Math 2374
Spring 2006
Final
May 8, 2006
Time Limit: 1 Hour

Name (Print): ______________________
Student ID: ______________________
Section Number: ______
Teaching Assistant: ______________________
Signature: ______________________

This exams contains 11 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. You are allowed to take one (doubled-sided) 8.5 inch $\times$ 11 inch sheet of notes into the exams.

Do not give numerical approximations to quantities such as $\sin 5$, π, or $\sqrt{2}$. However, you should simplify $\cos \frac{\pi}{2} = 0$, $e^0 = 1$, and so on.

The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. **All answers must be justified by valid mathematical reasoning, including the evaluation of definite and indefinite integrals**. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- **Mysterious or unsupported answers will not receive full credit**. Your work should be mathematically correct and carefully and legibly written.
- **A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit**; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive very little credit.

1	25 pts	
2	25 pts	
3	30 pts	
4	30 pts	
5	25 pts	
6	30 pts	
7	40 pts	
8	30 pts	
9	30 pts	
10	35 pts	
TOTAL	300 pts	

1. (25 points) Find the length of the curve C parameterized by $\vec{c}(t) = (3t^2, -t^3, 2t^3), \quad 0 \le t \le 1.$

2. (25 points) Find the equation for the tangent plane of the level set $f(x, y, z) = 2$ for the function $f(x, y, z) = 2x^2 - y^2 + 3z^2$ at the point $(x, y, z) = (2, 3, 1)$.

3. (30 points) Let S be the helicoid parameterized by

$$(x, y, z) = \vec{\Phi}(r, \theta) = (r\cos\theta, r\sin\theta, \theta)$$

for $0 \le r \le 2, 0 \le \theta \le 4\pi$, oriented with the upward unit normal vector (that is: $\vec{n} \cdot \vec{k} > 0$.) Let $\vec{F}$ be the vector field $\vec{F}(x, y, z) = \vec{k}$ for all (x, y, z). Find

$$\iint_S \vec{F} \cdot d\vec{S}.$$

4. (30 points) Let $f(x, y)$ be a differentiable function satisfying these conditions: (2,8) is a critical point of f, $f(2, 8) = -3$, and the Hessian matrix for f at (2,8) is

$$\begin{bmatrix} -3 & 1 \\ 1 & -2 \end{bmatrix}$$

   (i) (15 points) What is the second degree Taylor polynomial of f at (2,8)?

   (ii) (15 points) What does the second derivative test say about this critical point?

5. (25 points) Let the curve C, which is pictured below, be composed of the following closed curves, each of which is oriented counter-clockwise when viewed from the positive x-axis.
   C_1: $x = 10$, $(y-6)^2 + (z-6)^2 = 6^2$
   C_2: $x = 10$, $(y-9)^2 + (z-8)^2 = 1$
   C_3: $x = 10$, $(y-3)^2 + (z-8)^2 = 1$
   C_4: $x = 10$, $(y-6)^2 + (z-6)^2 = 1$
   C_5: $x = 10$, $z = (y-6)^2/4 + 2$ or $z = (y-6)^2/8 + 4$, for $0 < y < 10$.

   If $\vec{F}(x, y, z) = (2xy + xz^2, x^2, x^2 z)$, calculate $\int_C \vec{F} \cdot d\vec{s}$. (Hint: your answer should not involve extensive calculations.)

6. (30 points) Let W be the rectangular solid $[0,1] \times [0,4] \times [0,\frac{1}{2}]$ in (x,y,z)-space (i.e, W is defined by $0 \leq x \leq 1$, $0 \leq y \leq 4$, and $0 \leq z \leq \frac{1}{2}$). Write S for the boundary surface of W, oriented by the outward unit normal vector. Let $\vec{F}$ be the vector field

$$\vec{F}(x,y,z) = (x^2 + ze^{y^2})\,\vec{i} + (x \sin \pi z - xy)\,\vec{j} + (3z - xz + x^4 \ln y)\,\vec{k}.$$

Use the Divergence Theorem (Gauss' Theorem) to compute the flux integral

$$\iint_S \vec{F} \cdot d\vec{S}.$$

7. (40 points) Let $h : \mathbf{R}^2 \to \mathbf{R}$ be a differentiable function where $h(x, y)$ is the depth of a pool at point (x, y). We are given that $\vec{\nabla} h(1, 4) = (-0.1, 0.3)$.

   (i) (25 points) You wade in the pool so that your position at time t is $(x, y) = \vec{c}(t) = (\cos \pi t, 2t)$. What is the rate of change in depth you experience at time $t = 2$?

   (ii) (15 points) At the point $(x, y) = (1, 4)$, what is the slope of $h(x, y)$ in the direction of the vector $(3, 4)$?

8. (30 points) Calculate $\int_C \vec{F} \cdot d\vec{s}$, the circulation of the vector field $\vec{F}$ around C, where $\vec{F}(x,y) = ((x+y)\sin x + xy^2, \cos y - \cos x)$ and C is the counter-clockwise oriented boundary of the region $0 \le x \le 1$, $x^2 \le y \le \sqrt{x}$. (Hint: try converting the line integral to a double integral.)

9. (30 points) Let W be the solid upper hemisphere of radius 2: W consists of the points (x, y, z) satisfying $x^2 + y^2 + z^2 \leq 4$ and $z \geq 0$.

   (i) (20 points) Convert the integral

   $$\iiint_W \frac{x^2 + y^2}{x^2 + y^2 + z^2}\, dV$$

   to spherical coordinates.

   (ii) (10 points) Compute the integral.

10. (35 points) Let C_1 be the circle of radius 1 in the plane $z = 1$ and C_2 be the circle of radius 1 in the plane $z = 2$. These curves are parametrized by $\vec{c_1}(t) = (\cos t, \sin t, 1)$ and $\vec{c_2}(t) = (\cos t, \sin t, 2)$, for $0 \le t \le 2\pi$. If $\vec{F}$ is a vector field whose curl is

$$\vec{\nabla} \times \vec{F} = (1 - x^2 - y^2)(x\,\vec{i} + y\,\vec{j}) + (y\,\vec{i} - x\,\vec{j}) + xe^{3z}\,\vec{k},$$

show that the line integrals are equal:

$$\int_{C_1} \vec{F} \cdot d\vec{s} = \int_{C_2} \vec{F} \cdot d\vec{s}.$$