Math 2374
Fall 2010
Midterm 2
November 3, 2010
Time Limit: 1 hour

Name (Print): Solutions, 12/12/10
Student ID: Please report errors.
Section Number: ______
Teaching Assistant: ______________
Signature: ______________

This exam contains 8 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. You are allowed to take one-half of one (doubled-sided) 8.5 inch $\times$ 11 inch sheet of notes into the exam.

Do not give numerical approximations to quantities such as $\sin 5$, π, or $\sqrt{2}$. However, you should simplify $\cos\frac{\pi}{4} = \sqrt{2}/2$, $e^0 = 1$, and so on.

The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. **All answers must be justified by valid mathematical reasoning, including the evaluation of definite and indefinite integrals**. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- **Mysterious or unsupported answers will not receive full credit**. Your work should be mathematically correct and carefully and legibly written.
- **A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit**; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive from little to no credit.

TA sections:

Section	TA	Discussion time
011	Chen	T 9:05am
012	Chen	T 11:15am
013	Klein	T 1:25pm
014	Klein	T 3:35pm
015	Bu	T 4:40pm
016	Bu	T 6:45pm
021	Bashkirov	Th 8:00am
022	Bashkirov	Th 10:10am
023	He	Th 12:20pm
024	He	Th 2:30pm
025	Lee	Th 4:40pm
026	Lee	Th 6:45pm

1	25 pts	
2	25 pts	
3	20 pts	
4	20 pts	
5	25 pts	
6	25 pts	
TOTAL	140 pts	

1. (25 points) Find the work exerted by the vector field $F(x, y) = (x - y, x + y)$ on an object which travels once, counterclockwise, around the circle of radius 2 centered at $(0, 0)$.

This curve is parametrized by $c(t) = (2\cos t, 2\sin t)$
$0 \le t \le 2\pi$.

The work is

$$\int F \cdot ds = \int_0^{2\pi} F(c(t)) \cdot c'(t)\, dt$$

$$= \int_0^{2\pi} F(2\cos t, 2\sin t) \cdot (-2\sin(t), 2\cos(t))\, dt$$

$$= \int_0^{2\pi} (2\cos t - 2\sin t) \cdot (-2\sin t) + (2\cos t + 2\sin t) \cdot (2\cos t)\, dt$$

$$= 4\int_0^{2\pi} \cancel{-\cos t \cdot \sin t} + \sin^2 t + \cos^2 t + \cancel{\cos t \sin t}\, dt$$

$$= 4\int_0^{2\pi} 1\, dt$$

$$= 4 \cdot 2\pi = 8\pi.$$

OR

you can use Green's theorem

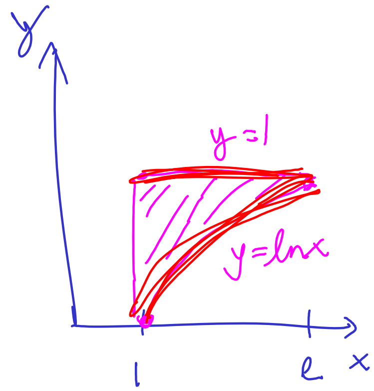
$$\int_C (x-y)\,dx + (x+y)\,dy = \iint (1+1)\,dA = 2 \cdot \text{area(circle)}$$

$$= 2 \cdot (\pi \cdot 2^2) = 8\pi.$$

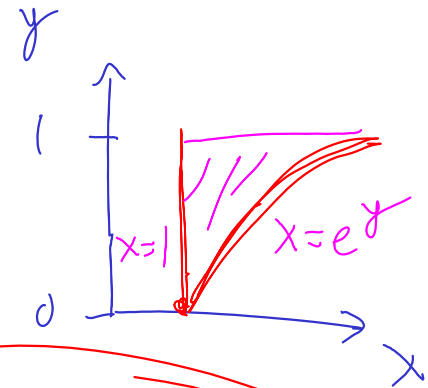
2. (25 points)
(a) (10 points) Re-express the following integral by changing the order of integration.

$$\int_1^e \int_{\ln x}^1 \frac{e^{(y^2)}}{x}\, dy\, dx$$

Region:



same as



so the integral becomes $\int_0^1 \int_1^{e^y} \frac{e^{y^2}}{x}\, dx\, dy$.

(25 points) (b) (15 points) Evaluate the integral.

$$\int_0^1 \int_1^{e^y} \frac{e^{y^2}}{x}\, dx\, dy$$

$$= \int_0^1 e^{y^2} \cdot \left[\ln x\right]\Big|_1^{e^y} dy$$

$$= \int_0^1 e^{y^2} \left(\ln(e^y) - \ln(1)\right) dy$$

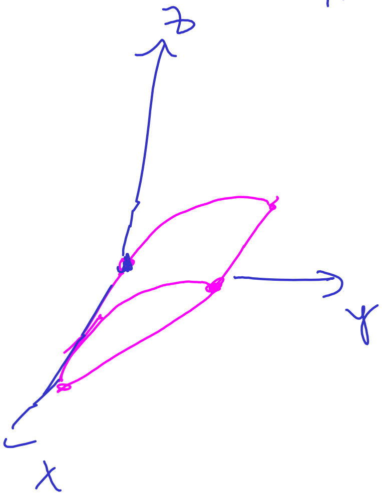
$$= \int_0^1 e^{y^2} \cdot y\, dy$$

sub: $y^2 = u$
$2y\, dy = du$

$$\int_{u=0}^{u=1} e^u \cdot \frac{du}{2} = \frac{1}{2}\left(e^u\right)\Big|_0^1 = \frac{1}{2}(e-1).$$

3. (20 points) (a) (10 points) Express the volume of the region enclosed by the surfaces $x = 0$, $x = 1$, $y = z^2$, and $y = z$ as a triple integral in terms of $dz\,dy\,dx$.

The curves $y=z^2$ and $y=z$ intersect when $z=z^2$, or $z=0,1$.



$$\int_0^1 \int_0^1 \int_y^{\sqrt{y}} dz\,dy\,dx$$

(b) (10 points) Find the volume of this region.

$$\int_0^1 \int_0^1 \int_y^{\sqrt{y}} dz\,dy\,dx$$

$$= \int_0^1 \int_0^1 (\sqrt{y} - y)\,dy\,dx$$

$$= \int_0^1 \left[\frac{2}{3}y^{3/2} - \frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^1 \left[\frac{2}{3} - \frac{1}{2}\right] dx = \frac{1}{6}$$

4. (20 points) Consider the vector field $F(x, y, z) = (x^2e^z, x^2y, y^2)$. Compute the quantity:

$$\operatorname{div}\Big((1, z, 0) \times \operatorname{curl}(F)\Big)$$

$$\operatorname{curl}(F) = \det \begin{pmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2e^z & x^2y & y^2 \end{pmatrix}$$

$$= \left(2y,\ x^2e^z,\ 2xy\right)$$

$$(1, z, 0) \times \operatorname{curl}(F) = \det \begin{pmatrix} i & j & k \\ 1 & z & 0 \\ 2y & x^2e^z & 2xy \end{pmatrix}$$

$$= \left(2xyz,\ -2xy,\ x^2e^z - 2yz\right)$$

$$\operatorname{div}\Big((1, z, 0) \times \operatorname{curl}(F)\Big)$$

$$= \left(2yz\right) + \left(-2x\right) + \left(x^2e^z - 2y\right)$$

$$= 2yz - 2x + x^2e^z - 2y$$

5. (25 points)
(a) (15 points) Find the arc length of the curve $c(t) = (2\sin^3 t, 2\cos^3 t)$ in the range $0 \le t \le \pi/2$.

$c'(t) = (6\sin^2 t \cdot \cos t, -6\cos^2 t \cdot \sin t)$ by the chain rule

$\|c'(t)\| = \sqrt{36 \sin^4 t \cos^2 t + 36 \cos^4 t \cdot \sin^2 t}$

$= \sqrt{36 \cdot \sin^2 t \cdot \cos^2 t \cdot (\sin^2 t + \cos^2 t)}$

$= \sqrt{36 \cdot \sin^2 t \cdot \cos^2 t} = 6|\sin t \cos t| = 3|\sin(2t)|$

so the length is

$\int_0^{\pi/2} \|c'(t)\| dt = \int_0^{\pi/2} 3|\sin(2t)| dt = \int_0^{\pi/2} 3\sin(2t)\, dt = -\frac{3}{2}\cos(2t)\Big|_0^{\pi/2}$

$= \left(\frac{3}{2} + \frac{3}{2}\right) = 3$.

(25 points) (b) (10 points) If this curve represents a wire with density at the point (x, y, z) given by $f(x, y) = y$, find the total mass of the wire.

Total mass is given by

$\int_0^{\pi/2} f(c(t)) \cdot \|c'(t)\| dt = \int_0^{\pi/2} (2\cos^3 t) \cdot (6 \sin t \cos t) dt$

$= \int_0^{\pi/2} 12 \cos^4 t \sin t \; dt$

Let $u = \cos t \quad du = -\sin t \; dt$

$= \int_1^0 -12 \; u^4 \; du = \left[-\frac{12}{5} u^5\right]_1^0 = \left[0 - \left(-\frac{12}{5}\right)\right] = \frac{12}{5}$

6. (25 points) Let $F(x, y) = (ye^{xy}+1, xe^{xy})$, and let C be the curve given by $c(t) = (e^{\cos(t)}, \sin^3(e^t))$ in the range $0 \le t \le 1$. Evaluate $\int_C F \cdot ds$.

(Hint: Show that F is a gradient vector field.)

F is the gradient of the function

$$g(x,y) = (e^{xy} + x)$$

(which you can find by integrating:

$$\int (ye^{xy}+1)\,dx = (e^{xy}+x) + c$$

constant in $\underline{x}$

$$\int \cancel{x} e^{xy}\,dy = e^{xy} + d$$

constant in $\underline{y}$)

Therefore,

$$\int_C F \cdot ds = g(c(1)) - g(c(0))$$

$$= g(e^{\cos(1)}, \sin^3(e^1)) - g(e^{\cos(0)}, \sin^3(e^0))$$

$$= g(e^{\cos 1}, \sin^3(e)) - g(e, \sin^3(1))$$

$$= \left[e^{e^{\cos 1} \cdot \sin^3(e)} + e^{\cos(1)} \right] - \left[e^{e \cdot \sin^3(1)} + e \right]$$