Second Midterm: Solutions

1. An effective way to solve this problem is to employ the "product rule" formula [1]

$$\operatorname{div}\,(f\cdot\operatorname{curl}\mathbf{G}) = f\cdot\operatorname{div}\mathbf{G} + \nabla f\cdot\mathbf{G}.$$

In our case, take $\mathbf{G}$ to be equal to $\operatorname{curl}\mathbf{F}$.

Exercise: Solve the problem, using this idea.

But the brute force approach still works pretty well here:

$$\begin{aligned}
\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz^2 & 1 & y^3zx \end{vmatrix} &= \big(3y^2zx - 0\big)\,\mathbf{i} - \big(y^3z - 4xz\big)\,\mathbf{j} + (0-0)\,\mathbf{k} \\
&= 3y^2zx\,\mathbf{i} + (4xz - y^3z)\,\mathbf{j} \\
f\cdot\operatorname{curl}\mathbf{F} &= (z+1)^5\cdot 3y^2zx\,\mathbf{i} + (z+1)^5\cdot(4xz-y^3z)\,\mathbf{j} \\
\operatorname{div}\,(f\cdot\operatorname{curl}\mathbf{F}) &= \frac{\partial}{\partial x}\Big((z+1)^5\cdot 3y^2zx\Big) + \frac{\partial}{\partial y}\Big((z+1)^5\cdot(4xz-y^3z)\Big) + \frac{\partial}{\partial y}\Big(0\Big) \\
&= (z+1)^5\cdot 3y^2z + (-3y^2z) + 0 = 0.
\end{aligned}$$

2. The work done by the force field $\mathbf{F}$ in moving a particle along the path $\mathbf{c}$ is equal to the line integral

$$W = \int_{\mathbf{c}} \mathbf{F}\Big(\mathbf{c}(t)\Big)\cdot\mathbf{c}'(t)\,dt.$$

In our case, in order to set up this integral, we need to figure out first, what force field is acting on the particle. For that, you need to recall Newton's second law of motion, which states that the force applied to a body produces a proportional acceleration. More precisely, $\mathbf{F} = m\mathbf{a}$, where m is the mass of the body and $\mathbf{a}$ is its acceleration vector.

We know that at any moment t $(0\le t\le 1)$ the coordinates of the particle are $\mathbf{c}(t) = (t^2, \sin t, \cos t)$. Hence, its acceleration $\mathbf{a}$ is equal to

$$\mathbf{c}''(t) = \Big(2, -\sin t, -\cos t\Big).$$

Now, we have all the data to set up the work integral:

$$\begin{aligned}
W &= \int_0^1 m\,\underbrace{\overbrace{\Big(2, -\sin t, -\cos t\Big)}^{\mathbf{F}=m\cdot\mathbf{c}''(t)}\cdot\Big(2t, \cos t, -\sin t\Big)}_{\text{the inner product}}\,dt \\
&= \int_0^1 m\cdot(4t - \sin t\,\cos t + \sin t\,\cos t)\,dt = \int_0^1 4mt\,dt = 2m\cdot t^2\Big|_0^1 = 2m.
\end{aligned}$$

[1] See Section 4.4, p.306.

3. Consider a thin wire which has the shape of a path $\mathbf{c}$. If $\rho(x, y)$ is a linear mass density (possible units: kg/m, lb/in etc) at the point (x, y) of the wire, then the path integral $M = \int_{\mathbf{c}} \rho(x, y)\, ds$ gives the total mass of the wire (in kg, lb etc).

In our case, we need to find $\mathbf{c}(t)$ first (a parametrization of the path). It is fairly easy: if we take $x = t$, then $y = t^2 + 1$. So $\mathbf{c}(t) = \left(t, t^2 + 1\right)$ as t varies from 1 to 4. We calculate

$$M = \int_{\mathbf{c}} \rho(x, y)\, ds = \int_1^4 \underbrace{t}_{x} \cdot \underbrace{\|\mathbf{c}'(t)\|\, dt}_{ds} = \int_1^4 t \cdot \|(1, 2t)\|\, dt = \int_1^4 t\sqrt{1 + 4t^2}\, dt.$$

Let $u = 1 + 4t^2$. Then $du = 8t\, dt$. Hence,

$$\int_1^4 t\sqrt{1 + 4t^2}\, dt = \int_5^{65} \frac{\sqrt{u}}{8}\, du = \frac{1}{12} u^{\frac{3}{2}}\Big|_5^{65} = \frac{1}{12}\left(65^{\frac{3}{2}} - 5^{\frac{3}{2}}\right).$$

4. If you try to evaluate the integral $\int_x^{\sqrt{x}} e^{\frac{x}{y}}\, dy$, you will stuck. It cannot be done by elementary means. In such a case, the first thing one should try, is to change the order of integration. After you sketch the domain of integration, reversing the order becomes easy:

$$\int_0^1 \int_x^{\sqrt{x}} e^{\frac{x}{y}}\, dy\, dx = \int_0^1 \int_{y^2}^{y} e^{\frac{x}{y}}\, dx\, dy.$$

Now we can calculate

$$\int_0^1 \int_{y^2}^{y} e^{\frac{x}{y}}\, dx\, dy = \int_0^1 \left(y e^{\frac{x}{y}}\right)\Big|_{y^2}^{y} = \int_0^1 \left(ye - ye^{y}\right) dy = e\int_0^1 y\, dy - \int_0^1 ye^{y}\, dy.$$

The first integral standing in the last expression is trivial. It is equal to $\frac{e}{2}$. The second one can be calculated using integration by parts. Take

$$u = y, \quad du = dy, \quad dv = e^y\, dy, \quad v = e^y.$$

Then

$$\int_0^1 \underbrace{y}_{u}\, \underbrace{e^y\, dy}_{dv} = \underbrace{y}_{u}\, \underbrace{e^y}_{v}\Big|_0^1 - \int_0^1 \underbrace{e^y}_{v}\, \underbrace{dy}_{du} = (e - 0) - e^y\Big|_0^1 = (e - 0) - (e - 1) = 1.$$

So, the given integral is equal to $\frac{e}{2} \cancel{+1}$. $\frac{e}{2} - 1$

5. A short solution.

Set up the integral with respect to the order $dz\,dy\,dx$:

$$\iiint_W z\,dx\,dy\,dz = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^1 z\,dz\,dy\,dx = \int_0^1 \int_0^{\sqrt{1-x^2}} z^2\Big|_0^1\,dy\,dx$$
$$= \int_0^1 \int_0^{\sqrt{1-x^2}} z^2\Big|_0^1\,dy\,dx = \frac{1}{2}\int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{1-x^2}} 1\,dy\,dx.$$

The latter integral is equal to the area of the domain D bounded by the line $y = 0$ and the curve $y = \sqrt{1-x^2}$ for $0 \le x \le 1$. You can figure out that D is actually the quarter of the unit disc. Hence, its area is equal to $\frac{\pi}{4}$. Therefore,

$$\iiint_W z\,dx\,dy\,dz = \frac{\pi}{8}.$$

A longer solution.

Set up the integral with respect to the order $dx\,dy\,dz$:

$$\iiint_W z\,dx\,dy\,dz = \int_0^1 \int_0^1 \int_0^{\sqrt{1-y^2}} z\,dx\,dy\,dz = \int_0^1 \int_0^1 zx\Big|_0^{\sqrt{1-y^2}}\,dy\,dz$$
$$= \int_0^1 \int_0^1 z\sqrt{1-y^2}\,dy\,dz = \left[y = \sin t,\ dy = \cos t\,dt\right]$$
$$= \int_0^1 \left(\int_0^{\frac{\pi}{2}} z\sqrt{1-\sin^2 t}\cos t\,dt\right) dz = \int_0^1 \left(\int_0^{\frac{\pi}{2}} z\cos^2 t\,dt\right) dz$$
$$= \int_0^1 z\left(\int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\,dt\right) dz = \int_0^1 z\left(\frac{t+\sin 2t}{2}\Big|_0^{\frac{\pi}{2}}\right) dz$$
$$= \int_0^1 z\cdot\frac{\pi}{4}\,dz = \frac{\pi}{4}\cdot\frac{z^2}{2}\Big|_0^1 = \frac{\pi}{8}.$$

6. The given contour C is closed. Hence, we may try to use the Green's Theorem here:

$$\int_C \underbrace{xy}_{P}\,dx + \underbrace{x^2}_{Q}\,dy = \int\!\!\int_\triangle \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy = \int_C \underbrace{xy}_{P}\,dx + \underbrace{x^2}_{Q}\,dy$$
$$= \int\!\!\int_\triangle 2x - x\,dx\,dy = \int\!\!\int_\triangle x\,dx\,dy,$$

where $\triangle$ is the triangle with the vertices $(0,0)$, $(2,0)$ and $(1,1)$.

The integral looks simple. So our approach will likely work just fine. The order of integration $dx\,dy$ is more preferable here, because otherwise (with respect to $dy\,dx$) you will need to split the triangle $\triangle$ into two smaller triangles as you set up the double integral as an iterated one (sketch a chart and see what happens).

Note that the sides of the triangle are lying on the lines $y = 0$, $x = 0$ and $x = 2 - y$. Now we are ready to evaluate the integral:

$$\int\!\!\int_\triangle x\,dx\,dy = \int_0^1 \int_y^{2-y} x\,dx\,dy = \int_0^1 \frac{x^2}{2}\Big|_y^{2-y}\,dy = \frac{1}{2}\int_0^1 4 - 4y\,dy = \frac{1}{2}(4y - 2y^2)\Big|_0^1 = 1.$$

It also can be solved by using line integrals. See this solution in a different file.