Entire solutions and a Liouville theorem for a class of parabolic equations on the real line

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Abstract. We consider a class of semilinear heat equations on $\mathbb{R}$, including in particular the Fujita equation

$$u_t = u_{xx} + |u|^{p-1}u, \quad x \in \mathbb{R}, \ t \in \mathbb{R},$$

where $p > 1$. We first give a simple proof and an extension of a Liouville theorem concerning entire solutions with finite zero number. Then we show that there is an infinite-dimensional set of entire solutions with infinite zero number.

Key words: Semilinear parabolic equations, entire solutions, Liouville theorems, zero number

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Contents

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1 Introduction

Consider the Fujita equation

$$u_t = u_{xx} + |u|^{p-1}u, \quad x \in \mathbb{R},\ t \in J, \tag{1.1}$$

where $p > 1$ and J is an open interval in $\mathbb{R}$. We are especially interested in the case $J = \mathbb{R}$; a solution of (1.1)—in this paper we only deal with classical solutions—is then called an *entire solution* of (1.1). The following Liouville-type theorem was proved in [2].

Theorem 1.1. *There exists no entire solution u of* (1.1) *such that for some $k \in \mathbb{N}$ one has $z(u(\cdot, t)) \le k$ for all $t \in \mathbb{R}$.*

Here z stands for the zero number functional: for a continuous function φ on $\mathbb{R}$, $z(\varphi)$ denotes the number (finite of infinite) of zeros of φ.

As discussed in detail in [2, 20], parabolic Liouville theorems, such as Theorem 1.1, are very useful in analysis of solutions of nonlinear parabolic equations. In combination with scaling and limiting arguments, they yield a priori bounds on global solutions and blowup rates for nonglobal solutions for a large class of semilinear equations whose nonlinearities have polynomial growth (see also [22] and references therein for numerous examples of such applications).

In Liouville theorems for multidimensional analogs of (1.1) (see [3, 19, 20, 21]), one typically considers positive solutions. The result concerning solutions with bounded zero number is relevant for one dimensional problems (or radial solutions in higher dimension). The reason for this is the monotonicity of the zero number along solutions. More specifically, consider a solution of

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R},\ t \in J, \tag{1.2}$$

where f is a locally Lipschitz function on $\mathbb{R}$ with $f(0) = 0$ and $J = (0, T)$ for some $T > 0$. If u satisfies the initial condition $u(\cdot, 0) = u_0$, where the function u_0 is sufficiently regular, say continuous and bounded, then $z(u(\cdot, t))$ is a nonincreasing function of $t \ge 0$. In particular, if $k := z(u_0) < \infty$, then

$z(u(\cdot,t) \le k$ for all $t > 0$ and any scaling procedures applied to the solution u lead to solutions with a bounded zero number.

In this note, we only consider bounded solutions of (1.1). We remark, however, that, as shown in [2], from Theorem 1.2 one can derive a similar Liouville theorem for radial solutions with bounded zero number of multidimensional equation

$$u_t - \Delta u = |u|^{p-1}u, \qquad x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{1.3}$$

with $1 < p < p_S$, where p_S is the critical Sobolev exponent:

$$p_S := \begin{cases} \dfrac{N+2}{N-2}, & \text{if } N \ge 3, \\ \infty, & \text{if } N \in \{1,2\}. \end{cases}$$

Also, a general argument given in [20] shows that once Theorem 1.2 is proved under the extra assumption that the solution u is bounded, a scaling-limiting procedure shows that the statement remains valid when the boundedness restriction is dropped.

Our first objective is to give a simple proof of Theorem 1.1 and extend the theorem to a broader class of equations (1.2). As we prove, the following condition on the nonlinearity is sufficient for the validity of the Liouville theorem:

$$uf(u) > 0 \quad (u \in \mathbb{R} \setminus \{0\}). \tag{1.4}$$

Theorem 1.2. *Assume that f is a locally Lipschitz function on $\mathbb{R}$ satisfying* (1.4). *Then there exists no bounded entire solution u of* (1.2) *such that for some $k \in \mathbb{N}$ one has $z(u(\cdot,t)) \le k$ for all $t \in \mathbb{R}$.*

Unlike Theorem 1.1, Theorem 1.2 is not valid in general without the boundedness assumption. For example, if f is globally Lipschitz, then the solution of the equation $\dot{\xi} = f(\xi)$ with $\xi(0) = 1$ is a positive entire (unbounded) solution of (1.2). We do not know if the theorem remains valid if one merely assumes that $z(u(\cdot,t))$ is finite for all t (and possibly unbounded as $t \to -\infty$).

Of course, the Liouville theorems 1.1, 1.2 are not valid without the assumption of the finiteness of the zero number. Periodic steady states are simple examples of nontrivial entire solutions with infinite zero number. Connecting orbits between steady states provide many more examples. While periodic steady states form just a two-parameter family, their initial data being the parameters, connecting orbits, as we show, constitute a set of infinite dimension. There are comprehensive studies of connecting orbits between steady states of one-dimensional parabolic problems on bounded intervals with Dirichlet or Neumann boundary conditions (see for example, [4, 8, 9]), and periodic boundary conditions can be related to these by the way of reflections. The structure of the connections in these problems is well understood if the nonlinearity f satisfies a dissipativity condition, requiring in particular the solutions of the Cauchy problem to be global and bounded. Superlinear equations like (1.1) lack the dissipativity, rending the results inapplicable, although a modification of the underlying techniques could still be useful. Here, we do not attempt to give a complete description of the connections; we investigate them just enough to be able to show that bounded entire solutions of (1.1) form a set of infinite Hausdorff dimension in the space $C_b(\mathbb{R}) := C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with the usual supremum norm. Namely, we prove the following result.

Theorem 1.3. *There is a set $M \subset C_b(\mathbb{R})$ with the following properties:*

(a) M consists of periodic functions and it contains C^1-submanifolds of $C_b(\mathbb{R})$ of arbitrarily large dimensions.

(b) For each $u_0 \in M$, there is a bounded entire solution $u(x,t)$ of (1.1) which is periodic in x and such that $u(\cdot, 0) = u_0$.

In addition to spatially periodic solutions, general equations of the form (1.2) are known to have other types of bounded entire solutions, such as traveling waves and other entire solutions constructed from traveling waves, see, for example, [6, 12, 17] and references therein. In equation (1.1), traveling waves do exist, but are unbounded (more comments on this can be found at the end of Section 4). We do not have a good general understanding of bounded entire solutions (with infinite zero number), even for the specific class of nonlinearities satisfying (1.4). In the following proposition, we just state a rather simple observation saying that, at least in a weak sense, each bounded entire solution of (1.2) connects two steady states.

Proposition 1.4. *Let f be a locally Lipschitz function. For every bounded entire solution u of (1.2) there exist steady states φ^-, φ^+ of (1.2) and sequences $t_n^\pm \in \mathbb{R}$, $n = 1, 2, \ldots$, such that $t_n^\pm \to \pm\infty$ and*

$$u(\cdot, t_n^-) \to \varphi^-, \qquad u(\cdot, t_n^+) \to \varphi^+ \tag{1.5}$$

in $L^\infty_{loc}(\mathbb{R})$ (that is, locally uniformly on $\mathbb{R}$). Moreover, if (1.4) *holds and $z(u(\cdot,t)) < \infty$ for some $t \in \mathbb{R}$, then necessarily $\varphi^+ \equiv 0$—in fact, $u(\cdot,t) \to 0$ in $L^\infty(\mathbb{R})$ (uniformly on $\mathbb{R}$) in this case.*

As we explain in more detail in Section 3, the existence of the limit steady states $\varphi^\pm$ follows from results of [10, 11] (note that condition (1.4) is not needed for this part of Proposition 1.4), while the conclusion on the uniform convergence to 0 follows from Theorem 1.2.

The rest of the paper is organized as follows. The proofs of Theorem 1.2 and Proposition 1.4 are given is Section 3. Theorem 1.3 is proved in Section 4. Section 2 contains preliminary material on steady states of (1.2), zero number, and various limit sets of bounded solutions of (1.2).

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2 Preliminaries

2.1 Steady states

The equation for the steady states of (1.2) is

$$u_{xx} + f(u) = 0, \quad x \in \mathbb{R}. \tag{2.1}$$

Under condition (1.4), the set of all solutions of (2.1) has a relatively simple structure, which we now describe (cp. Fig. 1).

The first order system corresponding to (2.1) is the Hamiltonian system

$$\begin{aligned} u' &= v, \\ v' &= -f(u). \end{aligned} \tag{2.2}$$

Its Hamiltonian is

$$H(u,v) := v^2/2 + F(u), \quad F(u) := \int_0^u f(s)\, ds.$$

Thus, every trajectory of (2.2) is contained in a level set of H. Note that these level sets are symmetric about the u-axis. Condition (1.4) implies that $(0,0)$ is the only equilibrium of (2.2), and it is the strict local minimum of the Hamiltonian. Therefore, all bounded orbits of (2.2) are periodic orbits (or, closed orbit). By the uniqueness of the equilibrium, every nonstationary periodic orbit contains the equilibrium $(0,0)$ in its interior domain (when the orbit is viewed as a Jordan curve). This in particular implies that every periodic solution of (2.1) has infinitely many zeros. Also, the equilibrium $(0,0)$ is a center: it has a neighborhood covered by periodic orbits of (2.2).

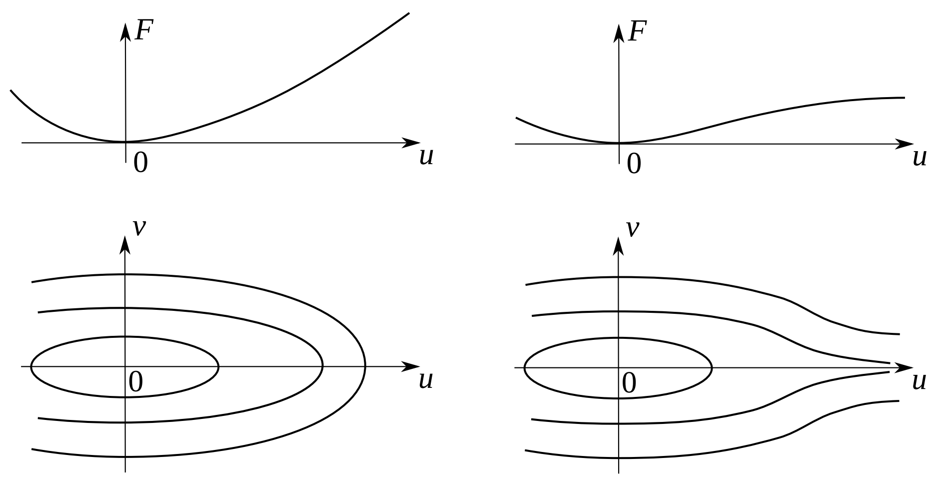

Figure 1: The graph of F and orbits of system (2.2). When $F(u)$ has infinite limits as $u \to \pm\infty$ (the figures on the left), all orbits of (2.2) are periodic. If $F(\infty) < \infty$ (the figures of the right) or $F(-\infty) > -\infty$, some orbits are unbounded.

For any C^1-function φ defined on an interval I we denote

$$\tau(\varphi) := \{(\varphi(x), \varphi'(x)) : x \in I\}. \tag{2.3}$$

If φ is a solution of (2.1), $\tau(\varphi)$ is the usual trajectory of the planar system (2.2).

For $\alpha \in \mathbb{R}$, let ϕ_α be the (maximally defined) solution of (2.1) with

$$\phi_\alpha(0) = 0, \quad \phi'_\alpha(0) = \alpha. \tag{2.4}$$

Lemma 2.1. *Assume* (1.4). *Then all solutions of* (2.1) *are global (their maximal existence interval is* $(-\infty, \infty)$*) and the following relations are valid:*

$$\bigcup_{\alpha\in\mathbb{R}} \tau(\phi_\alpha) = \mathbb{R}^2, \tag{2.5}$$

$$\lim_{|\alpha|\to\infty} \inf\{|(\xi,\eta)| : (\xi,\eta) \in \tau(\phi_\alpha)\} = \infty. \tag{2.6}$$

Proof. Let $u \not\equiv 0$ be a maximally defined solution of (2.1). Consider the corresponding trajectory $\tau(u)$ of system (2.2). Since the equilibrium $(0,0)$ is a center, $\tau(u)$ is disjoint from a neighborhood of $(0,0)$. If $\tau(u)$ intersects the u-axis at two different points, then, by the symmetry about the u-axis, $\tau(u)$ is a closed orbit and u is a periodic solution. Otherwise, u has at most one critical point. By (2.1) and (1.4), we have $u_{xx}(x) < 0$ when $u(x) > 0$, and $u_{xx}(x) > 0$ when $u(x) < 0$. This implies that u_x is bounded. Hence, (u, u_x) cannot blow up at any finite x, which means that the solution u is global.

To prove (2.5), take an arbitrary $(\xi,\eta) \in \mathbb{R}^2$ and let u be the (maximally defined) solution of (2.1) with $(u(0), u'(0)) = (\xi,\eta)$. We show that u has a zero, so that, for suitable $\alpha \in \mathbb{R}$,

$$\tau(\phi_\alpha) = \tau(u) \ni (\xi,\eta),$$

proving the desired conclusion. Suppose u has no zero; for definiteness suppose that $u > 0$ everywhere (the case $u < 0$ is analogous). Then, by (1.4), $u_{xx} = -f(u) < 0$; thus u is a strictly concave positive function on $\mathbb{R}$, which is absurd.

For the proof of (2.6), we use the fact that the Hamiltonian H is constant on the trajectory $\tau(\phi_\alpha)$. Thus, if $(\xi_\alpha, \eta_\alpha) \in \tau(\phi_\alpha)$ and $|\alpha| \to \infty$, then

$$H(\xi_\alpha, \eta_\alpha) = H(0, \alpha) = \alpha^2/2 \to \infty.$$

This and the continuity of H imply that $\{(\xi_\alpha, \eta_\alpha)\}_\alpha$ can have no finite accumulation point, proving (2.6). □

2.2 Limit sets

If u is a bounded solution of (1.2) on $J = (0, \infty)$, we consider two types of limit sets of u: the ω-limit set, defined by

$$\omega(u) := \{\varphi : u(\cdot, t_n) \to \varphi \text{ for some sequence } t_n \to \infty\}, \tag{2.7}$$

and the Ω-limit set, defined by

$$\Omega(u) := \{\varphi : u(\cdot + x_n, t_n) \to \varphi \text{ for some sequences } t_n \to \infty \text{ and } x_n \in \mathbb{R}\}. \tag{2.8}$$

The convergence in both these definitions is in $L^\infty_{loc}(\mathbb{R})$ (the locally uniform convergence).

The following results are standard consequences of the boundedness of u and parabolic regularity estimates (see [18], for example): if B is any of the sets $\omega(u)$, $\Omega(u)$, then B is a nonempty, compact, connected subset of $L^\infty_{loc}(\mathbb{R})$. Also, it is invariant in the following sense. If $\varphi \in B$, then there is an entire solution U of (1.2) such that $U(\cdot, 0) = \varphi$ and $U(\cdot, t) \in B$ for all $t \in \mathbb{R}$. Moreover, in (2.7), (2.8), one can take the convergence in $C^1_{loc}(\mathbb{R})$ and $\omega(u)$, $\Omega(u)$ are compact and connected in that space as well. Note also that $\Omega(u)$ is translation-invariant: with each $\varphi \in \Omega(u)$, $\Omega(u)$ contains all the translations $\varphi(\cdot + \xi)$, $\xi \in \mathbb{R}$, of φ. This and the compactness of $\Omega(u)$ imply that the set $\tau(\Omega(u))$ is compact in $\mathbb{R}^2$. Here and below, if $B \subset C^1(\mathbb{R})$, the set $\tau(B)$ is defined by

$$\tau(B) := \bigcup_{\varphi \in B} \tau(\varphi) \tag{2.9}$$

(with $\tau(\varphi)$ as in (2.3)). We remark that the set $\tau(\omega(u)) \subset \mathbb{R}^2$ is bounded, but it may not be closed.

The α-limit set of a bounded solution u of (1.2) on $J = (-\infty, 0)$ is defined analogously to $\omega(u)$:

$$\alpha(u) := \{\varphi : u(\cdot, t_n) \to \varphi \text{ for some sequence } t_n \to -\infty\}, \tag{2.10}$$

and it, too, is nonempty, compact, connected in $L^\infty_{loc}(\mathbb{R})$, and invariant in the same sense as above.

2.3 Zero number

Here we consider solutions of the linear equation

$$v_t = v_{xx} + a(x,t)v, \quad x \in \mathbb{R},\ t \in (s, T), \tag{2.11}$$

where $-\infty < s < T \leq \infty$, a is a bounded measurable function on $\mathbb{R} \times [s, T)$. Such an equation is solved by $v = u - \bar{u}$ if u, $\bar{u}$ are bounded solutions of the nonlinear equation (1.2); if $f(0) = 0$ one can take $\bar{u} \equiv 0$.

We recall the following theorem regarding the zero number $z(v(\cdot, t))$ (see [1, 5]).

Lemma 2.2. *Let $v \in C(\mathbb{R} \times [s, T))$ be a nontrivial solution of* (2.11) *on $\mathbb{R} \times (s, T)$. Then the following statements hold true:*

(i) *For each $t \in (s, T)$, all zeros of $v(\cdot, t)$ are isolated.*

(ii) *$t \mapsto z(v(\cdot, t))$ is a monotone nonincreasing function on $[s, T)$ with values in $\mathbb{N} \cup \{0\} \cup \{\infty\}$.*

(iii) *If for some $t_0 \in (s, T)$, the function $v(\cdot, t_0)$ has a multiple zero and $z(v(\cdot, t_0)) < \infty$, then for any $t_1, t_2 \in (s, T)$ with $t_1 < t_0 < t_2$ one has*

$$z(v(\cdot, t_1)) > z(v(\cdot, t_2)). \tag{2.12}$$

Thus, the zero number drops whenever a multiple zero occurs. In fact, the change in the zero number occurs locally near multiple zeros [1, 5]. This and the implicit function theorem imply the following result on the robustness of multiple zeros in solutions of (2.11) (see [7, Lemma 2.6] for details).

Lemma 2.3. *Assume that v is a nontrivial solution of* (2.11) *such that for some $s_0 \in (s, T)$, $x_0 \in \mathbb{R}$ one has $v(x_0, s_0) = v_x(x_0, s_0) = 0$. Assume further that for some $\delta, \epsilon > 0$, v_n is a sequence in $C^1([x_0 - \delta, x_0 + \delta] \times [s_0 - \epsilon, s_0 + \epsilon])$ which converges in this space to the function v. Then for all sufficiently large n the function $v_n(\cdot, t)$ has a multiple zero in $(x_0 - \delta, x_0 + \delta)$ for some $t \in (s_0 - \epsilon, s_0 + \epsilon)$.*

3 Proof of Theorem 1.2 and Proposition 1.4

In our proof of Theorem 1.2 the following Squeezing Lemma is crucial. Consider a bounded subset B of $C^1_b(\mathbb{R})$ which is *invariant* for (1.2): for each $\varphi \in B$ there is an entire solution u of (1.2) such that $u(\cdot, 0) = \varphi$ and $u(\cdot, t) \in B$ for all $t \in \mathbb{R}$. For example, B can be the orbit $\{u(\cdot, t) : t \in \mathbb{R}\}$ of a single bounded entire solution, or it can be equal to one of the limit sets $\omega(u)$, $\Omega(u)$ (or $\alpha(u)$) for some bounded solution u of equation (1.2) on $J = (0, \infty)$ (on $J = (-\infty, 0)$, in the case of $\alpha(u)$).

Let $\tau(B)$ be as in (2.9). Since B is a bounded subset of $C^1_b(\mathbb{R})$, $\tau(B)$ is a bounded subset of $\mathbb{R}^2$.

Lemma 3.1. *Let f be a locally Lipschitz function satisfying* (1.4). *Let $B \subset C^1_b(\mathbb{R})$ be a bounded set which is invariant for* (1.2). *Then the following statements are valid:*

(i) There is $\alpha \geq 0$ *such that*

$$\overline{\tau(B)} \cap (\tau(\phi_\alpha) \cup \tau(\phi_{-\alpha})) \neq \emptyset. \tag{3.1}$$

(ii) For all sufficiently large $\alpha > 0$, (3.1) *is not true (the intersection is empty).*

(iii) Set

$$\beta = \sup\{\alpha \geq 0 : (3.1) \text{ holds}\}. \tag{3.2}$$

Then the solution ϕ_β *is periodic and there are sequences* $\varphi_n \in B$ *and* $\xi_n \in \mathbb{R}$ *such that*

$$\varphi_n(\cdot + \xi_n) \to \phi_\beta \text{ in } C^1_{loc}(\mathbb{R}). \tag{3.3}$$

Moreover, if the set $\tau(B)$ *is closed in* $\mathbb{R}^2$*, then there is* $x_0 \in \mathbb{R}$ *such that* $\phi_\beta(\cdot - x_0) \in B$.

Proof. Since the closure $\overline{\tau(B)}$ is a nonempty compact set in $\mathbb{R}^2$, statements (i), (ii) follow directly from Lemma 2.1.

Let now β be as in (3.2). Using the compactness of $\overline{\tau(B)}$ and the continuity of the solutions of (2.2) with respect to initial data, one shows easily that (3.1) holds for $\alpha = \beta$. Therefore, either $\overline{\tau(B)} \cap \tau(\phi_\beta) \neq \emptyset$ or this holds with β replaced by $-\beta$. We assume the former, the latter being analogous (and in the end it turns out that $\tau(\phi_\beta) = \tau(\phi_{-\beta})$). Thus, there are sequences $\varphi_n \in B$, $x_n \in \mathbb{R}$, such that

$$(\varphi_n(x_n), \varphi_n'(x_n)) \to (\phi_\beta(x_0), \phi_\beta'(x_0)) \tag{3.4}$$

for some $x_0 \in \mathbb{R}$. By the invariance of B, for $n = 1, 2, \ldots$ there is an entire solution u_n of (1.2) such that $u_n(\cdot, 0) \equiv \varphi_n$ and $u_n(\cdot, t) \in B$ for all t. Using the boundedness of $B \subset C^1_b(\mathbb{R})$ and parabolic regularity estimates, we obtain, passing to subsequences if necessary, that

$$u_n(\cdot + x_n, \cdot) \to \tilde{u} \text{ in } C^1_{loc}(\mathbb{R}), \tag{3.5}$$

where $\tilde{u}$ is a bounded entire solution of (1.2). By (3.4),

$$\tilde{u}(0,0) = \phi_\beta(x_0), \quad \tilde{u}_x(0,0) = \phi_\beta'(x_0). \tag{3.6}$$

We claim that $v := \tilde{u} - \phi_\beta(\cdot + x_0) \equiv 0$. Indeed, assume that $v \not\equiv 0$. In view of (3.6), $v(\cdot, 0)$ has a multiple zero at $x = 0$. By Lemma 2.3, if $\alpha > \beta$ is sufficiently close to β, then there are y_0, t_0 (near 0) such that

$$\tilde{u}(y_0, t_0) = \phi_\alpha(y_0 + x_0), \quad \tilde{u}_x(y_0, t_0) = \phi_\alpha'(y_0 + x_0). \tag{3.7}$$

Now, from (3.5) and (3.7) it follows that

$$(\xi_n, \eta_n) := (u_n(y_0 + x_n, t_0), u_{n,x}(y_0 + x_n, t_0)) \to (\phi_\alpha(x_0 + y_0), \phi'_\alpha(x_0 + y_0)). \tag{3.8}$$

Since $u_n(\cdot, t_0) \in B$, we have $(\xi_n, \eta_n) \in \tau(B)$. Thus (3.8) shows that (3.1) holds, which contradicts the definition of β.

The identity $\tilde{u} \equiv \phi_\beta(\cdot + x_0)$ in particular implies that ϕ_β is bounded, hence it is a periodic solution of (2.1) (and $\tau(\phi_\beta) = \tau(\phi_{-\beta})$ by symmetry). By (3.5), (3.3) holds with $\xi_n := x_n - x_0$ (and $\varphi_n = u_n(\cdot, 0)$ as above).

Finally, assume that $\overline{\tau(B)} = \tau(B)$. In this case, one can take φ_n and x_n independent of n: $\varphi_n = \varphi$, $x_n = x_1$. Then $u := u_n$ is independent of n as well and $\phi_\beta(\cdot + x_0) \equiv \tilde{u} \equiv u(\cdot + x_1, \cdot)$. In particular, $\phi_\beta(\cdot + x_0 - x_1) = u(\cdot, 0) \in B$. □

Remark 3.2. Clearly, variants of Lemma 3.1 can be proved under different conditions than (1.4). For example, if (1.4) is only assumed to hold for sufficiently small $|u|$; then the statements of the lemma are valid provided one assumes that the invariant set B is contained in a sufficiently small neighborhood of the origin in $C^1_b(\mathbb{R})$.

Theorem 1.2 is a rather simple consequence of the Squeezing Lemma.

Proof of Theorem 1.2. Let f be a locally Lipschitz function satisfying (1.4). Assume that there is a bounded entire solution u of (1.2) such that for some $k \in \mathbb{N}$ one has $z(u(\cdot, t)) \le k$ for all $t \in \mathbb{R}$.

We apply Lemma 3.1 with $B := \{u(\cdot, t) : t \in \mathbb{R}\}$. This is a legitimate choice, for the boundedness of u and parabolic estimates give the boundedness of B in $C^1_b(\mathbb{R})$. Let β be as in (3.2). Obviously, $\beta = 0$ would give $u \equiv 0$, which is absurd due to the boundedness of the zero number. Thus, $\beta > 0$ and ϕ_β is a nonstationary periodic solution. Therefore, ϕ_β has infinitely many zeros. Of course, all of these zeros have to be simple due to the uniqueness for the initial value problems for (2.1).

Now, with our choice of the set B, for each of the functions φ_n in (3.3) one has $\varphi_n = u(\cdot, t_n)$ for some $t_n \in \mathbb{R}$. From (3.3) we now infer that $z(u(\cdot, t_n)) \to \infty$ as $n \to \infty$, in contradiction to the boundedness assumption. This contradiction rules out the existence of any entire solution u with the indicated property. □

Next we prove Proposition 1.4.

Proof of Proposition 1.4. Let u be a bounded entire solution of (1.2). A result of [10] (see also [11]) shows that $\omega(u)$ contains a steady state φ_+. Next, using the invariance of $\alpha(u)$, which is a nonempty compact subset of $C_b(\mathbb{R})$, we find a bounded solution $\tilde{u}$ of (1.2) with $\omega(\tilde{u}) \subset \alpha(u)$. Using [10] again, we find $\varphi^- \in \omega(\tilde{u}) \subset \alpha(u)$. The definitions of $\alpha(u)$, $\omega(u)$, imply that (1.5) holds for some sequences $\{t_n^\pm\}$.

Next assume that $z(u(\cdot,t))$ is finite for some (hence any) sufficiently large t. We apply Lemma 3.1 with $B = \Omega(u)$. If the periodic solution ϕ_β as in Lemma 3.1(iii) is nontrivial, we obtain a contradiction with the finiteness of $z(u(\cdot,t))$ as in the proof of Theorem 1.2. Thus, $\Omega(u) = \{0\}$, which clearly implies that $u(\cdot,t) \to 0$ in $L^\infty(\mathbb{R})$, as stated in Proposition 1.4. (Alternatively, in this part of the proof, one could apply Theorem 1.2 to the entire solutions in $\Omega(u)$.) $\square$

4 Entire solutions with infinite zero number

In this section, we take $f(u) := |u|^{p-1}u$. Using the symmetries and scaling invariance of (2.1)—$\lambda^{2/(p-1)}u(\lambda x)$ is a solution if $u(x)$ is—we can relate all the steady state solutions ϕ_α, $\alpha \in \mathbb{R}$, to ϕ_1 as follows: for each $\alpha \geq 0$ we have

$$\phi_\alpha(-x) = \phi_\alpha(x) = \alpha^{2/(p+1)}\phi_1(\alpha^{(p-1)/(p+1)}x) \quad (x \in \mathbb{R}), \qquad \phi_{-\alpha} = -\phi_\alpha. \tag{4.1}$$

In particular, all the ϕ_α are periodic solutions.

Proof of Theorem 1.3. Pick (the unique) $\alpha_0 > 0$ such that $\alpha_0^{(p-1)/(p+1)}$ is the first positive zero of ϕ_1 and set $\varphi := \phi_{\alpha_0}$. Then φ satisfies $\varphi(0) = 0 = \varphi(1)$ and $\varphi > 0$ in $(0,1)$. We first find a basic connecting orbit from φ to 0.

For a while, view φ as a steady state of the Dirichlet problem

$$\begin{aligned} u_t = u_{xx} + f(u), &\qquad x \in (0,1),\ t > 0; \\ u(0,t) = u(1,t) = 0, &\qquad t > 0. \end{aligned} \tag{4.2}$$

It follows from (4.1) that there are no steady states of this problem between φ and 0. Moreover, since $f'(0) = 0$, the trivial steady state is asymptotically (exponentially) stable for (4.2). In this situation, it is well known (see, for example, [16]) that there is a heteroclinic solution $\hat{u}$ from φ to 0. More precisely, $\hat{u}$ is an entire solution of (4.2) such that $0 < \hat{u} < \varphi$ and $\hat{u}_t < 0$ in $(0,1) \times \mathbb{R}$, and one has the following convergence properties in $C^1[0,1]$:

$$\lim_{t \to -\infty} \hat{u}(\cdot, t) = \varphi, \quad \lim_{t \to \infty} \hat{u}(\cdot, t) = 0. \tag{4.3}$$

We already know from the condition $f'(0) = 0$ that the convergence to 0 is exponential; we claim that the same is true for the convergence to φ. Since $\hat{u}_t < 0$, the rate of convergence of $\hat{u}(\cdot, t)$ to φ is determined by the principal eigenvalue of

$$v_{xx} + f'(\varphi(x))v + \mu v = 0, \quad x \in (0,1), \tag{4.4}$$

$$v(0) = v(1) = 0. \tag{4.5}$$

More specifically, the rate is exponential if the principal eigenvalue is not equal to zero (then it has to be negative, due to the instability of the steady state φ). We prove this by contradiction. Suppose the principal eigenvalue is equal to zero. Let v be the principal eigenfunction normalized by the condition $v'(0) = 1$ (for example). Consider now the function

$$w(x) := \frac{\partial}{\partial \alpha} \phi_\alpha(x)|_{\alpha=\alpha_0} = \frac{2}{p+1}\alpha_0^{-1}\varphi(x) + \frac{p-1}{p+1}\alpha_0^{-1}x\varphi'(x). \tag{4.6}$$

It satisfies the same linear equation as v—namely, equation (4.4) with $\mu = 0$—and $w(0) = 0$. Therefore, w is a scalar multiple of v. This and (4.5) give $w(1) = 0$. However, since $\varphi(1) = 0$ and $\varphi'(1) < 0$ (because φ is a nontrivial solution of (2.1)), (4.6) shows that $w(1) \neq 0$. This contradiction verifies that the principal eigenvalue of (4.4) is negative and the convergence in (4.3) is exponential in both cases.

For each $t \in \mathbb{R}$, we now take the odd extension and then the periodic extension of $\hat{u}(\cdot, t)$, denoting the resulting spatially periodic function by the same symbol $\hat{u}$. Using the fact that the nonlinearity $f(u) = |u|^{p-1}u$ is odd, one shows easily that $\hat{u}$ is an entire solution of (1.1). This solution and the steady state φ are 2-periodic in x, and they vanish at all $x \in \mathbb{Z}$. Moreover, $\hat{u}$ satisfies (4.3), where the convergence is in $C_b(\mathbb{R})$ and it is exponential in both cases.

Using the basic connection $\hat{u}$, we now establish the existence of many other connections from φ to 0. Take any positive integer n and consider $\hat{u}$ as an entire solution, connecting φ to 0, of the following Dirichlet problem:

$$\begin{aligned} u_t &= u_{xx} + f(u), && x \in (0,n),\ t > 0; \\ u(0,t) &= u(n,t) = 0, && t > 0. \end{aligned} \tag{4.7}$$

For this problem, too, 0 is an asymptotically stable steady state. On the other hand, φ is an unstable steady state of (4.7) with Morse index $m_n \geq n-1$. Here, the Morse index m_n refers to the number of negative eigenvalues of the eigenvalue problem

$$\begin{gathered} v_{xx} + f'(\varphi(x))v + \mu v = 0, \quad x \in (0,n), \\ v(0) = v(n) = 0. \end{gathered} \tag{4.8}$$

The relation $m_n \geq n-1$ follows, via standard Sturmian arguments, from the fact that φ' is a solution of (4.8) with $\mu = 0$ and it has n zeros in $[0,n]$.

Thus, φ, as a steady state of (4.7), has an m_n-dimensional (local) unstable manifold W_n^u (cp. [13, 14]). This is a C^1-submanifold of $C[0,n]$ with the following property. For each $u_0 \in W_n^u$ there is a solution of $u_t = u_{xx} + f(u)$ on $[0,n] \times (-\infty, 0]$ (*ancient solution*) satisfying Dirichlet boundary conditions such that $u(\cdot,0) = u_0$ and, as $t \to -\infty$, $u(\cdot,t) \to \varphi$ in $C[0,n]$, with exponential convergence. Also, from the fact that $\hat{u}$ approaches φ exponentially as $t \to -\infty$ it follows that $\hat{u}(\cdot,t_0)|_{[0,n]} \in W_n^u$ for all sufficiently large negative t_0. Fix any such t_0. Since $\hat{u}(\cdot,t) \to 0$ as $t \to \infty$, using the continuous dependence of the solutions of (4.7) on the initial data and the asymptotic stability of the trivial steady state, we find a neighborhood $\mathcal{U}_n$ of $\hat{u}(\cdot,t_0)|_{[0,n]}$ in W_n^u with the following property. For each $u_0 \in \mathcal{U}_n$, the solution u of (4.7) with $u(\cdot,0) = u_0$ approaches 0 as $t \to \infty$. Of course, since $\mathcal{U}_n \subset W_n^u$, the solution u extends to an entire solution (4.7) connecting φ and 0. Now, similarly as above with the solution of (4.2), we take the odd extensions and then the periodic extensions of these heteroclinic solutions. This yields a C^1-submanifold $M_n \subset C_b(\mathbb{R})$ (consisting of the extensions of the functions $u_0 \in \mathcal{U}_n$) of dimension m_n such that for each $\bar{u}_0 \in M_n$ there is an entire solution u of (1.1) satisfying $u(\cdot,0) = \bar{u}_0$ and

$$\lim_{t\to-\infty} u(\cdot,t) = \varphi, \quad \lim_{t\to\infty} u(\cdot,t) = 0,$$

with the convergence in $C_b(\mathbb{R})$. The entire solutions u constructed this way are odd and $2n$-periodic in x. Hence, we see that the conclusion of Theorem 1.3 holds with $M := \cup_n M_n$. □

Using similar arguments as in the proof of Theorem 1.3 and universal a priori estimates for global solutions of (4.7), one can show the existence of spatially periodic connections between nontrivial steady states of (1.1). For this, consider the unstable manifold W_n^u as in the above proof. Take the intersection of W_n^u with the boundary of the domain of attraction of 0 (0 is an asymptotically stable steady state of (4.7)). For any $u_0 \neq \varphi$ in this intersection, the solution u of (4.7) with $u(\cdot, 0) = u_0$ is global and bounded. This follows from universal a priori estimates for the solutions *in* the domain of attraction of 0, as given in [2] (see [2, Section 6] for more details on how such universal estimates can be applied in order to estimate solutions on the boundary of the domain of attraction). Standard convergence results for the Dirichlet problem (4.7) (see [15, 23]) then imply that $u(\cdot, t)$ approaches a steady state as $t \to \infty$. This steady state is nontrivial, for u_0 is not in the domain of attraction of 0. Since $u_0 \in W_n^u$, we obtain a heteroclinic solution between two nontrivial steady states of (4.7). Taking odd and periodic extensions of this solution, as in the proof above, we obtain a spatially periodic entire solution of (4.7) connecting two nontrivial steady states.

In conclusion of this section, we discuss briefly traveling waves of (1.1). A traveling wave is an entire solution of the form $u(x, t) = \psi(x - ct)$ for some $c \neq 0$. The equation for ψ, which guarantees that u is a solution of (1.1), is

$$\psi_{xx} + c\psi_x + |\psi|^{p-1}\psi = 0. \tag{4.9}$$

It can be proved—we omit all technical details in this remark—that for any $c \neq 0$ all solutions of this ordinary differential equation are global, meaning that their maximal existence interval is $(-\infty, \infty)$. It can further be shown that, with the exception of the trivial solution, these solutions are not periodic, in fact they are unbounded, and have infinitely many zeros. Thus, although traveling waves yield examples of entire solutions of (1.1) with infinite zero number which are not spatially periodic, they are unbounded at all times.

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