1. (30 points) Let $f(x,y) = \frac{x^3}{3} + \frac{y^3}{3} - 2x^2 - y^2$. Find all of the critical points of $f(x,y)$, and classify each critical point as a local maximum, local minimum, or saddle point.

$f_x = x^2 - 4x = 0$, $x = 4, 0$

$f_y = y^2 - 2y = 0$, $y = 0, 2$

Critical points are $(4,0)$ $(4,2)$

$(0,0)$ $(0,2)$.

$D = f_{xx} f_{yy} - (f_{xy})^2$

$= (2x-4)(2y-2) - 0$

$D(0,0) = 8 > 0$, $\frac{\partial^2 f}{\partial x^2}(0,0) \overset{=-4}{<} 0$, $(0,0)$ local max.

$D(0,2) = -4 \cdot 2 < 0$, $(0,2)$ saddle point.

$D(4,0) = 4 \cdot (0-2) < 0$, $(4,0)$ saddle point.

$D(4,2) = 4 \cdot 2 > 0$, $\frac{\partial^2 f}{\partial x^2}(4,2) = 4 > 0$,

$(4,2)$ local min.

2. (25 points) Let $\mathbf{F}(x, y) = (ye^{xy} + 2x, xe^{xy} - \cos y)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{s}$ where C is any curve from (1,0) to $(0, \pi/2)$.

Find f such that $F = \nabla f$.

$f_x = ye^{xy} + 2x$, $f = e^{xy} + x^2 + h_1(y)$

$f_y = xe^{xy} - \cos y$, $f = e^{xy} - \sin y + h_2(x)$.

Let $h_1(y) = -\sin y$

$h_2(x) = x^2$.

So, $f(x,y) = e^{xy} + x^2 - \sin y$

$$\int_C F \cdot d\vec{s} = \int_C \nabla f \cdot d\vec{s}$$

$$= f(0, \pi/2) - f(1, 0)$$

$$= (1 + 0 - 1) - (e^0 + 1 - 0)$$

$$= \underline{-2}$$

//

3. (25 points) A population of bacteria is living on a plate. The density of bacteria at the point (x, y) is given by the function $f(x, y) = e^{1-x^2-2y^2}$.

   (i) (15 points) At the point $(x, y) = (1, 0)$, at what rate does the density increase in the direction $(-1, 1)$. (In other words, what is the slope of f in that direction?)

   (ii) (10 points) At the point $(x, y) = (1, 0)$, in what direction does the density increase most rapidly?

(1)

$$\nabla f = \langle -2x\, e^{1-x^2-2y^2},\ -4y\, e^{1-x^2-2y^2} \rangle$$

$$\nabla f(1,0) \cdot \frac{(-1,1)}{\sqrt{(-1)^2+1^2}}$$

$$= \langle -2e^{1-1-0},\ 0 \rangle \cdot \frac{\langle -1, 1 \rangle}{\sqrt{2}}$$

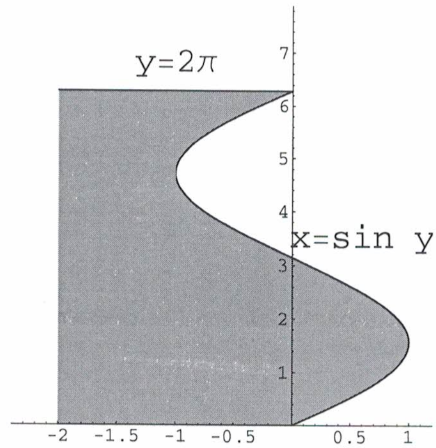
$$= \frac{2}{\sqrt{2}} = \underline{\sqrt{2}}$$

(2)

$$\nabla f(1,0) = \langle -2, 0 \rangle.$$

$\cancel{\#}$

4. (25 points) Let D be the region in the xy-plane defined by $0 \le y \le 2\pi$ and $-2 \le x \le \sin y$ as shown below.



Let ∂D be the counterclockwise oriented boundary of D. Compute $\int_{\partial D} \mathbf{F} \cdot d\mathbf{s}$, where $\mathbf{F}(x, y) = (e^{x^2}, \sin(y^2) - x^2)$.

By Green's Theorem.

$$\int_{\partial D} F \cdot ds = \iint \frac{\partial}{\partial x}(\sin(y^2) - x^2) - \frac{\partial}{\partial y}(e^{x^2})\, dA$$

$$= \int_0^{2\pi}\int_{-2}^{\sin y} -2x \, dx\, dy$$

$$= \int_0^{2\pi} -x^2 \Big|_{-2}^{\sin y} dy$$

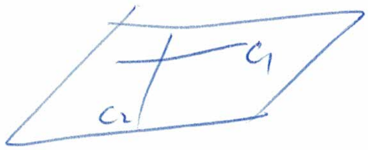
$$= \int_0^{2\pi} -\sin^2 y + 4 \, dy$$

$$= -\frac{1-\cos(2y)}{2} + 4y \Big|_0^{2\pi}$$

$$= -\pi + 8\pi = \underline{7\pi}$$

#

5. (15 points) Find an equation for the plane containing the line parametrized by $(x, y, z) = \mathbf{c}_1(t) = (1 + 2t, 2 - 3t, 3)$ and the line parametrized by $(x, y, z) = \mathbf{c}_2(t) = (-t, t + 3, 1 - 2t)$. Write your answer in the form $Ax + By + Cz + D = 0$.



point on plane $(1, 2, 3)$

normal vector $= \langle 2, -3, 0 \rangle \times \langle -1, 1, -2 \rangle$

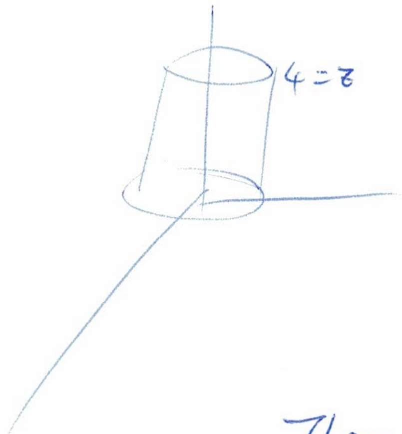
$= \langle 6, 4, -1 \rangle$

$6x + 4y - z - 11 = 0.$

6. (30 points) Let S be the surface which is the boundary of the cylindrical solid given by $x^2+y^2 \leq 9$ and $0 \leq z \leq 4$, with an outward pointing normal vector. Let $\mathbf{F}$ be the vector field given by

$$\mathbf{F} = \left(x, y, z^2(x^2+y^2)\right)$$

Set up and evaluate a triple integral which will give you the flux of $\mathbf{F}$ across S. You must use a method that would "always work," i.e. making up a triple integral which just happens to have the correct answer will not result in any credit.



~~By Gauss' theorem~~

flux of F across $= \iint F \cdot d\vec{S}$

By Gauss' Theorem.

$= \iiint_W \nabla \cdot F \, dV$

Then $\iiint \overbrace{\left(1 + 1 + (x^2+y^2)2z\right)}^{\nabla \cdot F} dV$

$= \iint_{x^2+y^2 \leq 9} \int_0^4 2 + 2z(x^2+y^2) \; dz \, dx \, dy$

$= \iint_{x^2+y^2 \leq 9} 8 + 16(x^2+y^2) \; dx \, dy$

polar coordinate

$= \int_0^{2\pi} \int_0^3 (8 + 16r^2) \, r \, dr \, d\theta$

$= \int_0^{2\pi} \left(4r^2 + 4r^4 \Big|_0^3 \right) d\theta$

$= 2\pi (36 + 324)$

$= \underline{720\pi}$ #

7. (20 points) Consider the surface defined by $z = f(x, y) = \sin(\pi xy) + x^2y - y^2 + 3$. Find the equation for the tangent plane at the point $(x, y, z) = (2, 1, 6)$.

$$z = f(2,1) + f_x(2,1)\,(x-2) + f_y(2,1)\,(y-1)$$

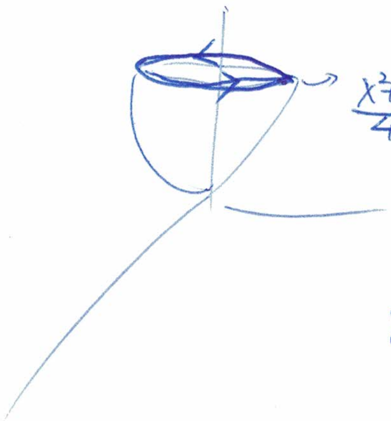
$$f_x = \pi y \cos(xy\pi) + 2xy, \quad f_x(2,1) = \pi + 4$$

$$f_y = \pi x \cos(xy\pi) + x^2 - 2y, \quad f_y(2,1) = 2\pi + 2.$$

$$z = 6 + (\pi+4)(x-2) + (2\pi+2)(y-1)$$

8. (30 points) Let S be the paraboloid $z = (x^2 + y^2)/4$ for $z <= 4$ oriented with upward normal vector. Use Stokes' Theorem to calculate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy^2 z\mathbf{i} - 4x^2 y\mathbf{j} + \frac{z-1}{x^2 + 2y^2 + 1}\mathbf{k}.$$



$\frac{x^2+y^2}{4} = z$, $z=4$, So, the boundary of S is $\frac{x^2+y^2}{4} = 4$, $x^2+y^2 = 4^2$,

i.e., $C(t) = (4\cos t, 4\sin t, 4)$, $0 \le t \le 2\pi$.

By Stokes' Theorem,

$$\iint_S (\nabla \times F) \cdot d\vec{S} = \int_C F \cdot d\vec{s}$$

$$= \int_0^{2\pi} \left\langle 4^4 \cos t \sin^2 t, -4^4 \cos^2 t \sin t, \frac{3}{4^2\cos^2 t + 2(4^2 \sin^2 t) + 1} \right\rangle \cdot \langle -4\sin t, 4\cos t, 0 \rangle$$

$$= \int_0^{2\pi} -4^5 \cos t \sin^3 t - 4^5 \cos^3 t \sin t \, dt$$

$$= \int_0^{2\pi} -4^5 \cos t \sin t \underbrace{(\cos^2 t + \sin^2 t)}_{1} \, dt$$

$$= 0. \ \#$$

9. (25 points) Let $\mathbf{r}(t) = \begin{bmatrix} \sin(2\pi t) \\ \cos(\pi t) \end{bmatrix}$ and $g : \mathbf{R}^2 \to \mathbf{R}$ be a function where $g(0,0) = 3$ and $\nabla g(0,0) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Find $F'(1/2)$ where $F(t) = g(\mathbf{r}(t))$.

$$F'(t) = \nabla g(r(t)) \cdot r'(t)$$

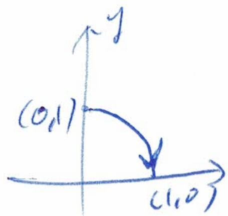
$$F'(\tfrac{1}{2}) = \nabla g(r(\tfrac{1}{2})) \cdot r'(\tfrac{1}{2})$$

$$= \nabla g\begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \langle 2\pi\cos(2\pi t), -\pi\sin(\pi t) \rangle \Big|_{t=\frac{1}{2}}$$

$$= \langle -2, 1 \rangle \cdot \langle -2\pi, -\pi \rangle$$

$$= \underline{3\pi} \ \#$$

10. (25 points) Find the value of the line integral $\int_C y dx - x dy$ along the quarter unit circle C from the point $(0, 1)$ to the point $(1, 0)$.



parametrize it, we get,

$$C(t) = (\sin t, \cos t), \quad 0 \le t \le \frac{\pi}{2}.$$

$$\int y dx - x dy = \int_C F \cdot ds, \text{ where } F = \langle y, -x \rangle$$

$$= \int_0^{\pi/2} \langle \cos t, -\sin t \rangle \cdot \langle \cos t, -\sin t \rangle dt$$

$$= \int_0^{\pi/2} (\cos^2 t + \sin^2 t) dt$$

$$= \pi/2$$

11. (25 points) Consider the surface defined by $z = f(x,y) = 2x^2 + xy + y^2 - 3$. Find the quadratic approximation of the surface (i.e., second-order Taylor polynomial of f) at the point $(x,y,z) = (1,2,5)$. Use your approximation to estimate the value of $f(0.8, 2.1)$.

$f(1,2) = 2+2+4-3 = 5$

$f_x = 4x + y \qquad f_x(1,2) = 6$

$f_y = x + 2y \qquad f_y(1,2) = 5$

$f_{xx} = 4$

$f_{yy} = 2$

$f_{xy} = 1,$

$$L_2(x,y) := 5 + 6(x-1) + 5(y-2) + \frac{1}{2}\Big[4(x-1)^2 + 2(y-2)^2 + 2(x-1)(y-2)\Big]$$

$$f(0.8, 2.1) \sim L_2(0.8, 2.1)$$

$$= 5 + 6(-0.2) + 5(0.1) + 2(-0.2)^2 + (0.1)^2 + (-0.2)(0.1)$$

$$= 5 - 1.2 + 0.5 + 0.08 + 0.01 - 0.02$$

$$= \underline{4.37}$$

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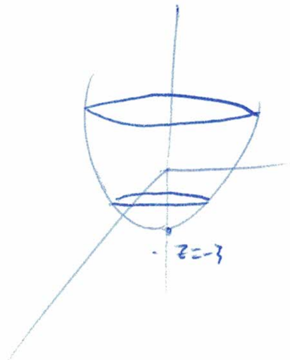
12. (25 points) Let S be the surface $z = x^2 + y^2 - 3$ for $-2 \le z \le 1$. Find the surface area of S.



$$\Phi(x,y) = (x, y, x^2+y^2-3),$$

$$-2 \le x^2+y^2-3 \le 1$$

$$\Rightarrow 1 \le x^2+y^2 \le 4$$

$$T_x = \langle 1, 0, 2x \rangle$$

$$T_y = \langle 0, 1, 2y \rangle$$

$$T_x \times T_y = \langle -2x, -2y, 1 \rangle$$

$$\iint_{1 \le x^2+y^2 \le 4} \| \langle -2x, -2y, 1 \rangle \| \, dx\, dy$$

$$= \iint_{1 \le x^2+y^2 \le 4} \sqrt{1 + 4(x^2+y^2)} \, dx\, dy$$

$$= \int_0^{2\pi} \int_1^2 \sqrt{1+4r^2} \, r \, dr\, d\theta$$

$$= \left(\frac{1}{8} \frac{2}{3} (1+4r^2)^{\frac{3}{2}} \Big|_1^2 \right) 2\pi$$

$$= 2\pi \frac{1}{12} \left(17^{\frac{3}{2}} - 5^{\frac{3}{2}} \right)$$

$$= \frac{\pi}{6} \left(17^{3/2} - 5^{3/2} \right)$$