Math 2374	**Name (Print):** Solutions, version 12/12/10
Fall 2010	**Student ID:** Please report errors!
Midterm 1	**Section Number:** ______
October 6, 2010	**Teaching Assistant:** ______
Time Limit: 1 hour	**Signature:** ______

This exam contains 9 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. You are allowed to take one-half of one (doubled-sided) 8.5 inch $\times$ 11 inch sheet of notes into the exam.

Do not give numerical approximations to quantities such as $\sin 5$, π, or $\sqrt{2}$. However, you should simplify $\cos\frac{\pi}{4} = \sqrt{2}/2$, $e^0 = 1$, and so on.

The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. **All answers must be justified by valid mathematical reasoning, including the evaluation of definite and indefinite integrals**. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- **Mysterious or unsupported answers will not receive full credit**. Your work should be mathematically correct and carefully and legibly written.
- **A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit**; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive from little to no credit.

1	20 pts	
2	16 pts	
3	22 pts	
4	12 pts	
5	16 pts	
6	12 pts	
7	22 pts	
8	20 pts	
TOTAL	140 pts	

1. (20 points) Consider the function $f(x,y) = (x^2 - 3, 3xy - y^3)$, and suppose $g(u,v) : \mathbb{R}^2 \to \mathbb{R}^3$ is a function whose matrix of partial derivatives at $(u,v) = (-2,-2)$ is given by

$$\mathbf{D}g(-2,-2) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 1 & 0 \end{bmatrix}$$

Find the matrix of partial derivatives of the function $g \circ f$ at the point $(1,2)$.

Chain rule says $D(g \circ f)(1,2) = Dg(f(1,2)) \cdot Df(1,2)$

$Df = \begin{bmatrix} 2x & 0 \\ 3y & 3x - 3y^2 \end{bmatrix}$ so $Df(1,2) = \begin{bmatrix} 2 & 0 \\ 6 & -9 \end{bmatrix}$

$f(1,2) = (-2,-2)$ so $Dg(f(1,2)) = Dg(-2,-2) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 1 & 0 \end{bmatrix}$ as given

So

$$D(g \circ f)(1,2) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 6 & -9 \end{bmatrix} = \begin{bmatrix} -4 & 9 \\ 10 & -18 \\ 2 & 0 \end{bmatrix}$$

2. (16 points) Suppose that we have a pool filled with fluid so that the temperature at position (x, y) is given by the function

$$T(x, y) = y + e^x(1 - y^2).$$

Find the direction that an amoeba located at position $(0, 2)$ should move in order to get warmer as quickly as possible. Write your answer in the form of a unit vector.

The direction we want is in the <u>same direction</u> as the gradient.

$$\nabla T(x,y) = \left(e^x(1-y^2),\ 1 + e^x(-2y) \right)$$

so

$$\nabla T(0,2) = (-3, -3)$$ is the correct direction.

To make this a unit vector, we divide by its length:

$$\sqrt{(-3)^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$$

which gives the unit vector

$$\frac{1}{3\sqrt{2}} \cdot (-3, -3) = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right).$$

3. (22 points) In this question, consider the function $f(x, y) = y^2 - x^3 + x$.

(a) Compute the tangent plane of the graph of $f(x, y)$ at the point $(x, y) = (1, 3)$. Write your answer in the form $Ax + By + Cz + D = 0$.

The graph is given by $z = y^2 - x^3 + x$

The formula for the tangent plane is

$$z = f(1,3) + \nabla f(1,3) \cdot (x-1, y-3).$$

$f(1,3) = 9$

$\nabla f(x,y) = (-3x^2+1, 2y)$ $\quad$ $\nabla f(1,3) = (-2, 6)$

so the equation is

$$z = 9 + (-2,6) \cdot (x-1, y-3)$$
$$= 9 + (-2x+2) + (6y-18)$$

or $0 = -2x + 6y - z - 7.$

(b) Find a linear approximation to $f(x, y)$ at the point $(x, y) = (1, 3)$. Use this linear approximation to estimate $f(0.9, 3.2)$.

The linear approximation is given by solving for z in the above:

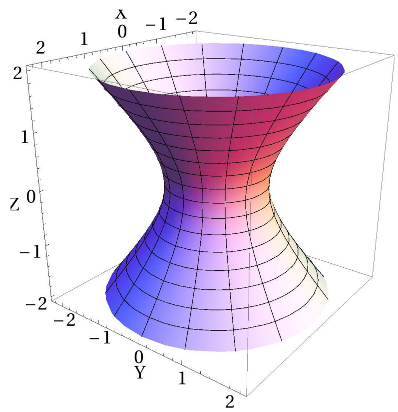
$$-2x + 6y - 7.$$

Plugging in $x = 0.9$, $y = 3.2$, we get

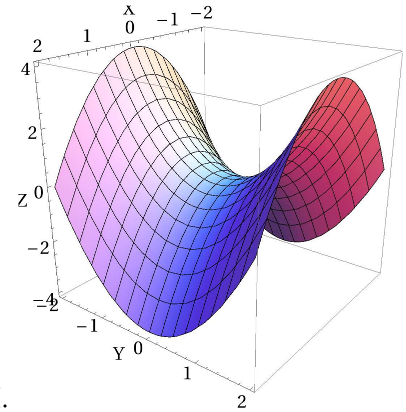
$$-2(0.9) + 6(3.2) - 7$$
$$= -1.8 + 19.2 - 7$$
$$= 10.4.$$

4. (12 points) Match the equation with its graph. Give reasons for your choice.

(a) $z = 2\sqrt{x^2+y^2} - 2$ (b) $z = y^2 - x^2$ (c) $x^2 + y^2 = 1 + z^2$ (d) $z = 2 - x^2$



I.



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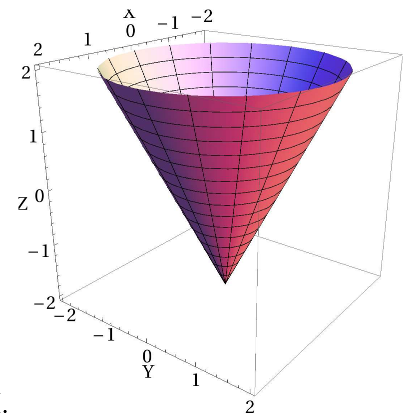

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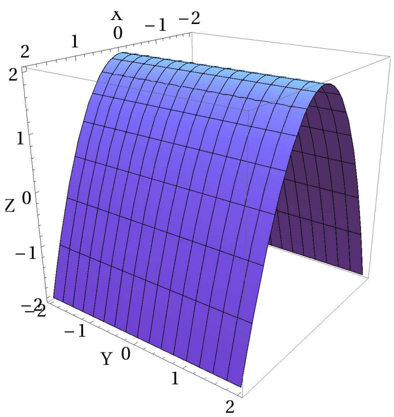


IV.

(a) matches III (any fixed z gives a circle of radius $\sqrt{\frac{z}{2}+1}$, only valid for $z \geq -2$)

(b) matches II (the sections where x is constant are parabolas $z = y^2 - x^2$)

(c) matches I (any fixed z gives a circle of radius $\sqrt{1+z^2}$)

(d) matches IV (the only graph independent of y)

These reasons are only examples.

5. (16 points) Consider the surface given by the equation $x^2 - y^2 + xz = -1$. Find the equation for the tangent plane at the point $(x, y, z) = (2, 1, -2)$. Write your answer in the form $Ax + By + Cz + D = 0$.

The equation for the tangent plane is

$$0 = \nabla f(2,1,-2) \cdot (x-2, y-1, z-(-2))$$

where $f(x,y,z) = x^2 - y^2 + xz + 1$.

$$\nabla f(x,y,z) = (2x+z, -2y, x)$$

$$\nabla f(2,1,-2) = (2, -2, 2)$$

so the equation is

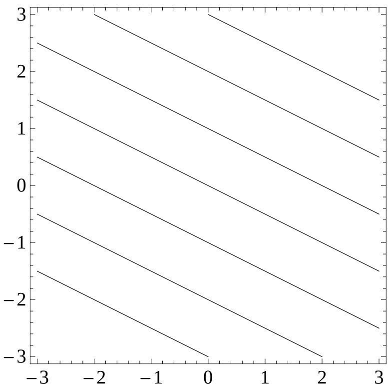
$$0 = (2,-2,2) \cdot (x-2, y-1, z+2)$$

$$0 = (2x-4) + (-2y+2) + (2z+4)$$

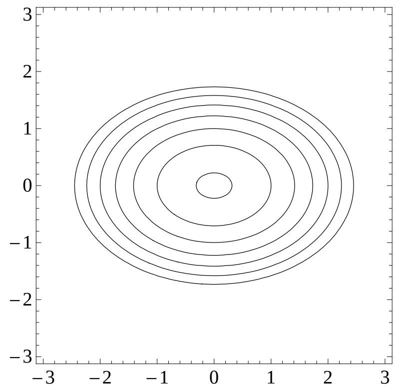
$$0 = 2x - 2y + 2z + 2$$

6. (12 points) Match the equation with its level curve plot. In each plot, the level curves $c = 0, 1, 2, 3, 4, 5, 6$ are shown. Give reasons for your choice.

(a) $f(x, y) = xy + 3$ (b) $f(x, y) = \frac{x}{2} + y + 3$
(c) $f(x, y) = \left(\frac{x}{2} + y\right)^2$ (d) $f(x, y) = x^2 + 2y^2$



I.



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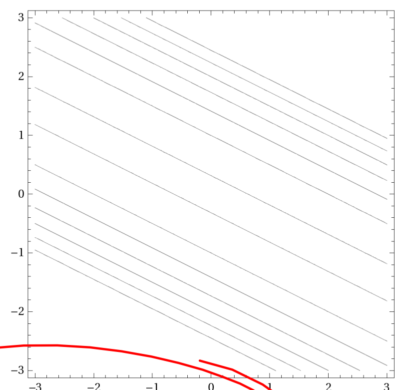
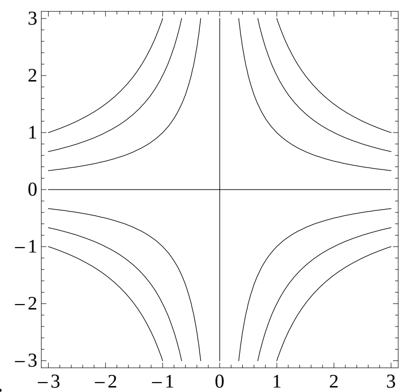


III.



IV.

(a) matches IV (level curves are $xy+3=c$, or $y = \frac{c-3}{x}$)

(b) matches I (level curves are lines $y = c-3-\frac{x}{2}$, evenly spaced)

(c) matches III (level curves are lines $y=\sqrt{c}-\frac{x}{2}$, spacing changes as c changes)

(d) matches II (level curves are ellipses $x^2+2y^2=c$)

7. (22 points) Assume $f(u, v)$ is a differentiable function, and define

$$g(x, y) = xy \cdot f(x + y, x - y).$$

(a) Compute the gradient ∇g.

Write $u = x+y$ $v = x-y$.

Then

$$\nabla g(x,y) = \left(\frac{\partial}{\partial x}\left(xy\cdot f(u,v)\right), \frac{\partial}{\partial y}\left(xy\cdot f(u,v)\right)\right)$$

$$= \left(y\cdot f(u,v) + xy\frac{\partial}{\partial x} f(u,v),\ x\cdot f(u,v) + xy\frac{\partial}{\partial y} f(u,v)\right)$$

$$= \left(y\cdot f(u,v) + xy\frac{\partial f}{\partial u}(u,v)\cdot \underbrace{\frac{\partial u}{\partial x}}_{1} + xy\frac{\partial f}{\partial v}(u,v)\cdot \underbrace{\frac{\partial v}{\partial x}}_{1},\ x\cdot f(u,v) + xy\frac{\partial f}{\partial u}(u,v)\cdot \underbrace{\frac{\partial u}{\partial y}}_{1} + xy\frac{\partial f}{\partial v}(u,v)\cdot \underbrace{\frac{\partial v}{\partial y}}_{-1}\right)$$

$$= \left(y\cdot f(u,v) + xy\frac{\partial f}{\partial u}(u,v) + xy\frac{\partial f}{\partial v}(u,v),\ x\cdot f(u,v) + xy\frac{\partial f}{\partial u}(u,v) - xy\frac{\partial f}{\partial v}(u,v)\right)$$

Replace u, v by x, y in the final solution!

(b) If $f(2, 0) = 0$, show that $||\nabla g(1, 1)||^2 = 2||\nabla f(2, 0)||^2$.

If $x=1$, $y=1$, then $u=2$ and $v=0$.

Substituting in we get

$$\nabla g(1,1) = \left(\cancelto{0}{f(2,0)} + \frac{\partial f}{\partial u}(2,0) + \frac{\partial f}{\partial v}(2,0),\ \cancelto{0}{f(2,0)} + \frac{\partial f}{\partial u}(2,0) - \frac{\partial f}{\partial v}(2,0)\right)$$

$$= \left(\frac{\partial f}{\partial u}(2,0) + \frac{\partial f}{\partial v}(2,0),\ \frac{\partial f}{\partial u}(2,0) - \frac{\partial f}{\partial v}(2,0)\right)$$

so its length-squared is

$$||\nabla g(1,1)||^2 = \left(\frac{\partial f}{\partial u}(2,0) + \frac{\partial f}{\partial v}(2,0)\right)^2 + \left(\frac{\partial f}{\partial u}(2,0) - \frac{\partial f}{\partial v}(2,0)\right)^2$$

$$= \left(\frac{\partial f}{\partial u}^2 + 2\frac{\partial f}{\partial u}\frac{\partial f}{\partial v} + \frac{\partial f}{\partial v}^2\right) + \left(\frac{\partial f}{\partial u}^2 - 2\frac{\partial f}{\partial u}\frac{\partial f}{\partial v} + \frac{\partial f}{\partial v}^2\right)$$

$$= 2\left(\frac{\partial f}{\partial u}^2 + \frac{\partial f}{\partial v}^2\right) = 2||\nabla f(2,0)||^2.$$

8. (20 points) Let $c : \mathbb{R} \to \mathbb{R}^2$ be a path defined by $c(t) = (8\sin(t) - \cos(2t), t - \sin(t))$. Find all points on the curve where the tangent vector is parallel to the x-axis.

$c'(t) = (8\cos(t) + 2\sin(2t), 1 - \cos(t))$.

This tangent vector is parallel to the x-axis if and only if its vertical component $(1-\cos t)$ is zero, i.e. $\cos(t) = 1$.

This happens precisely when t is a multiple of 2π,

i.e. $t = 2\pi n$ for $n \in \mathbb{Z}$.

Thus, the points on the curve whose tangent vector is parallel to the x-axis are (-1, 2pi n) for all integer n. <------ this is final solution!