Math 2374, Afternoon	**Name (Print):**	Solutions
Spring 2011	**Student ID:**	
Midterm 1	**Section Number:**	Please report
February 17, 2011	**Teaching Assistant:**	
Time Limit: 50 minutes	**Signature:**	errors.

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. You are allowed to take one-half of one (doubled-sided) 8.5 inch $\times$ 11 inch sheet of notes into the exam.

Do not give numerical approximations to quantities such as $\sin 5$, π, or $\sqrt{2}$. However, you should simplify $\cos\frac{\pi}{4} = \sqrt{2}/2$, $e^0 = 1$, and so on.

The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. **All answers must be justified by valid mathematical reasoning, including the evaluation of definite and indefinite integrals**. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- **Mysterious or unsupported answers will not receive full credit**. Your work should be mathematically correct and carefully and legibly written.
- **A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit**; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive from little to no credit.

1	25 pts	
2	30 pts	
3	30 pts	
4	30 pts	
5	20 pts	
TOTAL	140 pts	

1. (25 points) Suppose that atmospheric pressure at position (x, y, z) is given by the function

$$A(x, y, z) = 1 - x^2 - 2e^y z^2$$

If you were located at position $(2, 0, 1)$, find the direction that you would need to move in order to *decrease* the atmospheric pressure as quickly as possible. Write your answer in the form of a unit vector.

$\nabla A(x,y,z) = (-2x, -2e^y z^2, -4e^y z)$

$\nabla A(2,0,1) = \underline{(-4, -2, -4)}$

points in direction of fastest increase.

To convert to a unit vector:

$\|(-4, -2, -4)\| = \sqrt{16+4+16} = \sqrt{36} = 6$

so the unit vector is

$\cancel{\frac{1}{6}(-4, -2, -4) = \left(\frac{-2}{3}, \frac{-1}{3}, \frac{-2}{3}\right)}$

$$\frac{\langle 4, 2, 4 \rangle}{6} = \left\langle \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right\rangle$$

$\#$

2. (30 points) In this question, consider the function $f(x, y) = y - x^4 + 3x$.

(a) Compute the tangent plane of the graph $z = f(x, y)$ at the point where $(x, y) = (-1, 4)$. Write your answer in the form $Ax + By + Cz + D = 0$.

$\nabla f(x,y) = (-4x^3+3, 1)$ $\qquad f(-1,4) = 4 - 1 - 3 = 0$

$\nabla f(-1,4) = (7, 1)$

Tangent plane: $z = (7,1) \cdot (x-(-1), y-4) + 0$

$z = (7,1) \cdot (x+1, y-4)$

$z = 7x + 7 + y - 4$

$-7x - y + z - 3 = 0$

(b) Find a linear approximation to $f(x, y)$ at the point $(x, y) = (-1, 4)$. Use a linear approximation to estimate $f(-1.1, 4.2)$.

Linear approximation is

$f(x,y) \approx \boxed{0 + \overset{(7,1)}{\cancel{(4,1)}} \cdot (x-(-1), y-4)}$ from above

so $f(-1.1, 4.2) = \overset{(7,1)}{\cancel{(4,1)}} \cdot (-0.1, 0.2) = -0.5$

$= \cancel{0.1 + 0.2 = 0.3}$

3. (30 points) In this question, consider the parametrized curve (t^3, t, t^2).

(a) Give a parametrization of the tangent line at the point $t = -2$.

$c'(t) = (3t^2, 1, 2t)$ $\quad$ $c(-2) = (-8, -2, 4)$

$c'(-2) = (12, 1, -4)$

so the tangent line is

$d(s) = (-8, -2, 4) + s\cdot(12, 1, -4)$

$= (-8+12s, -2+s, 4-4s)$

(b) Find all times t where the tangent vector to this curve is parallel to the plane $x + 6z = 0$.

Normal vector $\vec{n} = (1, 0, 6)$ for this plane.

$c'(t)$ is parallel iff

$0 = \vec{n} \cdot c'(t)$

$0 = (1, 0, 6) \cdot (3t^2, 1, 2t)$

$0 = 3t^2 + 12t$

$0 = t(3t + 12)$

solutions only at $t = 0$ and $t = -4$

4. (30 points) Let $\overline{v}$ be a unit vector in the direction of $(3,4)$ and $\overline{w}$ a unit vector in the direction of $(-1,1)$. Suppose directional derivatives of the function $f(x,y)$ at the point $(0,0)$ are given by

$$D_{\overline{v}}f(0,0) = -1$$

and

$$D_{\overline{w}}f(0,0) = \sqrt{2}$$

Find the gradient $\nabla f(0,0)$.

$$\vec{v} = \frac{(3,4)}{\sqrt{3^2+4^2}} = \left(\frac{3}{5}, \frac{4}{5}\right) \qquad \vec{w} = \frac{(-1,1)}{\sqrt{(-1)^2+1^2}} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

If $\nabla f(0,0) = (a,b)$, then

$$-1 = D_{\vec{v}} f(0,0) = \vec{v} \cdot \nabla f(0,0) = \left(\frac{3}{5}, \frac{4}{5}\right) \cdot (a,b) = \left(\frac{3}{5}a + \frac{4}{5}b\right)$$

$$\sqrt{2} = D_{\vec{w}} f(0,0) = \vec{w} \cdot \nabla f(0,0) = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \cdot (a,b) = \left(\frac{-a+b}{\sqrt{2}}\right)$$

so $\quad 2 = -a+b \qquad a = b-2$

$$-1 = \frac{3}{5}a + \frac{4}{5}b = \frac{3}{5}(b-2) + \frac{4}{5}b = \frac{7}{5}b - \frac{6}{5}$$

$$\frac{7}{5}b = -1 + \frac{6}{5} = \frac{1}{5}$$

$$b = \frac{1}{7} \qquad a = b-2 = \frac{1}{7} - \frac{14}{7} = -\cancel{\frac{15}{7}} \qquad \frac{-13}{7}$$

$$\nabla f(0,0) = (a,b) = \left(\cancel{\frac{-15}{7}}\ \frac{-13}{7}, \frac{1}{7}\right)$$

5. (20 points) Consider the function $\cancel{f(x,y)}$ $f(x,y,z)$ $= (x+xy-y, xz-z^3)$, and suppose $g(u,v) : \mathbb{R}^2 \to \mathbb{R}^3$ is a function whose matrix of partial derivatives at $(u,v) = (3,1)$ is given by

$$\mathbf{D}g(3,1) = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

Find the matrix of partial derivatives of the function $g \circ f$ at the point $(2,1,1)$.

$$Df(x,y,z) = \begin{bmatrix} 1+y & x-1 & 0 \\ z & 0 & x-3z^2 \end{bmatrix} \qquad f(2,1,1) = (3,1)$$

$$Df(2,1,1) = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

so by the chain rule,

$$D(g \circ f)(2,1,1) = Dg(f(2,1,1)) \cdot Df(2,1,1)$$

$$= \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$