Math 2374	**Name (Print):** Solutions
Spring 2011	**Student ID:**
Midterm 2	**Section Number:** Please report
March 24, 2011	**Teaching Assistant:**
Time Limit: 50 minutes	**Signature:** errors

This exam contains 7 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated. You are allowed to take one-half of one (doubled-sided) 8.5 inch $\times$ 11 inch sheet of notes into the exam.

Do not give numerical approximations to quantities such as $\sin 5$, π, or $\sqrt{2}$. However, you should simplify $\cos \frac{\pi}{4} = \sqrt{2}/2$, $e^0 = 1$, and so on.

The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. **All answers must be justified by valid mathematical reasoning, including the evaluation of definite and indefinite integrals**. To receive full credit on a problem, you must show enough work so that your solution can be followed by someone without a calculator.
- **Mysterious or unsupported answers will not receive full credit**. Your work should be mathematically correct and carefully and legibly written.
- **A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit**; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- Full credit will be given only for work that is presented neatly and logically; work scattered all over the page without a clear ordering will receive from little to no credit.

1	30 pts	
2	30 pts	
3	20 pts	
4	30 pts	
5	30 pts	
TOTAL	140 pts	

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1. (30 points) (a) (15 points) Express the volume of the region enclosed by the surfaces $x = z^2 - 1$, $x = 0$, $y = 0$, and $y = 1$ as a triple integral in terms of $dy\,dx\,dz$.

The curves $x = z^2 - 1$ and $x = 0$ intersect when

$z = \pm 1$,

so the volume is

$$V = \int_{-1}^{1} \int_{z^2-1}^{0} \int_{0}^{1} 1 \; dy\,dx\,dz$$

(b) (15 points) Find the volume of this region.

$$\int_{-1}^{1} \int_{z^2-1}^{0} \int_{0}^{1} dy\,dx\,dz$$

$$= \int_{-1}^{1} \int_{z^2-1}^{0} y\Big|_0^1 \, dx\,dz$$

$$= \int_{-1}^{1} \int_{z^2-1}^{0} 1\,dx\,dz$$

$$= \int_{-1}^{1} x\Big|_{z^2-1}^{0} \, dz$$

$$= \int_{-1}^{1} 1 - z^2 \, dz$$

$$= z - \frac{z^3}{3}\Big|_{-1}^{1} = \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = 2 - \frac{2}{3} = \frac{4}{3}$$

2. (30 points) (a) (15 points) Find a function $f(x, y)$ whose gradient vector field is $F(x, y) = (\sin(y), x\cos(y))$.

$\frac{\partial f}{\partial x} = \sin y$ $\quad f(x,y) = \int \sin y \, dx = x\sin y + c$

↳ const in $\underline{x}$

$\frac{\partial f}{\partial y} = x\cos y$ $\quad f(x,y) = \int x\cos y \, dy = x \sin y + d$

↳ const in y

so $f(x,y) = \underline{x \sin y}$ is a solution

(b) (15 points) If $f(x, y, z) = y^2 + x^2 z + x^2 + 4$, find the line integral of the vector field ∇f over the curve $c(t) = (t - 1, t^2, t^4 - t^2)$ between $t = -1$ and $t = 1$.

By the fundamental theorem of calculus, this integral is

$f(c(1)) - f(c(-1))$

$= f(0,1,0) - f(-2,1,0)$

$= (1+4) - (1+4+4)$

$= -4$

3. (20 points) Consider the vector field $F(x, y, z) = (xe^y, xy^2z, xy^3e^x)$. Compute the quantities:
(a) (8 points) div(F)

$$\operatorname{div} F = \frac{\partial}{\partial x}(xe^y) + \frac{\partial}{\partial y}(xy^2z) + \frac{\partial}{\partial z}(xy^3e^x)$$

$$= (e^y) + (2xyz) + 0$$

$$= e^y + 2xyz$$

(b) (12 points) curl(F)

$$\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xe^y & xy^2z & xy^3e^x \end{vmatrix}$$

$$= \left(3xy^2e^x - xy^2,\ -y^3e^x - xy^3e^x,\ y^2z - xe^y\right)$$

4. (30 points) (a) (15 points) Using a path integral, find the arc length of the curve $c(t) = (t, 3, -t)$ in the range $0 \le t \le 4$.

$c'(t) = (1, 0, -1)$

$\|c'(t)\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$

$\text{Length} = \int_0^4 \sqrt{2}\, dt = \sqrt{2}t \Big|_0^4 = 4\sqrt{2}$

(b) (15 points) If this curve represents a wire with density at the point (x, y, z) given by $f(x, y, z) = x$, find the total mass of the wire.

Density at time t is $x(t) = t$

$\text{Total mass} = \int_0^4 (t)\|c'(t)\|\, dt = \int_0^4 t\sqrt{2} = \frac{\sqrt{2}}{2} t^2 \Big|_0^4 = 8\sqrt{2}$

5. (30 points) Use Green's theorem to find the line integral

$$\int (-xy)dx + (xy)dy$$

over a curve that moves, counterclockwise, around the boundary of the square $0 \leq x \leq 1$, $0 \leq y \leq 1$.

Method 1: It also can be solved by using line integrals. See this solution in a different file.

Method2: Green's theorem says that this integral is

$$\int_0^1 \int_0^1 \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(-xy)\, dx\,dy$$

$$= \int_0^1 \int_0^1 (y + x)\, dx\,dy$$

$$= \int_0^1 \left[xy + \frac{x^2}{2} \right]\Big|_0^1 dy$$

$$= \int_0^1 \left(y + \frac{1}{2}\right) dy$$

$$= \left(\frac{y^2}{2} + \frac{y}{2} \right)\Big|_0^1$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$