2-81\tau}{24}, \qquad r=-\frac{15}{32}\varepsilon^2,\end{aligned} \tag{2.10}$$

u = surface elevation; second-order model

$$\alpha=\kappa^2\frac{19-30\tau-45\tau^2}{360}, \quad \beta=\kappa\varepsilon\frac{5-6\tau}{12}, \quad \delta=\kappa\varepsilon\frac{23+15\tau}{24}, \quad r=-\frac{1}{8}\varepsilon^2, \tag{2.11}$$

u = surface elevation; Hamiltonian model

$$\alpha=0, \quad \beta=\frac{1-3\tau}{8}\kappa\varepsilon, \quad \delta=\frac{3(1-3\tau)}{8}\kappa\varepsilon, \quad r=\frac{5}{32}\varepsilon^2. \tag{2.12}$$

((2.12) corrects an error in [35, eqn. (4.28)].) It is interesting to note that none of these models is integrable, except the Hamiltonian model (2.10) for the horizontal velocity at the particular "magic depth"

$$\theta=\sqrt{\tfrac{11}{12}-\tfrac{3}{4}\tau}, \tag{2.13}$$

where the model turns out to be a fifth-order Korteweg-deVries equation. (This formula corrects a misprint in reference [35].)

The model

$$u_t+pu_x+\mu u_{xxx}+2quu_x+\alpha u_{xxxxx}=0 \tag{2.14}$$

arises in the study of water waves with surface tension in which the Bond number takes on the critical value $\tau=\frac{1}{3}$, where the Korteweg-deVries model no longer applies; cf. [18]. The particular case $p=\mu=0$ arises in both magneto-acoustics and nonlinear transmission lines; cf. [31], [49]. The equation

$$u_t+pu_x+\mu u_{xxx}+\alpha u_{xxxxx}-uu_{xxx}-2u_xu_{xx}=0 \tag{2.15}$$

was proposed by Benney [6] as one possible model for the interaction of short and long waves. Third-order models of the form

$$u_t+u_x+\mu u_{xxx}+2quu_x+\beta uu_{xxx}+\delta u_xu_{xx}+3ru^2u_x=0, \tag{2.16}$$

in which $\beta=2\delta\neq 0$, $r=0$, were proposed by Kunin [28, § 5.3] in his study of elastic media with microstructure. Note that the Hamiltonian model (2.12) for water waves is of this type, but with $\beta=3\delta\neq 0$, as are both second-order models (2.9), (2.11) at the particular Bond number $\tau=\frac{2}{15}\sqrt{30}-\frac{1}{3}\cong .3970$, and the Hamiltonian model (2.10) at depths $\theta^2=\frac{2}{3}$ or $\frac{5}{6}-\frac{1}{2}\tau$. Additional models of the form (2.1) have been derived for weakly nonlinear long waves in a stratified fluid [14] and free surface waves over rotational flows [12].

Incidentally, the theory of Kodama [25] shows that all such fifth-order equations with $\alpha\neq 0$, and $P(u)$ a cubic polynomial, can be recast asymptotically into canonical form as fifth-order Korteweg-deVries equations under an appropriate change of variables. Thus, in a certain sense, all the models (2.1), (2.2) are "approximately integrable," although this remark does not imply much in the way of rigorous results for them.

Very recently Ponce [39] has proved that the initial-value problem for (2.1), (2.2) is locally well posed in any Sobolev space $H^s(\mathbb{R})$ for any $s\geqq 4$. Specifically, Ponce proves the following result.

THEOREM 1. *For any* $u_0\in H^s(\mathbb{R})$ *with* $s\geqq 4$, *there exists a* $T>0$ *and a unique strong solution* $u(x,t)$ *in the space* $C([0,T],H^s)\cap L^2[0,T],H^{s+2}_{\text{loc}})$ *of the initial value problem* (2.1) *with* $u(x,0)=u_0(x)$.

3. Solitary wave solutions. We now review known results concerning solitary and other traveling wave solutions to particular models of the form (2.1). We begin by discussing the known explicit solutions to these equations.

First recall that the Korteweg-deVries equation, modified Korteweg-deVries equation, and the class of fifth-order Korteweg-deVries equations (2.5) all possess explicit sech^2 solitary wave solutions for all wave speeds $c>p=P'(0)$. The amplitude of these waves is proportional to the wave speed. If $q/\mu<0$, then the solitary wave is a wave of elevation, whereas if $q/\mu>0$ it is a wave of depression. The Sawada-Kotera equation (2.5) also admits sech^2 solitary wave solutions for all wave speeds $c>0$; cf. [30]. On the other hand, the Kaup equaton (2.6) has solitary wave solutions of the anomalous form

$$u(x,t)=\frac{2a^2(2\cosh 2\xi+1)}{(\cosh 2\xi+2)^2},\qquad \xi=ax-16a^5t. \tag{3.1}$$

Again, these exist for a range of wave speeds $c=16a^4>0$.

For the model (2.14) for water waves at critical surface tension, provided $\alpha\mu<0$, Yamamoto and Takizawa [48] produced an explicit solitary wave of depression in terms of a sech^4 function:

$$u(x,t)=-\frac{105\mu^2}{338\alpha q}\operatorname{sech}^4\left[\sqrt{-\frac{\mu}{52\alpha}}\left\{x+\left(p+\frac{36\mu^2}{169\alpha}\right)t\right\}\right]. \tag{3.2}$$

This solution was also derived by Hereman et al. [15] using a more systematic procedure, and, much later, also by Huang et al. [16]. This "anomalous" solitary wave solution is quite surprising; it only appears for one particular (positive) wave speed: $c = -36\mu^2/169\alpha$. It is unclear whether this solution has any physical meaning. (Other similar "anomalous" sech^2 solitary wave solutions will be determined for many of the models (2.1), (2.2) in § 6.) Of less direct relevance to our results, but still of interest, Hunter and Scheurle [17] proved the existence of traveling waves to the model (2.7) that bifurcate from Korteweg–deVries solitons, but are no longer decreasing as $|x| \to \infty$, having small but finite amplitude oscillatory tails.

Kawahara, [22], claims to numerically establish the existence of "oscillatory solitary wave" solutions to the model (2.14), and Nagashima [31], [32], in the case $p = \mu = 0$, "establishes" their existence experimentally (!). Also, Zufiria [51], in the context of the water wave problem, while more concerned with periodic traveling wave solutions, does investigate "approximate solitary waves" for this model and concludes that they are not unique. However, Amick and McLeod [3] have, using powerful complex-analytic methods, rigorously proved that the model (2.8) does *not* possess a solitary wave of elevation for $\alpha\mu > 0$, with α sufficiently small. (Note that this result does not exclude the exact solitary wave (3.1). See also Hunter and Scheurle [18] for a less rigorous version.) It appears to be quite difficult to extend this technique to the more general models considered in this paper, especially in view of the fact that, for certain models, solitary wave solutions do exist. Amick and McLeod's result implies that Kawahara and Zufiria's numerical solutions cannot be correct, and we propose an explanation for such numerical results in § 8. Indeed, many numerical procedures for finding such waves are, in our opinion, rather suspect, as most of the nonexistence results are of the "exponentially small" variety, i.e., to all orders in ε a solitary wave can be shown to exist, but one may suspect that exponentially small terms (like $e^{-1/\varepsilon}$) prevent its final establishment. See Byatt-Smith [11], Kruskal and Segur [27], [43], and Troy [44], for other problems of this type. Numerical schemes are hard pressed indeed to discover such exponentially small errors!

In the third-order model (2.16), which includes Kunin's third order models for elastic media and some of the water wave models, the equation for solitary waves can, in certain cases, be integrated directly, and one has the intriguing phenomenon of a *wave of maximal height*, reminiscent of the Stokes phenomenon (although the maximal height waves for these models exhibit cusps rather than corners). Indeed, for the full water wave problem, Amick and Toland [4], have proved the existence of monotone solitary wave solutions of small amplitudes up to a maximal height wave with a 120° corner for the problem in the absence of surface tension. (For large values of surface tension, meaning Bond number $\tau > \frac{1}{3}$, Amick and Kirchgässner [2] and Sachs [40] have proved the existence of monotone solitary wave solutions, while very recent results of Iooss and Kirchgässner [19], and Beale [5] demonstrate the existence of solitary wave solutions with damped oscillatory tails for $0 < \tau < \frac{1}{3}$). See also the papers of Wadati, Ichigawa, and Shimizu [45], and Kawamoto [23] for other types of model equations exhibiting limiting cusp waves. It is an interesting question as to whether any of the fifth-order models exhibit such phenomena. Also, the behavior of large amplitude waves (including the possibility of breaking) in these models is not known.

Finally, we mention papers by Yamamoto and Takizawa [47], [48], and Kano and Nakayama [20], which exhibit other types of traveling wave solutions, including periodic waves and solitary sech^2 waves approaching a nonzero asymptotic value as $x \to \pm\infty$. (These can, of course, always be transformed into "genuine" solitary wave solutions, with zero asymptotic limits, to a different model of the same basic form (2.1) by subtracting a suitable constant from u.)

Our own results include the following existence and nonexistence criteria.

On the one hand, we exhibit explicit conditions for a model of the form (2.1) to possess a sech^2 solitary wave solution. First, for such solutions to exist, $P(u)$ must necessarily be a cubic polynomial, (2.2). Interestingly, the parameter space $(\alpha, \beta, \delta, \mu, p, q, r)$ splits into five regions: three of these are relatively open subregions in which there are, respectively, two, one, or no exact sech^2 solitary wave solutions. In the first and second regions, most models have such a solution for a unique, or precisely two, possible wave speeds, similar to the anomalous sech^4 solution to the model (2.8). Secondly, we prove that there are two algebraic relations that must be satisfied by the coefficients in order for the model to admit a one-parameter family of sech^2 solitary wave solutions for a range of wave speeds. This family includes the higher-order Korteweg–deVries equations, (2.4), the Sawada–Kotera equation (2.6), and the Hamiltonian water wave model (2.10) at the particular depth (2.13), but also many other (presumably nonintegrable) equations as well. This leads to the two further regions, each of codimension 2, in which there is either a one-parameter family of sech^2 solitary wave solutions, or such a family plus a single anomalous sech^2 solitary wave solution. These results reconfirm the idea that solitary waves may arise independently of the model being integrable. Also, since the Kaup equation (2.7) admits a one-parameter family of solitary wave solutions for a range of wave speeds that are not sech^2 solutions, we must exercise a bit of caution in drawing unwarranted conclusions from this result!)

On the other hand, assuming $\mu\alpha q \neq 0$, and introducing a small parameter ε representing the departure of the models from the Korteweg–deVries equation, we prove that the only models that admit solitary wave solutions that are perturbations of the corresponding Korteweg–deVries solitons, and satisfy certain analyticity conditions, are the models that satisfy these same algebraic relations. Thus the only physically relevant solitary wave solutions that can exist are always given by sech^2 functions! In outline, our nonexistence result is proved in two basic steps, similar to earlier work of the first author on the nonexistence of breather solutions to a general class of nonlinear Klein–Gordon equations, including the φ^4 equation and the double sine-Gordon equation [24]. We first establish the existence of "solitary wave tails," i.e., traveling wave solutions that decay exponentially fast at either $+\infty$ or $-\infty$, by proving the convergence of the appropriate formal power series solution. The second step in the proof is to match this solution with a formal asymptotic expansion of the solution starting with the one soliton solution of the Korteweg–deVries equation obtained by omitting the fifth-order terms in the model. We then show that, by analyzing the poles of this solution in the complex plane, the second series cannot converge to a true solution, and so we conclude that such a solitary wave solution does not exist. The details will become clearer in the subsequent discussion.

4. The equation for traveling waves. We begin by recalling the elementary method for reducing the problem of traveling wave solutions to an evolution equation such as (2.1) to a connection problem for an ordinary differential equation. A traveling wave solution is just a solution of the particular form

$$u = u(\xi) = u(x - ct), \tag{4.1}$$

where c is the wave speed and $\xi = x - ct$ is the characteristic variable. Substituting the ansatz (4.1) into (2.1), we are led to look for solutions to the fifth-order ordinary differential equation

$$\alpha u''''' + (\beta u + \mu) u''' + \delta u' u'' + [P(u) - cu]' = 0, \tag{4.2}$$

where the primes indicate derivatives with respect to ξ. Any solution $u(\xi)$ of (4.2) thus provides a traveling wave solution to the original evolution equation (2.1). The ordinary differential equation (4.2) can be integrated once, so we effectively have a fourth-order equation

$$\alpha u'''' + (\beta u + \mu)u'' + \gamma u'^2 + Q(u) = 0, \tag{4.3}$$

where

$$Q(u) = P(u) - cu - d, \tag{4.4}$$

with d being a constant of integration.

Consider the case of a localized traveling wave solution, meaning one that is asymptotically small at large distances, so $u \to 0$ as $\xi \to \pm\infty$. Note that this requires $Q(0) = 0$, which fixes the constant of integration d. As it stands, (4.3) is still invariant under the group of translations in ξ (and so could be integrated once more, [36, § 2.5]) and the discrete reflection $\xi \mapsto -\xi$. One way to get rid of this ambiguity is to assume that the wave has its crest (or trough) at $\xi_0 = 0$, and is symmetric with respect to the crest, which means that u is an even function of ξ. Thus we have a fourth-order boundary value problem on the half line $\{\xi > 0\}$, with boundary conditions

$$u'(0) = u'''(0) = 0, \quad \text{and} \quad u(\xi) \to 0, \quad \xi \to +\infty. \tag{4.5}$$

As it stands, it is by no means obvious how to solve the nonlinear connection problem (4.3), (4.5); in particular, the two boundary conditions at $\xi = 0$ define too small a target to try to aim for with a standard shooting approach. This already strongly indicates that, barring exceptional circumstances, the existence of solitary wave solutions will be rare.

5. An equation for monotone solitary waves. We introduce an effective direct method for determining explicit "monotone" (see Definition 2 below) traveling wave solutions to general one-dimensional evolution equations, reducing the fourth-order boundary value problem (4.3), (4.5) on the half line to a (singular) third-order "initial-value problem." The method could also be used to effectively compute solitary and periodic waves (when they exist) numerically, although we have not tried to implement it. (In fact, the method was originally developed by the second author in a failed attempt to prove general existence results concerning solitary wave solutions to these models!) It draws its inspiration from a paper by Kano and Nakayama [20], in which they showed the existence of explicit periodic solutions involving combinations of elliptic functions to certain particular fifth-order models by proving that a suitable polynomial solution w to the reduced equation could be determined; see also Krishnan [26], where a similar method is applied to systems of Boussinesq type. Our method is much more direct and easier to implement than that of Hereman et al. [15].

DEFINITION 2. A *monotone solitary wave solution* is a localized traveling wave solution, i.e., $u \to 0$ as $\xi \to \pm\infty$, which is monotone on the open intervals $(-\infty, \xi_0)$, (ξ_0, ∞), and symmetric about the point ξ_0. The solitary wave is a *wave of elevation* (*depression*) if u is montone increasing (decreasing) on $(-\infty, \xi_0)$, in which case $u_0 = u(\xi_0)$ is called the *crest* (*trough*). A *monotone periodic wave solution* is a traveling wave solution which is periodic in ξ, is monotone on the intervals between crests and troughs, and is symmetric about any crest or trough. A *monotone kink wave solution* is a traveling wave solution which is monotone on the entire real line and approaches limiting values at large distances, so $u \to u_1$ as $\xi \to -\infty$, and $u \to u_2$ as $\xi \to \infty$.

Rather than try to look directly for the required solution $u(\xi)$, we assume that it can be reconstructed as the solution of the simple first-order ordinary differential equation

$$u'^2 = w(u), \qquad u' \equiv \frac{du}{d\xi}, \tag{5.1}$$

where $w(u)$ is a function to be determined. Clearly, once the function $w(u)$ is known, (5.1) can be solved explicitly for $u(\xi)$ by a simple quadrature:

$$\int_a^u \frac{dv}{\sqrt{w(v)}} = \xi + k. \tag{5.2}$$

Examples of solutions that have this form are the soliton and cnoidal wave solutions of the Korteweg-deVries equation [46, § 13.12], where the function $w(u)$ is a cubic polynomial. In particular, if $u(\xi)$ is a monotone function on a given interval, the function $w(u)$ is defined implicitly by the relation (5.1).

The key is the behavior of the function $w(u)$ near its zeros. A simple zero will correspond to a crest or a trough, while a double zero will provide an asymptotic exponential tail for $u(\xi)$ near ∞ or $-\infty$. Thus, a solitary wave solution will correspond to a positive solution $w(u)$ between a double zero at $u = 0$ and a simple zero at the crest or trough u_0. (See Fig. 1). Similarly, a periodic wave solution will correspond to a positive solution $w(u)$ between two consecutive simple zeros, (Fig. 2), while a kink solution has two consecutive double zeros, (Fig. 3). We thus have the following useful criterion for the existence of monotone traveling wave solutions to such models.

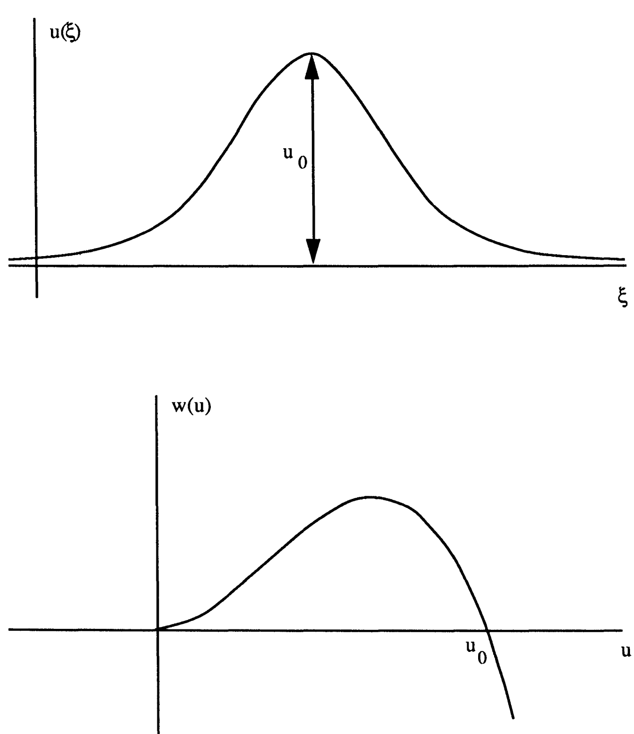

FIG. 1. *Solitary wave solution.*

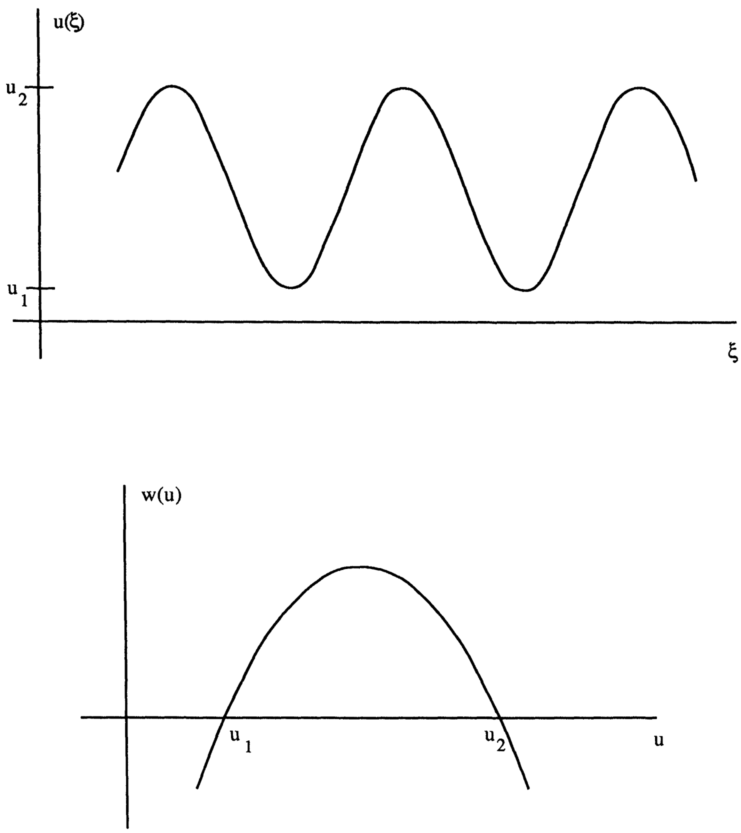

FIG. 2. *Periodic wave solution.*

PROPOSITION 3. *Let $w(u)$ be an analytic function of u, which is positive on the interval $u_0 < u < u_1$, with $w(u_0) = w(u_1) = 0$. Let $u(\xi)$ be the corresponding solution to the first-order ordinary differential equation* (5.1). *If u_0 and u_1 are simple zeros of w, then u is a monotone periodic traveling wave, oscillating between a peak u_1 and a trough u_0. If u_0 is a double zero and u_1 a simple zero of w, then u is a monotone solitary wave of elevation with peak u_1 and asymptotic value u_0 at $\pm\infty$. Conversely, if u_1 is a double zero and u_0 a simple zero of w, then u is a monotone solitary wave of depression with trough u_0 and asymptotic value u_1 at $\pm\infty$. Finally, if u_0 and u_1 are both double zeros of w, then u is a monotone kink wave with asymptotic values u_0, u_1 at $\pm\infty$ (either going from u_0 to u_1 or the reverse by the reflectional symmetry).*

Using the ansatz (5.1), we substitute into the ordinary differential equation for the traveling wave solution $u(\xi)$, and thereby obtain an ordinary differential equation for the function $w(u)$ of order *one less* than that for u. The goal is then to determine suitable solutions $w(u)$ (if any exist) of this reduced ordinary differential equation. Differentiating our basic equation (5.1), we find that, as long as $u' \neq 0$,

$$u'^2 = w,$$

$$u'' = \tfrac{1}{2} w',$$

$$u' u''' = \tfrac{1}{2} w w'',$$

$$u'''' = \tfrac{1}{2} w w''' + \tfrac{1}{4} w' w'',$$

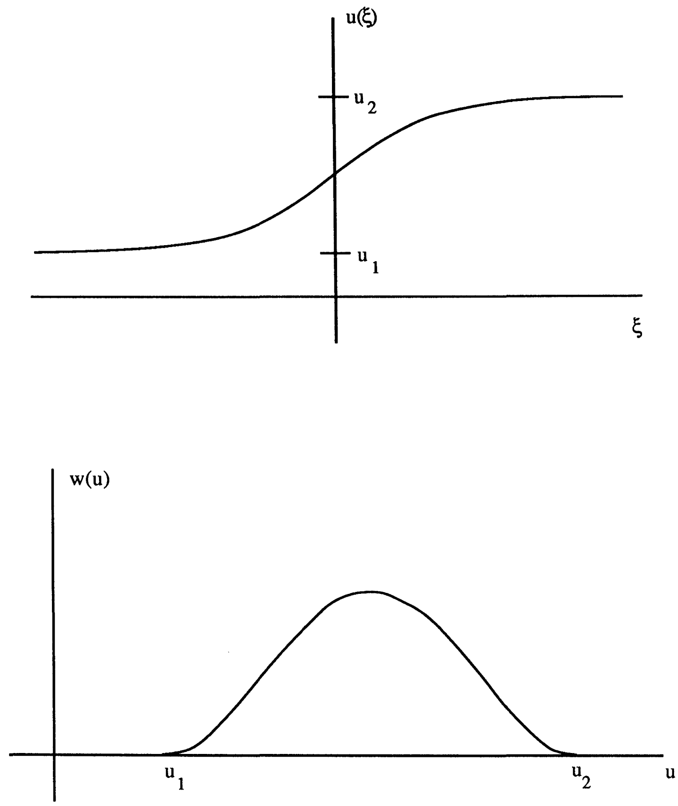

FIG. 3. *Kink wave solution.*

where the primes on w indicate derivatives with respect to u. Substituting into (4.3), we deduce that w must satisfy the third-order ordinary differential equation

$$\frac{\alpha}{4}\{2ww''' + w'w''\} + \frac{1}{2}(\beta u + \mu)w' + \gamma w + Q(u) = 0. \tag{5.3}$$

Any solution to (5.3) will implicitly determine a special traveling wave solution to the original wave equation (2.1) via the integral (5.2). In particular, for a monotone solitary wave solution to the original equation, we need to find a solution $w(u)$ to (5.3) satisfying the initial conditions

$$w(0) = w'(0) = 0, \qquad w''(0) > 0, \tag{5.4}$$

is positive, $w(u) > 0$, for u between 0 and $a \neq 0$, and

$$w(a) = 0, \quad w'(a) \neq 0, \quad w''(a) < \infty. \tag{5.5}$$

In this case a will be the amplitude (crest or trough depending on the sign) of the solitary wave.

6. Exact solitary wave solutions. In certain special cases, we can use the representation (5.1) to easily find exact sech^2 solitary wave solutions to our original evolution equation (2.1). For a solitary wave solution of the specific form

$$u(x, t) = a\,\text{sech}^2\,\lambda(x - ct), \qquad \lambda > 0, \tag{6.1}$$

the corresponding function $w(u)$ must be a cubic polynomial:

$$w(u)=4\lambda^2\left(u^2-\frac{1}{a}u^3\right)=\rho u^2+\sigma u^3, \tag{6.2}$$

where

$$\rho=4\lambda^2>0, \qquad \sigma=-\frac{4\lambda^2}{a}\neq 0 \tag{6.3}$$

are constants to be determined from the equation. Note that since $a=-\rho/\sigma$, we see that $\sigma<0$ gives a wave of elevation, and $\sigma>0$ a wave of depression. Substituting (6.2) into (5.3), we first deduce that $Q(u)$ (and hence $P(u)$) must be a cubic polynomial,

$$Q(u)=(p-c)u+qu^2+ru^3, \tag{6.4}$$

with $Q(0)=0$; cf. (2.2), (4.4). Moreover, the coefficients ρ and σ must satisfy three polynomial equations:

$$\alpha\rho^2+\mu\rho+(p-c)=0, \tag{6.5}$$

$$15\alpha\rho\sigma+2(\beta+\gamma)\rho+3\mu\sigma+2q=0, \tag{6.6}$$

$$15\alpha\sigma^2+(3\beta+2\gamma)\sigma+2r=0, \tag{6.7}$$

arising as the coefficients of the powers of u in (5.3). The fact that the solution ρ of the *indicial equation* (6.5) must be positive places certain inequality constraints on the wave speed c depending on the relative signs of the coefficients α, μ. As long as we also have a nonzero solution σ to (6.7), then (6.6) imposes a single compatibility condition on all the coefficients of the evolution equation (2.1) and the wave speed c. As we will see, this implies that there are three open regions in parameter space (coordinated by $\alpha, \beta, \gamma, \mu, p, q, r$), where the model (2.1), (2.2) has precisely 0, 1, or 2 sech^2 solitary wave solutions, for a particular value of the wave speed c.

For a special five-parameter family of models, there is actually a continuum of sech^2 solitary wave solutions for all sufficiently large wave speeds. Note that according to (6.5), ρ will depend on the wave speed c, whereas (6.7) implies that σ does not. Therefore, if the compatibility condition (6.6) is to hold for a range of wave speeds, the coefficient of ρ and the constant term must lead to the same equation for σ. We conclude that the models for which this occurs are those for which

$$(\beta+\gamma)\mu=5q\alpha \quad \text{and} \quad 15\alpha r=\beta(\beta+\gamma). \tag{6.8}$$

In particular, the four-parameter family of integrable fifth-order Korteweg-deVries equations (2.5), and the Sawada-Kotera equation, (2.6), both satisfy these constraints. However, these do not exhaust all the models satisfying the constraints (6.8); presumably most of the others are not integrable. (Although the Kaup equation, (2.7), has a continuum of solitary wave solutions, they are not of sech^2 type, and so it is in a different class.)

For these particular models, the nature of the sech^2 solitary waves, which comes from an elementary analysis of the conditions for (6.5), (6.7) to admit real solutions ρ, σ, and the resulting signs, is of interest. Since the wave amplitude is given by the formula $a=3\mu\rho/(2q)$, and $\rho>0$, if $q\mu>0$, then the solitary wave is a wave of elevation, whereas if $q\mu<0$ it is a wave of depression, as in the Korteweg-deVries case (2.3). Substituting into (6.5), we deduce the following quadratic equation relating wave speed and amplitude:

$$c=\frac{4\alpha q^2}{9\mu^2}a^2+\frac{2}{3}qa+p, \qquad \text{sign } a=\text{sign } q\mu. \tag{6.9}$$

If $\alpha\mu > 0$, then there is a unique solitary wave for each *supercritical* wave speed $c > p$. However, if α and μ have opposite signs, then besides these supercritical sech^2 solutions, there is a nonzero sech^2 solitary wave at the critical wave speed $c = p$, and *two* distinct sech^2 solitary waves for the range of *subcritical* wave speeds between p and $p - \mu^2/(4\alpha)$, reducing to a single wave of amplitude $a^* = -3\mu^2/(4\alpha q)$ at the limiting wave speed $c^* = p - \mu^2/(4\alpha)$. Figures 4 and 5 graph the different possible relationships (6.9) between wave speed and amplitude for the one-parameter family of sech^2 solutions to the models satisfying (6.8).

The elementary observation that a model of the form (2.1) can admit more than one distinct solitary wave solution for a given wave speed does not appear to be well known, even for the integrable fifth-order Korteweg–deVries models. In this particular case, this result can also be detected using the associated scattering problem as follows. The Lax pair for such an equation takes the form $L_t = [B, L]$, where L is the usual second-order Schrödinger operator, and $B = \mu B_3 + \alpha B_5 = \mu L_+^{3/2} + \alpha L_+^{5/2}$ is a linear combination of the third- and fifth-order operators giving the homogeneous third- and fifth-order Korteweg–deVries equations. The eigenvalue for the soliton is constant, and the associated norming constant has the time dependence $m(t)^2 = m(0)^2 \exp[8\mu t\eta^3 - \alpha\eta^5]$. The corresponding wave speed is then $c = (8\mu\eta^3 - \alpha\eta^5)/2\eta$. Thus, we can clearly have ranges of wave speeds for which there are two distinct sech^2 solitons traveling at the same speed. Note that the corresponding two-soliton solution for two such waves represents a bound state with two humps traveling at the same

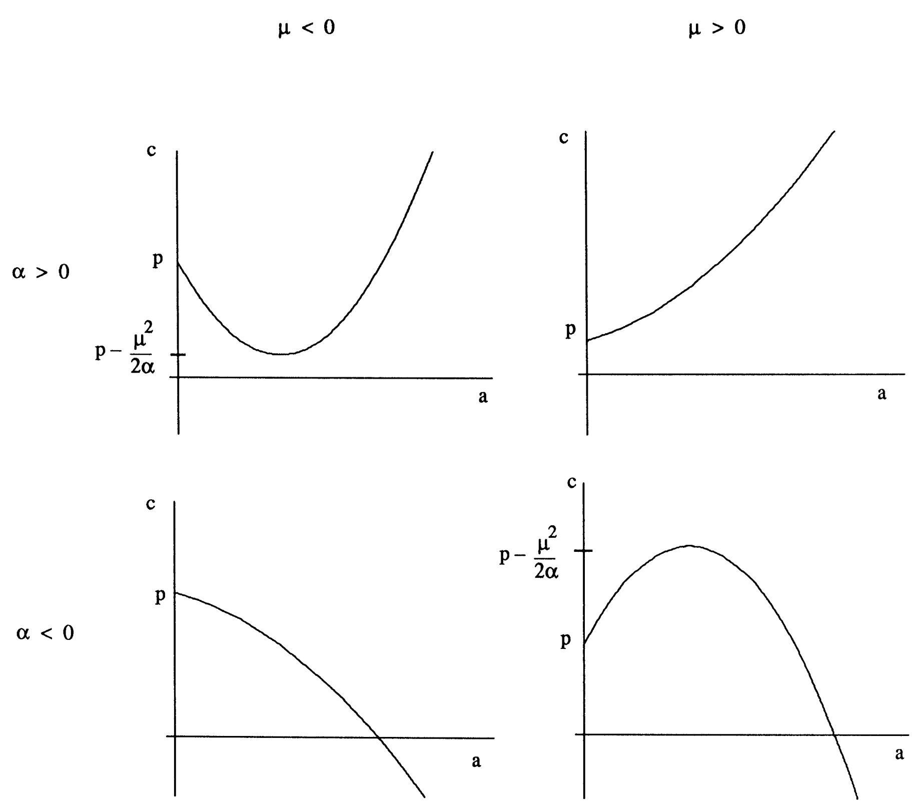

FIG. 4. *Wave speeds and amplitudes for $q\mu > 0$.*

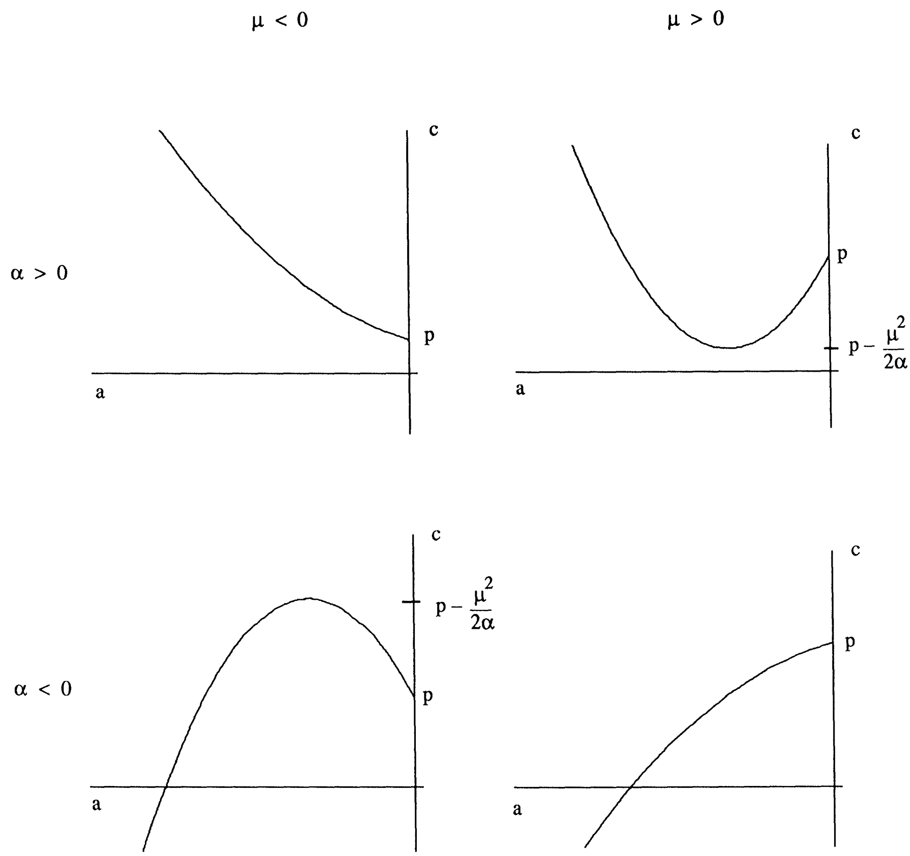

FIG. 5. *Wave speeds and amplitudes for* $q\mu < 0$.

speed. This phenomenon is reminiscent of the construction of bound states to the sine-Gordon equation, consisting of several solitons with phases having real parts with the same speed, the sine-Gordon breather being an example. However, the present property is much stronger, and its appearance for the fifth-order Korteweg-deVries equation is, we believe, a new observation. Note that similarly, we can arrange bound states for linear combinations of higher-order Korteweg-deVries equations to have any number of desired humps traveling in tandem.

Let us summarize our general results completely characterizing models admitting exact sech^2 solitary wave solutions. The different possibilities are: 0, 1, or 2 exact sech^2 solitary wave solutions, a one-parameter family of sech^2 solutions, or a one-parameter family along with a single additional exact sech^2 solution. The first three occur on relatively open subsets of parameter space, whereas the latter two occur on parts of the boundaries between these subsets.

THEOREM 4. *Consider the model evolution equation* (2.1), *assuming* $\alpha \neq 0$. *If* $P(u)$ *is not a cubic polynomial, then the model has no exact* sech^2 *solitary wave solutions. If* $P(u)$ *is given by* (2.2), *then we define*

$$\zeta = (3\beta + 2\gamma)^2 - 120\alpha r, \tag{6.10}$$

so that (6.7) *has* 0, 1, *or* 2 *real roots*

$$\sigma_1, \sigma_2 = \frac{-(3\beta + 2\gamma) \pm \sqrt{\zeta}}{30\alpha}, \tag{6.11}$$

according to whether ζ is negative, zero, or positive. If $r \neq 0$, the real roots are nonzero; if $r = 0$, one root, namely $\sigma_1 = -(3\beta + 2\gamma)/(15\alpha)$, is nonzero unless $\beta = -\frac{3}{2}\gamma$ also. Then the model (2.1), (2.2) *will have* 0, 1, *or* 2 *exact* sech^2 *solitary wave solutions for each nonzero real root σ_i*, which also satisfies

$$\text{(6.12)} \qquad \nu_i = 15\alpha\sigma_i + 2(\beta + \gamma) \neq 0, \qquad \rho_i = \frac{3\mu\sigma_i + 2q}{\nu_i} > 0.$$

Finally, if $(\beta + \gamma)\mu = 5q\alpha$ and $15\alpha r = \beta(\beta + \gamma)$, then the model has a one-parameter family of exact sech^2 *solitary wave solutions valid for a range of wave speeds corresponding to the first root $\sigma_1 = -2(\beta + \gamma)/(15\alpha)$. Moreover, if $\gamma \neq 0$, the second root $\sigma_2 = -\beta/(15\alpha)$ gives rise to a single additional exact* sech^2 *solitary wave solution provided ρ_2, as defined by* (6.12), *is positive.*

Example 5. The only possible water wave model which has a one-parameter family of exact sech^2 solitary wave solutions, i.e., satisfies the conditions (6.8), is the Hamiltonian model (2.10) at the particular depth (2.13). Otherwise, these models all fail to have families of sech^2 solitary wave solutions of the requisite type. However, Theorem 4 implies that many of the water wave models admit one or two anomalous sech^2 solitary wave solutions. The precise numerical values for which the different possibilities occur are rather strange; we will just summarize the results, which were deduced with the help of MATHEMATICA. First, in the case of the second-order depth model (2.9) provided $\alpha \neq 0$, i.e., except for the particular Bond number $\tau = (2\sqrt{30} - 5)/15 \cong .3970$, the model admits a single exact sech^2 solitary wave solution unless $3\beta + 2\gamma = 0$, which occurs when $\tau = (73 - 36\theta^2)/51$. For

$$0 \leqq \tau < \frac{2\sqrt{30} - 5}{15} \quad \text{or} \quad \tau > \frac{73 - 36\theta^2}{51}$$

the anomalous solitary wave is a wave of elevation, while for

$$\frac{2\sqrt{30} - 5}{15} < \tau < \frac{73 - 36\theta^2}{51}$$

it is a wave of depression.

Similarly, for the second-order surface model (2.11) there are one or two exact sech^2 solitary wave solutions provided $\alpha \neq 0$, and $\zeta > 0$, which requires

$$0 \leqq \tau < \frac{4\sqrt{19866} - 249}{333} \cong .9453, \qquad \tau \neq \frac{2\sqrt{30} - 5}{15} \cong .3970.$$

On the range

$$\frac{\sqrt{85} + 5}{30} \cong .4740 < \tau < \frac{\sqrt{23377} - 91}{102} \cong .6068,$$

there are two anomalous solitary wave solutions; otherwise, there is just one. In all cases, these are waves of elevation. The Hamiltonian depth model (2.10) also admits exact solitary wave solutions for various ranges of values of the Bond number and depth, but the results are too complicated to warrant inclusion here. We are not sure of the physical significance (if any) of such exact solutions.

7. Existence of solitary wave tails. We now turn to the consideration of more general types of solitary wave solutions. We begin by proving the existence of "solitary wave tails," meaning solutions to the ordinary differential equation (4.3) for traveling waves with the correct asymptotic behavior at $+\infty$. First, let

$$Q(u)=\sum_{m=1}^{\infty} q_m u^m \tag{7.1}$$

be the power series expansion of Q at $u=0$. (Note that $Q(0)=0$ is necessary for the existence of an asymptotically decreasing solution to (4.3).) If $P(u)$ is a cubic of the form (2.2), then

$$q_1=p-c, \quad q_2=q, \quad q_3=r, \quad q_m=0, \quad m>3, \tag{7.2}$$

where c is the wave speed.

DEFINITION 6. A *solitary wave tail* is an exponentially decreasing solution $u(\xi)$ to the equation for traveling waves with asymptotic expansion

$$u(\xi)\sim u_1\,e^{-\theta\xi}+u_2\,e^{-2\theta\xi}+u_3\,e^{-3\theta\xi}+\cdots, \tag{7.3}$$

with $\theta>0$, which converges for ξ sufficiently large.

Of course, we can also discuss solitary wave tails at $\xi=-\infty$, but these are found by using the reflectional symmetry replacing ξ by $-\xi$. We can also consider "oscillatory solitary wave tails," i.e., convergent expansions of the form (7.3) with θ complex and Re $\theta>0$. Our convergence proof will work more or less the same way in this case, but we will just concentrate on the real exponentials for simplicity.

The existence of such an expansion leads to immediate restrictions on the exponent θ and the coefficients in the model. These result from an analysis of the balance equations obtained by substituting (7.3) into (4.3), and equating terms in the various exponentials $e^{-k\theta\xi}$, $k=1,2,3,\cdots$. The first few of these are easily found.

$$e^{-\theta\xi}: \quad (\alpha\theta^4+\mu\theta^2+q_1)u_1=0, \tag{7.4}$$

$$e^{-2\theta\xi}: \quad (16\alpha\theta^4+4\mu\theta^2+q_1)u_2+[(\beta+\gamma)\theta^2+q_2]u_1^2=0, \tag{7.5}$$

$$e^{-3\theta\xi}: \quad (81\alpha\theta^4+9\mu\theta^2+q_1)u_3+[(5\beta+4\gamma)\theta^2+2q_2]u_1u_2+q_3u_1^3=0. \tag{7.6}$$

Since $u_1\neq 0$, (but is otherwise arbitrary), the first balance equation leads immediately to the *indicial equation*

$$\alpha\theta^4+\mu\theta^2+q_1=0. \tag{7.7}$$

The existence of positive real solutions θ to the indicial equation (7.7) places constraints on the coefficients α, μ, q_1 of the linearized model so that exponentially decaying solutions can exist; see Theorem 7 below. Assuming these hold, we eliminate q_1 using (7.7), and the balance equation resulting from the coefficient of $e^{-n\theta\xi}$ takes the form

$$((n^2+1)\alpha\theta^4+\mu\theta^2)u_n=\Psi_n, \tag{7.8}$$

where Ψ_n is a (complicated) polynomial involving the coefficients of the equation and the previous coefficients $u_1,\cdots,u_{n-1}$. Therefore, as long as the *nonresonance* condition

$$(n^2+1)\alpha\theta^2+\mu\neq 0, \qquad n=2,3,\cdots, \tag{7.9}$$

holds for the root θ of the indicial equation, we can solve recursively for all the coefficients u_n, $n=1,2,\cdots$, in the expansion (7.3) and thereby determine a formal solitary wave tail for the equation. Note that if α and μ have the same sign, then the nonresonance condition (7.9) automatically holds. The resonant case is quite intriguing, but we have not investigated it in any detail, and we leave it aside in what follows.

Note in particular, if $u(\xi)=a\,\mathrm{sech}^2\,\lambda\xi$, then

$$\theta=-2\lambda, \quad u_1=4a, \quad u_2=-8a, \quad u_3=12a. \tag{7.10}$$

Substituting (7.10) into the three balance equations (7.4), (7.5), (7.7), and using (7.2), (6.3), we recover our earlier three equations (6.5), (6.6), (6.7), relating the equation parameters and the solitary wave parameters a, λ. Thus, we can deduce our earlier parameter restrictions for the existence of sech^2 solitary waves by an alternative procedure based on the asymptotic expansion at ∞. However, in contrast to the earlier direct method, this does not prove that the sech^2 wave is actually a solution to (4.3), since we must also verify the higher-order balance equations. Remarkably, these are all satisfied; see § 8. This observation strongly indicates that only the first three balance equations are important for solitary waves, a fact borne out in the following section.

THEOREM 7. *Consider the model* (2.1), *and let* $Q(u)=P(u)-cu-P(0)$. *If any one of the conditions* (a) $\alpha Q'(0)<0$, (b) $\alpha\mu<0$ *and* $Q'(0)=0$, (c) $\alpha\mu<0$ *and* $4\alpha Q'(0)=\mu^2$, *or* (d) $\alpha=0$ *and* $\mu Q'(0)<0$, *then there exists a unique solitary wave tail* (7.3) *provided the nonresonance condition* (7.9) *holds. If* $0<4\alpha Q'(0)<\mu^2$ *and* $\alpha\mu<0$, *then, again provided the nonresonance condition* (7.9) *holds, there are two solitary wave tails. In all other cases there are no convergent analytic exponentially decreasing solitary wave tails.*

The conditions of Theorem 7 place restrictions on the possible wave speeds c for which there is any possibility of a solitary wave solution decaying exponentially fast to 0 at $\pm\infty$. In the case $\alpha\mu>0$, for a unique asymptotic tail, we need the usual condition that the wave speed be supercritical: $c>p=P'(0)$. (For the water wave models, this gives the standard result that the wave speed of a solitary wave (if it exists) must be larger than 1.) However, if α and μ have opposite signs, there is the possibility of *nonunique* solitary wave tails for some subcritical wave speeds $c<p$. Indeed, this corresponds precisely to what we observed in § 5 for the cases where explicit sech^2 solutions exist.

Proof of Theorem 7. Rather than work with the formal asymptotic expansion for $u(\xi)$ directly, it turns out to be simpler to employ the method introduced in § 5. We let $w(u)=u'^2$ and prove that there is a convergent power series expansion

$$w(u)=\sum_{k=2}^{\infty} w_k u^k = w_2u^2+w_3u^3+\cdots, \tag{7.11}$$

for w at $u=0$, which solves the third-order equation (5.3) with the initial conditions

$$w(0)=w'(0)=0, \qquad w''(0)=2w_2>0. \tag{7.12}$$

It is easy to express the coefficients w_k of w in terms of the coefficients u_i of u; in particular, $w_2=\theta^2$. Clearly, proving the existence of such an analytic solution w will imply that the corresponding solution $u(\xi)$ will have a convergent series expansion (7.3), which is exponentially decreasing as $\xi\to\infty$. Substituting (7.11) into (5.3), we find that the only constant term is $Q(0)$, which must necessarily vanish. The terms involving the first power of u give our by now familiar *indicial equation*

$$\alpha w_2^2+\mu w_2+q_1=0; \tag{7.13}$$

cf. (6.5), (7.7). Assuming that we have a positive solution w_2 to (7.13) (cf. the hypotheses of the theorem), we construct the corresponding power series for w recursively. The coefficient of u^m, $m\geqq 2$, in (5.3) is

$$\sum_{\substack{i+j=m+4\\ i\geqq 3,\, j\geqq 3}} \frac{\alpha(j-1)(j-2)}{2}[jw_{i-1}w_j+iw_iw_{j-1}]+\frac{\beta m w_m+\mu(m+1)w_{m+1}}{2}+\gamma w_m+q_m=0.$$

Extracting the terms involving w_{m+1} from the sum, we find the recurrence relation

$$w_{m+1}=-\frac{\alpha\sum_{k=3}^{m}k(k-1)(m+k-1)w_kw_{m-k+3}+2(\beta m+2\gamma)w_m+4q_m}{2(m+1)[\alpha w_2(m^2+1)+\mu]}. \tag{7.14}$$

Since $w_2=\theta^2$, the denominator does not vanish owing to the nonresonance condition (7.9), so we can continue to implement the recurrence relation (7.14), and thus construct a formal series solution to (5.3) with the prescribed initial conditions (7.12). We now need to prove convergence, which will follow from the next lemma.

LEMMA 8. *Let $w_2=\theta^2$ be a positive root to the indicial equation* (7.13). *Assume that the nonresonance condition* (7.9) *holds, and let w_m, $m\geqq 3$, satisfy the recurrence relation* (7.14). *Then there exist positive constants A and M such that*

$$|w_m|\leqq\frac{AM^{m-3}}{m^2},\qquad m\geqq 3. \tag{7.15}$$

Proof. Given the convergent power series expansion (7.1) for Q, we know that there exists a number $R>1$ such that the coefficients of the expansion satisfy the inequality

$$|q_m|\leqq R^m\quad\text{for all } m\geqq 1. \tag{7.16}$$

The nonresonance condition implies that there exists a constant $K>0$ such that the inequality

$$m^2+m\leqq 2K|\alpha w_2(m^2+1)+\mu| \tag{7.17}$$

is valid for all $m\geqq 3$. Thus, we have the following estimate on the denominator of (7.14):

$$2(m+1)|\alpha w_2(m^2+1)+\mu m|\geqq\frac{(m+1)^2m}{K}. \tag{7.18}$$

Define the following constants:

$$A=9|w_3|,\qquad M=K\max\left\{\pi^2\alpha A,\frac{2}{3}(|\beta|+|\gamma|),\frac{4R^3}{A},\frac{R}{K}\right\}. \tag{7.19}$$

A straightforward induction, starting at $m=3$, will prove the validity of (7.15). We estimate all of the terms in the numerator of (7.14) in turn. For the summation, we have

$$\begin{aligned}\sum_{k=3}^{m}k(k-1)(m+k-1)|w_k||w_{m-k+3}|&\leqq\sum_{k=3}^{m}\frac{A^2M^{m-3}k(k-1)(m+k-1)}{k^2(m-k+3)^2}\\&\leqq A^2M^{m-3}\sum_{k=3}^{m}\frac{m+k}{(m-k)^2}\\&\leqq A^2M^{m-3}\sum_{j=0}^{m-3}\frac{2m-j}{j^2}\\&\leqq\frac{\pi^2A^2mM^{m-3}}{3}\\&\leqq\frac{AmM^{m-2}}{3K\alpha}.\end{aligned}$$

For the next two terms, we find, since $m\geqq 3$,

$$\begin{aligned}2(|\beta|m+2|\gamma|)|w_m|&\leqq\frac{2A(|\beta|m+2|\gamma|)M^{m-3}}{m^2}\\&\leqq\frac{2Am(|\beta|+|\gamma|)M^{m-3}}{9}\leqq\frac{AmM^{m-2}}{3K},\end{aligned}$$

and, by (7.16),

$$4|q_m| \leqq 4R^m \leqq 4R^3 M^{m-3} \leqq \frac{AmM^{m-2}}{3K},$$

both following from the definition (7.19) of M. Substituting these three estimates and (7.18) into (7.14) easily proves the inductive step for the inequality (7.15).

8. Nonexistence of solitary waves. Having dealt with existence of explicit solitary wave solutions to particular types of the general model (2.1), we now turn our attention to a nonexistence result. We begin by explicitly introducing the small parameter ε into our model, and restrict our attention from the beginning to models in which $P(u)$ is a cubic polynomial. However, this restriction is inessential, and, coupled with the results from Theorem 4, we can deduce that only in this case is there any possibility of suitable solitary wave solutions existing. In the physical models of the form (2.1), (2.2), there is a small parameter ε, relative to which the translation coefficient p has order 1, the Korteweg-deVries terms have coefficients μ, q of order ε, and the fifth-order terms have coefficients α, β, γ (or δ), and r of order ε^2. We also assume that μ, q, and α are all nonzero, so that the model is truly fifth-order, and, moreover, reduces to a Korteweg-deVries equation when the $O(\varepsilon^2)$ terms are neglected. We are interested in the behavior of solutions in the limit $\varepsilon \to 0$, but this is rather trivial without further rescaling since all the terms except the translation will scale out, and everything will reduce to zero. Rather than this, we need to introduce a rescaling of the equation in which the fifth-order terms still have order ε^2, but the translation and Korteweg-deVries terms are of order 1, and compare these solutions in the $\varepsilon \to 0$ limit. In terms of the physical limit, then, we expect the solutions to be order ε^2 perturbations of the corresponding Korteweg-deVries solutions, which are themselves of order ε. Note that, in this limit, the velocity of a Korteweg-deVries soliton has order $c = p + O(\varepsilon^2)$.

We begin with the once-integrated equation for traveling waves (4.3), which, using (2.2), we write in the form

$$(p-c)u + \mu u'' + qu^2 + \alpha u'''' + \beta uu'' + \gamma u'^2 + ru^3 = 0. \tag{8.1}$$

Introduce the scaling

$$\xi = \varepsilon\eta, \quad u = \kappa^2 v, \quad c - p = \kappa^2 s, \tag{8.2}$$

where ε, κ are small parameters, and $s \neq 0$. Rewriting (8.1) for $v = v(\eta)$, we have

$$\varepsilon^2 \mu v'' + \kappa^2(qv^2 - sv) + \varepsilon^4 \alpha v'''' + \kappa^2\varepsilon^2(\beta vv'' + \gamma v'^2) + \kappa^4 rv^3 = 0. \tag{8.3}$$

The condition that the rescaled equation (8.3) possess solutions having the proper expansions in powers of $e^{-\eta}$ at $\eta = +\infty$ is that the rescaled indicial equation

$$s\kappa^2 = \varepsilon^2(\mu + \alpha\varepsilon^2), \tag{8.4}$$

relating the two scaling parameters, hold. This allows us to eliminate κ and rewrite the traveling wave equation in terms of the single small parameter ε:

$$\begin{aligned} &v'' - v + \frac{q}{s}v^2 + \varepsilon^2\left[\frac{\alpha}{\mu}(v'''' - v) + \frac{\alpha q}{s\mu}v^2 + \frac{1}{s}(\beta vv'' + \gamma v'^2) + \frac{\mu r}{s^2}v^3\right] \\ &\qquad + \varepsilon^4\left[\frac{\alpha}{s\mu}(\beta vv'' + \gamma v'^2) + 2\frac{r\alpha}{s^2\mu}v^3\right] + \varepsilon^6\frac{\alpha^2 r}{s^2\mu^2}v^3 = 0. \end{aligned} \tag{8.5}$$

PROPOSITION 9. *There exists a formal asymptotic solution to* (8.5) *of the form*

$$v(\varepsilon, \eta) \sim v_0(\eta) + \varepsilon^2 v_1(\eta) + \varepsilon^4 v_2(\eta) + \cdots, \tag{8.6}$$

in which

$$v_0(\eta)=\frac{3s}{2q}\operatorname{sech}^2\frac{\eta}{2}, \tag{8.7}$$

and each $v_j=P_j(v_0)$ *is a polynomial in* $\operatorname{sech}^2(\eta/2)$, *with* $P_j(0)=0$.

Remark. The expansion (8.6) will formally represent the proposed solitary wave solution to the original model reducing to the Korteweg–deVries soliton, (8.7), in the limit $\varepsilon\to 0$. Thus each $v_j(\eta)$ satisfies the condition that it describe a solitary wave; in particular, it is an exponentially decreasing function of $\eta\in\mathbb{R}$. The numerically observed solitary wave solutions [22], [31], [50] can, we believe, be explained by the existence of this nonconvergent formal series. Indeed, a numerical code would be an approximation to a finite truncation of the series (8.6), which would appear to be a numerical approximation to a genuine solitary wave. But owing to the ultimate nonconvergence of the series, the numerically observed solitary wave solution cannot, in fact, be considered to approximate any actual solution to the ordinary differential equation (8.5).

Proof. Note first that (8.7) is the unique even, decaying solution to the zeroth-order equation

$$v_0''-v_0+\frac{q}{s}v_0^2=0. \tag{8.8}$$

To avoid complications in the subsequent formulae, it helps to introduce a further rescaling

$$\zeta=\frac{\eta}{2},\qquad V(\zeta)=\frac{2q}{3s}v(2\zeta), \tag{8.9}$$

in terms of which (8.5) takes the form

$$\begin{aligned}&\tfrac14 V''-V+\tfrac32 V^2+\varepsilon^2[\hat\alpha\{\tfrac{1}{16}V''''-V+\tfrac32 V^2\}+\hat\beta VV''+\hat\gamma V'^2+\hat r V^3]\\&\quad+\varepsilon^4\hat\alpha[\hat\beta VV''+\hat\gamma V'^2+2\hat r V^3]+\varepsilon^6\hat\alpha^2\hat r V^3=0,\end{aligned} \tag{8.10}$$

where

$$\hat\alpha=\frac{\alpha}{\mu},\quad \hat\beta=\frac{3\beta}{8\mu},\quad \hat\gamma=\frac{3\gamma}{8\mu},\quad \hat r=\frac{9\mu r}{4q^2}. \tag{8.11}$$

The solution $V(\xi)$ will have a formal asymptotic expansion

$$V(\zeta)\sim V_0(\zeta)+\varepsilon^2V_1(\zeta)+\varepsilon^4V_2(\zeta)+\cdots, \tag{8.12}$$

with leading term $V_0(\zeta)=\operatorname{sech}^2\zeta$.

Using the abbreviation $S(\zeta)$ for $\operatorname{sech}^2\zeta$, we group here a few formulae that are elementary, but which will be required in the sequel:

$$S'^2=4S^2(1-S),\qquad S''=4S-6S^2, \tag{8.13}$$

$$\frac{d^2}{d\zeta^2}S^m=mS^m[4m-(4m+2)S]. \tag{8.14}$$

Iterating (8.14) yields

$$\begin{aligned}\frac{d^4}{d\zeta^4}S^m&=16m^4S^m-16m(2m+1)(2m^2+2m+1)S^{m+1}\\&\quad+4m(m+1)(2m+1)(2m+3)S^{m+2}.\end{aligned} \tag{8.15}$$

Consider the particular Schrödinger operator

$$L \equiv -\frac{d^2}{d\zeta^2} + 4 - 12S(\zeta). \tag{8.16}$$

We note that $-12S(\zeta)$ is a three-soliton potential (cf. [33]), so that the spectrum of (8.16) consists of the eigenvalues $\{-5, 0, 3\}$ and a continuous spectrum $\{\lambda \geqq 4\}$; moreover, zero is a simple eigenvalue, with eigenfunction $S'(\zeta)$, which is odd. Thus, L is invertible on even functions in L^2. Also, (8.14) implies

$$L(S^m) = mS^m[4(1-m^2) + (4m^2 + 2m - 12)S]. \tag{8.17}$$

Together, these facts imply the following.

LEMMA 10. *The differential equation*

$$Lf = S^2 P(S), \qquad P \text{ a polynomial} \tag{8.18}$$

has a unique even solution which has the form $f = SQ(S)$, where Q is a polynomial.

Now, inserting the expansion (8.12) in (8.10), each coefficient of ε^{2k} results in an equation of the form

$$\tfrac{1}{4}V_k'' - V_k + 3SV_k = F_k(\zeta),$$

or, in view of (8.16)

$$L(V_k) = -4F_k(\zeta). \tag{8.19}$$

One can see by induction that V_k must have the form $SP_k(S)$, where P_k is a polynomial. Indeed, according to Lemma 10, we need only prove that $F_k(\zeta)$ has the form $S^2R_k(S)$, where R_k is a polynomial in S. This results from the following:

(i) The remaining terms in V^2 have the form V_iV_{k-i}, $1 \leqq i \leqq k-1$, and, by the induction hypothesis, each V_i has the form $SP_i(S)$;

(ii) The coefficient of ε^{2k} in the terms ε^2V^2, ε^2V^3, ε^4V^3, and ε^6V^3 is similarly determined from $V_0, \cdots, V_{k-1}$;

(iii) V'^2 is a sum of terms of the form $P(S)'Q(S)'$, and S'^2 has S^2 as a factor by (8.13);

(iv) VV'' has S^2 as a factor by (8.13) again;

(v) (8.15) shows that $\frac{1}{16}V_{\zeta\zeta\zeta\zeta} - V$ also has the form $S^2R(S)$ if $V = SP(S)$.

Therefore, we have proved that there exists a formal series solution to (8.10) of the form

$$V(\zeta) \sim \operatorname{sech}^2 \zeta + \sum_{k=1}^{\infty} \varepsilon^k P_k(\operatorname{sech}^2 \zeta), \tag{8.20}$$

where the P_k are polynomials, $P_k(0) = 0$. This completes the proof of Proposition 9.

PROPOSITION 11. *If the expansion* (8.6) *converges to a holomorphic function in ε and* $\operatorname{sech}^2 \eta/2$ *for $\eta \to \infty$, and ε near zero, then its associated solitary wave tail is a translate of the exponentially decaying tail previously constructed in Lemma* 7.

Proof. By hypothesis, we have a convergent expansion for the tail of the form

$$v(\varepsilon, \eta) = a_1(\varepsilon)\, e^{-\eta} + a_2(\varepsilon)\, e^{-2\eta} + \cdots. \tag{8.21}$$

We must show that $a(\varepsilon) = a_1(\varepsilon)$ never vanishes so that we may replace η by $\eta + \log a(\varepsilon)$ to obtain the series

$$\tilde{v}(\varepsilon, \eta) = e^{-\eta} + b_2(\varepsilon)\, e^{-2\eta} + \cdots, \tag{8.22}$$

which can be compared to the previous form of the tail. To achieve this, we assume $a(\varepsilon_0)=0$ for some ε_0 (possibly complex). Since (8.21) must solve (8.5), the series argument from § 7 immediately shows that in this case, all the coefficients vanish at the point ε_0, $a_k(\varepsilon_0)=0$, and hence $v(\varepsilon_0, \eta)=0$ vanishes for all η. We show that this implies that every ε derivative $(\partial^n v/\partial\varepsilon^n)(\varepsilon_0, \eta)=0$ of v also vanishes at the point ε_0, for all η which, by the holomorphy assumption, ensures $v(\varepsilon, \eta)\equiv 0$, which is impossible since $v_0(\eta)\not\equiv 0$.

Note first that if $v(\varepsilon_0, \eta)$ vanishes for all η, so do all its η-derivatives; therefore, the first ε-derivative $z(\eta)=v_\varepsilon(\varepsilon_0, \eta)$ solves the linear ordinary differential equation

$$z''-z+\varepsilon_0^2\frac{\alpha}{\mu}\{z''''-z\}=0, \tag{8.23}$$

since all the nonlinear terms vanish at ε_0. Moreover, since $v(\varepsilon, \eta)$ is holomorphic, we also have that $z\to 0$ exponentially fast at infinity. But it is easy to see (e.g., by using the Fourier transform) that (8.23) has no nonzero L^2 solutions. Similarly, an easy induction proves that each derivative $z=(\partial^n v/\partial\varepsilon^n)(\varepsilon_0, \eta)$ also solves (8.23), and must, therefore, also be identically zero. This completes the proof and demonstrates the connection between our two series solutions.

Now, by analysis of the analyticity properties of the solutions to our earlier balance equations for the coefficients in the expansion (8.6) we deduce our final nonexistence result.

THEOREM 12. *Suppose* (8.5) *possesses a series solution* (8.6), *which is holomorphic, convergent on a region of the form*

$$|\varepsilon|^2<\left|\frac{\mu}{5\alpha}\right|+\kappa_0, \qquad |e^{-\eta}|<\kappa_1, \tag{8.24}$$

for $\kappa_0, \kappa_1>0$. *Then the equation necessarily satisfies the constraints* (6.8) *and thus has a one-parameter family of exact* sech^2 *solutions.*

Remark. The exact sech^2 solutions are clearly holomorphic in a region of the indicated form (8.24) provided κ_1 is chosen sufficiently small.

Proof. We begin by writing (8.5) in the more convenient form

$$\begin{aligned} &v''+\varepsilon^2\tilde{\alpha}v''''-(1+\varepsilon^2\tilde{\alpha})v \\ &\quad =-\tilde{q}(1+\varepsilon^2\tilde{\alpha})\{v^2+\varepsilon^2[\tilde{\beta}vv''+\tilde{\gamma}v'^2+(1+\varepsilon^2\tilde{\alpha})\tilde{q}\tilde{r}v^3]\}, \end{aligned} \tag{8.24$'$}$$

where

$$\tilde{\alpha}=\frac{\alpha}{\mu}, \quad \tilde{q}=\frac{q}{s}, \quad \tilde{\beta}=\frac{\beta}{q}, \quad \tilde{\gamma}=\frac{\gamma}{q}, \quad \tilde{r}=\frac{\mu r}{q^2}. \tag{8.25}$$

We substitute the expansion (8.21) into (8.24′) to compute the balance equations; cf. (7.4), (7.5), (7.6), for the coefficients a_i. The indicial equation, i.e., the terms in $e^{-\eta}$, are already balanced by design. The terms in $e^{-2\eta}$ lead to the equation

$$3(1+5\varepsilon^2\tilde{\alpha})a_2=-\tilde{q}(1+\varepsilon^2\tilde{\alpha})\{(1+5\varepsilon^2\tilde{\alpha})+\varepsilon^2(\tilde{\beta}+\tilde{\gamma}-5\tilde{\alpha})\}a_1^2. \tag{8.26}$$

Thus, a_2 will have poles at $\varepsilon^2=1/(5\tilde{\alpha})$, contradicting the hypothesis of the theorem, unless $\tilde{\beta}+\tilde{\gamma}=5\tilde{\alpha}$, which, in view of (8.25), is the same as the first condition in (6.8). Assuming this holds, and using (8.26) to solve for a_2, the remaining terms in $e^{-3\eta}$ lead to the further balance equation

$$\begin{aligned} &8(1+10\varepsilon^2\tilde{\alpha})a_3 \\ &\quad =\tfrac{2}{3}\tilde{q}^2(1+\varepsilon^2\tilde{\alpha})^2\{(1+10\varepsilon^2\tilde{\alpha})+\tfrac{1}{2}\varepsilon^2(5\tilde{\beta}+4\tilde{\gamma}-3\tilde{r}-20\tilde{\alpha})\}a_1^3. \end{aligned} \tag{8.27}$$

Thus, a_3 will have poles at $\varepsilon^2 = 1/(10\tilde{\alpha})$, unless $5\tilde{\beta}+4\tilde{\gamma}=3\tilde{r}+20\tilde{\alpha}$, which, in view of the previous condition reduces to $\tilde{\beta}=3\tilde{r}$, and, by (8.25) is the same as the second condition in (6.8); therefore, the expansion will be holomorphic in the indicated domain if and only if the conditions (6.8) hold and the equation admits exact sech^2 solutions. This completes the proof of Theorem 12.

The assumption of analyticity in Theorem 12 parallels that of [24]. It is likely that the constant $\mu/(5\alpha)$ in the domain (8.24) can be replaced by any positive constant $\varepsilon_0 > 0$, as the following argument plausibly indicates. Set, for simplicity,

$$a_1 = \frac{-6}{\tilde{q}(1+\varepsilon^2\tilde{\alpha})}.$$

Then the nth balance equation can, by a simple induction, be shown to take the form

$$(n^2-1)(1+(n^2+1)\varepsilon^2\tilde{\alpha})a_n = \frac{6n+\Phi_n}{\tilde{q}(1+\varepsilon^2\tilde{\alpha})}, \tag{8.28}$$

where each Φ_n is a rational function in ε, with poles at $\varepsilon^2 = -1/((k^2+1)\tilde{\alpha})$, for $k = 2, 3, \cdots, n-1$, and which vanishes identically if the sech^2 conditions (6.8) hold. In order that the expansion (8.6), and hence the a_i depend analytically on ε in some neighborhood of $\varepsilon = 0$, these coefficients cannot have complex poles accumulating at $\varepsilon = 0$. Thus, for n sufficiently large, each $\Phi_n + 6n$ must vanish at $\varepsilon^2 = -1/((n^2+1)\tilde{\alpha})$. This infinite collection of polynomial conditions seems highly unlikely in the absence of (6.8). Indeed, we can straightforwardly reduce the size of the domain (8.24) by an involved analysis of the first few of the rational functions Φ_n for n small, perhaps using MATHEMATICA, but we have not tried to implement this.

Note finally that the proof of Theorem 12 can be readily extended to include the case when $P(u)$ is an analytic function, in which case the hypotheses imply that $P(u)$ must be a cubic polynomial also. Indeed, by the above arguments, analyticity of (8.6) in a region (8.24) implies that not only the first three coefficients $p = p_1$, $q = p_2$, $r = p_3$, in the Taylor expansion of $P(u) = \sum p_n u^n$ satisfy (8.6), but, moreover, a simple induction will then show that all remaining coefficients must vanish if the poles in the general recursion relation (8.28) are to cancel, so that $p_n = 0$ for $n \geqq 4$. We leave the remaining details to the reader, and conclude this section by summarizing our basic nonexistence result in a convenient unscaled form.

THEOREM 13. *Consider an evolution equation of the form*

$$u_t + [\varepsilon\mu u_{xx} + \varepsilon^2(\alpha u_{xxxx} + \beta u u_{xx} + \gamma u_x^2) + P(u,\varepsilon)]_x = 0, \tag{8.29}$$

where ε is a small parameter, α, β, γ, μ are constants, and P is an analytic function of the form

$$P(u,\varepsilon) = pu + \varepsilon q u^2 + \varepsilon^2 r u^3 + \varepsilon^2 u^4 R(u,\varepsilon), \tag{8.30}$$

where p, q, r are constants, and R is analytic. Assume $q\mu \neq 0$, so that the $O(\varepsilon)$ terms are of Korteweg-deVries type. Then the model has a solitary wave solution of the form $u = u(x-ct, \varepsilon)$ with speed $c = p + \varepsilon^2 s + \cdots$, which has a formal expansion of the form

$$u = \varepsilon\varphi_0[\sqrt{\varepsilon}(x-ct)] + \varepsilon^3\varphi_1[\sqrt{\varepsilon}(x-ct)] + \varepsilon^5\varphi_2[\sqrt{\varepsilon}(x-ct)] + \cdots, \tag{8.31}$$

which reduces to the Korteweg-deVries soliton $\varphi_0(\eta) = \{(3s)/(2q)\}\operatorname{sech}^2\eta/2$ in the limit. Assume that the expansion (8.31) *converges to an analytic function in a complex domain of the form $|\varepsilon|^2 < |\mu/(5\alpha)| + \kappa$, $\kappa > 0$, $x - ct \gg 0$. Then, necessarily, $R = 0$; so $P(u,\varepsilon)$ is a cubic polynomial in u, and the coefficients of* (8.29), (8.30) *are related by the conditions*

$$(\beta+\gamma)\mu = 5q\alpha \quad \text{and} \quad 15\alpha r = \beta(\beta+\gamma), \tag{8.32}$$

which guarantee the existence of a one-parameter family of exact sech^2 *solitary wave solutions to the model.*

In summary, then, the models (2.1) which admit a one-parameter family of exact sech^2 solitary wave solutions are distinguished by the analyticity properties of their solutions. This result is in direct analogy with those of [24], in which the linear, sine-, and sinh-Gordon equations were distinguished among all one-dimensional Klein-Gordon equations by similar types of analyticity properties. However, our result is more revealing of the general method in that we no longer distinguish, by the smoothness properties of their solutions, just integrable equations, but rather those having particular explicit solutions. The method used here and in [24] is rather general, and is applicable to a wide variety of similar problems.

9. Conclusions and further work. We have been able to prove, under certain reasonable hypotheses, the nonexistence of solitary wave solutions to most fifth-order evolution equations that arise as models for nonlinear water waves. This is very strange, since most of the water wave models, except for the model (2.10) at the particular depth (2.13), where the Hamiltonian model is a fifth-order Korteweg-deVries equation, do not satisfy the requisite conditions (6.8) on the coefficients in the equation. Thus, by trying to do better in modeling real solitary water waves, which are known to exist [4], we, in a sense, do worse. The Korteweg-deVries model does have solitary wave (soliton) solutions that do a reasonably good job approximating solitary water waves [7], [8], [13]. But trying to get a more accurate model by retaining terms in ε^2 leaves us with *no* solitary wave solutions at all! Of course, this is not really an unequivocal problem since presumably the model does do a reasonable job approximating the solitary water waves for times on the order of $1/\varepsilon^2$ (the Kortweg-deVries model being accurate for times on the order of $1/\varepsilon$). Nevertheless, the results of this paper should give one pause in the noncritical application of naïve perturbation expansions as a means for deriving model equations.

This leads us to wonder about the following questions: what happens to initial conditions corresponding to solitary water waves as the time $t \to +\infty$? We expect that small amplitude waves decay by dispersion or radiation, whereas it is plausible that larger waves may even break. Is there a wave of maximal height? How do they behave under collision—specifically do they emerge unscathed as true solitons [33], or is there a small, but nonzero nonelastic effect, as in the BBM equation, [9]? It appears that there is a need for good numerical integration procedures to study these models in more detail. However, these must be long time accurate, and take into account exponentially small effects. For Hamiltonian models, some form of symplectic integrator [10] might be a good bet for investigating these questions. There is a lot of work remaining to be done in this direction.

Acknowledgments. It is a great pleasure to thank Jerry Bona, Stuart Hastings, Bryce McLeod, John Toland, and Bill Troy, all of whom made valuable comments on earlier attempts on this problem.

REFERENCES

[1] C. J. AMICK, L. E. FRAENKEL, AND J. F. TOLAND, *On the Stokes conjecture for the wave of extreme form*, Acta Math., 148 (1982), pp. 193–214.

[2] C. J. AMICK AND K. KIRCHGÄSSNER, *Solitary water waves in the presence of surface tension*, in Dynamical Problems in Continuum Physics, J. L. Bona et al., eds., Springer-Verlag, New York, 1987.

[3] C. J. AMICK AND J. B. MCLEOD, *A singular perturbation problem in water waves*, Stability and Appl. Anal. of Continuous Media, to appear.

[4] C. J. AMICK AND J. F. TOLAND, *On solitary waves of finite amplitude*, Arch. Rational Mech. Anal., 76 (1981), pp. 9–95.

[5] J. T. BEALE, *Exact solitary water waves with capillary ripples at infinity*, Comm. Pure Appl. Math., 44 (1991), pp. 211–257.

[6] D. J. BENNEY, *A general theory for interactions between short and long waves*, Stud. Appl. Math., 56 (1977), pp. 81–94.

[7] J. L. BONA, W. G. PRITCHARD, AND L. R. SCOTT, *A comparison of solutions of two model equations for long waves*, in Fluid Dynamics in Astrophysics and Geophysics, N. R. Lebovitz, ed., Lectures in Appl. Math., Vol. 20, American Mathematical Society, Providence, RI, 1983, pp. 235–267.

[8] ———, *An evaluation of a model equation for water waves*, Philos. Trans. Roy. Soc. London, 302 (1981), pp. 457–510.

[9] ———, *Solitary-wave interaction*, Phys. Fluids, 23 (1980), pp. 438–441.

[10] P. J. CHANNEL AND C. SCOVEL, *Symplectic integration of Hamiltonian systems*, Nonlinearity, 3 (1990), pp. 231–259.

[11] J. G. BYATT-SMITH, *On the existence of homoclinic and heteroclinic orbits for differential equations with a small parameter*, Nonlinearity, to appear.

[12] K. W. CHOW, *A second order solution for the solitary wave in a rotational flow*, Phys. Fluids, A1 (1989), pp. 1235–1239.

[13] W. CRAIG, *An existence theory for water waves, and the Boussinesq and Korteweg-deVries scaling limits*, Comm. in Partial Differential Equations, 10 (1985), pp. 787–1003.

[14] J. GEAR AND R. GRIMSHAW, *A second order theory for solitary waves in shallow fluids*, Phys. Fluids, 26 (1983), pp. 14–29.

[15] W. HEREMAN, P. P. BANERJEE, A. KORPEL, G. ASSANTO, A. VAN IMMERZEELE, AND A. MEERPOEL, *Exact solitary wave solutions of nonlinear evolution equations using a direct algebraic method*, J. Phys. A, 19 (1986), pp. 607–628.

[16] G-X. HUANG, S-Y. LUO, AND X-X. DAI, *Exact and explicit solitary wave solutions to a model equation for water waves*, Phys. Lett. A, 139 (1989), pp. 373–374.

[17] J. K. HUNTER AND J. SCHEURLE, *Existence of perturbed solitary wave solutions to a model equation for water waves*, Phys. D, 32 (1988), pp. 253–268.

[18] ———, *Nonexistence of solitary wave solutions of a model equation for water waves*, preprint, 1989.

[19] G. IOOSS AND J. KIRCHGÄSSNER, *Bifurcation d'ondes solitaires en présence d'une faible tension superficielle*, C.R. Acad. Sci. Paris, 311 (1990), pp. 265–268.

[20] K. KANO AND T. NAKAYAMA, *An exact solution of the wave equation* $u_t + uu_x - u_{(5x)} = 0$, J. Phys. Soc. Japan, 50 (1981), pp. 361–362.

[21] D. J. KAUP, *On the inverse scattering problem for cubic eigenvalue problems of the class* $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$, Stud. Appl. Math. 62 (1980), pp. 189–216.

[22] T. KAWAHARA, *Oscillatory solitary waves in dispersive media*, J. Phys. Soc. Japan, 33 (1972), pp. 260–264.

[23] S. KAWAMOTO, *Cusp soliton solutions of the Ito-type coupled nonlinear wave equation*, J. Phys. Soc. Japan, 53 (1984), pp. 1203–1205.

[24] S. KICHENASSAMY, *Breather solutions of the nonlinear wave equation*, Comm. Pure Appl. Math., 44 (1991), pp. 789–818.

[25] Y. KODAMA, *On integrable systems with higher order corrections*, Phys. Lett., 107A (1985), pp. 245–249.

[26] E. V. KRISHNAN, *An exact solution of the classical Boussinesq equation*, J. Phys. Soc. Japan, 51 (1982), pp. 2391–2392.

[27] M. KRUSKAL AND H. SEGUR, *Asymptotics beyond all orders in a model of dendritic crystals*, preprint, 1987.

[28] I. A. KUNIN, *Elastic Media with Microstructure* I, Springer-Verlag, New York, 1982.

[29] T. R. MARCHANT AND N. F. SMYTH, *The extended Korteweg-deVries equation and the resonant flow of a fluid over topography*, J. Fluid Mech., 221 (1990), pp. 263–288.

[30] Y. MATSUNO, *Properties of conservation laws of nonlinear evolution equations*, J. Phys. Soc. Japan, 59 (1990), pp. 3093–3100.

[31] H. NAGASHIMA, *Experiment on solitary waves in the nonlinear transmission line described by the equation* $\partial u/\partial\tau + u\partial u/\partial\xi - \partial^5 u/\partial\xi^5 = 0$, J. Phys. Soc. Japan, 47 (1979), pp. 1387–1388.

[32] H. NAGASHIMA AND M. KUWAHARA, *Computer simulation of solitary waves of the nonlinear wave equation* $u_t + uu_x - u_{(5x)} = 0$, J. Phys. Soc. Japan, 50 (1981), pp. 3792–3800.

[33] A. C. NEWELL, *Solitons in Mathematics and Physics*, CBMS-NSF Regional Conference Series in Applied Mathematics, Vol. 48, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1985.

[34] P. J. OLVER, *Hamiltonian perturbation theory and water waves*, Contemp. Math., 28 (1984), pp. 231–249.

[35] ———, *Hamiltonian and non-Hamiltonian models for water waves*, in Trends and Applications of Pure Mathematics to Mechanics, P. G. Ciarlet and M. Roseau, eds., Lecture Notes in Physics No. 195, Springer-Verlag, New York, 1984, pp. 273–290.

[36] P. J. OLVER, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Math. 107, Springer-Verlag, New York, 1986, pp. 140–142.

[37] G. PETIAU, *Sur des fonctions d'ondes d'un type nouveau, solutions d'équatons non linéaires généralisant l'équation des ondes de la mécanique ondulatoire*, Comptes Rendus Acad. Sci. Paris A-B, 244 (1957), pp. 1890–1893.

[38] ———, *Sur un généralisation non linéaire de la mécanique ondulatoire et les propriétés des fonctions d'onde correspondantes*, Nuovo Cimento, 9 (1958), pp. 542–568.

[39] G. PONCE, *Lax pairs and higher order models for water waves*, preprint, 1990.

[40] R. SACHS, *On the existence of small amplitude solitary waves with strong surface tension*, J. Differential Equations, 90 (1991), pp. 31–51.

[41] K. SAWADA AND T. KOTERA, *A method for finding N-soliton solutions of the K.d.V. equation and K.d.V.-like equation*, Progr. Theoret. Phys., 51 (1974), pp. 1355–1367.

[42] H. SEGUR, *Solitons and the inverse scattering transform*, Topics in Ocean Physics, 80 (1982), pp. 235–277.

[43] H. SEGUR AND M. KRUSKAL, *Nonexistence of small amplitude breather solutions in ϕ^4 theory*, Phys. Rev. Lett., 58 (1987), pp. 747–750.

[44] W. C. TROY, *Nonexistence of monotonic solutions in a model of dendritic growth*, Quart. Appl. Math., 48 (1990), pp. 209–215.

[45] M. WADATI, Y. H. ICHIGAWA, AND T. SHIMIZU, *Cusp soliton of a new integrable nonlinear evolution equation*, Progr. Theoret. Phys., 64 (1980), pp. 1959–1967.

[46] G. B. WHITHAM, *Linear and Non-linear Waves*, John Wiley, New York, 1974.

[47] Y. YAMAMOTO AND É. I. TAKIZAWA, *Solutions of nonlinear differential equation* $\partial u/\partial t + (45/2)\delta^2 u^2(\partial u/\partial x) - \partial^5 u/\partial x^5 = 0$, J. Phys. Soc. Japan, 50 (1981), pp. 1055–1056.

[48] ———, *On a solution of nonlinear time-evolution equation of fifth order*, J. Phys. Soc. Japan, 50 (1981), pp. 1421–1422.

[49] K. YOSHIMURA AND S. WATANABE, *Chaotic behaviour of nonlinear evolution equation with fifth order dispersion*, J. Phys. Soc. Japan, 51 (1982), pp. 3028–3035.

[50] V. E. ZAKHAROV, *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, J. Appl. Mech. Tech. Phys., 2 (1986), pp. 190–194.

[51] J. A. ZUFIRIA, *Symmetry breaking in periodic and solitary gravity-capillary waves on water of finite depth*, J. Fluid Mech, 184 (1987), pp. 183–206.