Dynamics of nonnegative solutions of one-dimensional reaction-diffusion equations with localized initial data. Part II: Generic nonlinearities

H. Matano

Meiji Institute for Advanced Study of Mathematical Sciences,

Meiji University, 4-21-1 Nakano, Tokyo 164-8525

P. Poláčik*

School of Mathematics, University of Minnesota,

Minneapolis, MN 55455

Abstract

Abstract. We consider the Cauchy problem

$$\begin{aligned} &u_t = u_{xx} + f(u), && x \in \mathbb{R},\ t > 0, \\ &u(x,0) = u_0(x), && x \in \mathbb{R}, \end{aligned}$$

where f is a C^1 function on $\mathbb{R}$ with $f(0) = 0$, and u_0 is a nonnegative continuous function on $\mathbb{R}$ whose limits at $\pm\infty$ are equal to 0. Assuming that the solution u is bounded, we study its asymptotic behavior as $t \to \infty$. In the first part of this study, we proved a general quasiconvergence result: as $t \to \infty$, the solution approaches a set of steady states in the topology of $L^\infty_{loc}(\mathbb{R})$. In this paper, we show that under certain generic, explicitly formulated conditions

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on the nonlinearity f, the solution necessarily converges to a single steady state φ in $L^\infty_{loc}(\mathbb{R})$. Then, under the same conditions, we describe the *global* asymptotic shape of the solution: the graph of $u(\cdot,t)$ has a top part close to the graph of φ and two sides taking shapes of "terraces" moving in the opposite directions with precisely determined speeds.

Key words: Parabolic equations on $\mathbb{R}$, localized initial data, convergence, quasiconvergence, traveling fronts, propagating terraces

AMS Classification: 35K15, 35K57, 35B40

Contents

1 Introduction

In this paper, we continue our study, initiated in [15], of the Cauchy problem

$$u_t = u_{xx} + f(u), \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

where f is a locally Lipschitz function on $\mathbb{R}$ with $f(0) = 0$, and u_0 is a nonnegative function in $C_0(\mathbb{R})$. Here and below $C_0(\mathbb{R})$ stands for the space of continuous functions on $\mathbb{R}$ whose limits at $\pm\infty$ exist and are both equal to 0.

For any u_0 in $C_0(\mathbb{R})$, or, more generally in $C_b(\mathbb{R}) := C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we denote by $u(\cdot, t, u_0)$ the unique solution of (1.1), (1.2) and by $T(u_0) \in (0, \infty]$ its maximal existence time. In this paper, a solution always refers to a classical solution which is bounded on $\mathbb{R} \times J$ for any compact subinterval J of its existence interval. If u is bounded on $\mathbb{R} \times [0, T(u_0))$, then necessarily $T(u_0) = \infty$, that is, the solution is *global*. We examine the large-time behavior of bounded solutions from several points of view.

Our first goal has been to describe the large time behavior of bounded solutions in a localized topology. For that aim, we define the ω-limit set of a bounded solution u as follows:

$$\omega(u) := \{\varphi : u(\cdot, t_n) \to \varphi \text{ for some sequence } t_n \to \infty\}. \tag{1.3}$$

Here the convergence is in $L^\infty_{loc}(\mathbb{R})$, that is, the locally uniform convergence. By standard parabolic regularity estimates, the trajectory $\{u(\cdot, t) : t \geq 1\}$ of the bounded solution u is relatively compact in $L^\infty_{loc}(\mathbb{R})$. This implies that $\omega(u)$ is nonempty, compact, and connected in $L^\infty_{loc}(\mathbb{R})$. Moreover, it attracts the solution in the following sense

$$\operatorname{dist}_{L^\infty_{loc}(\mathbb{R})}(u(\cdot, t), \omega(u)) \to 0 \text{ as } t \to \infty. \tag{1.4}$$

The distance is taken with respect to a metric of $L^\infty_{loc}(\mathbb{R})$ (which is a metrizable locally convex space); but, due to parabolic regularity, one can also take the distance in $C^2_{loc}(\mathbb{R})$. If $u = u(\cdot, \cdot, u_0)$, we also use the symbol $\omega(u_0)$ for $\omega(u)$.

We proved in [15] that under the current assumptions on u_0, namely, $u_0 \in C_0(\mathbb{R})$ and $u_0 \geq 0$, if the solution $u(\cdot, \cdot, u_0)$ is bounded, then it is *quasiconvergent*, by which we mean that $\omega(u)$ consists of steady states of

(1.1). We remark that this result is not true in general if one merely assumes that $u_0 \in C_0(\mathbb{R})$ (see [19] for counterexamples). On a related note, if the function u_0 has limits at $\pm\infty$, but the limits are distinct, then the solution $u(\cdot,\cdot,u_0)$ is quasiconvergent if bounded, with no additional conditions on f or u_0 (see [18]).

It is a natural question if, or under what conditions, the quasiconvergence result can be improved to the convergence of the solution to a single steady state. As proved in [7], an example of such a condition is that—in addition to $u_0 \geq 0$—u_0 has compact support. In [15], we gave several more general sufficient conditions, all formulated in terms of u_0 (see also the survey paper [20] for an overview of other convergence results for (1.1) and references). Our first goal in this paper is to clarify if the convergence of bounded solutions can be guaranteed by some sort of generic conditions on the nonlinearity, assuming no extra condition on u_0. This is not a trivial question—unlike, for instance, in the case of equations on bounded intervals—as nonconstant steady states of (1.1) occur in continua, regardless of the assumptions on f. The conditions we impose on f are generic in suitable spaces of C^1-functions, but are also explicitly formulated (see Section 2.1). If $f'(0) < 0$ (and the generic conditions are in effect), we prove that for any $u_0 \in C_0(\mathbb{R})$ with $u_0 \geq 0$ the solution $u(\cdot,\cdot,u_0)$ is convergent if bounded. If $f'(0) > 0$, we do not have such a general convergence result; in this case we need to make additional assumptions on u_0 as well (it is sufficient that $u_0 = v + w$, where v is symmetrically decreasing and w is nonnegative with compact support).

Having addressed the problem of convergence of bounded solutions in the localized topology, our second main objective is to examine the *global* shape of the solution $u(\cdot,t,u_0)$ at large times. For this, a different notion of the limit set is more appropriate. We define the Ω-limit set of a bounded solution u as follows:

$$\Omega(u) := \{\varphi : u(\cdot + x_n, t_n) \to \varphi \text{ for some sequence } t_n \to \infty \text{ and } x_n \in \mathbb{R}\}. \tag{1.5}$$

The convergence here is also in $L^\infty_{loc}(\mathbb{R})$, as in the definition of $\omega(u)$. Thus, while we still look at the shape of $u(\cdot,t_n)$ in bounded intervals, the intervals are not fixed and can be shifted around arbitrarily as $t_n \to \infty$. We sometimes denote $\Omega(u(\cdot,\cdot,u_0))$ simply by $\Omega(u_0)$.

Both $\Omega(u)$ and $\omega(u)$ provide interesting and relevant information about the large-time behavior of the solution u. The set $\Omega(u)$ yields a more global picture—and, obviously, $\omega(u) \subset \Omega(u)$—but it does not tell us what the solution looks like in fixed spatial intervals as $t \to \infty$; that is the role of $\omega(u)$.

In [15], we gave a partial description of $\Omega(u)$. Specifically, we described what we call *the top of* $\Omega(u)$ (see Section 3.2 for definition). We showed that it consists of steady states: a constant steady state and, possibly, all translations of an even symmetrically decreasing steady state. This partial description alone has interesting consequences (see [15]), and, moreover, for some nonlinearities, such as the bistable nonlinearity, the top of $\Omega(u)$ is actually the same as $\Omega(u)$.

It is clear that in general $\Omega(u)$ does not consist of steady states. For example, [11, Theorem 3.2] yields solutions whose shape is described in terms of two (bistable) fronts traveling in the opposite directions; the profile functions of these fronts as well as their limit constant steady states are included in $\Omega(u)$. One could naturally ask if it is true that $\Omega(u)$ always consists of steady states and traveling fronts. This is indeed a result we are after, but, again, we stress the importance of the assumption $u_0 \geq 0$ here. The result is not valid for a general $u_0 \in C_0(\mathbb{R})$. Indeed, as indicated in [19], other types of entire solutions, such as those found in [5, 16], can occur even in the smaller set $\omega(u)$. In this paper, we give a complete description of $\Omega(u)$ under our generic conditions on f, assuming only that $u_0 \in C_0(\mathbb{R})$, $u_0 \geq 0$, and in the case $f'(0) > 0$ also that u_0 has compact support. We show that $\Omega(u)$ does indeed consist entirely of steady states and traveling fronts of (1.1). All these traveling fronts come from a certain minimal propagating terrace (see Section 2.1 for the definition), which depends on the solution u only via $\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})}$.

Having the results on $\omega(u)$ and $\Omega(u)$ at hand, we are able to describe the global shape of the solution $u(\cdot,t,u_0)$ for large t. Namely, we show that the graph of $u(\cdot,t,u_0)$ has three parts: the "top" and two "sides" (see Figure 1 below). The top is either flat and close to a horizontal line, or is close to the graph of a symmetric and symmetrically decreasing steady state of (1.1). The two sides have the shapes of "terraces" having plateaus at the same heights and interfaces moving in the opposite directions with precisely determined speeds.

Such a general and precise description of the global shape of the solutions is new even for nonnegative solutions with compact initial support. However, several related results can be found in the literature, some of them even in higher-dimensional equations

$$u_t = \Delta u + f(u), \quad x \in \mathbb{R}^N,\, t > 0. \tag{1.6}$$

As part of his studies in [26], Rossi considered nonnegative solutions with compact initial support of equations (1.6) with multistable nonlinearities. He established the existence of nearly radially symmetric plateaus in the graph of such solutions; the speeds with which the plateaus expand come from a minimal propagating terrace for a one dimensional equation. Radial nonnegative and spatially decaying solutions of multistable equations (1.6) have been studied in [8]. Assuming that the solution propagates to a constant steady state, its approach to a radial propagating terrace (with a flat top) has been established in [8]. A similar result has been proved in an independent paper [24], where, moreover, gradient systems of equations have been considered.

Most closely related to our work is another recent preprint of Risler, [23], which we have learned of while the present paper was being completed. For one-dimensional equations and gradient systems satisfying certain generic assumptions, he considers there solutions which approach a stable constant steady state as $|x| \to \infty$. His main result shows, similarly as one of our main theorems, that the graph of the solution at large times is composed of three parts: two sides given by propagating terraces, and a middle part given by a "standing terrace" (the middle part of the graph can still move, but it has an asymptotically vanishing speed). The convergence of the "middle part," of the solution is not considered in the paper (and on that account, the generic conditions are slightly weaker in [23] than our generic conditions) and neither is the case when the solution convergences to an unstable constant steady state as $|x| \to \infty$. Also, our result is more precise in that it shows the approach of the sides of the solution to the propagating terrace with what could be called the asymptotic phase (see statements (b),(c) of Theorem 2.11 below for the meaning of this). The techniques in [23] are very different—and, given that they apply to gradient systems, have more general scope—than ours.

The proofs of our results have three main components. First, we give a complete description of $\Omega(u)$, as detailed above. The top of $\Omega(u)$ having already been described in [15], we examine $\Omega(u)$ below its top. We employ some results on the large time behavior of front-like solutions proved in [22] (see Section 3.3 of the present paper). Each "side" of the graph of the solution $u(\cdot, t, u_0)$ is sandwiched by two front-like solutions. The asymptotics of these front-like solutions and a Liouville theorem for entire solutions yield the desired description of $\Omega(u)$ and at the same time show that the sides of the graph of $u(\cdot, t, u_0)$ are attracted to propagating terraces. When $f'(0) < 0$, the sandwiching estimates are relatively easy. Convergence properties of the minimal propagating terrace allow us to adapt Fife-McLeod type estimates [11]; we employ propagating terraces in such estimates is a similar way traveling fronts are used in [11]. If $f'(0) > 0$, the estimates are not so straightforward. To deal with the extra difficulties, we first modify $f(u)$ for $u < 0$, $u \approx 0$ in a suitable way so that, in particular, f has a bistable interval $[a_0, a_2]$ containing 0 in its interior. This facilitates an application of the sandwiching estimates in $\{u : u > a_0\}$, from which we obtain a description of $\Omega(u)$, except for its "bottom part." The missing information on $\Omega(u)$ is then obtained by a different technique—very similar to the technique of spatial trajectories used in [22].

With the description of $\Omega(u)$ and the sides of the graph of $u(\cdot, t, u_0)$ at hand, we next address the problem of convergence of the solution in $L^\infty(\mathbb{R})$. If the top $\Omega(u)$ is flat, the locally uniform converge of $u(\cdot, t, u_0)$ to a constant is proved easily. The real issue here lies with the case when the top of $\Omega(u)$ is given by the shifts of a symmetrically decreasing steady state: we need to show that $u(\cdot, t, u_0)$ settles down at one of those shifts as time approaches infinity. For this we use a reflection argument relying heavily on convergence properties of the propagating terraces attracting the sides of the graph of $u(\cdot, t, u_0)$.

Finally, building on the previous results—description of $\Omega(u)$ and of the sides of the graph of $u(\cdot, t, u_0)$; and the convergence result, which tells us what happens on the top of the graph of $u(\cdot, t, u_0)$—we provide a detailed description of how the global graph of the solution looks like at large times.

The remainder of the paper is organized as follows. The precise formulations of our main theorems are given in Section 2.2 and their proofs in Section 4. Section 3 contains preliminary results needed for the proofs. In the appendix, we prove the genericity of our hypotheses. For that purpose, we also prove there several useful results concerning the dependence of the speeds of traveling fronts on the nonlinearity f.

2 Hypotheses and main results

Throughout the paper, our standing hypotheses of f are as follows:

(H) $f \in C^1(\mathbb{R})$, $f(0) = 0$, and f' is bounded on $\mathbb{R}$.

We have added the global Lipschitz continuity of f just for convenience; it is at no cost to generality as we are dealing with bounded solutions.

We next formulate our remaining hypotheses and discuss their genericity. Then we state our main theorems.

2.1 Generic hypotheses

Set

$$F(u) = \int_0^u f(s)\, ds. \tag{2.1}$$

As in [15], $\tilde{\Gamma}$ stands for the set of all critical points of F in $[0, \infty)$ which are "left-global" maximizers of F:

$$\tilde{\Gamma} := \Big\{ \gamma \geq 0 : f(\gamma) = 0 \text{ and } F(\gamma) \geq F(v) \text{ for all } v \in [0, \gamma] \Big\}. \tag{2.2}$$

The left-global maximizers are the only zeros of f that are relevant for our description of the large-time behavior of bounded solutions of (1.1). Trivially, $0 \in \tilde{\Gamma}$.

We now list three new hypotheses (G1)–(G3). The first two of them concern the left-global maximizers; the last one concerns a class of minimal propagating terraces. We give the definition of a minimal propagating terrace right after the hypotheses.

(G1) Each left-global maximizer of F is strict: if $\gamma \in \tilde{\Gamma}$, then $F(\gamma) > F(v)$ for all $v \in [0, \gamma)$.

(G2) If $\gamma \in \tilde{\Gamma}$, then γ is a nondegenerate critical point of F: $f'(\gamma) \neq 0$.

(G3) If $\gamma \in \tilde{\Gamma}$, $\gamma > 0$, and $\{(\phi_I, c_I) : I \in \mathcal{N}\}$ is the minimal propagating terrace for the interval $[0, \gamma]$, then $c_I \neq c_J$ for any $I, J \in \mathcal{N}$ with $I \neq J$.

To define the minimal propagating terrace, we first introduce, following [27], the notion of a minimal system of waves. If ϕ is a C^1 function on $\mathbb{R}$, we set

$$\tau(\phi) = \{(\phi(x), \phi_x(x)) : x \in \mathbb{R}\}. \tag{2.3}$$

Recall that a traveling front of (1.1) with speed c and profile ϕ is a solution of (1.1) of the form $U(x,t) = \phi(x - ct)$, where ϕ is a decreasing solution of

$$\phi'' + c\phi' + f(\phi) = 0, \quad x \in \mathbb{R}. \tag{2.4}$$

Definition 2.1. For any $\gamma > 0$ with $f(\gamma) = 0$, a $[0, \gamma]$-system of waves of (1.1) is a continuous function R on $[0, \gamma]$ with the following properties:

(i) $R(0) = R(\gamma) = 0, \quad R(u) \le 0 \;\; (u \in [0, \gamma])$;

(ii) If $I = (a, b) \subset [0, \gamma]$ is a nodal interval of R, that is, a connected component of the set $R^{-1}(-\infty, 0)$, then there are $c \in \mathbb{R}$ and a decreasing solution ϕ of (2.4) such that $\phi(-\infty) = b$, $\phi(\infty) = a$, and

$$\{(u, R(u)) : u \in (a, b)\} = \tau(\phi). \tag{2.5}$$

Thus, the graph of R between its successive zeros is given by the trajectory of the profile of a traveling front.

Definition 2.2. A $[0, \gamma]$-system of waves R_0 is said to be *minimal* if for an arbitrary $[0, \gamma]$-system of waves R one has

$$R_0(u) \le R(u) \quad (u \in [0, \gamma]).$$

As shown in [27, Theorem 1.3.2], a minimal $[0, \gamma]$-system of waves exists for any interval $[0, \gamma]$ with $f(\gamma) = f(0) = 0$; obviously, by definition, it is unique. Of course, there is nothing special about the interval $[0, \gamma]$ in these considerations; any other interval whose boundary points are zeros of f can be treated the same way.

Given any $\gamma > 0$ with $f(\gamma) = 0$, take the corresponding minimal $[0, \gamma]$-system of waves R_0 and denote by $\mathcal{N}$ the (countable) set of all nodal intervals of R_0. Since R_0 is single valued, for each interval $I = (a, b)$ in $\mathcal{N}$ the speed $c = c_I$ and the solution $\phi = \phi_I$ in Definition 2.1(ii) are determined uniquely if we postulate

$$\phi(0) = \frac{a + b}{2}. \tag{2.6}$$

This way we obtain the *families of speeds* and *profile functions* corresponding to R_0:

$$\{c_I : I \in \mathcal{N}\}, \; \{\phi_I : I \in \mathcal{N}\}. \tag{2.7}$$

Consider now the family of traveling fronts $U_I(x, t) = \phi_I(x - c_I t)$, $I \in \mathcal{N}$. As in [10, 22], we refer to this family as the *minimal propagating terrace for the interval* $[0, \gamma]$, or the *minimal* $[0, \gamma]$*-propagating terrace.* When convenient, we write the minimal $[0, \gamma]$-propagating terrace a little differently as $\{(\phi_I, c_I) : I \in \mathcal{N}\}$.

In general, the set $\mathcal{N}$ may be infinite; and positive, negative, and zero speeds may be included in $\{c_I : I \in \mathcal{N}\}$. However, if (G1), (G2) are in effect and $\gamma \in \tilde{\Gamma}$, then the set $\mathcal{N}$ is finite and the speeds are all positive. The proofs of these results and the results in the next proposition can be found in [22, Section 3.2]. Note that assumption (G1) in particular rules out the occurrence of a standing wave (a traveling front with zero speed) in the minimal propagating terrace.

We define a natural ordering on $\mathcal{N}$:

$$I_1 < I_2 \quad \text{if } I_1 = (a_1, b_1),\ I_2 = (a_2, b_2) \text{ and } b_1 \le a_2, \tag{2.8}$$

and write $I_1 \le I_2$ if $I_1 = I_2$ or $I_1 < I_2$. Since two different nodal intervals of R_0 cannot overlap, $\mathcal{N}$ in simply ordered by this relation. Using the above notation, the following statements are valid.

Proposition 2.3. *If $\gamma \in \tilde{\Gamma}$ and* (G1), (G2) *hold, then set $\mathcal{N}$ is finite, $R_0^{-1}\{0\} \subset \tilde{\Gamma}$, and for any $I, J \in \mathcal{N}$ with $I < J$ one has $c_I \ge c_J > 0$.*

Our hypothesis (G3) requires that the relation between c_I and c_J be strict: $c_I > c_J$ if $I < J$.

The hypotheses (G1)–(G3) are generic: in "reasonable" topological spaces of C^1-functions, the set of all f satisfying (G1)–(G3) is residual (it contains a countable intersection of open and dense sets). A specific genericity result is stated in the next proposition. For definiteness, we choose the space $C^1_b(\mathbb{R})$ of continuous functions on $\mathbb{R}$ which are bounded together with their first derivatives (we assume a standard norm on $C^1_b(\mathbb{R})$, say $\|f\| = \|f\|_{L^\infty(\mathbb{R})} + \|f'\|_{L^\infty(\mathbb{R})}$). Other spaces of (not necessarily bounded) C^1-functions f can be considered similarly.

Proposition 2.4. *Let $X := \{f \in C^1_b(\mathbb{R}) : f(0) = 0\}$ and equip it with the norm induced from $C^1_b(\mathbb{R})$. Then there is a residual set $\mathcal{F}$ in X such that for each $f \in \mathcal{F}$ the conditions* (G1)-(G3) *are all satisfied.*

The proof is given in the appendix.

2.2 Main theorems

Throughout this section, we assume that the standing hypotheses (H) as well as hypotheses (G1)–(G3) are satisfied. Also, we assume that u_0 is a fixed arbitrarily chosen nonnegative function in $C_0(\mathbb{R})$.

If γ is a zero of f, a *ground state of* (1.1) *at level* γ refers to a steady state φ of (1.1) such that $\varphi > \gamma$ and $\varphi(x) \to \gamma$ as $|x| \to \infty$. Any ground state is symmetric (even) about some center $\xi \in \mathbb{R}$ and is decreasing in (ξ, ∞). In the phase plane of equation (2.4) with $c = 0$, the ground state corresponds to a homoclinic orbit to the equilibrium $(\gamma, 0)$.

Our first result is a convergence theorem. Note that hypothesis (G2) in particular implies that $f'(0) < 0$ or $f'(0) > 0$. In the first case, we will prove the convergence of the solution $u(\cdot, \cdot, u_0)$ with no additional conditions on u_0:

Theorem 2.5. *Assume that $f'(0) < 0$. If the solution $u(\cdot, \cdot, u_0)$ is bounded, then $\omega(u_0) = \{\varphi\}$—that is, $\lim_{t\to\infty} u(\cdot, t, u_0) = \varphi$ in $L^\infty_{loc}(\mathbb{R})$—where φ is a steady state of* (1.1) *and, more specifically, there is $\gamma \in \tilde{\Gamma}$ such that either $\varphi \equiv \gamma$ or φ is a ground state at level $\gamma \in \tilde{\Gamma}$.*

If $f'(0) > 0$, we do not have such a convergence theorem with no additional conditions on u_0. For reference, we state here our convergence result from [15]:

Theorem 2.6. *Assume that $u_0 = v + w$, where $v, w \in C(\mathbb{R})$, v is even and nonincreasing on $(0, \infty)$ (possibly $v \equiv 0$), and w is nonnegative and has compact support. If the solution $u(\cdot, \cdot, u_0)$ is bounded, then the same conclusion as in Theorem 2.5 holds.*

We remark that this theorem applies to all $f \in C^1(\mathbb{R})$ with $f(0) = 0$; in particular, the generic conditions are not needed in it. The theorem was previously proved in [7, 9] for nonnegative initial data with compact support and in [17] for symmetrically decreasing initial data.

We now consider the larger limit set, $\Omega(u_0)$.

We need some more notation. Let

$$\begin{aligned} &\gamma_{max}(u_0) := \max K, \\ &\text{where } K := \{\gamma \in \tilde{\Gamma} : \gamma \leq \varphi \text{ for some } \varphi \in \Omega(u_0)\}. \end{aligned} \tag{2.9}$$

It follows from (G2), that if the solution $u(\cdot, \cdot, u_0)$ is bounded, then K is a finite nonempty set; so the maximum is well defined.

The case $\gamma_{max}(u_0) = 0$ was considered in our previous paper. In [15, Theorem 2.5] we proved the following uniform-convergence result:

Theorem 2.7. *Assume that $f'(0) < 0$. If the solution $u(\cdot, \cdot, u_0)$ is bounded and $\gamma_{max}(u_0) = 0$, then $\lim_{t\to\infty} u(\cdot, t, u_0) = \psi$ in $L^\infty(\mathbb{R})$, where $\psi \equiv 0$ or ψ is a ground state at level* 0*. In particular, $\Omega(u_0) = \{0\} \cup \{\psi(\cdot - \xi) : \xi \in \mathbb{R}\}$.*

Remark 2.8. It is easy to show (see the beginning of Section 4) that in the case $f'(0) > 0$ one necessarily has $\gamma_{max}(u_0) > 0$.

We now consider the case $\gamma_{max}(u_0) > 0$. Let $\{(\phi_I, c_I) : I \in \mathcal{N}\}$ be the minimal $[0, \gamma_{max}(u_0)]$-propagating terrace. For some k we have $\mathcal{N} = \{I_1, \dots, I_k\}$ with $I_j = (a_j, a_{j+1})$, $j = 1, \dots, k$, where

$$a_j \in \tilde{\Gamma} \quad (j = 1, \dots, k+1), \qquad 0 = a_1 < a_2 < \dots < a_{k+1} = \gamma_{max} \tag{2.10}$$

(cp. Proposition 2.3). To simplify the notation, we drop one index: $\phi_j := \phi_{I_j}$, $c_j := c_{I_j}$, $j = 1, \dots, k$. Also, for any j, we define

$$\tilde{\phi}_j(x) := \phi_j(-x) \quad (x \in \mathbb{R}).$$

The following two theorems use the above notation. The first one gives a complete description of $\Omega(u_0)$.

Theorem 2.9. *Assume that the solution $u(\cdot, \cdot, u_0)$ is bounded and $\gamma_{max}(u_0) > 0$. If $f'(0) > 0$, assume also that u_0 has compact support. Then*

$$\Omega(u_0) = \{\psi(\cdot - \xi) : \xi \in \mathbb{R}\} \cup \{a_j : j = 1, \dots, a_{k+1}\} \\ \cup \{\phi_j(\cdot - \xi) : \xi \in \mathbb{R}\} \cup \{\tilde{\phi}_j(\cdot - \xi) : \xi \in \mathbb{R}\}, \tag{2.11}$$

where either $\psi \equiv \gamma_{max}(u_0)$ or ψ is a ground state at level $\gamma_{max}(u_0)$.

Remark 2.10. We will also show that the constant γ in Theorems 2.5, 2.6 is equal to $\gamma_{max}(u_0)$.

In our final theorem, we use the minimal $[0, \gamma_{max}(u_0)]$-propagating terrace and the limit steady state ψ from Theorems 2.5, 2.6 to describe the shape of the solution at large times.

Theorem 2.11. *Assume that the solution $u(\cdot, \cdot, u_0)$ is bounded and one has $\gamma_{max}(u_0) > 0$. If $f'(0) > 0$ assume also that u_0 has compact support. Then there are C^1 functions $\zeta_1^+, \dots, \zeta_k^+$, $\zeta_1^-, \dots, \zeta_k^-$ defined on an interval (t_0, ∞) such that the following statements are valid:*

(a) $\lim_{t\to\infty} (\zeta_j^\pm)'(t) = 0 \quad (j = 1, \dots, k)$*;*

(b) *If $f'(0) < 0$, then $\zeta_j^\pm(t)$ has a finite limit $\eta_j^\pm$ as $t \to \infty$ for all $j \in \{1, \dots, k\}$.*

(c) *If $f'(0) > 0$, then $\zeta_j^{\pm}(t)$ has a finite limit $\eta_j^{\pm}$ as $t \to \infty$ for all $j \in \{1, \dots, k\} \setminus \{1\}$.*

(d) *As $t \to \infty$, one has*

$$
\begin{aligned}
u(x,t) &- (\psi(x) - \gamma) \\
&- \Bigg(\sum_{j=1,\dots,k} \phi_j(x - c_j t - \zeta_j^+(t)) - \sum_{j=1,\dots,k} a_{j+1} \\
&\qquad + \sum_{j=1,\dots,k} \tilde{\phi}_j(x + c_j t + \zeta_j^-(t)) - \sum_{j=1,\dots,k-1} a_{j+1} \Bigg) \to 0,
\end{aligned}
\tag{2.12}
$$

where $\gamma = \gamma_{max}(u_0)$, ψ as in Theorems 2.5, 2.6, and the convergence is uniform with respect to $x \in \mathbb{R}$.

Remark 2.12. In the case $f'(0) > 0$, Theorems 2.9, 2.11 are not valid if the assumption of u_0 having compact support is dropped. Indeed, with $f'(0) > 0$, the bottom part of the minimal propagating terrace is the traveling front $\phi_1(x - c_1 t)$ with minimal speed for the interval $[0, a_2]$ (see [22] for more details on this). Theorems 2.9, 2.11 tell us that in its bottom part (between 0 and a_2) the graph of $u(\cdot, t, u_0)$ approaches the graphs of ϕ_1 and $\tilde{\phi}_1$. If u_0 is merely required to decay to zero, then many different behaviors of the bottom part of $u(\cdot, t, u_0)$ may occur, just as in well-known examples with front-like solutions (see [12, 14, 28]). Also, as with front-like solutions (cp. [4, 13]), in the case $f'(0) > 0$ the convergence of $\zeta_1(t)$ is not to be expected. This is why we have the condition $j \neq 1$ in statement (c).

According to (2.12), for large t the graph of the function $u(\cdot, t, u_0)$ is as in Figure 1. It has three parts: the "top" and two "sides." The top is either flat and close to the line $\{u = \gamma\}$ (the top figure) or is close to the graph of a ground state at level γ (the bottom figure). The two sides have the shapes of "terraces" having their flat parts at the same levels and the interfaces moving in the opposite directions with speeds $c_1 > \cdots > c_k$. In addition, statements (b) and (c) show that each interface, except for the bottom ones in the case $f'(0) > 0$, approaches a single traveling front as $t \to \infty$.

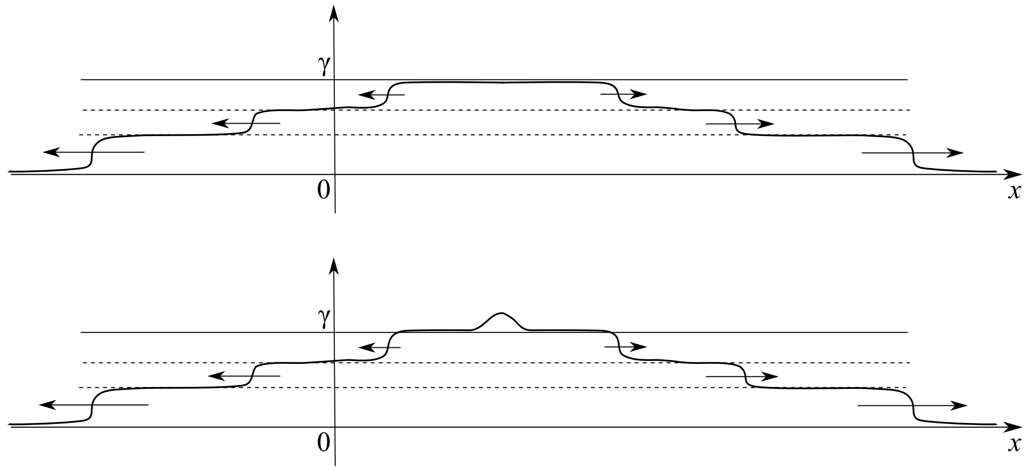

Figure 1: Two possibilities as to how the graph of $u(\cdot, t, u_0)$ can look like at large times (for $k = 3$).

3 Preliminaries

This section consists of three parts. First, we summarize basic properties of the zero-number functional. Then we recall the invariance and other properties of the ω and Ω-limit sets and state a Liouville theorem for entire solutions. In the last part, we recall or prove some useful properties of minimal propagating terraces.

At several places below we use the well known elementary fact that the function $q_x^2(x)/2 + F(q(x))$ is constant for any solution q of

$$q'' + f(q) = 0, \quad x \in \mathbb{R}. \tag{3.1}$$

3.1 Zero number

Here we consider solutions of the linear equation

$$v_t = v_{xx} + cv_x + a(x,t)v, \quad x \in \mathbb{R},\ t \in (s,T), \tag{3.2}$$

where $-\infty \le s < T \le \infty$ and a is a bounded measurable function. Note that if u, $\bar{u}$ are bounded solutions of a nonlinear equation (1.1) (or equation (3.4) considered below) with a Lipschitz nonlinearity, then the difference $v = u - \bar{u}$ and the derivative u_x satisfy a linear equation (3.2).

For an interval $I = (a, b)$, with $-\infty \le a < b \le \infty$, we denote by $z_I(v(\cdot, t))$ the number, possibly infinite, of all zeros $x \in I$ of the function $x \to v(x, t)$. If $I = \mathbb{R}$, we usually omit the subscript $\mathbb{R}$:

$$z(v(\cdot, t)) := z_{\mathbb{R}}(v(\cdot, t)).$$

The following intersection-comparison principle holds (see [1, 6]).

Lemma 3.1. *Let v be a nontrivial solution of* (3.2) *and $I = (a, b)$, where $-\infty \le a < b \le \infty$. Assume that the following conditions are satisfied:*

(c1) *if $b < \infty$, then $v(b, t) \ne 0$ for all $t \in (s, T)$,*

(c2) *if $a > -\infty$, then $v(a, t) \ne 0$ for all $t \in (s, T)$.*

Then the following statements hold true:

(i) *For each $t \in (s, T)$, all zeros of $v(\cdot, t)$ are isolated. In particular, if $a > -\infty$ and $b < \infty$, then $z_I(v(\cdot, t)) < \infty$ for all $t \in (s, T)$.*

(ii) *$t \mapsto z_I(v(\cdot, t))$ is a monotone nonincreasing function on (s, T) with values in $\mathbb{N} \cup \{0\} \cup \{\infty\}$.*

(iii) *If for some $t_0 \in (s, T)$, the function $v(\cdot, t_0)$ has a multiple zero in I and $z_I(v(\cdot, t_0)) < \infty$, then for any $t_1, t_2 \in (s, T)$ with $t_1 < t_0 < t_2$ one has*

$$z_I(v(\cdot, t_1)) > z_I(v(\cdot, t_2)). \tag{3.3}$$

If (3.3) holds, we say that $z_I(v(\cdot, t))$ drops in the interval (t_1, t_2).

Remark 3.2. It is clear that if the assumptions of Lemma 3.1 are satisfied and for some $s_0 \in (s, T)$ one has $z_I(v(\cdot, s_0)) < \infty$, then $z_I(v(\cdot, t))$ can drop at most finitely many times in (s_0, T) and if it is constant on (s_0, T), then $v(\cdot, t)$ has only simple zeros in I for each $t \in (s_0, T)$.

The following result, which is a version of Lemma 3.1 for time-dependent intervals, is derived easily from Lemma 3.1 (cp. [2, Section 2]).

Lemma 3.3. *Let v be a nontrivial solution of* (3.2) *and $I(t) = (a(t), b(t))$, where $-\infty \le a(t) < b(t) \le \infty$ for $t \in (s, T)$. Assume that the following conditions are satisfied:*

(c1) *Either* $b \equiv \infty$ *or* b *is a (finite) continuous function on* (s,T)*. In the latter case,* $v(b(t),t) \neq 0$ *for all* $t \in (s,T)$*.*

(c2) *Either* $a \equiv -\infty$ *or* a *is a continuous function on* (s,T)*. In the latter case,* $v(a(t),t) \neq 0$ *for all* $t \in (s,T)$*.*

Then statements (i), (ii) of Lemma 3.1 are valid with I, a, b *replaced by* $I(t)$, $a(t)$, $b(t)$, *respectively; and statement (iii) of Lemma 3.1 is valid with all occurrences of* $z_I(v(\cdot,t_j))$, $j = 0,1,2$, *replaced by* $z_{I(t_j)}(v(\cdot,t_j))$, $j = 0,1,2$, *respectively.*

The following lemma—stating persistence of multiple zeros—is a reformulation of [7, Lemma 2.6].

Lemma 3.4. *Assume that* v *is a nontrivial solution of* (3.2) *such that for some* $s_0 \in (s,T)$ *the function* $v(\cdot,s_0)$ *has a multiple zero at some* x_0*:* $v(x_0,s_0) = v_x(x_0,s_0) = 0$*. Assume further that for some* $\delta > 0$*,* v_n *is a sequence in* $C^1([x_0-\delta,x_0+\delta]\times[s_0-\delta,s_0+\delta])$ *which converges in this space to* v*. Then for all sufficiently large* n *the function* $v_n(\cdot,t)$ *has a multiple zero in* $(x_0-\delta,x_0+\delta)$ *for some* $t \in (s_0-\delta,s_0+\delta)$*.*

3.2 Limit sets, entire solutions, and a Liouville theorem

Consider the following equation

$$u_t = u_{xx} + cu_x + f(u), \qquad x \in \mathbb{R},\ t > 0, \tag{3.4}$$

where f satisfies (H) and $c \in \mathbb{R}$ is a fixed parameter.

The ω and Ω-limit sets of a bounded solution u of (3.4) is defined as in (1.3) and (1.5), respectively. It will be useful to remember that if u is a bounded solution of (1.1), then the function $\tilde{u}(x,t) := u(x+ct,t)$ is a bounded solution of (3.4). Clearly, u and $\tilde{u}$ have the same initial value at $t = 0$ and $\Omega(u) = \Omega(\tilde{u})$.

An *entire solution* of (3.4) refers to a solution defined for each $t \in \mathbb{R}$. We now recall well-known invariance properties of the limit sets of bounded solutions of (3.4), which yield entire solutions (cp. [15, Section 2], where more details are given).

If u is a bounded solution of (3.4), then the following is true for $M = \omega(u)$ as well as for $M = \Omega(u)$. If $\varphi \in M$, then there is an entire solution U of

(3.4) such that $U(\cdot,0)=\varphi$ and $U(\cdot,t)\in M$ for each $t\in\mathbb{R}$. Moreover, if $\{(x_n,t_n)\}$ is a sequence in $\mathbb{R}^2$ such that $t_n\to\infty$ and $u(x_n+\cdot,t_n+\cdot)\to\varphi$ in $L^\infty_{loc}(\mathbb{R})$ (we take $x_n=0$ for all n if $K=\omega(u)$), then, possibly after replacing $\{(x_n,t_n)\}$ by a subsequence, one has

$$D^{2,1}u(\cdot+x_n,\cdot+t_n)\to D^{2,1}U, \tag{3.5}$$

uniformly on each compact set in $\mathbb{R}^2$, where $D^{2,1}u$ stands for (u,u_x,u_{xx},u_t).

In addition, $\Omega(u)$ has the translation invariance property: if $\varphi\in\Omega(u)$, then $\Omega(u)$ contains the $L^\infty_{loc}(\mathbb{R})$-closure of the translation group orbit of φ, $\{\varphi(\cdot+\xi):\xi\in\mathbb{R}\}$. This follows directly from the definition of $\Omega(u)$. In particular, if the limit $\gamma:=\varphi(\infty)$ (or $\gamma:=\varphi(-\infty)$) exists, then $\gamma\in\Omega(u)$.

Recall that, given a nonnegative function $u_0\in C_0(\Omega)$, $u(\cdot,\cdot,u_0)$ stands for the solution of (1.1), (1.2). Assuming that the solution is bounded, we now consider its Ω-limit set, $\Omega(u_0)$. The top of $\Omega(u_0)$ was defined in [15] as follows:

$$\mathcal{T}_\Omega(u_0):=\{\varphi\in\Omega(u_0):\varphi\geq\gamma_{max}(u_0)\}. \tag{3.6}$$

Here $\gamma_{max}(u_0)$ is as in (2.9). It is clear from (2.9) that

$$\gamma_{max}(u_0)\in\tilde{\Gamma},\quad \mathcal{T}_\Omega(u_0)\neq\emptyset.$$

We will use the following properties of $\mathcal{T}_\Omega(u_0)$.

Lemma 3.5. *Assume that $u_0\in C_0(\Omega)$, $u_0\geq 0$, and the solution $u(\cdot,\cdot,u_0)$ is bounded. Then the following statements are valid.*

(i) Either $\mathcal{T}_\Omega(u_0)=\{\gamma_{max}(u_0)\}$ or there is a ground state ψ at level $\gamma_{max}(u_0)$ such that

$$\mathcal{T}_\Omega(u_0)=\{\gamma_{max}(u_0)\}\cup\{\psi(\cdot+\xi):\xi\in\mathbb{R}\}. \tag{3.7}$$

(ii) For each $\varphi\in\Omega(u_0)$ one has either $\varphi\in\mathcal{T}_\Omega(u_0)$ or $\varphi<\gamma_{max}(u_0)$.

Proof. Statement (i) is the content of [15, Theorem 2.4].

To prove statement (ii), take any $\varphi\in\Omega(u_0)\setminus\mathcal{T}_\Omega(u_0)$. We first show that $\varphi\leq\gamma_{max}(u_0)$. Suppose this is not true. Then, since $\varphi\notin\mathcal{T}_\Omega(u_0)$, φ is a nonconstant function such that $\varphi-\gamma_{max}(u_0)$ changes sign.

We claim that φ is necessarily a steady state of (1.1), either a ground state at some level $\gamma\geq 0$ or a strictly monotone solution of (3.1). This can be proved in a similar way as it was done in [15, Section 4] for $\varphi\in\mathcal{T}_\Omega(u_0)$, but the arguments need some modifications, which we now specify.

Crucial for the proof in [15] is Lemma 4.1 which says in particular that if $\varphi \in \mathcal{T}_\Omega(u_0)$ is nonconstant and q is a periodic solution of (3.1), then $\tau(\varphi) \cap \tau(q) = \emptyset$ (recall that $\tau(\phi)$ is defined in (2.3)). This result remains valid, with a nearly identical proof, if it is merely assumed that $\varphi \in \Omega(u_0)$ but the following assumption on q is added: $\max q > \gamma_{max}(u_0)$. The next step in the proof in [15] is Proposition 4.4, where the previous result is used to prove that any $\varphi \in \mathcal{T}_\Omega(u_0)$ is a steady state, either a constant, a ground state at some level $\gamma \geq 0$, or a strictly monotone solution of (3.1) (other conclusions of [15, Proposition 4.4] are not needed here). This result is also valid for any $\varphi \in \Omega(u_0)$ such that $\varphi - \gamma_{max}(u_0)$ changes sign, but this is where a more significant modification of the arguments is needed. Consider, as in the proof of [15, Proposition 4.4], the entire solution U of (1.1) with $U(\cdot, 0) = \varphi$ and $U(\cdot, t) \in \Omega(u)$ for all $t \in \mathbb{R}$. For all t in some interval $(-\delta, \delta)$ with $\delta > 0$, the function $U(\cdot, t) - \gamma_{max}(u_0)$ changes sign. Take now the following set

$$K := \{(U(x,t), U_x(x,t)) : x \in \mathbb{R},\, t \in (-\delta, \delta)\} = \bigcup_{t \in (-\delta,\delta)} \tau(U(\cdot, t)). \qquad (3.8)$$

It can be proved that there is a steady state $\tilde{\varphi}$ of (1.1), either a ground state at some level $\gamma \geq 0$ or a strictly monotone solution of (3.1), such that

$$K \cap \{(\xi, \eta) \in \mathbb{R}^2 : \xi > \gamma_{max}(u_0)\} \subset \tau(\tilde{\varphi}). \qquad (3.9)$$

The arguments for this are essentially the same as in the proof of [15, Proposition 4.4], where a more global version of (3.9) is given. In the present case, we have to take the intersection in (3.9) because of the extra condition $\max q > \gamma_{max}(u_0)$ imposed above on q. Taking $x_0 \in \mathbb{R}$ so that $U(x_0, 0) = \varphi(x_0) > \gamma_{max}(u_0)$, it follows from (3.9) that for some $\epsilon > 0$ one has

$$(U(x,0), U_x(x,0)) \in \tau(\tilde{\varphi}) \quad (x \in (x_0 - \epsilon, x_0 + \epsilon)).$$

To conclude the proof of the claim, we just refer to the local unique continuation result stated in Lemma 3.6 below (a global version of this result was sufficient in [15]).

Having proved our claim, we first find a contradiction in the case when φ is a strictly monotone solution of (3.1). Consider its limits $\gamma^\pm := \varphi(\pm\infty)$. Clearly, $\gamma_{max}(u_0)$ is contained in the open interval with the end points γ^-, γ^+. Using elementary properties of solutions of equation (3.1), in particular the fact that $\varphi_x^2(x)/2 + F(\varphi(x))$ is a constant function, one shows easily that $f(\gamma^{\pm}) = 0$ and $F(\gamma^-) = F(\gamma^+) > F(\gamma_{max}(u_0))$. On the other hand, $\gamma_{max}(u_0) \in \tilde{\Gamma}$ implies that $F(\gamma_{max}) \geq F(\min\{\gamma_-, \gamma_+\})$ and we have a contradiction. The case when φ is a ground state at some level γ leads to a contradiction in a similar way. This time we have

$$\gamma = \varphi(\pm\infty) < \gamma_{max}(u_0) < \beta := \max \varphi.$$

Elementary properties of solutions of (3.1) imply that $F(u) < F(\gamma) = F(\beta)$ for all $u \in (\gamma, \beta)$. Using this with $u = \gamma_{max}(u_0)$, we obtain a contradiction to $\gamma_{max}(u_0) \in \tilde{\Gamma}$.

The inequality $\varphi \leq \gamma_{max}(u_0)$ is thus proved. To prove the strict inequality, consider the entire solution U of (1.1) such that $U(\cdot, 0) = \varphi$ and $U(\cdot, t) \in \Omega(u_0)$ for each $t \in \mathbb{R}$. Applying what we have proved above with φ replaced by $\tilde{\varphi} := U(\cdot, -1)$—note that $\tilde{\varphi} \notin \mathcal{T}_\Omega(u_0)$, otherwise $\varphi \in \mathcal{T}_\Omega(u_0)$ by the comparison principle—we obtain $\tilde{\varphi} \leq \gamma_{max}(u_0)$, $\tilde{\varphi} \not\equiv \gamma_{max}(u_0)$. The comparison principle now gives the desired inequality $\varphi = U(\cdot, 0) < \gamma_{max}(u_0)$. $\square$

The following unique continuation-type result was used in the previous proof and will be used again below.

Lemma 3.6. *Let U be a solution of* (1.1) *on $\mathbb{R} \times (t_1, t_2)$ for some $t_1 < t_2$ and let $\tilde{\varphi}$ be a steady state of* (1.1)*. Assume that for some $t_0 \in (t_1, t_2)$, $x_0 \in \mathbb{R}$, and $\epsilon > 0$ one has*

$$(U(x, t_0), U_x(x, t_0)) \in \tau(\tilde{\varphi}) \quad (x \in (x_0 - \epsilon, x_0 + \epsilon)). \tag{3.10}$$

Then there is a constant c such that $U \equiv \tilde{\varphi}(\cdot + c)$.

Proof. This is a reformulation of [22, Lemma 6.10] (cp. formula (6.32) in [22]). $\square$

We will also use the following Liouville property for entire solutions sandwiched between two shifts of a traveling front.

Lemma 3.7. *Let* (G1), (G2) *hold. Suppose that for some $\gamma \in \tilde{\Gamma}$, the traveling front $\phi_I(x - c_I t)$ belongs to the minimal $[0, \gamma]$-propagating terrace. Let $U(x, t)$ be an entire solution of* (1.1) *such that for some $\xi_1, \xi_2 \in \mathbb{R}$ one has*

$$\phi_I(x - c_I t - \xi_1) \leq U(x, t) \leq \phi_I(x - c_I t - \xi_2) \quad ((x, t) \in \mathbb{R}^2). \tag{3.11}$$

Then there is $\xi \in \mathbb{R}$ such that $U(x, t) = \phi_I(x - c_I t - \xi)$ for all $x \in (x, t) \in \mathbb{R}^2$.

Proof. This is a special case of [21, Theorem 3.1] (which, in its turn, is based on Theorems 3.1, 3.5 of [3]). $\square$

3.3 Propagating terraces, their extensions, and attraction of front-like solutions

In this section, we prove some useful technical results concerning minimal propagating terraces. We also recall some results on approach of front-like solutions to minimal propagating terraces in the generic case. Our standing hypotheses (H) as well as all hypotheses (G1)–(G3) are assumed to be valid here.

First, we recall some results on traveling fronts.

Lemma 3.8. *Suppose that $\gamma_1, \gamma_2 \in \tilde{\Gamma}$, $0 < \gamma_1 < \gamma_2$, and $(\gamma_1, \gamma_2) \cap \tilde{\Gamma} = \emptyset$. Then there exist a ground state ψ at level γ_1 and a traveling front $\phi(x - ct)$ with the following properties: $\psi(0) = max_{x\in\mathbb{R}}\psi(x) < \gamma_2$, $\phi' < 0$, the range of ϕ is the interval (γ_1, γ_2), and*

$$c \int_{-\infty}^{\infty} (\phi'(x))^2 \, dx = F(\gamma_2) - F(\gamma_1) > 0. \tag{3.12}$$

Proof. Conditions (G1), (G2) and the assumption $(\gamma_1, \gamma_2) \cap \tilde{\Gamma} = \emptyset$ imply that there is a unique $\beta \in (\gamma_1, \gamma_2)$ such that $F(\beta) = F(\gamma_1)$, $F(u) < F(\gamma_1)$ for all $u \in (\gamma_1, \beta)$, and $f(u) > 0$ for all $u \in [\beta, \gamma_2)$. In this situation, the existence of a ground state and a traveling front with the indicated properties is a standard, well-known property (see, for example, [27]). Due to the nondegeneracy conditions required in (G2), the following limits exists with an exponential rate: $\phi(-\infty) = \gamma_2$, $\phi(\infty) = \gamma_1$, $\phi'(\pm\infty) = 0$. In particular, the improper integral in (3.12) is finite. Relation (3.12) is found by multiplying the equation $\phi'' + c\phi' + f(\phi) = 0$ by ϕ' and integrating. □

We now rewrite relation (3.12) in more convenient way using the following notation. If ϕ is a strictly monotone C^1 function on an interval J, then p^ϕ is a function defined on the range of ϕ as follows:

$$p^\phi(u) = \phi_x(x), \text{ where } x \in J \text{ is the unique point with } \phi(x) = u. \tag{3.13}$$

In other words, the graph of the function $u \mapsto p^\phi(u)$ coincides with $\tau(\phi)$ (cp. (2.3)).

Lemma 3.9. *Under the hypotheses of Lemma 3.8 and in the above notation, we have*

$$c \int_{\gamma_1}^{\gamma_2} |p^\phi(u)| \, du = F(\gamma_2) - F(\beta), \tag{3.14}$$

where $\beta = \psi(0)$. Moreover, there is a positive constant d determined only by $f\big|_{[\gamma_1,\beta]}$ such that

$$\int_{\gamma_1}^{\gamma_2} |p^{\phi}(u)|\, du > d. \tag{3.15}$$

Proof. First note that $F(\gamma_1) = F(\beta)$. This is a consequence of the relations $\psi(-\infty) = \gamma_1$, $\psi'(-\infty) = 0$, $\psi'(0) = 0$, and the fact that the function $\psi_x^2(x)/2 + F(\psi(x))$ is constant. Relation (3.14) is now obtained from (3.12) via the substitution $u = \phi(x)$:

$$\int_{-\infty}^{\infty} \phi'(x)\phi'(x)\, dx = -\int_{\gamma_1}^{\gamma_2} p^{\phi}(u)\, du.$$

To prove the second statement of the lemma, we use the well-known fact, easily proved by a phase-plane analysis of (2.4) (see [27]), that the trajectory $\tau(\phi)$ does not intersect the trajectory $\tau(\psi)$ and stays in the exterior of the loop formed by the homoclinic orbit $\tau(\psi)$ and its limit equilibrium $(\gamma_1, 0)$. In terms of the functions p^{ψ}, p^{ϕ}, this means that

$$p^{\phi}(u) < p^{\psi}(u) < 0 \quad (u \in (\gamma_1, \beta)).$$

Thus, (3.15) holds with

$$d := \int_{\gamma_1}^{\beta} |p^{\psi}(u)|\, du.$$

Since ψ depends only on $f\big|_{[\gamma_1,\beta]}$, the second conclusion of the lemma is proved. $\square$

In the following two results, we consider extensions of the minimal propagating terraces for an interval to a larger interval.

Lemma 3.10. *Let γ_1, γ_2, ϕ, c be as in Lemma 3.8. Let $\{(\phi_I, c_I) : I \in \mathcal{N}\}$ be the minimal $[0,\gamma_1]$-propagating terrace. If $c < \min_{I\in\mathcal{N}} c_I$, then, replacing ϕ by a shift of ϕ so that $\phi(0) = (\gamma_1 + \gamma_2)/2$, the union*

$$\{(\phi_I, c_I) : I \in \mathcal{N}\} \cup \{(\phi, c)\} \tag{3.16}$$

is the minimal $[0,\gamma_2]$-propagating terrace.

Proof. By (G2), we have $f'(\gamma_1) < 0$, $f'(\gamma_2) < 0$. Therefore, c is the uniquely determined speed for traveling fronts with range (γ_1, γ_2) and ϕ is the unique profile function for such a front, up to translations. In particular, the set $\{(\phi, c)\}$ is the minimal $[\gamma_1, \gamma_2]$-propagating terrace. The conclusion now follows directly from the next lemma. $\square$

The following lemma is a little more general than needed in the previous proof.

Lemma 3.11. *Assume that* $0 = a_1 < a_2 < \cdots < a_{k+1} = \gamma$ *are elements of* $\tilde{\Gamma}$ *and for* $i = 1, \ldots, k$ *the following holds:* $\phi_i(x - c_i t)$ *is a traveling front with* $\phi_i(-\infty) = a_{i+1}$, $\phi_i(\infty) = a_i$, $\phi_i(0) = (a_i + a_{i+1})/2$, *and* c_i *is the (unique or) minimal speed for such traveling fronts. If* $c_1 \geq c_2 \geq \cdots \geq c_k$, *then*

$$\phi_i(x - c_i t), \quad i = 1, \ldots, k,$$

is the minimal $[0, \gamma]$*-propagating terrace.*

Proof. This can be proved by induction in k. If $k = 1$, the conclusion is obvious if the speed for the interval $[0, a_2]$ is unique. If the speed is not unique (which may occur if 0 is unstable for the equation $\dot{\xi} = f(\xi)$), the minimality of the speed is equivalent to $\phi_1(x - c_1 t)$ being the minimal $[0, a_2]$-propagating terrace (consisting of just one traveling front); this is a consequence of Theorems 1.3.8 and 1.3.14 of [27]. Thus the conclusion holds in this case as well.

For the induction argument, we assume that $k \geq 2$ and

$$\phi_i(x - c_i t), \quad i = 1, \ldots, k - 1,$$

is the minimal $[0, a_k]$-propagating terrace. If the minimal $[0, \gamma]$-propagating terrace is not given by (3.16), then it necessarily contains a front $\tilde{\phi}(x - \tilde{c} t)$ with range $(a, \gamma) \subset (0, \gamma)$ containing a_k. In particular, there is $x_0 \in \mathbb{R}$ such that $\tilde{\phi}(x_0) = a_k$ and $\tilde{\phi}'(x_0) < 0$. Under such conditions, statements (i) and (ii) of [22, Lemma 3.17] imply that the range of $\tilde{\psi}$ cannot be contained in $(0, \gamma)$ and we have a contradiction. Thus (3.16) is the minimal $[0, \gamma]$-propagating terrace. $\square$

The next lemma is analogous to Lemma 3.10 (the extension is at the bottom), and it can be proved similarly.

Lemma 3.12. *Assume that $f'(0) > 0$, $\gamma \in \tilde{\Gamma}$, and $\gamma > 0$. Let $\{(\phi_I, c_I) : I \in \mathcal{N}\}$ be the minimal $[0,\gamma]$-propagating terrace; and let $k \in \mathcal{N}$ and $b > 0$ be such that $\mathcal{N} = \{I_1, \dots, I_k\}$, $I_1 < \cdots < I_k$ (cp. (2.8)), and $I_1 = (0, b)$. Assume further that for some $a < 0$ with $f(a) = 0$, $f'(a) < 0$, there is a traveling front $\phi(x - ct)$ with range $I_1' := (a, b)$ such that $c > c_{I_2}$ (this requirement is void if $k = 1$). Then*

$$\{(\phi_{I_j}, c_{I_j}) : j \in \{1, \dots, k\},\, j \neq 1\} \cup \{(\phi, c)\}$$

is the minimal $[a, \gamma]$-propagating terrace.

In the next two propositions, we recall some results on approach of front-like solutions of (1.1) to propagating terraces. We consider initial data $u_0 \in C(\mathbb{R})$ satisfying

$$0 \le u_0 \le \gamma, \quad \lim_{x\to-\infty} u_0(x) = \gamma, \quad \lim_{x\to\infty} u_0(x) = 0, \tag{3.17}$$

for some $\gamma \in \tilde{\Gamma}$. By the comparison principle, the corresponding solution of (1.1), (1.2) is global and takes values between 0 and γ.

Proposition 3.13. *Assume that $u_0 \in C(\mathbb{R})$ satisfies (3.17) for some $\gamma \in \tilde{\Gamma}$. If $f'(0) > 0$, assume also that $u_0 \equiv 0$ on some interval (m, ∞). Let $\{(\phi_I, c_I) : I \in \mathcal{N}\}$ be the $[0,\gamma]$-minimal propagating terrace. Then*

$$\Omega(u) = \{\phi_I(\cdot - \xi) : I \in \mathcal{N},\, \xi \in \mathbb{R}\} \cup \{a_I : I \in \mathcal{N}\} \cup \{\gamma\}, \tag{3.18}$$

where a_I stands for the left end-point of the interval I.

Proof. If $f'(0) < 0$, the statement is a special case of [22, Corollary 2.10]; if $f'(0) > 0$, it is a special case of [22, Corollary 2.18]. □

For the next proposition we introduce the following notation. With γ and $\mathcal{N}$ as in Proposition 3.13, there is $k \in \mathbb{N}$ such that $\mathcal{N} = \{I_1, \dots, I_k\}$ with $I_j = (a_j, a_{j+1})$, $j = 1, \dots, k$, where

$$a_j \in \tilde{\Gamma} \quad (j = 1, \dots, k+1), \qquad 0 = a_1 < a_2 < \cdots < a_{k+1} = \gamma. \tag{3.19}$$

We set

$$\phi_j := \phi_{I_j},\ c_j := c_{I_j} \quad (j = 1, \dots, k). \tag{3.20}$$

Proposition 3.14. *Assume the hypotheses of Proposition 3.13 to be satisfied. There are C^1 functions $\zeta_1, \ldots, \zeta_k$ defined on an interval (t_0, ∞) such that the following statements are valid:*

(a) $\lim_{t\to\infty}(\zeta_j)'(t) = 0 \quad (j = 1, \ldots, k)$*;*

(b) *If $f'(0) < 0$, then $\zeta_j(t)$ has a finite limit η_j as $t \to \infty$ for all $j \in \{1, \ldots, k\}$.*

(c) *If $f'(0) > 0$, then $\zeta_j(t)$ has a finite limit η_j as $t \to \infty$ for all $j \in \{1, \ldots, k\} \setminus \{1\}$.*

(d) *As $t \to \infty$, one has, using the notation as in Proposition 3.13,* (3.19), *and* (3.20)*:*

$$u(x,t) - \left(\sum_{j=1,\ldots,k} \phi_j(x - c_j t - \zeta_j(t)) - \sum_{j=1,\ldots,k-1} a_{j+1} \right) \to 0, \quad (3.21)$$

where the convergence is uniform with respect to $x \in \mathbb{R}$.

Proof. The existence of functions ζ_j such that statements (a), (d) hold are proved, under weaker conditions, in Theorems 2.11 and 2.19 of [22]. The convergence properties (b) and (c) are established (under the generic conditions (G1)-(G3)) in [22, Section 3]. We remark that in the case $f'(0) < 0$ the convergence in (3.21) is exponential (in this case, the theorem also follows from a more general result given [25, Theorem 2.2]). $\square$

Remark 3.15. The following observation will be useful below. Under the assumptions and notation of Proposition 3.14, let $\{(x_n, t_n)\}$ be a sequence in $\mathbb{R}^2$ such that $t_n \to \infty$ and and one of the following conditions is satisfied:

(a) $x_n - c_j t_n \to \bar{\eta} \in \mathbb{R}$ for some $j \in \{1, \ldots, k\}$; in the case $f'(0) > 0$ it is also required that $j \geq 2$.

(b) $|x_n - c_i t_n| \to \infty$ (as $n \to \infty$) for all $i \in \{1, \ldots, k\}$.

Then (a) implies that $u(x+x_n, t+t_n) \to \phi_j(x - c_j t + \bar{\eta} + \eta_j)$ locally uniformly with respect to $(x,t) \in \mathbb{R}^2$; and (b) implies that $u(x+x_n, t+t_n) \to a_i$ locally uniformly with respect to $(x,t) \in \mathbb{R}^2$ for some $i \in \{1, \ldots, k\}$. This is shown easily using Proposition 3.14 and the relations $c_1 > \cdots > c_k$ (cp. Proposition 2.3 and hypothesis (G3)).

4 Proofs of the main theorems

In this section, we assume that hypotheses (H) and (G1)–(G3) hold. Also, we assume that $u_0 \not\equiv 0$ is any nonnegative function in $C_0(\mathbb{R})$ such that the solution $u(\cdot, \cdot, u_0)$ is bounded; u_0 is assumed be fixed throughout the section.

With u_0 fixed, we simplify the notation by setting

$$\gamma_{max} := \gamma_{max}(u_0), \quad \mathcal{T}_\Omega := \mathcal{T}_\Omega(u_0)$$

(see (2.9), (3.6) for the definitions of $\gamma_{max}(u_0)$ and $\mathcal{T}_\Omega(u_0)$). Further, we reserve the symbol u for the solution of (1.1), (1.2):

$$u(x,t) = u(x,t,u_0) \quad (x \in \mathbb{R}, t \geq 0); \tag{4.1}$$

solutions of (1.1) with different initial conditions will always be denoted differently.

Since $u_0 \geq 0$, $u_0 \not\equiv 0$, we have $u(\cdot, t) > 0$ for all $t > 0$. Also, it is well known that the space $C_0(\mathbb{R})$ is invariant for (1.1), hence

$$\lim_{x \to \pm\infty} u(x,t) = 0 \quad (t \geq 0). \tag{4.2}$$

Recall from Lemma 3.5 that

$$\mathcal{T}_\Omega = \{\gamma_{max}\} \cup \{\psi(\cdot + \xi) : \xi \in \mathbb{R}\}, \tag{4.3}$$

where $\psi \equiv \gamma_{max}$ or ψ is a ground state at level γ_{max}.

Consider first the case $\gamma_{max} = 0$; that is $\mathcal{T}_\Omega = \Omega(u)$. Note that $f'(0) > 0$ is ruled out in this case. Indeed, (4.3) implies that either $u(\cdot, t) \to 0$ uniformly on $\mathbb{R}$ as $t \to \infty$, or there is ground state at level 0. It is well known that the former cannot hold for the positive solution $u(\cdot, t)$ if $f'(0) > 0$. Likewise, the latter is impossible if $f'(0) > 0$. Thus, due to (G2), we have $f'(0) < 0$. In this case, Theorem 2.5 follows from Theorem 2.7 and there is nothing to be proved in Theorems 2.9 and 2.11.

Henceforth, we assume that $\gamma_{max} > 0$.

We also fix the following notation. Let $\{(\phi_I, c_I) : I \in \mathcal{N}\}$ be the minimal $[0, \gamma_{max}]$-propagating terrace. For some integer k we have $\mathcal{N} = \{I_1, \ldots, I_k\}$ and $I_j = (a_j, a_{j+1})$, $j = 1, \ldots, k$, where

$$a_j \in \tilde{\Gamma} \quad (j = 1, \ldots, k+1), \qquad 0 = a_1 < a_2 < \cdots < a_{k+1} = \gamma_{max}. \tag{4.4}$$

We set

$$\phi_j := \phi_{I_j},\ c_j := c_{I_j} \quad (j = 1, \dots, k). \tag{4.5}$$

Recall from Section 2.1 that

$$c_1 > \dots > c_k > 0. \tag{4.6}$$

4.1 A modification of the nonlinearity

To facilitate an effective use of the results concerning front-like solutions, as given in Section 3.3, we modify the nonlinearity f in a suitable way outside the range of the solution u.

It follows from Lemma 3.5(ii) and the definition of $\Omega(u)$ that

$$S := \limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\mathbb{R})} = \|\psi\|_{L^\infty(\mathbb{R})},$$

with ψ as in (4.3). Therefore, given any $\epsilon > 0$, we may assume without loss of generality that

$$u(\cdot,t) < S + \epsilon \quad (t \geq 0). \tag{4.7}$$

Indeed, this is achieved when u_0 is replaced by $u(\cdot,t_1)$ with sufficiently large t_1, which has no effect on the validity of our theorems concerning the large-time behavior of $u(\cdot,t)$.

In view of (4.7), we can freely modify f in $(S+\epsilon,\infty)$ without effecting the validity of the theorems. We want to make the modification in such a way that, after the modification, (G1)–(G3) are still satisfied and there is $\hat\gamma \in \tilde\Gamma$ such that

$$\hat\gamma > S + \epsilon, \quad (\gamma_{max}, \hat\gamma) \cap \tilde\Gamma = \emptyset, \tag{4.8}$$

and the traveling front $\phi(x-ct)$ with range $(\gamma_{max},\hat\gamma)$ (cp. Lemma 3.8) normalized by the condition $\phi(0) = (\gamma_{max}+\hat\gamma)/2$ belongs to the minimal $[0,\hat\gamma]$-propagating terrace. By Lemma 3.10, a sufficient condition for the last requirement is that

$$c < c_k, \tag{4.9}$$

with c_k as in (4.6).

Note that if ψ is a ground state at level γ_{max}, then $S = \max\psi$ and we have $F(\gamma_{max}) = F(S)$, $f(S) > 0$, and $F(S) > F(u)$ for all $u \in (\gamma_{max}, S)$ (see the proof of Lemma 3.8). Thus, if $\epsilon > 0$ is sufficiently small, then

$$f(u) > 0 \quad (u \in [S, S+\epsilon]), \qquad \tilde\Gamma \cap (\gamma_{max}, S+\epsilon] = \emptyset. \tag{4.10}$$

In this case, it easy to modify f in $(S+\epsilon,\infty)$ so that it has a unique zero $\hat{\gamma}$ in $(S+\epsilon,\infty)$ and $f'(\hat{\gamma})<0$. It is clear from (4.10) that (G1), (G2) remain valid after the modification; and if (4.9) holds, then (G3) remains valid as well. By Lemma 3.9, to achieve (4.9), we just need to make sure that the modified f satisfies

$$F(\hat{\gamma})-F(S)=\int_S^{\hat{\gamma}} f(u)\,du<\frac{\min_{I\in\mathcal{N}} c_I}{d}:=d_0,$$

where d (as well as $\min_{I\in\mathcal{N}} c_I$) is a positive constant determined only by $f\big|_{[0,S]}$. It is easy to make such a modification, provided $\epsilon>0$ is so small that

$$F(S+\epsilon)-F(S)=\int_S^{S+\epsilon} f(u)\,du<d_0/2,$$

which can be assumed from the start.

If $\psi\equiv\gamma_{max}$, then, of course, $S=\gamma_{max}$ and, due to (G1), we have $f'(S)<0$. In this case, for all sufficiently small $\epsilon>0$ we have

$$f(u)<0 \quad (u\in[S,S+\epsilon]). \tag{4.11}$$

Modifying f in $(S+\epsilon,\infty)$ we achieve that there is $\gamma_2\in\tilde{\Gamma}$ with $\gamma_2>\gamma_{max}$ and the hypotheses of Lemma 3.8 are satisfied with $\gamma_1:=\gamma_{max}$. Lemma 3.8 in particular yields a ground state ψ at the level γ_1 with $\psi<\gamma_2$. One more modification of f, this time in $(\max\psi,\infty)$ similarly as above, produces a function satisfying all the desired conditions.

It will also be convenient to assume (again with no loss of generality) that $f(u)\neq 0$ for $u>\hat{\gamma}$ (this means that $f(u)<0$ for $u>\hat{\gamma}$, since $f'(\hat{\gamma})<0$).

We summarize what we have achieved by the above modifications of f as follows (the traveling front with range $(\gamma_{max},\hat{\gamma})$ is denoted by ϕ_{k+1} and its speed by c_{k+1}).

(M1) There exist $\hat{\gamma}\in\tilde{\Gamma}$ and a traveling front $\phi_{k+1}(x-c_{k+1}t)$ of (1.1) with range $(\gamma_{max},\hat{\gamma})$ such that

$$(\gamma_{max},\hat{\gamma})\cap\tilde{\Gamma}=\emptyset, \qquad f(u)<0 \quad (u>\hat{\gamma}), \tag{4.12}$$

and

$$\{(\phi_j,c_j): j=1,\dots,k+1\}$$

is the minimal $[0,\hat{\gamma}]$-propagating terrace. Moreover, for some $\delta>0$ one has

$$u(x,t)<\hat{\gamma}-\delta \quad (x\in\mathbb{R},\ t\geq 0). \tag{4.13}$$

If $f'(0) > 0$, we also make a modification of the nonlinearity in $(-\infty, 0)$ (this, of course, has no effect on the hypotheses (G1)–(G3) or the nonnegative solution u). What we want to achieve is the following.

(M2) There exist $a_0 < 0$ and a traveling front $\phi_0(x - c_0 t)$ of (1.1) with range (a_0, a_2) such that $f(a_0) = 0 > f'(a_0)$, $f < 0$ in $(a_0, 0)$, and the union

$$\{(\phi_j, c_j) : j \in \{1, \dots, k\},\ k \neq 1\} \cup \{(\phi_0, c_0)\}$$

is the minimal $[a_0, \gamma_{max}]$-propagating terrace.

For example, taking $q := f'(0)$, we can redefine f in $(-\infty, 0)$ such that for $u \in (-1, 0)$ one has $f(u) = u(a_0 - u)q/a_0$, where $a_0 \in (-1, 0)$ is a parameter. Then f is monostable $(0, a_2)$ and bistable in (a_0, a_2). Here "bistable" is understood in a broader sense: the end points a_0, a_2 are stable and there exists a front from a_0 to a_2 (the existence of the front is shown in the paragraph preceding Lemma 5.4 in the appendix). Also, c_1 is the minimal speed for the traveling fronts with range $I_1 = (0, a_2)$ (see [22, Theorem 2.4]; this theorem is a consequence of Theorems 1.3.8 and 1.3.14 of [27]). In the appendix (see Lemma 5.4), we show that as $a_0 \nearrow 0$ the speed c_0 of the traveling front with range (a_0, a_2) approaches c_1. In particular, due to (4.6), if $|a_0|$ is sufficiently small then $c_0 > c_2$. It then follows from Lemma 3.12 that all conditions in (M2) are satisfied.

In the remainder of Section 4, unless specified otherwise, we assume that, in addition to (G1)–(G3), (M1) is satisfied, and in the case $f'(0) > 0$ also (M2) is satisfied.

4.2 Sandwiching by front-like solutions

Our basic strategy for proving our main theorems in the case $f'(0) < 0$, and partly in the case $f'(0) > 0$, is to sandwich the graph of solution $u(\cdot, t)$ below γ_{max} by the graphs of two front-like solutions. We then use the convergence results for front-like solutions (Section 3.3) and the Liouville theorem stated in Lemma 3.7, to make conclusions on $\Omega(u)$. Our further considerations will build on the latter.

Finding suitable sandwiching front-like solutions, in particular the one at the bottom, requires a little bit of work. For an upper estimate, choose any decreasing function $u_0^+ \in C(\mathbb{R})$ such that

$$u_0^+(-\infty) = \hat{\gamma}, \quad u_0^+(\infty) = 0, \quad u_0^+ \geq u_0. \tag{4.14}$$

Such a choice is clearly possible due to (4.13) and the assumption that $u_0 \in C_0(\mathbb{R})$. We denote by u^+ the solution of (1.1), (1.2) with u_0 replaced by u_0^+. By the comparison principle, we have the following upper estimate:

$$u(x,t) < u^+(x,t) \quad (x \in \mathbb{R},\, t > 0). \tag{4.15}$$

For a lower estimate, we need some preparation. If $f'(0) < 0$, we set $a_0 := 0$ $(=a_1)$. If $f'(0) > 0$, we take a_0 as in (M2). We choose a monotone nonincreasing function $u_0^- \in C(\mathbb{R})$ such that for some $m_0 \in \mathbb{R}$ one has

$$u_0^-(-\infty) = \gamma_{max}, \qquad u_0^-(x) = a_0 \quad (x > m_0). \tag{4.16}$$

Denote by u^- the solution of (1.1), (1.2) with u_0 replaced by u_0^-. Then, by the comparison principle, for each $t > 0$ we have $\gamma_{max} > u^-(\cdot,t) > a_0$ and $u^-(\cdot,t)$ is a decreasing function. We now need the following technical result.

Lemma 4.1. *There are positive constants M, ϵ_0, $\mu < \nu$, and t_0 such that*

$$u(x,t) \geq \gamma_{max} - Me^{-\nu t} \quad (t \geq t_0,\ |x| \leq \epsilon_0 t) \tag{4.17}$$

and the function $\tilde{u}^-(x,t) := u^-(x - e^{-\mu t}, t) - e^{-\nu t}$ is a subsolution of equation (1.1)*, that is,*

$$\tilde{u}_t^- \leq \tilde{u}_{xx}^- + f(\tilde{u}^-), \quad x \in \mathbb{R},\ t > t_0. \tag{4.18}$$

Proof. Estimate of the form (4.17) is proved in [9, Lemma 2.6] for solutions of (1.1) with initial data

$$\tilde{u}_0 = \begin{cases} 0 & \text{for } |x| \geq R, \\ \theta & \text{for } |x| < R, \end{cases} \tag{4.19}$$

where $\theta \in (0, \gamma_{max})$ is such that $f > 0$ on $[\theta, \gamma_{max}]$ and $R = R(\theta)$ is sufficiently large. Assumptions in [9, Lemma 2.6] are $f'(\gamma_{max}) < 0$ (which in particular makes the choice of θ possible) and $F(v) < F(\gamma_{max})$ for all $v \in [0, \gamma_{max}]$. They are both satisfied in the present situation, as $\gamma_{max} \in \tilde{\Gamma}$ and (G2) is assumed.

It is clear from the fact that $\gamma_{max} \in \Omega(u)$ that a shift of the function $\tilde{u}_0$ of the form (4.19) can be placed below $u(\cdot, t_0)$ if t_0 is sufficiently large. The comparison principle and the lower estimate from [9, Lemma 2.6] then imply that (4.17) holds for suitably adjusted positive constants M, ν, and t_0.

Estimate (4.18) is proved by standard Fife-McLeod arguments (see [11]). First we apply Proposition 3.14 to the solution u^- (replacing 0 and a_1 by a_0 in the case $f'(0) > 0$). Using parabolic regularity estimates and the relations $\phi_j' < 0$, $j = 2, \ldots, k+1$, $\phi_0' < 0$, one proves easily that given any $\epsilon > 0$ there are positive constants σ and t_1 such that the following alternative holds for all $x \in \mathbb{R}$, $t \geq t_1$: either $|u^-(x,t) - a_j| < \epsilon$ for some $j \in \{2, \ldots, k\} \cup \{0\}$ or else $u_x^- < -\delta$. We choose ϵ so small that

$$\varsigma := \max\{f'(u) : |u - a_j| < \epsilon \text{ for some } j \in \{2, \ldots, k+1\} \cup \{0\}\} < 0. \tag{4.20}$$

Then a straightforward computation similar to that in [11], shows that if μ, ν are positive constants with $\mu < \nu$ and ν is sufficiently small and t_0 is large enough, then (4.18) holds. We now adjust the constants μ, ν, $\epsilon_0 > 0$, and t_0, such that both estimates (4.17) and (4.18) are valid with the same ν. $\square$

Remark 4.2. Further adjusting the constants ν and t_0, we may assume that (4.17) and (4.18) are valid and $M = 1$.

Obviously, Lemma 4.1 and Remark 4.2 remain valid if $u^-(x,t)$ is replaced by a spatial shift $u^-(x+\xi, t)$. We choose ξ positive and sufficiently large so that

$$\tilde{u}^-(x+\xi, t_0) = u^-(x+\xi - e^{-\mu t_0}, t_0) - e^{-\nu t_0} < 0 < u(x, t_0) \quad (x \geq 0). \tag{4.21}$$

This is possible since $u^-(\infty, t) = a_0 \leq 0$ for all t. Since $u^- < \gamma_{max}$ everywhere, using (4.17) (with $M = 1$) and (4.18) we obtain

$$\tilde{u}^-(x+\xi, t) < \gamma_{max} - e^{-\nu t} \leq u(x,t) \quad (t \geq t_0,\ 0 \leq x \leq \epsilon_0 t). \tag{4.22}$$

In view of (4.21), (4.22), the comparison principle gives

$$u^-(x+\xi - e^{-\mu t}, t) - e^{-\nu t} = \tilde{u}^-(x+\xi, t) < u(x,t) \quad (t \geq t_0,\ x \geq 0). \tag{4.23}$$

This is our lower estimate.

4.3 Proof of Theorem 2.9

We first use estimates (4.15), (4.23) to prove the following lemma.

Lemma 4.3. *The following statements are valid.*

(i) If $f'(0) < 0$ and $\varphi \in \Omega(u) \setminus \mathcal{T}_\Omega$, then either $\varphi \equiv a_j$ for some $j \in \{1, \dots, k+1\}$, or $\varphi \equiv \phi_j(\cdot - \xi)$ for some $j \in \{1, \dots, k\}$ and $\xi \in \mathbb{R}$, or else $\varphi \equiv \phi_j(\xi - \cdot)$ for some $j \in \{1, \dots, k\}$ and $\xi \in \mathbb{R}$.

(ii) If $f'(0) > 0$, $\varphi \in \Omega(u) \setminus \mathcal{T}_\Omega$, and $\varphi(x_0) > a_2$ for some $x_0 \in \mathbb{R}$, then either $\varphi \equiv a_j$ for some $j \in \{2, \dots, k+1\}$, or $\varphi \equiv \phi_j(\cdot - \xi)$ for some $j \in \{1, \dots, k\} \setminus \{1\}$ and $\xi \in \mathbb{R}$, or else $\varphi \equiv \phi_j(\xi - \cdot)$ for some $j \in \{1, \dots, k\} \setminus \{1\}$ and $\xi \in \mathbb{R}$.

Proof. Fix any $\varphi \in \Omega(u) \setminus \mathcal{T}_\Omega$; if $f'(0) > 0$ assume also that $\varphi(x_0) > a_2$ for some $x_0 \in \mathbb{R}$. There is a sequence $\{(x_n, t_n)\}$ with $t_n \to \infty$ such that $u(\cdot + x_n, t_n) \to \varphi$ in $L^\infty_{loc}(\mathbb{R})$. Passing to a subsequence, we may assume that

$$u(\cdot + x_n, \cdot + t_n) \to U \quad \text{in } L^\infty_{loc}(\mathbb{R}^2) \tag{4.24}$$

where U is an entire solution of (1.1) (cp. (3.5)) and, moreover, $x_n \to \infty$, or $x_n \to -\infty$, or else $x_n \to x_\infty \in \mathbb{R}$. Actually, the last possibility cannot occur, for estimate (4.17) would imply that $\varphi \geq \gamma_{max}$, in contradiction to our assumption that $\varphi \notin \mathcal{T}_\Omega$.

Consider, the case $x_n \to \infty$. Passing to a further subsequence of (x_n, t_n), we may assume (in addition to all the above properties) that one of the following 3 possibilities occurs (recall that c_{k+1} is as in (M1)):

(p1) $x_n \leq c_{k+1} t_n$ for all $n = 1, 2, \dots$,

(p2) $x_n > c_1 t_n$ for all $n = 1, 2, \dots$,

(p3) there is a fixed $j \in \{1, \dots, k\}$ such that $x_n \in (c_{j+1} t_n, c_j t_n]$ for $n = 1, 2, \dots$;

and, at the same time, one of the following 2 possibilities occurs:

(a) $x_n - c_j t_n \to \bar\eta \in \mathbb{R}$ for some $j \in \{1, \dots, k+1\}$

(b) $|x_n - c_i t_n| \to \infty$ (as $n \to \infty$) for all $i \in \{1, \dots, k+1\}$.

Now, from (4.15), (4.23), we obtain that for all sufficiently large n the following estimates hold:

$$\begin{aligned} u^-(x + x_n + \xi - e^{-\mu t_n}, t + t_n) - e^{-\nu t_n} \\ < u(x + x_n, t + t_n) < u^+(x + x_n, t + t_n) \\ (x \geq -x_n,\ t > -t_n/2). \end{aligned} \tag{4.25}$$

We apply Proposition 3.14 and Remark 3.15 to both u^- (with the interval $[a_0, \gamma_{max}]$ in place of $[0, \gamma]$) and u^+ (with the interval $[0, \hat{\gamma}]$ in place of $[0, \gamma]$).

Assume first that (a) holds. Observe that if $f'(0) > 0$, then necessarily $j \geq 2$. Indeed, if (a) held with $j = 1$, then from the upper estimate in (4.25) and Proposition 3.14 applied to u^+ we would obtain that

$$\varphi(x) = \lim_{n\to\infty} u(x + x_n, t_n) \leq a_2$$

in contradiction to our assumption on φ. For a similar reason, in both cases $f'(0) > 0$ and $f'(0) < 0$, we have $j \leq k$, for (a) with $j = k+1$ would in particular imply that $c_k t_n - x_n \to \infty$ and then from the lower estimate in (4.25) and Proposition 3.14 applied to u^- we would obtain that

$$\varphi(x) = \lim_{n\to\infty} u(x + x_n, t_n) \geq a_k = \gamma_{max},$$

contradicting the assumption $\varphi \notin \mathcal{T}_\Omega$.

From Remark 3.15 (applied to both to both u^- and u^+) and (4.25), we obtain, upon taking limits, that for some constants $\eta^\pm$ one has

$$\phi_j(x - c_j t + \eta^-) \leq U(x,t) \leq \phi_j(x - c_j t + \eta^+) \quad (x \in \mathbb{R},\, t \in \mathbb{R}). \tag{4.26}$$

This and Lemma 3.7 imply that $U(x,t)$ is a shift of $\phi_j(x - c_j t)$, in particular $\varphi = U(\cdot, 0) \equiv \phi_j(\cdot - \xi)$ for some $\xi \in \mathbb{R}$.

Next assume that (b) holds. Using Remark 3.15 and (4.25) similarly as above, we obtain that for some $i \in \{1, \dots, k\} \setminus \{1\} \cup \{0\}$ and $\ell \in \{1, \dots, k+1\}$ one has

$$a_i \leq U(x,t) \leq a_\ell \quad (x \in \mathbb{R},\, t \in \mathbb{R}), \tag{4.27}$$

where a_i, a_ℓ are the limits, as $n \to \infty$, of the functions $u^-(x + x_n + \xi, t + t_n)$, $u^+(x + x_n, t + t_n)$, respectively. We just need to show that $\ell = i$. This is clear from (b) (and Proposition 3.14 applied to $u^\pm$) if (p3) holds, as well as if (p2) holds and $f'(0) < 0$ (in this case $a_0 = a_1 = 0$). We show that the remaining cases cannot occur. Indeed, if (p1) holds, then in particular $c_k t_n - x_n \to \infty$, which implies that $i = k$ so $a_i = \gamma_{max}$. But then $\varphi = U(\cdot, 0) \geq \gamma_{max}$, which is impossible as $\varphi \notin \mathcal{T}_\Omega$. Similarly, if $f'(0) > 0$ and (p2) holds, then in particular, $x_n - c_2 t_n \to \infty$ which implies $\ell \leq 2$. But then $\varphi = U(\cdot, 0) \leq a_2$, which is impossible due to our assumption on φ in the case $f'(0) > 0$. Thus the conclusions of Lemma 4.3 hold in all cases when $x_n \to \infty$.

When $x_n \to -\infty$, the proof is completely analogous and can in fact be brought to the case $x_n \to \infty$ by the transformation $x \mapsto -x$. One proves this way that φ is one of the constants a_j, or the functions $\phi_j(\cdot - \xi)$, as claimed in the lemma. The proof is now complete. $\square$

Completion of the proof of Theorem 2.9 in the case $f'(0) < 0$. Assume that $f'(0) < 0$. Lemmas 4.3 and 3.5(i) show one desired inclusion: (2.11) holds with the equality sign replaced by the inclusion sign "$\subset$". On the other hand, it is clear from (4.17) and (4.2) that for each $\xi \in [0, \gamma_{max})$ (the translation invariant set) $\Omega(u)$ contains functions $\varphi^{\pm}$ with $\varphi^{\pm}(0) = \xi$, $\varphi_x^+(0) \leq 0$, $\varphi_x^-(0) \geq 0$. Using this and the inclusion just proved above, we see that the opposite inclusion must hold as well, completing the proof of (2.11). $\square$

What we are lacking in the case $f'(0) > 0$ is any information on the functions $\varphi \in \Omega(u)$ which take values between 0 and a_2. Clearly, the above estimates provide no such information, as the bottom parts of the functions $u^{\pm}$ in our estimates (4.25) propagate with different speeds. One could instead try to use a sandwiching argument directly in $[0, \infty)$—without introducing the bistable interval $[a_0, a_2]$ by the modification of f—choosing sandwiching functions $u_0^{\pm}$ that vanish near $x = \infty$ and applying Proposition 3.14 to each of them. This would not provide any additional information either, as Proposition 3.14 has no convergence result for the function $\zeta_1(t)$ in the case $f'(0) > 0$. Thus the solutions $u^{\pm}$ could still "drift apart" at their bottom parts, albeit with a sublinear speed. Taking the limits in estimates similar to (4.25) would then not lead to estimate (4.26) with $j = 1$. It seems that the only way to obtain such an estimate at the bottom level is to take from the start u^- as a shift of u^+, and ensure that (4.25) is valid in some relevant regions so that taking limits does give (4.26) with $j = 1$. We were able to make this idea work in some cases, but not in general. We use a different method, very similar to the technique of spatial trajectories used in [22].

In the remainder of this subsection, we assume that $f'(0) > 0$ and the support of u_0 is compact. Having already proved Lemma 4.3, condition (M2) will no longer be needed here. Instead, it will be convenient to temporally re-modify f in $(-\infty, 0)$ so that $f > 0$ there and f is Lipschitz on $\mathbb{R}$ (we loose the C^1 property of f at $u = 0$, but this is of no concern in this part of the paper).

We need some more preparation. First, recall that if $\kappa > 0$ is so large that $(-\kappa, \kappa)$ contains the support of u_0, then for all $t > 0$ one has

$$u_x(\cdot, t) > 0 \text{ in } (-\infty, -\kappa), \quad u_x(\cdot, t) < 0 \text{ in } (\kappa, \infty) \tag{4.28}$$

(see [7], for example.) Next, we draw the following simple conclusion from the upper estimate (4.15):

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}:\,|x|\geq(c_1+\epsilon)t} u(x,t) = 0 \quad (\epsilon > 0). \tag{4.29}$$

To show this, apply Proposition 3.14 to u^+ (the solution in (4.15)) and use the fact that for all large enough t one has $(c_1 t + \epsilon t > c_1 t + \zeta_1(t)$ (this follows form $\zeta_1'(t) \to 0)$. This way, one obtains a part of estimate (4.29), namely, the estimate with $|x|$ replaced by x. The other part of (4.29) is obtained by applying the above result to the solution $u(-x,t)$.

For the next result, we recall that $\tau(\phi)$ stands for the planar trajectory of ϕ (cp. (2.3)), and if ϕ is strictly monotone and of class C^1, the function p^ϕ is defined in (3.13). For the function ϕ_1, we have

$$\tau(\phi_1) = \{(u, p^{\phi_1}(u)) : u \in (0, a_2)\}. \tag{4.30}$$

We consider solutions q of (2.4), for some $c \in \mathbb{R}$, satisfying the initial conditions

$$q(0) = \xi, \quad q'(0) = \eta. \tag{4.31}$$

Lemma 4.4. *There is $\epsilon > 0$ such that if $\varphi \in \Omega(u)$ satisfies $\varphi(x_0) \in (a_2-\epsilon, a_2)$ for some $x_0 \in \mathbb{R}$, then $\varphi'(x_0) \neq 0$ and the following statements are valid:*

(i) If $\varphi'(x_0) < 0$, then there is $y_1 \in \mathbb{R}$ such that $\varphi \equiv \phi_1(\cdot - y_1)$.

(ii) If $\varphi'(x_0) > 0$, then there is $y_1 \in \mathbb{R}$ such that $\varphi \equiv \phi_1(y_1 - \cdot)$.

The following technical result, whose proof is given at the end of this subsection, will be used in the proof of Lemma 4.4.

Lemma 4.5. *If $\epsilon > 0$ is sufficiently small, then for any $\xi \in (a_2 - \epsilon, a_2)$ and $\eta \in (-\infty, 0]$ with $\eta \neq p^{\phi_1}(\xi)$, there is $c \in \mathbb{R}$ such the solution q of (2.4), (4.31) has the following property. For all $y \in \mathbb{R}$ and $t > 0$ the zero number $z(u(\cdot + ct + y, t) - q)$ is finite, and there are y_0, $t_1 > 0$ such that one of the following statements holds:*

(a) $q' < 0$ and

$$z(u(\cdot + ct + y, t) - q) = 1 \quad (t > 0,\ |y| \geq y_0);$$

(b) $\sup_{x\in\mathbb{R}} q(x) < a_2$, $c > c_1$, *and*

$$z_{(-ct-y,\infty)}(u(\cdot + ct + y, t) - q) = 1 \quad (t > 0,\ |y| \geq y_0).$$

Remark 4.6. If (a) holds, then the function $u(\cdot + ct + y, t) - q$ has a unique zero for all $t > 0$ and $|y| \geq y_0$; by Lemma 3.1, this zero has to be simple. If (b) holds, then there is $\bar{t}$ such that for all $t > \bar{t}$, and $|y| \geq y_0$, the unique zero of $u(\cdot + ct + y, t) - q$ in $(-ct - y, \infty)$ has to be simple. This is a consequence of Lemma 3.3, which applies to the variable interval $(-ct - y, \infty)$ because for all sufficiently large t we have $u(0, t) - q > 0$. This follows from the first relation in (b) and the fact that $\liminf_{t\to\infty} u(0, t) > \gamma_{max} \geq a_2$ (cp. (4.17)).

Proof of Lemma 4.4. Let $\epsilon > 0$ be as in Lemma 4.5. Assume that $\varphi \in \Omega(u)$ and $\varphi(x_0) \in (a_2 - \epsilon, a_2)$ for some $x_0 \in \mathbb{R}$. First, we prove that if $\varphi'(x_0) \neq 0$, then statements (i), (ii) hold. Then we rule out the possibility $\varphi'(x_0) = 0$.

For definiteness, assume that $\varphi'(x_0) < 0$, the case $\varphi'(x_0) > 0$ is analogous. We claim that

$$(\varphi(x), \varphi'(x)) \in \tau(\phi_1) \quad (x \approx x_0). \tag{4.32}$$

By Lemma 3.6, inclusion (4.32) and the fact that $\varphi = U(\cdot, 0)$ for some entire solution of (3.4) imply that $\varphi \equiv \phi_1(\cdot - y_1)$ for some y_1. Thus, by proving the claim we will have proved that statement (i) holds.

We go by contradiction. Assume (4.32) is not true for some $x = \tilde{x}_0$ which is so close to x_0 that $\xi := \varphi(\tilde{x}_0) \in (a_2 - \epsilon, a_2)$ and $\eta := \varphi'(\tilde{x}_0) < 0$. In terms of ξ and η, (4.32) not being true for $x = \tilde{x}_0$ means that $\eta \neq p^{\phi_1}(\xi)$. Replacing φ by a shift (which is still an element of $\Omega(u)$), we may assume that $\tilde{x}_0 = 0$.

With (ξ, η) as above, let c, y_0, and q be as in Lemma 4.5. Denote $\tilde{u}(x, t) := u(x + ct, t)$; this is a solution of equation (3.4) which is also solved by q. Since $\varphi \in \Omega(u) = \Omega(\tilde{u})$, there are sequences $x_n \in \mathbb{R}$, $t_n \to \infty$ such that

$$D^{2,1}\tilde{u}(\cdot + x_n, \cdot + t_n) \to D^{2,1}U \text{ in } L^\infty_{loc}(\mathbb{R}^2), \tag{4.33}$$

where U is an entire solution of (3.4) with $U(\cdot, 0) = \varphi$. Consider the function $v = U - q$, which is a solution of a linear equation (3.2). The definitions of ξ, η, and q imply that $v(\cdot, 0)$ has a multiple zero at $x = 0$. Therefore, by Lemma 3.4, there is a sequence $(\bar{x}_n, \tau_n)$ converging to $(0, 0)$ such that for all sufficiently large n the function

$$\tilde{u}(\cdot + x_n, t_n + \tau_n) - q = u(\cdot + c(t_n + \tau_n) + x_n, t_n + \tau_n) - q$$

has a multiple zero $\bar{x}_n$.

To find a contradiction, we distinguish two cases: $|x_n| \to \infty$ and $x_n \to \hat{x} \in \mathbb{R}$ (one of them occurs, after passing to a subsequence of $\{(x_n, t_n)\}$).

If $x_n \to \hat{x} \in \mathbb{R}$, then, by parabolic estimates and boundedness of the solution, the sequence $\tilde{u}(\cdot + \hat{x}, \cdot + t_n) - q$ converges to the same limit, $U - q$, as $\tilde{u}(\cdot + x_n, \cdot + t_n) - q$ (the convergence is in $C^1_{loc}(\mathbb{R}^2)$). Hence, applying Lemma 3.4 similarly as above, we obtain that for all sufficiently large n there is $\bar{\tau}_n \in (-1, 1)$ such that $\tilde{u}(\cdot + \hat{x}, \bar{\tau}_n + t_n) - q$ has a multiple zero. Now, $\bar{\tau}_n + t_n \to \infty$, while the zero number $z(\tilde{u}(\cdot + \hat{x}, t) - q)$ is finite due to Lemma 4.5. Thus, we have a contradiction to the properties of the zero number (see Remark 3.2).

Now assume that $|x_n| \to \infty$. In particular, for all sufficiently large n we have $|x_n| > y_0$, where y_0 is as in Lemma 4.5. If condition (a) in Lemma 4.5 holds, the existence of a multiple zero of the function $\tilde{u}(\cdot + x_n, \bar{\tau}_n + t_n) - q$ gives immediately a contradiction to Remark 4.6. If (b) holds, we obtain a contradiction similarly, we just need to show that the multiple zero $\bar{x}_n$ belongs to the interval $(-x_n - c(\bar{\tau}_n + t_n), \infty)$. We appeal to (4.33) and the facts that $U(\cdot, 0) = \varphi$ and $\bar{x}_n \to 0$. It is a consequence of the relations $\varphi(0) < a_2$ and $\varphi'(0) < 0$ that for all sufficiently large n one has

$$u(\bar{x}_n + x_n + ct_n, t_n) < a_2 - \delta, \quad u_x(\bar{x}_n + x_n + ct_n, t_n) < 0,$$

where δ is a positive constant. This, in conjunction with (4.17) and (4.28), implies the desired relation $\bar{x}_n + x_n + ct_n > 0$, and we have a contradiction to Remark 4.6.

It remains to rule out the possibility $\varphi'(x_0) = 0$. Suppose it holds. We claim that then $\varphi' \equiv 0$ on a neighborhood of x_0. Indeed, if not, then for some $\tilde{x}_0 \approx x_0$ we have $\varphi(\tilde{x}_0) \in (a_2 - \epsilon, a_2)$ and $\varphi'(\tilde{x}_0) \neq 0$. But then, as just shown above, one of the statements (i), (ii) applies, which in particular gives $\varphi'(x_0) \neq 0$—a contradiction.

Note that $\varphi_x(x) = 0$ for $x \approx x_0$ implies that, in fact, $\varphi_x \equiv 0$ on $\mathbb{R}$. To see this, apply Lemma 3.1 to the function $U_x(x, t)$, where U is the entire solution with $U(\cdot, 0) = \varphi$. Thus, to complete the proof, we need to show that if $\xi \in (a_2 - \epsilon, a_2)$, then (the constant function) ξ does not belong to $\Omega(u_0)$. Suppose it does: for some sequences $x_n \in \mathbb{R}$, $t_n \to \infty$ we have $u(\cdot + x_n, t_n) \to \xi$ in $L^\infty_{loc}(\mathbb{R})$. By (4.17), we have $|x_n| \to \infty$. Passing a subsequence, and replacing $u(x, t)$ by $u(-x, t)$ if necessary, we may assume that $x_n \to \infty$.

With ξ as above and $\eta := 0$, let $c > 0$, $y_0 > 0$, and q be as in Lemma 4.5. Note that statement (b) holds in this case ($q'(0) = \eta = 0$ rules out (a)). Also, for all large enough n we have $x_n < (c - \epsilon_0)t_n$ where $\epsilon_0 := (c - c_1)/2 > 0$. This follows from (4.29) and the relation $c > c_1$ (cp. statement (b)).

Set now $\tilde{u}(x,t) := u(x+ct,t)$ and $\tilde{x}_n := x_n - ct_n$, so that $\tilde{u}(\cdot+\tilde{x}_n, t_n) \to \xi$. Passing to a subsequence, we have $\tilde{u}(\cdot + \tilde{x}_n, \cdot + t_n) \to U$ in $C^1_{loc}(\mathbb{R}^2)$, where $U = U(t)$ is the solution of the equation $\dot{U} = f(U)$ with the initial condition $U(0) = \xi$ (this is the entire solution corresponding to $\xi \in \Omega(\tilde{u})$). As in the previous part of the proof, Lemma 3.4 implies that there is a sequence $(\bar{x}_n, \tau_n)$ converging to $(0,0)$ such that for all sufficiently large n the function $\tilde{u}(\cdot + \tilde{x}_n, t_n + \tau_n) - q = u(\cdot + c(t_n + \tau_n) + \tilde{x}_n, t_n + \tau_n) - q$ has a multiple zero $\bar{x}_n$. We have, for all large enough n, $\tilde{x}_n = x_n - ct_n < -\epsilon_0 t \to -\infty$. Hence, for all large enough n, $|\tilde{x}_n| > y_0$. Also, since $(\bar{x}_n, \tau_n) \to (0,0)$, for all large enough n, we have $\bar{x}_n > -x_n - c\tau_n = -(c(t_n + \tau_n) + \tilde{x}_n)$, meaning that the multiple zero $\bar{x}_n$ is in the interval $(-(c(t_n + \tau_n) + \tilde{x}_n), \infty)$. We thus have a contradiction to Remark 4.6. The proof of Lemma 4.4 is now complete. $\square$

Completion of the proof of Theorem 2.9 in the case $f'(0) > 0$. With ϵ as in Lemma 4.4, fix $\theta \in (a_2 - \epsilon, a_2)$. From (4.28), (4.17), and (4.2) it follows that for all sufficiently large t there are uniquely determined points $\zeta^-(t) < -\kappa$, $\zeta^+(t) > \kappa$ such that $u(\pm(\zeta^\pm(t) + c_1 t), t) = \theta$. Moreover, one has $u_x(\pm(\zeta^\pm(t) + c_1 t), t) < 0$. We claim that

$$\lim_{t\to\infty} u(\cdot + c_1 t + \zeta^+(t), t) = \phi_1(\cdot + \xi) \quad \text{in } L^\infty_{loc}(\mathbb{R}), \tag{4.34}$$

and

$$\lim_{t\to\infty} u(\cdot - c_1 t - \zeta^-(t), t) = \phi_1(\xi - \cdot) \quad \text{in } L^\infty_{loc}(\mathbb{R}), \tag{4.35}$$

where $\xi \in \mathbb{R}$ is the unique point with $\phi_1(\xi) = \theta$. We only give a proof of (4.34), the proof of (4.35) is analogous.

It is sufficient to prove that given any sequence $\{t_n\}$ with $t_n \to \infty$, passing to a subsequence one achieves that (4.34) holds with t replaced by t_n. We can always choose the subsequence in such a way that $u(\cdot + c_1 t_n + \zeta^+(t_n), t_n) \to \varphi$ for some $\varphi \in \Omega(u)$ (the convergence is in $C^1_{loc}(\mathbb{R})$). Clearly, $\varphi(0) = \theta$, $\varphi'(0) \le 0$. By Lemma 4.4, $\varphi \equiv \phi_1(\cdot + \xi)$ for some $\xi \in \mathbb{R}$. Our claim is proved.

Now, from (4.28), (4.2) we deduce that the convergence in (4.34) holds uniformly on any interval (λ, ∞), $\lambda \in \mathbb{R}$. Consequently, for any $\varphi \in \Omega(u)$ such that $\varphi(x_0) \le \theta$ and $\varphi'(x_0) \le 0$ for some x_0, one has $\varphi \equiv \phi_1(\cdot + \xi)$ for some $\xi \in \mathbb{R}$. In an analogous way one proves, using (4.35), that for any

$\varphi \in \Omega(u)$ such that $0 < \varphi(x_0) \le \theta$ and $\varphi'(x_0) \ge 0$ for some x_0, one has $\varphi \equiv \phi_1(\xi - \cdot)$ for some $\xi \in \mathbb{R}$. Since $\theta \in (a_2 - \epsilon, a_2)$ is arbitrary, we see that in fact any $\varphi \in \Omega(u)$ taking values in $(0, a_2)$ is a shift of ϕ_1 or $\tilde{\phi}_1$. This and Lemma 4.3(ii) imply that (2.11) holds with the equality sign replaced by the inclusion sign $\subset$. The opposite inclusion is proved the same way as in the case $f'(0) < 0$. Theorem 2.9 is proved. □

Remark 4.7. The statement of Theorem 2.9 can be made a little more precise. Assume that $\varphi \in \Omega(u) \setminus \mathcal{T}_\Omega$ and $\{(x_n, t_n)\}$ is a sequence with $t_n \to \infty$ such that $u(\cdot + x_n, t_n) \to \varphi$ in $L^\infty_{loc}(\mathbb{R})$ (and in $C^1_{loc}(\mathbb{R})$). We know from (2.11) that if $\varphi'(0) = 0$, then $\varphi \equiv a_j$ for some $j \in \{1, \dots, k\}$, and if $\varphi'(0) \neq 0$, then φ is a shift of one of the functions ϕ_j, $\tilde{\phi}_j$, $j \in \{1, \dots, k\}$. Now, more precisely, if $\varphi'(0) < 0$, then necessarily $x_n \to \infty$ and φ is a shift of one of the functions ϕ_j, $j \in \{1, \dots, k\}$; and if $\varphi'(0) > 0$, then necessarily $x_n \to -\infty$ and φ is a shift of one of the functions $\tilde{\phi}_j$, $j \in \{1, \dots, k\}$. This follows from estimates (4.15), (4.23) (and analogous estimates with shifts of the reflected functions $u^\pm(-x, t)$) if $a_2 < \varphi(0) < \gamma_{max}$, or $0 \le \varphi(0) < a_2$ and $f'(0) < 0$. If $0 \le \varphi(0) < a_2$ and $f'(0) > 0$, it follows from (4.17) and (4.28). The point is that it is impossible that $u(\cdot + x, t_n)$ approaches ϕ_j along a sequence of points $x = x_n$ and $\tilde{\phi}_j$ along a sequence of points $x = y_n$ with $y_n > x_n$. (Such a possibility is not ruled out by (2.11) alone.)

Proof of Lemma 4.5. It is proved in [22]—see Corollary 3.18 in [22]—that if $\xi \in (0, a_2)$ is arbitrary and $\eta < p^{\phi_1}(\xi)$, then there is $c \in \mathbb{R}$ such that the solution q of (2.4), (4.31) satisfies the following relations for some $x_1 < x_2$:

$$\begin{aligned} &q(x) \in (0, \hat{\gamma}) \quad (x \in (x_1, x_2)), \\ &q(x_1) = \hat{\gamma}, \ q(x_2) = 0, \\ &q'(x) < 0 \quad (x \in [x_1, x_2]). \end{aligned} \tag{4.36}$$

From the assumptions that $f < 0$ on $(\hat{\gamma}, \infty)$ (cp. (M1)) and $f > 0$ on $(-\infty, 0)$ (this is our temporary re-modification of f), one shows easily—or, see formula (6.2) in [22]—that $q' < 0$ on $\mathbb{R}$, hence $q < 0$ on (x_2, ∞), $q > \hat{\gamma}$ on $(-\infty, x_1)$. Since $0 \le u(\cdot, t) \le \hat{\gamma}$ for all $t \ge 0$, Lemma 3.1 implies that the zero number $z(u(\cdot + ct + y, t) - q)$ is finite and greater than or equal to 1 for all $y \in \mathbb{R}$ and $t > 0$. Also, since u_0 has compact support, it is clear that the zero number is equal to 1 for $t = 0$—hence, by the monotonicity, for any $t > 0$—if $|y|$ is sufficiently large. Statement (a) holds in this case.

It is also proved in [22] that if $\xi \in (0, a_2)$ is sufficiently close to a_2 and $\eta \in (p^{\phi_1}(\xi), 0]$, then there is $c > \max\{c_1, 2\sqrt{f'(0)}\}$ such that the solution q of (2.4), (4.31) satisfies the following relations for some $x_0 \in \mathbb{R}$ and $\lambda > 0$:

$$\begin{aligned} &0 < q(x) < a_2 \quad (x \in (x_0, \infty)), \\ &q(x_0) = 0, \;\; q'(x_0) > 0, \\ &\lim_{x\to\infty} (q(x), q'(x)) \to (0,0), \\ &\lim_{x\to\infty} \frac{q'(x)}{q(x)} = -\lambda. \end{aligned} \tag{4.37}$$

See Proposition 3.25(iii) in [22] (for the relation $c > c_1$, see also Part 3 of the proof [22, Proposition 3.25]). The last property in (4.37) implies—see [22, Remark 3.6(iii)]—that for some $x_1 \in \mathbb{R}$ one has

$$q(x), \; -q_x(x) > e^{-\lambda x/2} \quad (x \geq x_1). \tag{4.38}$$

On the other hand, the assumption that u_0 has compact support implies—see [22, Lemma 6.4], for example—that given any $t > 0$, there is a constant $C = C(t)$ such that

$$u(x,t), \; |u_x(x,t)| \leq Ce^{-\lambda x} \quad (x \in \mathbb{R}). \tag{4.39}$$

Similarly as above, the relation $f > 0$ on $(-\infty, 0)$ implies that $q < 0$ on $(-\infty, x_0)$. Using relations (4.39), (4.38), and $u \geq 0$, and applying Lemma 3.1, we obtain that $z(u(\cdot + ct + y, t) - q)$ is finite for all $y \in \mathbb{R}$ and $t > 0$.

We next show that (b) holds. The relation $\sigma := \sup_{x\in\mathbb{R}} q(x) < a_2$ follows from (4.37) and the relation $q < 0$ on $(-\infty, x_0)$. Due to (4.17), we have $u(0,t) > \sigma$ for all $t \geq t_1$, if t_1 is sufficiently large. In view of this and relations (4.39) and (4.38), Lemma 3.3 is applicable. We obtain that the zero number in (b) is (finite) and nonincreasing in t for $t \geq t_1$. Clearly, the zero number is always at least 1. Now, fixing any large enough $t = t_1$, relations (4.39), (4.38), and $u(0, t_1) > \sigma$ imply that

$$z_{(-ct_1 - y, \infty)}(u(\cdot + ct_1 + y, t_1) - q) = 1 \tag{4.40}$$

if y is sufficiently large and negative; while relations (4.38) and $q'(x_0) > 0$ imply that (4.40) holds if y is sufficiently large and positive. Hence, (b) follows from the monotonicity of the zero number. $\square$

4.4 The sides of the graph of $u(\cdot,t)$

In this section, we give a part of the proof of Theorem 2.11. Namely, we describe the graph of $u(\cdot,t)$ below the line $\{u = \gamma_{max}\}$.

We start with the following result.

Lemma 4.8. *Let $j \in \{1,\dots,k\}$ and $\theta \in (a_j, a_{j+1})$. Then there exist $s_0 > 0$ and a C^1 functions $\zeta^{\pm}(t)$ defined on (s_0,∞) such that*

$$\big(\theta - u(x + c_j t + \zeta^+(t), t)\big)x < 0 \quad (t > s_0,\, x \neq 0,\, x + c_j t + \zeta^+(t) > 0), \tag{4.41}$$

$$\big(\theta - u(x - c_j t - \zeta^-(t), t)\big)x > 0 \quad (t > s_0,\, x \neq 0,\, x - c_j t - \zeta^-(t) < 0), \tag{4.42}$$

$(\zeta^{\pm})'(t) \to 0$ as $t \to \infty$, and

$$\lim_{t\to\infty} u(\cdot \pm (c_j t + \zeta^{\pm}(t)), t) = \phi_j(\cdot \pm x_0) \;\; in\ C^1_{loc}(\mathbb{R}), \tag{4.43}$$

where x_0 is the unique point with $\phi_j(x_0) = \theta$.

Proof. Fix an arbitrary $\theta \in (a_j, a_{j+1})$. We prove the existence and the indicated properties for the function ζ^+; the proof of ζ^- is analogous.

Theorem 2.9 and the relations $\phi_j' < 0$, $j = 1,\dots,k$, imply that if t is large enough, then $u_x(x,t) \neq 0$ whenever $u(x,t) = \theta$. If $u(x,t) = \theta$ and x is large enough, then actually $u_x(x,t) < 0$; this follows from Remark 4.7. By (4.17) and (4.2), for all large enough t there is at least one zero of $u(\cdot + c_1 t, t) - \theta$ in $(-c_j t, \infty)$; we denote by $\zeta^+(t)$ the minimal of the zeros in $(-c_j t, \infty)$. By (4.17), $c_1 t + \zeta^+(t) \to \infty$ as $t \to \infty$. Therefore, for large t, there can be no other zero x of $u(x + c_1 t, t) - \theta$ in $[\zeta^+(t), \infty)$, for at least one of them would have $u_x(x + c_1 t, t) \geq 0$, contradicting the above note on u_x. Thus, $x = \zeta^+(t)$ is the unique solution of $u(x + c_1 t, t) = \theta$ with $x > -c_1 t$. The uniqueness and (4.17) clearly imply (4.41) if s_0 is large enough. By the implicit function theorem, ζ^+ is a C^1-function.

With x_0 defined as in the lemma, one has $u(c_1 t + \zeta^+(t), t) = \theta = \phi_j(x_0)$. Now, any sequence $t_n \to \infty$ has a subsequence such that $u(\cdot + c_1 t_n + \zeta^+(t_n), t_n)$ converges in $C^1_{loc}(\mathbb{R})$ to an element $\varphi \in \Omega(u)$ with $\varphi(0) = \theta$, $\varphi'(0) \leq 0$. By Theorem 2.9, $\varphi \equiv \phi_j(\cdot + x_0)$. Since this is true, with the same limit, for any sequence $\{t_n\}$, we have proved (4.43).

It remains to show that $(\zeta^+)'(t) \to 0$ as $t \to \infty$. This is rather standard; we just repeat an argument from the proof of Lemma 6.14 in [22]. To simplify the notation, set $\tilde{u}(x,t) := u(x + c_1 t, t)$. Recall from Section 3.2 that any sequence $t_n \to \infty$ can be replaced by a subsequence such that $\tilde{u}(\cdot + \zeta^+(t_n), \cdot + t_n)$ converges in $C^1_{loc}(\mathbb{R}^2)$ to an entire solution U of equation (3.4). By (4.43), we have $U(\cdot, 0) = \phi_j(\cdot + x_0)$. Since ϕ_j is a steady state of (3.4), we have $U \equiv \phi_j$, by uniqueness and backward uniqueness for (3.4). Thus, the convergence in $C^1_{loc}(\mathbb{R}^2)$ yields

$$(\tilde{u}(\cdot + \zeta^+(t_n), \cdot + t_n), \tilde{u}_x(\cdot + \zeta^+(t_n), \cdot + t_n), \tilde{u}_t(\cdot + \zeta^+(t_n), \cdot + t_n)) \\ \to (\phi_j(\cdot + x_0), \phi_j'(\cdot + x_0), 0).$$

Since this is true for any sequence $t_n \to \infty$, the convergence takes place with t_n replaced by t, with $t \to \infty$. In particular, at $x = 0$ we have

$$\big(\tilde{u}(\zeta^+(t), t), \tilde{u}_x(\zeta^+(t), t), \tilde{u}_t(\zeta^+(t), t)\big) \to (\theta, \phi_j'(x_0), 0), \tag{4.44}$$

as $t \to \infty$. By the definition of ζ^+, $\tilde{u}(\zeta^+(t), t) = \theta$. Differentiating this relation, we obtain $\tilde{u}_x(\zeta^+(t), t)(\zeta^+)'(t) + u_t(\zeta^+(t), t) = 0$. Since $\phi_j'(x_0) \neq 0$, from (4.44) we conclude that $(\zeta^+)'(t) \to 0$ as $t \to \infty$.

□

We now define functions $\zeta_j^\pm$ for which Theorem 2.11 will be proved. Given $j \in \{1, \dots, k\}$, set $\theta := (a_j + a_{j+1})/2$ and $\zeta_j^\pm := \zeta^\pm$ with $\zeta^\pm$ as in Lemma 4.8. All these functions are defined on some interval (t_0, ∞). Note that the choice of θ implies that $x_0 = 0$ in Lemma 4.8.

By Lemma 4.8, the functions $\zeta_j^\pm$ satisfy statement (a) of Theorem 2.11. The following lemma is needed for the proof of statement (d).

Lemma 4.9. *Let $\zeta_j^\pm$ be defined as above. As $t \to \infty$, we have, for any $\ell > 0$,*

$$\sup_{x \in (c_k t - \ell, \infty)} \left| u(x,t) - \left(\sum_{j=1,\dots,k} \phi_j(x - c_j t - \zeta_j^+(t)) - \sum_{j=1,\dots,k-1} a_{j+1} \right) \right| \to 0, \tag{4.45}$$

$$\sup_{x \in (-\infty, -c_k t + \ell)} \left| u(x,t) - \left(\sum_{j=1,\dots,k} \tilde{\phi}_j(x + c_j t + \zeta_j^+(t)) - \sum_{j=1,\dots,k-1} a_{j+1} \right) \right| \to 0. \tag{4.46}$$

Proof. We only prove (4.45), (4.46) being analogous. We derive this result from Lemma 4.8 applied with $\theta := (a_j + a_{j+1})/2$ and $j = 1, \dots, k$.

Suppose that (4.45) is not true: there are sequences $x_n \in (\ell, \infty)$, $t_n \to \infty$ such that for $n = 1, 2 \dots$ the value of the modulus in (4.45) evaluated at $u(x_n, t_n)$ is bounded from below by some $\epsilon > 0$. Since $c_1 > \cdots > c_k$ and $\zeta_j'(t) \to 0$ as $t \to \infty$ for $j = 1, \dots, k$, we have $c_{j+1}t + \zeta_{j+1}(t) < c_j t + \zeta_{j+1}(t)$, $j = 1, \dots, k-1$, for all large enough t. Therefore, passing to a subsequence of (x_n, t_n), we may assume that one of the following 3 possibilities occurs (recall that c_{k+1} is as in (M1)):

(p1) $x_n \le c_k t_n + \zeta_k(t_n)$ for all $n = 1, 2, \dots,$

(p2) $x_n > c_1 t_n + \zeta_1(t_n)$ for all $n = 1, 2, \dots,$

(p3) there is a fixed $i \in \{1, \dots, k-1\}$ such that $x_n \in (c_{i+1}t_n + \zeta_{i+1}(t_n), c_i t_n + \zeta_i(t_n)]$ for all $n = 1, 2, \dots$.

Assume the case that (p3) occurs. Passing to a further subsequence, we may assume that one of the following 3 possibilities occurs:

(a1) $c_i t_n + \zeta_i(t_n) - x_n \to \bar{\eta} \in \mathbb{R}$,

(a2) $x_n - c_{i+1}t_n - \zeta_{i+1}(t_n) \to \bar{\eta} \in \mathbb{R}$,

(a3) $|x_n - c_j t_n - \zeta_j(t_n)| \to \infty$ (as $n \to \infty$), $j = i, i+1$.

If (a1) holds, then by Lemma 4.8 and the definition of ζ_i, we have

$$u(x_n, t_n) - \phi_i(x_n - c_i t_n - \zeta_i(t_n)) \to 0. \tag{4.47}$$

Also,

$$\phi_j(x_n - c_j t_n - \zeta_j(t_n)) \to \begin{cases} \phi_j(-\infty) = a_{j+1} & \text{if } j < i, \\ \phi_j(\infty) = a_j & \text{if } j > i. \end{cases} \tag{4.48}$$

Therefore the limit of the modulus in (4.45) evaluated at $u(x_n, t_n)$ is zero and we have a contradiction. Similarly one obtains a contradiction if $(a2)$ holds.

Assume that (a3) holds. Then (4.48) holds and also

$$\phi_j(x_n - c_j t_n - \zeta_j(t_n)) \to \phi_j(-\infty) = a_j. \tag{4.49}$$

Passing to a subsequence we may assume that $u(\cdot + x_n, t_n) \to \varphi$, where $\varphi \in \Omega(u)$, hence either φ is one of the constants $a_1, \dots, a_{k+1}$ or $\varphi = \phi_j(\cdot - y)$

for some $j \in \{1, \dots, k\}$ and $y \in \mathbb{R}$ (the latter follows from Remark (4.7) and the fact that $x_n \to \infty$, cp. (p3)). By (p3), the definition of the ζ_j, and Lemma 4.8, $(a_j + a_{j+1})/2 < u(x_n, t_n) < (a_{j+1} + a_{j+2})/2$. Hence, necessarily, $\varphi = a_j$. Using this in conjunction with (4.48) and (4.49), we obtain that, again, the limit of the modulus in (4.45) evaluated at $u(x_n, t_n)$ is zero, which is a contradiction. The cases (p1), (p2) can be considered similarly and we omit these details. $\square$

We now show that the functions $\zeta_j^{\pm}$ satisfy statements (b) and (c) of Theorem 2.11:

Lemma 4.10. *The function $\zeta_j^{\pm}(t)$ has a finite limit $\eta_j^{\pm}$ as $t \to \infty$ for all $j \in \{1, \dots, k\} \setminus \{1\}$. If $f'(0) < 0$, then $\zeta_1^{\pm}(t)$ has a finite limit limit $\eta_1^{\pm}$ as $t \to \infty$, as well.*

We will only prove the convergence of the functions ζ_j^{+}; the proof for ζ_j^{-} is analogous. For the remainder of this section, we fix $j \in \{1, \dots, k\}$, assuming that in the case $f'(0) > 0$ we have $j > 1$ (if such j exists).

Notice that the convergence $\zeta_j^{\pm}(t) \to \eta_j^{\pm}$ is equivalent to the following property of the function $u(x + c_j t, t)$:

$$u(\cdot + c_j t, t) \to \phi_j(\cdot - \eta^{+}) \text{ in } L^{\infty}_{loc}(\mathbb{R}). \tag{4.50}$$

This follows from the definition of ζ_j^{+} (we have (4.43) with $\zeta = \zeta_j^{+}$).

Our arguments for the proof of (4.50) are just minor modifications of the arguments given in the proof of Theorem 2.22 in [22, Section 6.9] (similar arguments can also be found in [25]). Therefore we will omit some straightforward computations.

The idea is to relate the function $\tilde{u}(x, t) := u(x + c_j t, t)$ to a solution of an asymptotically autonomous equation of the form

$$v_t = v_{xx} + c_j v_x + f(v) + h(x, t), \quad x \in \mathbb{R}, \ t > 0. \tag{4.51}$$

Here h is a uniformly continuous function on $\mathbb{R} \times [0, \infty)$ such that for some positive constants κ and σ one has

$$\|h(\cdot, t)\|_{L^{\infty}(\mathbb{R})} \leq \kappa e^{-\sigma t} \quad (t \geq 0). \tag{4.52}$$

Note that $\tilde{u}$ solves (4.51) with $h \equiv 0$.

Solutions of (4.51) have the following convergence property (see [22, Lemma 6.23]):

Lemma 4.11. *Under the above assumptions on h, assume that v is a solution of* (4.51) *such that*

$$\inf_{\eta\in\mathbb{R}} \|v(\cdot,t) - \phi_j(\cdot - \eta)\|_{L^\infty(\mathbb{R})} \to 0, \quad \text{as } t \to \infty. \tag{4.53}$$

Then there are $\eta \in \mathbb{R}$ and $\vartheta > 0$ such that

$$\lim_{t\to\infty} e^{\vartheta t} \|v(\cdot,t) - \phi_j(\cdot - \eta)\|_{L^\infty(\mathbb{R})} = 0. \tag{4.54}$$

Proof of Lemma 4.10. By Lemma 4.11, to prove (4.50) it is sufficient to find functions h and v such that the hypotheses of Lemma 4.11 are satisfied and for some positive constants $t_0 > 0$, δ one has

$$v(x,t) = \tilde{u}(x,t) \quad (-\delta t < x < \delta t,\ t \geq t_0). \tag{4.55}$$

To find such functions v and h, we need some estimates on the function u. Define c_j^- and c_j^+ as follows:

$$c_j^- = \frac{c_{j+1} + c_j}{2} \tag{4.56}$$

(recall that c_{k+1} appears in the minimal $[0,\hat{\gamma}]$-propagating terrace) and

$$c_j^+ = \begin{cases} \infty & \text{if } j = 1 \text{ (and } f'(0) < 0), \\ \dfrac{c_{j-1} + c_j}{2} & \text{if } j > 1. \end{cases} \tag{4.57}$$

Obviously, $c_j^- < c_j < c_j^+$.

We claim that for some positive constants s, ϵ, c_0, and ς the following estimates are valid:

$$u(x,t) \leq a_{j+1} + c_0 e^{-\varsigma t} \quad (x > (c_j^- - \epsilon)t,\ t \geq s), \tag{4.58}$$
$$u(x,t) \geq a_{j+1} - c_0 e^{-\varsigma t} \quad (0 < x < (c_j^- + \epsilon)t,\ t \geq s), \tag{4.59}$$

and

$$u(x,t) \geq a_j - c_0 e^{-\varsigma t} \quad (0 < x < (c_j^+ + \epsilon)t,\ t \geq s), \tag{4.60}$$
$$u(x,t) \leq a_j + c_0 e^{-\varsigma t} \quad (x > (c_j^+ - \epsilon)t,\ t \geq s). \tag{4.61}$$

Indeed, as shown in [22, Section 6.9], such estimates are valid (without the constraints $0 < x$) for front-like solutions, such as the solutions u^+, u^- in our upper and lower estimates (4.15), (4.23) (the generic assumptions (G1)–(G3) are important for this). Using those estimates together with (4.15), (4.23) and adjusting the constants, we obtain estimates (4.58)–(4.61).

As in [22, Section 6.9], using (4.58)-(4.61) and parabolic estimates, one obtains, possibly after adjusting the constants c_0, ς, and s, that

$$|u_x(x,t)| \leq c_0 e^{-\varsigma t} \quad (x \in (c_j^- t - 1, c_j^- t] \cup [c_j^+ t, c_j^+ t + 1),\ t \geq s). \tag{4.62}$$

We now define functions v and h as in [22]. Let ρ be a smooth function on $\mathbb{R}$ such that $0 \leq \rho \leq 1$, $\rho \equiv 0$ on $(-\infty, 0)$, and $\rho \equiv 1$ on $(1, \infty)$. Define first a function w on $\mathbb{R} \times (s, \infty)$ by

$$w(x,t) = \begin{cases} \big(1 - \rho(x - (c_j^- t - 1))\big) a_{j+1} + \rho(x - (c_j^- t - 1)) u(x,t) & (x \leq c_j t), \\ \big(1 - \rho(x - c_j^+ t)\big) u(x,t) + \rho(x - c_j^+ t) a_j & (x \geq c_j t). \end{cases}$$

It is understood here that $\rho(-\infty) = 0$, so in the case $c_j^+ = \infty$, we have $w(x,t) = u(x,t)$ on $(c_j t, \infty)$. Notice that

$$v(x,t) := w(x + c_j t, t) = \begin{cases} a_{j+1} & (x < (c_I^- - c_I)t - 1), \\ \tilde{u}(x,t) & ((c_j^- - c_j)t < x < (c_j^+ - c_j)t), \\ a_j & (x > (c_I^+ - c_I)t + 1). \end{cases} \tag{4.63}$$

In particular, (4.55) holds with a sufficiently small δ and sufficiently large t_0.

Now set

$$h(x,t) := v_t(x,t) - v_{xx}(x,t) - c_j v_x(x,t) - f(v_t(x,t)),$$

so that v obviously satisfies equation (4.51). It is straightforward to verify (cp. [22]) that h is uniformly continuous and (4.27) holds for some constants κ and σ.

Finally, using (4.50), (4.63), (4.58)-(4.61), and the relations $\phi_j(-\infty) = a_{j+1}$, $\phi_j(\infty) = a_j$, one shows easily that

$$\lim_{t \to \infty} \|v(\cdot, t) - \phi_j(\cdot - \zeta_j(t))\|_{L^\infty(\mathbb{R})} = 0. \tag{4.64}$$

Thus v, h satisfy all hypotheses of Lemma 4.11. As noted at the beginning of the proof, this and (4.55) imply the conclusion of Lemma 4.10. $\square$

4.5 Proof of Theorems 2.5 and 2.11

In the previous section, we have defined functions $\zeta_j^{\pm}$ and proved that they satisfy statements (a), (b), (c) of Theorem 2.11, as well as the conclusions of Lemmas 4.9 and 4.8 (with $\theta = (a_j + a_{j+1})/2$, $\zeta^{\pm} = \zeta_j^{\pm}$). The following relations, where t_0 is assumed to be sufficiently large, follow directly from Lemma 4.8 (cp. (4.41), (4.42)):

$$\begin{aligned} &u(x,t) > (a_j + a_{j+1})/2 \;\; (t > t_0,\, -c_j t - \zeta_j^-(t) < x < c_j t + \zeta_j^+(t)) \\ &u(x,t) < (a_j + a_{j+1})/2 \;\; (t > t_0,\, x \in (-\infty, -c_j t - \zeta_j^-(t)) \cup (c_j t + \zeta_j^+(t), \infty)). \end{aligned} \tag{4.65}$$

We now prove the convergence result, Theorem 2.5, in the case $f'(0) < 0$. The proof depends heavily on the convergence property of the functions $\zeta_k^{\pm}$.

Proof of Theorem 2.5. Assume $f'(0) < 0$. To simplify the notation, set $\eta^{\pm} := \eta_k^{\pm}$ (with $\eta_k^{\pm}$ as in Theorem 2.11(b)).

Let ψ be as in (4.3). If $\psi \equiv \gamma_{max}$, it follows immediately from the definition of γ_{max} (see (2.9)) and (4.17) that $\omega(u) = \{\gamma_{max}\}$. In this case, we are done. Henceforth we assume that ψ is a ground state at level γ_{max}. We prove the convergence of $u(\cdot,t)$ to a shift of ψ by showing the stabilization of certain critical points of $u(\cdot,t)$ as $t \to \infty$.

An application of Lemma 3.1 shows that for all large t the function $\gamma_{max} - u(\cdot,t)$ has a finite number of zeros all of them simple. Let $\alpha^-(t)$, $\alpha^+(t)$ be the first and last of these zeros, respectively. Obviously, $u(x,t) < \gamma_{max}$ for $x \in \mathbb{R} \setminus (\alpha^-(t), \alpha^+(t))$. We consider the critical points of $u(\cdot,t)$ in the interval $x \in (\alpha^-(t), \alpha^+(t))$:

$$M(t) := \{x \in \mathbb{R} : x \in (\alpha^-(t), \alpha^+(t)),\; u_x(x,t) = 0\}. \tag{4.66}$$

Since $\pm u_x(\alpha^{\pm}(t), t) < 0$, we infer from Lemma 3.3 that for all sufficiently large t the number of zeros of $u_x(\cdot,t)$ in $(\alpha^-(t), \alpha^+(t))$ is finite and independent of t and of all of these zeros are simple. By the implicit function theorem, for some $m \in \mathbb{N}$ and $t_0 > 0$ we have

$$M(t) = \{\vartheta_1(t), \dots, \vartheta_m(t)\} \quad (t > t_0), \tag{4.67}$$

where $\vartheta_1(t) < \cdots < \vartheta_m(t)$ are C^1 functions of t.

We now show that the set $M(t)$ is bounded uniformly in t. For $\lambda \in \mathbb{R}$, we define a function $V_\lambda u$ by

$$V_\lambda u(x,t) = u(2\lambda - x, t) - u(x,t). \tag{4.68}$$

Note that $v := V_\lambda u$ satisfies a linear equation (3.2) and $v(\lambda, t) = 0$ for all t. Take $\lambda = c_k t_1 + \zeta_k^+(t_1)$ where $t_1 > t_0$ is so large that

$$2(c_k t_1 + \zeta_k^+(t_1)) > \zeta_k^+(t) - \zeta_k^-(t) \quad (t \geq t_1).$$

Such a choice is certainly possible, since $c_k > 0$ and the $\zeta_k^\pm(\infty)$ are finite. Observe that

$$2\lambda - (c_k t + \zeta_k^+(t)) > -c_k t - \zeta_k^-(t) \quad (t \geq t_1). \tag{4.69}$$

Without loss of generality, we may also assume that

$$\lambda < c_k t + \zeta_k^+(t) \quad (t > t_1).$$

Indeed, if this is not valid, we simply replace t_1 by the maximal t satisfying $\lambda = c_k t + \zeta_k^+(t)$. Consider the function $v = V_\lambda u$ on the domain

$$Q := \{(x, t) : t > t_1, \lambda < x < c_k t + \zeta_k^+(t)\}.$$

Relations (4.69), (4.65) (with $j = k$) and $v(\lambda, t) = 0$ imply that $v \geq 0$ on the parabolic boundary of Q. By the maximum principle, we have $v > 0$ in Q, and, by the Hopf boundary principle, $2u_x(\lambda, t) = -v_x(\lambda, t) < 0$ for all $t \geq t_1$.

Relation (4.65) with $j = k$ yields

$$\vartheta_m(t) < \alpha^+(t) < c_k t + \zeta_k^+(t) \quad (t > t_0). \tag{4.70}$$

In particular, at at $t = t_1$ we have

$$\vartheta_m(t) < \lambda \quad (= c_k t_1 + \zeta_k^+(t_1)). \tag{4.71}$$

Since $u_x(\lambda, t) < 0$, (4.71) continues to hold for all $t \geq t_1$.

Using analogous arguments, one shows that $\vartheta_1(t)$ is bounded from below by a constant independent of t; hence, the set $M(t)$ is uniformly bounded. In conjunction with (4.3), this implies that, first,

$$\omega(u) \subset \{\psi(\cdot + \xi) : \xi \in J\}, \tag{4.72}$$

where J is a bounded interval, and, second, that $M(t)$ consists of just one point (for the latter, one uses (4.72) and the fact that in the definition of $\omega(u)$ one can replace the space $L^\infty_{loc}(\mathbb{R})$ by $C^2_{loc}(\mathbb{R})$).

We claim that given any $\lambda \in \mathbb{R} \setminus \{(\eta^+ - \eta^-)/2\}$, the following holds for all sufficiently large t:

$$u_x(\lambda, t) = -\frac{1}{2} V_\lambda u(x,t) \neq 0. \tag{4.73}$$

This implies that the unique element $\vartheta_1(t)$ of $M(t)$ has a limit $\xi \in J$ as $t \to \infty$ and, consequently, $\omega(u) = \{\psi(\cdot + \xi)\}$. So the proof of Theorem 2.5 will be complete once once we prove our claim.

Consider again the the function $v = V_\lambda u$, this time in the domain

$$Q := \{(x,t) : t > t_1, -c_k t - \zeta_k^-(t) < x < c_k t + \zeta_k^+(t)\},$$

where t_1 is large enough so that $(\lambda, t) \in Q$ for all $t > t_1$. Notice that since $\lambda \neq (\eta^+ - \eta^-)/2$, we have, possibly after making t_1 larger,

$$2\lambda + c_k t + \zeta_k^-(t) \neq c_k t + \zeta_k^+(t) \quad (t \geq t_1).$$

Consequently, by (4.65), (4.68),

$$v(-c_k t - \zeta_k^-(t), t) \neq 0 \text{ and } v(c_k t + \zeta_k^+(t), t) \neq 0 \quad (t \geq t_1).$$

Thus, we may apply Lemma 3.3 to conclude that for all large enough t the function $v(\cdot, t)$ has only a finite number of zeros in $(-c_k t - \zeta_k^-(t), c_k t + \zeta_k^+(t))$, all of them simple. Since λ is one of the zeros, (4.73) is proved. $\square$

Remark 4.12. It will be useful below to stress that the fact that $M(t)$ consists of a single point $\vartheta_1(t)$ (in the case that ψ is a ground state) means that $u_x(\cdot, t) > 0$ in $(\alpha^-(t), \vartheta_1(t))$ and $u_x(\cdot, t) < 0$ in $(\vartheta_1(t), \alpha^+(t))$. This and the convergence of $u(\cdot, t)$ to $\psi(\cdot + \xi)$ imply that for any sequence $\{x_n\}$ with $|x_n| \to \infty$ one has $\limsup u(x_n, t_n) \leq \gamma_{max}$.

Proof of Theorem 2.11. We have already proved above that statements (a), (b), (c) of Theorem 2.11 hold. We only need to prove statement (d). We derive it from Theorems 2.5, 2.6, and Lemma 4.9 in a very similar fashion as Lemma 4.9 was proved. Take $\gamma = \gamma_{max}$ in (2.12). Assume that the convergence (2.12) fails along a sequence (x_n, t_n) with $t_n \to \infty$. Passing to a subsequence, we may assume that one of the following possibilities occurs:

(p1) the sequence $\{x_n\}$ is bounded,

(p2) ($x_n \to \infty$ and) the sequence $\{x_n - c_k t_n - \zeta_k^+(t_n)\}$ is bounded from below,

(p3) ($x_n \to -\infty$ and) the sequence $\{x_n + c_k t_n + \zeta_k^- k(t_n)\}$ is bounded from above,

(p4) $x_n \to \infty$ and $x_n - c_k t_n - \zeta_k^+(t_n) \to -\infty$,

(p5) $x_n \to -\infty$ and $x_n + c_k t_n + \zeta_k^- k(t_n) \to \infty$.

The possibilities (p1), (p2), (p3) lead to a contradiction via a simple application of Theorems 2.5, 2.6 (in the case (p1)) or Lemma 4.9 (in the cases (p2), (p3)). The two cases (p4) and (p5) are analogous, we consider just one of them. Assume that (p4) holds. Then, for $j = 1, \dots, k$,

$$\phi_j(x_n - c_j t_n - \zeta_j^+(t_n)) \to \phi_j(-\infty) = a_{j+1},$$
$$\tilde{\phi}_j(x_n + c_j t_n + \zeta_j^-(t_n)) \to \tilde{\phi}_j(\infty) = \phi_j(-\infty) = a_j.$$

Also,

$$\psi(x_n) \to \psi(\infty) = \gamma_{max}.$$

Therefore, in (2.12) we have $\psi(x_n) - \gamma_{max} \to 0$ and the expression in the parentheses, evaluated at $(x,t) = (x_n, t_n)$, approaches a_{k+1}.

Now, passing to a further subsequence, we may assume that $u(x_n, t_n) \to a$ for some $a \in \mathbb{R}$, and $u(\cdot + x_n, t_n) \to \varphi$ for some $\varphi \in \Omega(u)$ with $\varphi(0) = a$. From (p4) it follows that $a \le \gamma_{max} = a_{k+1}$ (this is trivial if $\psi \equiv \gamma_{max}$, otherwise see Remark 4.12) and $a \ge (a_k + a_{k+1})/2$ (see (4.65). By Theorem 2.9, φ is one of the constants $a_1, \dots, a_{k+1}$, or one of the functions $\varphi = \phi_j(\cdot - y)$, $j \in \{1, \dots, k\}$, $y \in \mathbb{R}$ (cp. Remark 4.7), or else $\varphi = \psi(\cdot - y)$ for some $y \in \mathbb{R}$. Due to the above restrictions on $a = \varphi(0)$, we necessarily have $\varphi \equiv a_{k+1}$. Thus the limit in (2.12) is 0 and we have a contradiction.

Theorem 2.11 is now proved. $\square$

5 Appendix

This section is devoted to the proofs of Proposition 2.4 (the genericity of conditions (G1)-(G3)) and the result at the end of Subsection 4.1. In both of them, we use a min-max characterization of the speeds of traveling fronts, as given in [27]. To recall the characterization, we need to introduce some notation.

Consider an interval $[a, b]$, with $f(a) = f(b) = 0$. Let C_a be the set of all absolutely continuous functions on $[a, b]$ satisfying the conditions

$$\rho(a) = 0, \qquad \rho(u) > 0 \quad (u \in (a, b]).$$

We call any such function a test function for the interval $(a, b]$. For $\rho \in C_a$, set

$$\psi^*(\rho) = \sup \left(\rho'(u) + \frac{f(u)}{\rho(u)} \right),$$

where supremum is taken over those $u \in (a, b)$ for which the derivative $\rho'(u)$ exists. Let now

$$\omega^* := \inf_{\rho \in C_a} \psi^*(\rho).$$

Similarly, the set of test functions for the interval $[a, b)$, C_b, is the set of all absolutely continuous functions on $[a, b]$ satisfying the conditions

$$\rho(b) = 0, \qquad \rho(u) > 0 \quad (u \in [a, b)).$$

Given $\rho \in C_b$, we define

$$\psi_*(\rho) = \inf \left(\rho'(u) + \frac{f(u)}{\rho(u)} \right),$$

taking the infimum over all those $u \in (a, b)$ for which the derivative $\rho'(u)$ exists. Let

$$\omega_* := \sup_{\rho \in C_a} \psi_*(\rho).$$

When needed, we will write $\psi^*(\rho, f)$, $\omega^*(f)$, $\psi_*(\rho, f)$, $\omega_*(f)$ to indicate the dependence of these functional on f.

The following results from [27, Theorem 1.3.14] show the relation of the functionals ω^*, ω_* to speeds of traveling fronts of (1.1):

(C1) If $f'(a) < 0$, $f'(b) < 0$, and a traveling front with range (a, b) exists, then for its unique speed c one has $\omega_* = c = \omega^*$.

(C2) If $f'(a) > 0 > f'(b)$ and a traveling front with range (a, b) exists, then the minimal speed c for such traveling fronts is given by $c = \omega_*$ (also, the speed has to be positive in this case).

Assume now that f satisfies the following conditions on an interval $[a, b]$:

$$f(a) = f(b) = 0, \ f'(a) \neq 0, \ f'(b) < 0, \tag{5.1}$$

$$\int_a^u f(s)\, ds < \int_a^b f(s)\, ds \quad (u \in (a, b)). \tag{5.2}$$

Note that(5.2) guarantees the existence of a traveling front with range (a, b) and speed $c \geq 0$. We examine the dependence of the speed on f. Two results on this dependence will be needed in the proof of the genericity of condition (G3): continuity and strict monotonicity.

The continuity of the minimal speed with respect to the nonlinearity can be proved in a number of different ways and is surely well-known to the experts, but we were unable to locate a reference. We give a simple proof, relying on the above min-max characterization of the speeds.

Lemma 5.1. *Assume* (5.1), (5.2). *If* $f_n \in C^1[a, b]$, $n = 1, 2, \dots$ *is a sequence of functions vanishing at* a *and* b *such that* $f_n \to f$ *in* $C^1[a, b]$, *then for any* $\epsilon > 0$ *there is* n_0 *such that for all* $n \geq n_0$ *one has*

$$\omega_*(f) - \epsilon < \omega_*(f_n), \quad \omega^*(f_n) < \omega^*(f) + \epsilon. \tag{5.3}$$

Remark 5.2. Relations (5.3) can also be proved in a slightly more general situation when $f_n \in C^1[a-1, b+1]$, $n = 1, 2, \dots$ is a sequence of functions converging in $C^1[a-1, b+1]$ to f; a_n, b_n are sequences converging to a, b, respectively, such that $f(a_n) = f(b_n) = 0$ for $n = 1, 2, \dots$; and the functionals $\omega^*(f_n)$, $\omega_*(f_n)$ are considered relative to the corresponding interval $[a_n, b_n]$. This case can be easily reduced to the fixed interval case by using suitable shifts and scalings of the u-variable.

Proof of Lemma 5.1. We prove the result for ω_*; the arguments for ω^* are analogous. Set $c := \omega_*(f)$ and fix any $\epsilon > 0$. There is a test functions $\rho \in C_b$ such that $\psi_*(\rho, f) > c - \epsilon/2$. In fact, the test function can be chosen such that it is of class C^1 on $[a, b]$ and $\rho'(b) < 0$ (see Lemma 5.3(i) below). It is then clear that

$$\rho'(u) + \frac{f_n(u)}{\rho(u)} \to \rho'(u) + \frac{f(u)}{\rho(u)}$$

uniformly on $[a, b]$, and therefore $\psi_*(\rho, f_n) > \psi_*(\rho, f) - \epsilon/2 > c - \epsilon$ for all large enough n. The result for ω_* follows immediately from this. $\square$

The following lemma provides some useful test functions; the first one was used in the previous proof, the second one will be needed below.

Lemma 5.3. *Assume* (5.1), (5.2) *and let* $c := \omega_*(f)$.

(i) For any $\tilde{c} < c$, *there is* $\rho \in C_b \cap C^1[a, b]$ *such that* $\rho'(b) < 0$ *and*

$$\rho'(u) + \frac{f(u)}{\rho(u)} = \tilde{c} \quad (u \in [a, b)). \tag{5.4}$$

(ii) Let $f'(a) < 0$ *(so* $c := \omega_*(f) = \omega^*(f)$*). Suppose that* $\tilde{f} \in C^1[a, b]$ *satisfies* $\tilde{f} \leq f$ *and* $\tilde{f} \not\equiv f$. *Then for any* $\tilde{c} < c$ *sufficiently close to* c *there is* $\rho \in C_a \cap C^1[a, b]$ *such that*

$$\rho'(u) + \frac{\tilde{f}(u)}{\rho(u)} \leq \tilde{c} \quad (u \in (a, b]). \tag{5.5}$$

In particular, $\omega^*(\tilde{f}) < c = \omega^*(f)$.

We remark, that a monotonicity of the functional $\omega^*(f)$ is obvious from its definition: $\omega^*(\tilde{f}) \leq \omega^*(f)$, if $\tilde{f} \leq f$. The point of Lemma 5.3(ii) is that the strict relation $\omega^*(\tilde{f}) < \omega^*(f)$ holds, unless $\tilde{f} \equiv f$.

Proof of Lemma 5.3. As noted above, there is a traveling front $\phi(x - ct)$ of (1.1) with range (a, b), and c is the (unique or) minimal speed for such traveling fronts. The function $p := p^\phi$ (cp. (3.13)) satisfies the following relations: $p < 0$ on (a, b), $p(a) = p(b) = 0$, $p'(a) < 0 < p'(b)$, and

$$p'(u) + \frac{f(u)}{p(u)} = -c \quad (u \in (a, b)) \tag{5.6}$$

(see [27, Sections 1.1.1, 1.1.2] or [22, Section 3.1]). Now, for any $\tilde{c} < c$ there is a C^1 function $\tilde{p}$ on $[a, b]$ such that $\tilde{p} < p$ on $[a, b)$, $\tilde{p}(b) = 0$, $\tilde{p}'(b) > 0$, and

$$\tilde{p}'(u) + \frac{f(u)}{\tilde{p}(u)} = -\tilde{c} \quad (u \in [a, b)) \tag{5.7}$$

(see [22, Section 3.1]). Clearly, the function $\rho := -\tilde{p}$ gives a test function as needed in statement (i).

We next prove statement (ii). First, we modify $\tilde{f}$, increasing it if needed near the points a, b, so that, in addition to the original assumptions, one has $\tilde{f} \equiv f$ on some small intervals $[a, a+\delta]$, $[b-\delta, b]$ with $\delta > 0$. Due to the obvious monotonicity of the left-hand side of (5.5) in $\tilde{f}$, proving the conclusion for the modified nonlinearity $\tilde{f}$ will also prove it for the original one (with the same function $\rho \in C_a$).

If $\tilde{c} < c$, $\tilde{c} \approx c$, then there is a unique solution $\tilde{p}$ of

$$\tilde{p}'(u) + \frac{f(u)}{\tilde{p}(u)} = -\tilde{c}, \quad u \in (a, a+\delta], \tag{5.8}$$

such that $\tilde{p} < 0$ on $(a, a+\delta]$ and $\tilde{p}(a) = 0$ (see [22, Section 3.1]; the result is in fact a version of the saddle point property for the planar systems associated with (2.4) with c replaced by $\tilde{c}$). Moreover, one has $\tilde{p}(u) \to p(u)$ as $\tilde{c} \nearrow c$, uniformly for $u \in [a, a+\delta]$. Of course, we may replace f by $\tilde{f}$ in (5.8), as the functions coincide in $(a, a+\delta]$. We now extend $\tilde{p}$ as a solution of the equation

$$\tilde{p}'(u) + \frac{\tilde{f}(u)}{\tilde{p}(u)} = -\tilde{c} \tag{5.9}$$

to its maximal existence interval in $(a, b]$—it is required that $\tilde{p}$ be negative on such interval. Since $\tilde{p}(a+\delta) \to p(a+\delta)$ as $\tilde{c} \nearrow c$, using the continuity of solutions of (5.9) with respect to initial data and a simple comparison argument for such solutions (using the assumption that $\tilde{f} < f$ on an interval), one shows easily that for all $\tilde{c} < c$ close enough to c the solution $\tilde{p}$ is defined up to $u = b - \delta$ and $\tilde{p}(b-\delta) < p(b-\delta) - \epsilon$ for some $\epsilon > 0$ independent of $\tilde{c}$. We claim that the last relation implies that $\tilde{p}(u) < 0$ on $[b-\delta, b]$. To show this, we observe that, analogously to equation (5.8), equation

$$\tilde{p}'(u) + \frac{f(u)}{\tilde{p}(u)} = -\tilde{c} \quad u \in [b-\delta, b), \tag{5.10}$$

has for all $\tilde{c} < c$, $\tilde{c} \approx c$ a unique solution $\hat{p}$ with $\hat{p} < 0$ on $[b-\delta, b)$ and $\hat{p}(b) = 0$; and one has $\hat{p}(u) \to p(u)$ as $\tilde{c} \nearrow c$ uniformly on $[b-\delta, b]$. In particular, if $\tilde{c} < c$ is close enough to c, $\tilde{p}(b-\delta) < p(b-\delta) - \epsilon < \hat{p}(b-\delta)$. This implies that $\tilde{p}(u) < \hat{p}(u)$ for $u \in [b-\delta, b)$ and the uniqueness property of $\hat{p}$ implies that the inequality holds at $u = b$, too.

Thus, $\tilde{p}$ is negative on $(a, b]$ and solves there equation (5.9). Consequently, the function $\rho = -\tilde{p}$ satisfies (5.5). $\square$

We now take on the proof of Proposition 2.4. For $n = 1, 2, \ldots$, let

$$\mathcal{F}_n := \{f \in X : \text{(G1)–(G3) hold for all } \gamma \in \tilde{\Gamma} \cap [0, n]\}.$$

We prove that the sets $\mathcal{F}_n$ are open and dense in X, so the set

$$\mathcal{F} := \{f \in X : \text{(G1)–(G3) hold}\} = \bigcap_{n=1}^{\infty} \mathcal{F}_n$$

is residual in X. It is a simple exercise, which we omit, to prove that the set

$$\mathcal{G}_n := \{f \in X : \text{(G1), (G2) hold for all } \gamma \in \tilde{\Gamma} \cap [0, n]\}$$

is open and dense in X. It is thus sufficient to prove that for any n the set $\mathcal{F}_n$ is open and dense in $\mathcal{G}_n$.

Proof of the openness. To prove the openness, fix any $f \in \mathcal{F}_n$. We show that any $\tilde{f} \in X$ sufficiently close to f also belongs to $\mathcal{F}_n$. Denote by $\tilde{\Gamma}(\tilde{f})$ the set defined as $\tilde{\Gamma}$ with f replaced by $\tilde{f}$. Pick $\epsilon > 0$ such that $\tilde{\Gamma}(f) \cap [0, n] = \tilde{\Gamma}(f) \cap [0, n+2\epsilon)$. Clearly, if $\tilde{f}$ is close enough to f, then the set $\tilde{\Gamma}(\tilde{f}) \cap [0, n+\epsilon)$ has the same (finite) number of elements as $\tilde{\Gamma}(f) \cap [0, n]$, and the elements of the former set are small perturbations of the elements of the latter set.

Fix now any element $\gamma \in \tilde{\Gamma}(f) \cap (0, n]$. Let $\{(\phi_j, c_j) : j = 1, \dots, k\}$ be the minimal $[0, \gamma]$-propagating terrace for equation (1.1) and let $I_j = (a_j, a_{j+1})$ be the range of the profile function ϕ_j. Thus,

$$a_j \in \tilde{\Gamma}(f) \quad (j = 1, \dots, k+1), \qquad 0 = a_1 < a_2 < \dots < a_{k+1} = \gamma. \quad (5.11)$$

Let $0 = \tilde{a}_1 < \tilde{a}_2 < \dots < \tilde{a}_{k+1} = \tilde{\gamma}$ denote the corresponding nearby elements of $\tilde{\Gamma}(\tilde{f}) \cap [0, n+\epsilon)$, for $\tilde{f} \approx f$. We claim that if $\tilde{f}$ is close enough to f, then the following statement is valid:

For $j = 1, \dots, k$ there is a traveling front $\tilde{\phi}_j(x - \tilde{c}_j t)$ of the perturbed equation

$$u_t = u_{xx} + \tilde{f}(u), \qquad x \in \mathbb{R},\ t > 0, \quad (5.12)$$

with range $(\tilde{a}_j, \tilde{a}_{j+1})$ and the speeds of these traveling fronts satisfy the relations $\tilde{c}_1 > \dots > \tilde{c}_k$.

Indeed, the existence of such traveling fronts for $\tilde{f} \approx f$ follows from the fact that $f \in \mathcal{G}_n$. If $f'(0) > 0$ (which implies $\tilde{f}'(0) > 0$), we always take the front $\tilde{\phi}_1(x - \tilde{c}_1 t)$ with the minimal speed for the interval $(0, \tilde{a}_2)$. Now, the assumption that $f \in \mathcal{F}_n$ gives $c_1 > \dots > c_k$. In view of the above characterizations (C1)–(C2), Lemma 5.1 and Remark 5.2 imply that the same relations are valid for the speeds $\tilde{c}_1, \dots, \tilde{c}_k$ if $\tilde{f}$ is close enough to f. This proves the claim.

Observe that the claim implies that $\tilde{f}$ and γ satisfy (G3). Simply replace the $\tilde{\phi}_j$ by suitable shifts so that $\tilde{\phi}_j(0) = (\tilde{a}_j + \tilde{a}_{j+1})/2$. By Lemma 3.11, the relations $\tilde{c}_1 > \cdots > \tilde{c}_k$ guarantee that $\{(\tilde{\phi}_j, \tilde{c}_j) : j = 1, \ldots, k\}$ is the minimal $[0, \gamma]$-propagating terrace for (5.12). Thus these relations show that (G3) holds.

Since there are only finitely many elements in $\gamma \in \tilde{\Gamma}(f) \cap (0, n]$, the above arguments show the openness of the set $\mathcal{F}_n$. □

Proof of the density. To prove the density of $\mathcal{F}_n$ in $\mathcal{G}_n$, fix any $f \in \mathcal{G}_n$ and $\epsilon > 0$. We need to find $\tilde{f} \in \mathcal{G}_n$, such that $\tilde{f} \in \mathcal{F}_n$ and $\|f - \tilde{f}\|_X < \epsilon$.

We have, for some k,

$$\tilde{\Gamma} \cap [0, n] = \{0, \gamma_1, \ldots, \gamma_k\}, \quad \text{with } 0 < \gamma_1 < \cdots < \gamma_k \leq n.$$

We will find $\tilde{f}$ by making small successive perturbations of f in small left neighborhoods of the points $\gamma_1, \ldots, \gamma_k$ only. In particular, any such perturbation $\tilde{f}$ will have the same set $\tilde{\Gamma}(\tilde{f}) \cap [0, n]$ as $\tilde{\Gamma} \cap [0, n]$, where $\tilde{\Gamma}(\tilde{f})$ has the same meaning as in the proof of the openness.

The goal that we want to achieve by such perturbations is that for $j = 1, \ldots, k$ condition (G3) is satisfied by $\tilde{f}$ and γ_j. For $j = 1$ this is automatic, no perturbation is needed. Proceeding by induction, suppose that the goal has been achieved for all j up to $j = \ell - 1$ for some $\ell \leq k$. Clearly, this is unaffected by any subsequent perturbation of $\tilde{f}$ outside the interval $[0, \gamma_{\ell-1}]$. We show that a suitable small perturbation of $\tilde{f}$ in a left neighborhood of γ_ℓ—where $\tilde{f} > 0$ due to $f'(\gamma_\ell) < 0$—yields a new function $\tilde{f}$ such that (G3) is satisfied by $\tilde{f}$ and γ_ℓ. This will complete the induction argument and thereby the proof of the density of $\mathcal{F}_n$.

Let $\{(\phi_j, c_j) : j = 1, \ldots, m\}$ be the minimal $[0, \gamma_\ell]$-propagating terrace for (5.12). Set $\gamma := \phi_m(\infty)$; so $\gamma \in \tilde{\Gamma}$, $\gamma < \gamma_\ell$, and the range of ϕ_m is the interval (γ, γ_ℓ). Clearly, $\{(\phi_j, c_j) : j = 1, \ldots, m-1\}$ is the the minimal $[0, \gamma]$-propagating terrace for (5.12). Therefore, the induction hypothesis gives $c_1 > \cdots > c_{m-1}$. Also, $c_{m-1} \geq c_m$ (cp. Proposition 2.3). If $c_{m-1} > c_m$, (G3) is satisfied by $\tilde{f}$, γ_ℓ and we are done. Assume $c_{m-1} = c_m$. Lemma 5.3(ii) facilitates an arbitrarily small local perturbation of $\tilde{f}$ in a left neighborhood of γ_ℓ after which the following holds for the new function $\tilde{f}$. Equation (5.12) has traveling front $\tilde{\phi}_m(x - \tilde{c}_m t)$ with range (γ, γ_ℓ) such that $\tilde{\phi}_m(0) = (\gamma + \gamma_\ell)/2$ and $\tilde{c}_m < c_{m-1}$. It follows from Lemma 3.11 that

$$\{(\phi_j, c_j) : j = 1, \ldots, m-1\} \cup \{(\tilde{\phi}_m, \tilde{c}_m)\}$$

the minimal $[0, \gamma_\ell]$-propagating terrace for (5.12). This and the inequalities $c_1 > \cdots > c_{m-1} > \tilde{c}_m$ show that (G3) is satisfied by $\tilde{f}$, γ_ℓ. The proof of the density is now complete. $\square$

We now prove a result needed at the end of Subsection 4.1. With $q := f'(0) > 0$, and $a_2 > 0$, $c_1 > 0$ as in (4.4), (4.5), we assume that

$$f(u) = \frac{q}{a_0} u(a_0 - u) \quad (u \in [a_0, 0)), \tag{5.13}$$

where $a_0 < 0$ is a parameter. Then f is bistable in (a_0, a_2), and we denote by $c(a_0)$ the (unique) speed of the traveling front with range (a_0, a_2). The front indeed exists, for

$$\int_{a_0}^{u} f(s)\, ds < \int_{a_0}^{a_1} f(s)\, ds$$

due to (5.13) and the existence of the front $\phi_1(x - c_1 t)$ with range $(0, a_2)$. Here is the result we used in Subsection 4.1.

Lemma 5.4. *As $a_0 \nearrow 0$, one has $c(a_0) \to c_1$.*

Proof. Let $\omega_*(0, a_2)$, $\omega_*(a_0, a_2)$ denote the functional ω_* relative the intervals $(0, a_2)$, (a_0, a_2), respectively. One has $\omega_*(0, a_2) \geq \omega_*(a_0, a_2)$ (see [27, Theorem 1.2.8]). As noted at the end of Subsection 4.1, c_1 is the minimal speed for the traveling fronts with range $(0, a_2)$. Therefore, by (C2), $c_1 = \omega_*(0, a_2) > 0$. Given any $\epsilon \in (0, c_1)$, there is a test function ρ for the interval $[0, a_2)$ such that $\psi_*(\rho) > c_1 - \epsilon > 0$. With $\rho_0 := \rho(0) > 0$, we extend ρ to the interval $[a_0, a_2]$ by setting

$$\rho(u) = \frac{\rho_0}{2} - \frac{\rho_0}{2a_0}(u - a_0), \quad u \in [a_0, 0). \tag{5.14}$$

This way, ρ becomes a test function for the interval $[a_0, a_2)$, too. Now

$$\begin{aligned} \inf_{u \in (a_0, 0)} \left(\rho'(u) + \frac{f(u)}{\rho(u)} \right) &= -\frac{\rho_0}{2a_0} + \inf_{u \in (a_0, 0)} \frac{2qu(a_0 - u)}{\rho_0(2a_0 - u)} \\ &\geq -\frac{\rho_0}{2a_0} - \frac{2q|a_0|}{2\rho_0} \to \infty, \end{aligned}$$

as $a_0 \nearrow 0$. Therefore, if $a_0 < 0$ is sufficiently close to 0, the extended test function ρ satisfies $\psi_*(\rho) > c_1 - \epsilon$ (relative to the interval $[a_0, a_2)$). Consequently, $c(a_0) = \omega_*(a_0, a_2) > c_1 - \epsilon$, for all $a_0 < 0$ sufficiently close to 0. This and the relation $\omega_*(0, a_2) \geq \omega_*(a_0, a_2)$ verify the validity of the statement of Lemma 5.4. $\square$

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