1. (25 points) Find the length of the curve C parameterized by $\vec{c}(t) = (3t^2, -t^3, 2t^3)$, $0 \le t \le 1$.

$$c' = (6t, -3t^2, 6t^2)$$

$$\int_0^1 \| \sqrt{(6t)^2 + (-3t^2)^2 + (6t^2)^2} \, dt$$

$$= \int_0^1 \sqrt{36t^2 + 45t^4} \, dt$$

$$= \int_0^1 t \sqrt{36 + 45t^2} \, dt$$

$$= \frac{2}{3} \frac{1}{90} (36 + 45t^2)^{\frac{3}{2}} \Big|_0^1$$

$$= \frac{2}{3} \frac{1}{90} \left[(36+45)^{\frac{3}{2}} - 36^{\frac{3}{2}} \right]$$

$$= \frac{1}{135} (9^3 - 6^3)$$

$$= \frac{513}{135}$$

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2. (25 points) Find the equation for the tangent plane of the level set $f(x, y, z) = 2$ for the function $f(x, y, z) = 2x^2 - y^2 + 3z^2$ at the point $(x, y, z) = (2, 3, 1)$.

$\nabla f(2,3,1) \cdot (x-2,\ y-3,\ z-1) = 0.$

$\nabla f = \langle 4x,\ -2y,\ 6z \rangle$

$\nabla f(2,3,1) = (8,\ -6,\ 6)$

So,

$\langle 8, -6, 6 \rangle \cdot (x-2,\ y-3,\ z-1) = 0$

$8x - 6y + 6z - 16 + 18 - 6 = 0$

$8x - 6y + 6z - 4 = 0.$

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3. (30 points) Let S be the helicoid parameterized by

$$(x, y, z) = \vec{\Phi}(r, \theta) = (r\cos\theta, r\sin\theta, \theta)$$

for $0 \le r \le 2, 0 \le \theta \le 4\pi$, oriented with the upward unit normal vector (that is: $\vec{n} \cdot \vec{k} > 0$.) Let $\vec{F}$ be the vector field $\vec{F}(x, y, z) = \vec{k}$ for all (x, y, z). Find

$$\iint_S \vec{F} \cdot d\vec{S}.$$

$F = \langle 0, 0, 1 \rangle$

$T_r = \langle \cos\theta, \sin\theta, 0 \rangle$

$T_\theta = \langle -r\sin\theta, r\cos\theta, 1 \rangle$

$T_r \times T_\theta = \langle \sin\theta, -\cos\theta, r \rangle$

$\int_0^{4\pi}\int_0^2 F(\Phi(r, \theta)) \cdot \langle \sin\theta, -\cos\theta, r \rangle \, dr\, d\theta$

$= \int_0^{4\pi}\int_0^2 \langle 0, 0, 1 \rangle \cdot \langle \sin\theta, -\cos\theta, r \rangle \, dr\, d\theta$

$= \int_0^{4\pi} \frac{1}{2} r^2 \Big|_0^2 \, d\theta$

$= 4\pi \cdot 2 = \underline{8\pi}$

4. (30 points) Let $f(x, y)$ be a differentiable function satisfying these conditions: (2,8) is a critical point of f, $f(2, 8) = -3$, and the Hessian matrix for f at (2,8) is

$$\begin{bmatrix} -3 & 1 \\ 1 & -2 \end{bmatrix}$$

means $\begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(2,8) & \frac{\partial^2 f}{\partial x \partial y}(2,8) \\ \frac{\partial^2 f}{\partial x \partial y}(2,8) & \frac{\partial^2 f}{\partial y^2}(2,8) \end{bmatrix}$

(i) (15 points) What is the second degree Taylor polynomial of f at (2,8)?

(ii) (15 points) What does the second derivative test say about this critical point?

Since (2,8) is a critical point, $\frac{\partial f}{\partial x}(2,8) = 0 = \frac{\partial f}{\partial y}(2,8)$

(1) $L_2(x,y) = f(2,8) + f_x(x-2) + f_y(y-8)$

$$+\frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(2,8)(x-2)^2 + \frac{\partial^2 f}{\partial y^2}(2,8)(y-8)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(2,8)(x-2)(y-8)\right]$$

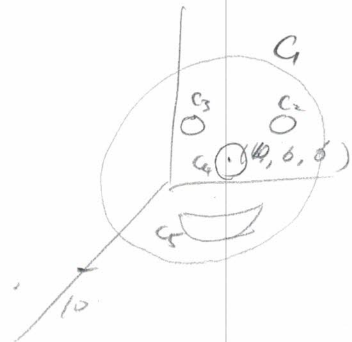
$$= -3 + 0 + 0 + \frac{1}{2}\left[-3(x-2)^2 + (-2)(y-8)^2 + 2 \cdot 1\,(x-1)(y-8)\right]$$

(2) $D(2,8) = (-3)(-2) - 1 = 5 > 0$

$\frac{\partial^2 f}{\partial x^2}(2,8) = -3 < 0$

So, f has local max. at (2,8)

5. (25 points) Let the curve C, which is pictured below, be composed of the following closed curves, each of which is oriented counter-clockwise when viewed from the positive x-axis.
C_1: $x = 10$, $(y-6)^2 + (z-6)^2 = 6^2$
C_2: $x = 10$, $(y-9)^2 + (z-8)^2 = 1$
C_3: $x = 10$, $(y-3)^2 + (z-8)^2 = 1$
C_4: $x = 10$, $(y-6)^2 + (z-6)^2 = 1$
C_5: $x = 10$, $z = (y-6)^2/4 + 2$ or $z = (y-6)^2/8 + 4$, for $0 < y < 10$.
If $\vec{F}(x,y,z) = (2xy + xz^2, x^2, x^2 z)$, calculate $\int_C \vec{F} \cdot d\vec{s}$. (Hint: your answer should not involve extensive calculations.)



By Stokes' Theorem,

$$\int_{C_j} \vec{F} \cdot d\vec{s} = \iint_{S_j} (\nabla \times F) \cdot d\vec{S}$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ 2xy+xz^2 & x^2 & x^2 z \end{vmatrix} = (0, 2xz - 2xz, 2x - 2x)$$

$$= \langle 0, 0, 0 \rangle$$

So, $\iint_{S_j} (\nabla \times F) \cdot d\vec{S} = 0$

6. (30 points) Let W be the rectangular solid $[0,1] \times [0,4] \times [0,\frac{1}{2}]$ in (x,y,z)-space (i.e, W is defined by $0 \le x \le 1$, $0 \le y \le 4$, and $0 \le z \le \frac{1}{2}$). Write S for the boundary surface of W, oriented by the outward unit normal vector. Let $\vec{F}$ be the vector field

$$\vec{F}(x,y,z) = (x^2 + ze^{y^2})\,\vec{\imath} + (x\sin\pi z - xy)\,\vec{\jmath} + (3z - xz + x^4 \ln y)\,\vec{k}.$$

Use the Divergence Theorem (Gauss' Theorem) to compute the flux integral

$$\iint_S \vec{F} \cdot d\vec{S}.$$

By Gauss' Theorem

$$\iint_S F \cdot d\vec{s} = \iiint_W \nabla \cdot F \, dV$$

$$= \int_0^1 \int_0^4 \int_0^{\frac{1}{2}} (2x - x - x + 3) \, dz\,dy\,dx$$

$$= \int_0^1 \tfrac{1}{2}(3) \cdot 4 \, dx$$

$$= \underline{6.}$$

7. (40 points) Let $h : \mathbf{R}^2 \to \mathbf{R}$ be a differentiable function where $h(x, y)$ is the depth of a pool at point (x, y). We are given that $\vec{\nabla} h(1, 4) = (-0.1, 0.3)$.

   (i) (25 points) You wade in the pool so that your position at time t is $(x, y) = \vec{c}(t) = (\cos \pi t, 2t)$. What is the rate of change in depth you experience at time $t = 2$?

   (ii) (15 points) At the point $(x, y) = (1, 4)$, what is the slope of $h(x, y)$ in the direction of the vector $(3, 4)$?

$c(2) = (\cos(2\pi), 4) = (1, 4)$

(1)

$D[h(c(t))]$

chain rule

$= \nabla h(c(2)) \cdot c'(2)$

$\nabla h(c(2)) = \nabla h(1, 4) = (-0.1, 0.3)$

$c'(t) = (-\pi \sin(\pi t), 2)$, $c'(2) = (0, 2)$

$\nabla h(1, 4) \cdot c'(2) = 0 + 0.6 = \underline{0.6}$

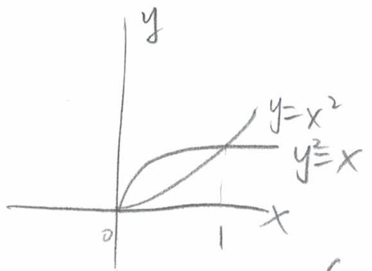
(2)

unit vector

$\nabla h(1, 4) \cdot \frac{(3, 4)}{\sqrt{3^2 + 4^2}} = (-0.1, 0.3) \cdot (\frac{3}{5}, \frac{4}{5})$

$= \frac{0.9}{5} = \underline{0.18}.$

8. (30 points) Calculate $\int_C \vec{F} \cdot d\vec{s}$, the circulation of the vector field $\vec{F}$ around C, where $\vec{F}(x, y) = ((x+y)\sin x + xy^2, \cos y - \cos x)$ and C is the counter-clockwise oriented boundary of the region $0 \le x \le 1$, $x^2 \le y \le \sqrt{x}$. (Hint: try converting the line integral to a double integral.)



Green's Theorem

$$\int_C F \cdot ds = \iint \frac{\partial}{\partial x}(\cos y - \cos x) - \frac{\partial}{\partial y}((x+y)\sin x + y^2 x)\, dA$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \sin x - \sin x - 2xy \;dy\, dx$$

$$= \int_0^1 -xy^2 \Big|_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 -x(x - x^4)\, dx$$

$$= -\tfrac{1}{3}x^3 + \tfrac{1}{6}x^6 \Big|_0^1$$

$$= \underline{-\tfrac{1}{6}}$$

#

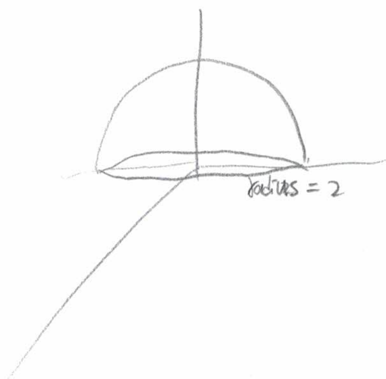
9. (30 points) Let W be the solid upper hemisphere of radius 2: W consists of the points (x, y, z) satisfying $x^2 + y^2 + z^2 \leq 4$ and $z \geq 0$.

   (i) (20 points) Convert the integral

$$\iiint_W \frac{x^2 + y^2}{x^2 + y^2 + z^2}\, dV$$

   to spherical coordinates.

   (ii) (10 points) Compute the integral.



(1) $\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} \frac{\rho^2 \sin^2\phi}{\rho^2} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

(2)

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} \rho^2 \sin^3\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \left(\frac{1}{3}\rho^3 \sin^3\phi \Big|_0^2 \right) d\phi\, d\theta$$

$$= \frac{8}{3} \int_0^{2\pi}\int_0^{\pi/2} \sin\phi(1-\cos^2\phi)\, d\phi\, d\theta$$

$$= \frac{8}{3} \int_0^{2\pi} \left(-\cos\phi \Big|_0^{\pi/2} + \frac{\cos^3\phi}{3} \Big|_0^{\pi/2} \right) d\theta$$

$$= \frac{8}{3}\, 2\pi \left(1 - \frac{1}{3} \right)$$

$$= \frac{16\pi}{3} \cdot \frac{2}{3} = \frac{32\pi}{9}$$

A

10. (35 points) Let C_1 be the circle of radius 1 in the plane $z = 1$ and C_2 be the circle of radius 1 in the plane $z = 2$. These curves are parametrized by $\vec{c_1}(t) = (\cos t, \sin t, 1)$ and $\vec{c_2}(t) = (\cos t, \sin t, 2)$, for $0 \le t \le 2\pi$. If $\vec{F}$ is a vector field whose curl is

$$\vec{\nabla} \times \vec{F} = (1 - x^2 - y^2)(x\,\vec{\imath} + y\,\vec{\jmath}) + (y\,\vec{\imath} - x\,\vec{\jmath}) + xe^{3z}\,\vec{k}, \quad = \langle (1-x^2-y^2)x + y, \; (1-x^2-y^2)y - x, \; xe^{3z} \rangle$$

show that the line integrals are equal:

$$\int_{C_1} \vec{F} \cdot d\vec{s} = \int_{C_2} \vec{F} \cdot d\vec{s}.$$

By Stokes' Theorem,

$$\int_{C_1} F \cdot ds = \iint_{S_1} (\nabla \times F) \cdot d\vec{S}, \quad S_1 \text{ is the surface } z=1,\ x^2+y^2 \le 1$$

$$\int_{C_2} F \cdot ds = \iint_{S_2} (\nabla \times F) \cdot d\vec{S}, \quad S_2 \text{ is the surface } z=2,\ x^2+y^2 \le 1$$

On S_1, $\Phi_1(r,\theta) = (r\cos\theta, r\sin\theta, 1)$

$T_r = (\cos\theta, \sin\theta, 0)$

$T_\theta = (-r\sin\theta, r\cos\theta, 0)$.

$T_r \times T_\theta = (0, 0, r)$

$$\int_0^{2\pi}\int_0^1 (\nabla \times F)(\Phi(r,\theta)) \cdot (0,0,r)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r\cos\theta\, e^3\, dr\, d\theta = 0.$$

On S_2, $\Phi_2(r,\theta) = (r\cos\theta, r\sin\theta, 2)$, $T_r \times T_\theta = (0,0,r)$

$$\int_0^{2\pi}\int_0^1 r\cos\theta\, e^{3\cdot 2}\, dr\, d\theta = 0.$$