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Bifurcations and Dynamics of Spiral Waves

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Summary. In several chemical systems such as the Belousov-Zhabotinsky reaction or the catalysis on platinum surfaces, transitions from meandering spiral waves to more complicated patterns have been observed. Seemingly key to the dynamics of spiral waves is the Euclidean symmetry group SE(N). In this article, it is shown that the dynamics near meandering spiral waves or other patterns is determined by a finite-dimensional vector field which has a certain skew-product structure over the group SE(N). This generalizes our earlier work on center-manifold theory near rigidly-rotating spiral waves to meandering spirals. In particular, for meandering spirals, it is much more sophisticated to extract the aforementioned skew-product structure since spatio-temporal rather than only spatial symmetries have to be accounted for. Another difficulty is that the action of the Euclidean symmetry group on the underlying function space is not differentiable, and in fact may be discontinuous. Using this center-manifold reduction, Hopf bifurcations and periodic forcing of spiral waves are then investigated. The results explain the transitions to patterns with two or more temporal frequencies which have been observed in various experiments and numerical simulations.

1. Introduction

Spiral waves arise as stable spatio-temporal patterns in various chemical systems. They have been observed experimentally, for instance, in Belousov-Zhabotinsky reactions [5, 19, 30, 34] and in the catalysis on platinum surfaces [21]. These patterns can be roughly divided into the following categories. Spiral waves may rotate rigidly, that is, they are equilibria in a rotating frame. In the original frame, rigidly-rotating spirals are periodic in time. On the other hand, spirals may either drift or else meander, that is, they are time-periodic in a suitable moving or rotating frame, respectively. In the original frame, drifting spirals are modulated travelling waves, while meandering waves are quasi-periodic in time.

In several experiments and numerical simulations, transitions from spiral waves or other patterns to more complicated waves have occurred. The dynamics near rigidly-rotating spiral waves and their transition to meandering and drifting spirals has been studied extensively; see, for instance, [3, 30]. The motion of spiral waves on a large cylinder has been investigated in [31]. The transition from planar meandering spirals to invariant tori with seemingly three frequencies was investigated in [23] by calculating the Fourier spectrum of the waves. Winfree and others [7, 16, 22] observed patterns in models posed on three-dimensional space in numerical simulations. The effect of periodic forcing on rigidly-rotating spiral-wave solutions has been investigated experimentally in the catalysis on platinum surfaces [21], and in the Belousov-Zhabotinsky reaction [5, 34]. The authors of [29] and [34] also considered periodic forcing of meandering spiral waves and found invariant three-dimensional tori. They observed that the frequencies of the solutions on the torus and the external forcing may lock.

Seemingly key to the dynamical behavior of spiral waves and their bifurcations is the Euclidean symmetry of the plane or the three-dimensional space. Here, the (special) Euclidean symmetry group $\mathrm{SE}(N)$ of $\mathbb{R}^N$ consists of all translations and rotations. Barkley [4] was the first who noticed the relevance of this group to the understanding of spiral waves. He proposed that the dynamics of planar rigidly-rotating spirals is governed by an equivariant vector field on the group $\mathrm{SE}(2)$. Phenomenologically, he could then interpret the transition to meandering or drifting spiral waves as a Hopf bifurcation. Indeed, numerically, Barkley verified the crossing of a pair of simple eigenvalues through the imaginary axis; the rest of the spectrum (except for the eigenvalues enforced by symmetry) is strictly contained in the left half-plane. Similarly, Mantel and Barkley [20] described periodic forcing of meandering spirals by investigating periodically-forced equivariant equations on the group $\mathrm{SE}(2)$.

The purpose of the present article is to corroborate the role of the Euclidean symmetry group by establishing a rigorous link between certain modeling assumptions and the description of the dynamics of spiral waves by equivariant vector fields on the Euclidean symmetry group. We can then explain the aforementioned phenomena observed in experiments and numerical simulations rigorously rather than heuristically.

Chemical systems are traditionally modeled by reaction-diffusion systems on suitable domains. Here, the main modeling assumption is that the domain is actually unbounded, that is, the governing equation is posed on either the plane or the three-dimensional space. We will discuss in Sect. 9 whether and in what sense this hypothesis is justified. The symmetry group $\mathrm{SE}(N)$ then acts on functions $u(x)$ according to

$$((R,S)u)(x) := u(R^{-1}(x-S)) \qquad\qquad x \in \mathbb{R}^N \quad ,$$

where $R \in \mathrm{SO}(N)$ is a rotation and $S \in \mathbb{R}^N$ a translation. In other words, the group element (R, S) first rotates the pattern described by the function u using R, and then shifts it to S. In particular, we may interpret rigidly-rotating spiral waves as *relative equilibria*, that is, as solutions whose time-orbit is contained in the group orbit of its initial value. On the other hand, drifting and meandering spirals can be interpreted as *relative periodic orbits*; after one time period, the solution is contained in the group orbit of its initial value. In the spirit of equivariant bifurcation theory [13], the idea is to prove the existence of a smooth, $\mathrm{SE}(N)$-invariant center manifold near such spiral waves and to discuss the flow on this manifold.

Unfortunately, there are some major technical obstacles which have to be resolved before obtaining such a center manifold. Firstly, the group $\mathrm{SE}(N)$ is not compact, and therefore many standard results are not applicable. Secondly, and more importantly, the aforementioned $\mathrm{SE}(N)$-action is *not* differentiable on reasonable function spaces such as $L^2(\mathbb{R}^N)$ or the space $C^0_{\mathrm{unif}}(\mathbb{R}^N)$ of bounded, uniformly continuous functions on $\mathbb{R}^N$. On the latter space, the action is not even strongly continuous; a counterexample is provided by the rotations acting on the function $u(x_1, x_2) = \cos(x_1)$. Therefore, there is no a priori reason why a center manifold should exist which is at the same time smooth and invariant under the group $\mathrm{SE}(N)$.

In one of the author's doctoral thesis [33], Hopf bifurcations and period forcing of planar rigidly-rotating spirals were investigated using Lyapunov-Schmidt reduction, that is, without deriving equations-of-motion. In [25, 26], we presented a rigorous center-manifold reduction near relative equilibria for general non-compact groups. Fiedler et al. [9] clarified the skew-product structure of vector fields on center manifolds associated with relative equilibria having compact isotropy. In addition, conditions for drifting have been derived in this paper. Simultaneously, using a formal center-bundle construction, Golubitsky et al. [12] investigated Hopf bifurcation of ℓ-armed planar spiral waves and derived conditions for the existence of drifting multi-armed spirals. Normal forms for the reduced equation on the center manifold have been given in [10]. Ashwin and Melbourne [1] studied equivariant maps on non-compact groups. They derived conditions for drifting using group theory. Existence of spiral waves has recently been established in [27, 28] using a Ginzburg-Landau reduction.

Relative periodic orbits of compact groups have been investigated by Field [11]. In this article, relative periodic solutions for non-compact groups are considered. In particular, we extend the aforementioned results derived in [9, 26] so that they also apply to relative periodic solutions. These solutions exhibit a much richer structure since spatio-temporal symmetries rather than only spatial symmetries have to be accounted for. Therefore, though the extension of the center-manifold theorem is not particularly original, it is more sophisticated to extract the skew-product structure. With these methods at hands, we can then explain various experiments and numerical simulations of spiral waves. Specifically, we consider Hopf bifurcations of meandering spirals to invariant three-dimensional tori, the dynamics of spiral waves on cylindrical surfaces, and periodic forcing of spirals and scroll waves.

As mentioned above, we consider reaction-diffusion equations of the form

$$u_t = d\Delta u + F(u,\mu) \qquad x \in \mathbb{R}^N,\ N = 2,3 \tag{1.1}$$

on the plane or in three-dimensional space. The matrix d is diagonal with non-negative entries, and F is a smooth nonlinearity. The function $u : \mathbb{R}^N \to \mathbb{R}^M$ can be interpreted as a vector of spatially dependent concentrations of chemical species. External control parameters are incorporated into the parameter μ. Changes of these parameters may lead to bifurcations. Equation (1.1) is well-posed on the space $C^0_{\mathrm{unif}}(\mathbb{R}^N, \mathbb{R}^M)$ of bounded, uniformly continuous functions. In addition, (1.1) is equivariant with respect to the Euclidean group $\mathrm{SE}(N)$. The (special) Euclidean group $\mathrm{SE}(N)$ is the semi-direct product $\mathrm{SO}(N)\dot{+}\mathbb{R}^N$ of the orthogonal group $\mathrm{SO}(N)$ and the group $\mathbb{R}^N$ of translations with the composition

$$(R,S)(\tilde{R},\tilde{S}) = (R\tilde{R}, S + R\tilde{S}) \tag{1.2}$$

on the product $\mathrm{SO}(N)\dot{+}\mathbb{R}^N$. The Lie algebra $\mathrm{se}(N)$ is the product of the space $\mathrm{so}(N)$ of skew-symmetric matrices, which generate the rotations, and the space $\mathbb{R}^N$ generating the translations. We denote elements in the Lie algebra by $(r,s) \in \mathrm{so}(N) \times \mathbb{R}^N = \mathrm{se}(N)$.

Suppose that $u_*(t)$ is a relative periodic solution with period T, that is,

$$u_*(T) = (R_*, S_*)u_*(0)$$

for some $(R,S) \in \mathrm{SE}(N)$. Furthermore, assume that the group $\mathrm{SE}(N)$ acts continuously on u_*. We denote the linearization of the time-T map $\Phi_T(u)$ associated with (1.1) evaluated at $u_*(0)$ by $\mathrm{D}\Phi_T(u_*)$. We assume that the set of elements in the spectrum of $(R_*, S_*)^{-1}\mathrm{D}\Phi_T(u_*)$ which have modulus equal or bigger than one consists of finitely many, isolated eigenvalues with finite multiplicity. Let E^{cu}_* be the associated generalized center-unstable eigenspace.

In the first step, it is shown that $\mathrm{SE}(N)$ actually acts smoothly on u_* and on all elements in the eigenspace E^{cu}_*. Therefore, even though the group acts discontinuously on the space C^0_{unif}, the aforementioned spectral hypothesis enforces smoothness of the group action on u_* and E^{cu}_*. The proof requires results on strongly-continuous group actions on Banach spaces; in particular, we show that the group acts differentiably on a dense subspace. This generalizes earlier results by Dancer [6] for compact groups. The proof given here seems to be simpler even for compact groups. We then prove the existence of a smooth center-unstable manifold which is invariant under the semiflow and the group $\mathrm{SE}(N)$. Hence, the infinite-dimensional dynamical system near the relative periodic orbit is reduced to some ordinary differential equation on the center-unstable manifold.

In the second step, we identify the skew-product structure of the vector field on the center manifold. The flow on the manifold can be represented as a dynamical system

$$\begin{aligned} \dot{R} &= R\,r(v,\theta,\mu) & \dot{S} &= R\,s(v,\theta,\mu) \\ \dot{v} &= f_N(v,\theta,\mu) & \dot{\theta} &= f_\Theta(v,\theta,\mu) \end{aligned}$$

on the space $\mathrm{SE}(N) \times V_* \times \mathbb{R}$. Here, $(R, S) \in \mathrm{SE}(N)$ is in the group, $\theta \in \mathbb{R}$ corresponds to the coordinate in the time direction, and $v \in V_*$ is in a complement of the tangent space of $\mathrm{SE}(N)u_*$ and the time derivative $\partial_t u_*(0)$ in E_*^{cu}. Furthermore, the function $(r, s)(v, \theta, \mu)$ has values in the Lie algebra $\mathrm{se}(N)$. There are further restrictions on the vector field enforced by the spatio-temporal symmetries of the relative periodic solution. The skew-product structure manifests itself in the fact that the equations for (v, θ) decouple from the equations on the group $\mathrm{SE}(N)$ as a consequence of $\mathrm{SE}(N)$-equivariance.

Summarizing, a systematic and rigorous procedure is developed which allows us to derive equations-of-motion near relative periodic orbits. The main difficulty is that not only spatial but also spatio-temporal symmetries of the relative periodic solution have to be taken into account. We point out that the aforementioned results are formulated in an abstract functional-analytic set-up which includes semilinear parabolic equations equivariant under arbitrary finite-dimensional, and possibly non-compact, Lie groups.

The paper is organized as follows. Section 2 contains the center-manifold reduction for autonomous equations, while periodic forcing is considered in Sect. 3. Section 4 contains a short excursion on linear representations of finite-dimensional Lie groups on Banach spaces. In Sect. 5 and 6, we discuss the regularity and spectral hypotheses in more detail. Applications to spiral waves are then given in Sect. 7 and 8. Finally, conclusions and open problems are discussed in Sect. 9. Sections 7 and 8 are self-contained, so that readers interested mainly in the applications can skip the other sections.

2. Center Manifolds near Relative Periodic Orbits

The main results of this section, Theorems 2.2 and 2.9, establish the reduction to a finite-dimensional center manifold and the characterization of the vector field on the manifold as a skew-product flow.

2.1. The Center-Manifold Reduction

Consider the autonomous semilinear differential equation

$$u_t = -Au + F(u) \tag{2.1}$$

on a Banach space X. We assume that A is sectorial with dense domain $D(A)$. The nonlinearity F is a C^k-function from X^α to X for some $k \geq 3$ and some $\alpha \in [0, 1)$, and X^α is the domain of the fractional power A^α; see, for instance, [15]. We set $Y = X^\alpha$. Equation (2.1) generates a local C^k-semiflow Φ_t on Y. We remark that parameters can always be incorporated as additional components with trivial dynamics.

Let G be a finite-dimensional, possibly non-compact Lie group with Lie algebra $\mathrm{alg}(G) = T_{\mathrm{id}}G$. We write $\exp(\xi)$ for the exponential map from $\mathrm{alg}(G)$ to G. The adjoint action of an element $g \in G$ on the Lie algebra is given by

$$\mathrm{Ad}_g\,\xi := g\xi g^{-1} = \frac{\mathrm{d}}{\mathrm{d}t} g \exp(\xi t) g^{-1}\Big|_{t=0} \in \mathrm{alg}(G) \ .$$

Let $\rho : G \to \mathrm{GL}(Y)$, $g \mapsto \rho_g$, be a faithful and isometric representation of G in the space of bounded, invertible operators. In other words, we require $\|\rho_g\| = 1$ for all $g \in G$. Note that we do *not* assume continuity or smoothness of the map ρ. The group orbit of an element $u \in Y$ is $Gu = \{\rho_g u;\ g \in G\}$.

The structure we require is G-equivariance of the semiflow generated by (2.1)

$$\Phi_t(\rho_g u) = \rho_g \Phi_t(u)$$

for all $t > 0$ and $g \in G$. We are interested in solutions for which time and group orbit are related. Examples of such solutions are relative periodic solutions which, after one period of time, are contained in the group orbit of their initial value.

More precisely, suppose that $u_* \in Y$ is relative periodic with period one, that is,

$$\Phi_1(u_*) = \rho_{g_*} u_*$$

for some $g_* \in G$, and $\Phi_t(u_*) \notin Gu_*$ for $t \in (0,1)$. Its isotropy subgroup is defined by $H_* = \{h \in G;\ \rho_h u_* = u_*\}$. The *relative periodic orbit* $\mathcal{O}_*$ itself is given as the time orbit of the group orbit of u_*

$$\mathcal{O}_* = \{\rho_g \Phi_t(u_*);\ g \in G,\ t \in \mathbb{R}\} \ .$$

We shall investigate the dynamics and possible bifurcations of u_* using a center-manifold reduction near its orbit $\mathcal{O}_*$. Hence, a hypothesis on the spectrum of the linearization about u_* is needed.

Hypothesis (S) *Assume that* $\{\lambda \in \mathbb{C};\ |\lambda| \geq 1\}$ *is a spectral set for the spectrum* $\mathrm{spec}(L_*)$ *of the operator*

$$L_* := \rho_{g_*}^{-1} \mathrm{D}\Phi_1(u_*) \in \mathcal{L}(Y)$$

with associated spectral projection $P_* \in \mathcal{L}(Y)$ *such that the generalized eigenspace* $E_*^{\mathrm{cu}} := \mathcal{R}(P_*)$ *is finite-dimensional.*

In other words, there are only finitely many elements with norm equal or bigger than one in the spectrum of L_*, and these elements are isolated in $\mathrm{spec}(L_*)$ and have finite multiplicity. Let $E_*^{\mathrm{s}} := \mathcal{N}(P_*)$ be the stable subspace. Finally, we assume certain regularity properties.

Hypothesis (R)

(i) *The function* $G \to Y$, $g \mapsto \rho_g u_*$ *is* C^k.

(ii) *For any neighborhood* U *of* id *in* G, *there is a* $\delta > 0$ *such that* $|\rho_g u_* - u_*| \geq \delta$ *for all* $g \in G \setminus (H_* U)$. *Here,* $H_* U = \{hg;\ h \in H_*,\ g \in U\}$.

(iii) *Considered as elements of* $\mathcal{L}(Y)$, *the operators* $\rho_g P_*$ *and* $P_* \rho_g$ *are* C^{k-1} *in* $g \in G$.

(iv) *The tangent space* $T_{u_*}(Gu_*)$ *is contained in the center-unstable eigenspace* E_*^{cu}.

Hypotheses (R)(i) and (ii) imply that the group orbit $\mathcal{O}_*$ of u_* is a smooth embedded manifold which is diffeomorphic to G/H_*. Note that the isotropy H_* is closed due to Hypothesis (R)(i). Hypothesis (R)(iii) requires in particular that the group acts differentiably on elements in $E_*^{\rm cu}$. The last assumption (R)(iv) is true if the group has an invariant metric. In particular, it is met for compact groups and also for the Euclidean group SE(N); see Lemma 6.2. For general non-compact groups, however, it is not necessarily satisfied; as pointed out in [12, Remark after Proposition A.3], the affine group provides a counterexample. We show in Sect. 5 that Hypothesis (R) is a consequence of the spectral assumption (S) under some mild additional assumptions on the group G.

We start with the definition of a center manifold.

Definition 2.1. Let u_* be a relative periodic solution. We say that (2.1) has a G-invariant center-manifold $M_*^{\rm cu}$ associated with u_* if the following is true. There exists a G-invariant, locally semiflow-invariant manifold $M_*^{\rm cu}$ contained in Y. The manifold $M_*^{\rm cu}$ is of class C^{k-1}, the vector field on $M_*^{\rm cu}$ is C^{k-2}, and the action of G on $M_*^{\rm cu}$ is C^{k-1}-smooth. Furthermore, there is a $\delta > 0$ such that $M_*^{\rm cu}$ contains all solutions which stay in the δ-neighborhood of the relative periodic orbit $\mathcal{O}_*$ for all negative times. Its tangent space at the point $\rho_g \Phi_t(u_*)$ is $\rho_g {\rm D}\Phi_t(u_*) E_*^{\rm cu}$. Finally, $M_*^{\rm cu}$ is locally exponentially attracting.

We emphasize that the action of G on the center manifold is smooth though it may act discontinuously on the whole space.

In general, the center manifold may contain unstable directions. Inverting the smooth, equivariant flow on the center-manifold, it is straightforward to construct a center-manifold tangent to the subspace of $E_*^{\rm cu}$ corresponding to center directions. Thus, with a slight abuse of notation, we use the term center manifold even when unstable directions are included.

Recall that a continuous action of a Lie group G on a manifold M is called *proper* if the map $(g, u) \rightarrow (\rho_g u, u) \in M \times M$ maps closed sets into closed sets and preimages of points are compact. We can then state the existence result.

Theorem 2.2. *Assume that Hypotheses (S) and (R) are met for the relative periodic solution $u_* \in Y$ of (2.1). There exists then a G-invariant center manifold $M_*^{\rm cu}$. Moreover, the G-action on $M_*^{\rm cu}$ is proper provided H_* is compact.*

The proof of the theorem is given in the next section.

2.2. The Center Manifold for the Poincaré Map

Theorem 2.2 will be proved by applying the graph transform to the Poincaré map associated with a suitable section transverse to the time direction. Since we want to preserve G-equivariance, this section should be invariant under the group G.

First, it is shown that the center-unstable eigenspace is invariant under the isotropy group H_*. If the operator $L_* = \rho_{g_*}^{-1} {\rm D}\Phi_1(u_*)$ were H_*-equivariant, the center-unstable eigenspace would be H_*-invariant as a consequence of the definition of P_*. Note that $\mathrm{D}\Phi_1(u_*)$ is H_*-equivariant. Furthermore, since $\rho_h \Phi_1(u_*) = \Phi_1(\rho_h u_*) = \Phi_1(u_*)$ and $\Phi_1(u_*) = \rho_{g_*} u_*$, we have $\rho_{g_*}^{-1} \rho_h \rho_{g_*} u_* = u_*$. Hence, as sets, $g_* H_* = H_* g_*$. However, in general, g_* does not commute with all elements in H_*, and L_* is then not equivariant under H_*.

Lemma 2.3. *The spectral projection P_* associated with L_* is equivariant with respect to the isotropy group H_* of u_*. In particular, the center-unstable eigenspace E_*^{cu} is H_*-invariant.*

Proof. It suffices to show that $\mathcal{N}(P_*) = E_*^{\mathrm{s}}$ and $\mathcal{R}(P_*) = E_*^{\mathrm{cu}}$ are invariant under H_*. On account of the spectral hypothesis (S), there are constants C_1, C_2, and $\eta_{\mathrm{s}} < \eta_{\mathrm{cu}} < 1$ such that

$$\|L_*^\ell(\mathrm{id} - P_*)\| \le C_1 \eta_{\mathrm{s}}^\ell, \quad \|(L_*|_{E_*^{\mathrm{cu}}})^{-\ell} P_*\| \le C_2 \frac{1}{\eta_{\mathrm{cu}}^\ell} \tag{2.2}$$

for all $\ell \in \mathbb{N}$.

Suppose that there is an element w such that

$$|L_*^\ell w| \le C \eta_{\mathrm{s}}^\ell |w| \tag{2.3}$$

for some C independent of $\ell \in \mathbb{N}$. We then conclude

$$|P_* w| = |(L_*|_{E_*^{\mathrm{cu}}})^{-\ell} P_* L_*^\ell w| \le C_2 \frac{1}{\eta_{\mathrm{cu}}^\ell} |L_*^\ell w| \le C C_2 \left(\frac{\eta_{\mathrm{s}}}{\eta_{\mathrm{cu}}}\right)^\ell |w| \ ,$$

and therefore $P_* w = 0$ since $\eta_{\mathrm{s}} < \eta_{\mathrm{cu}}$.

The idea is now to show that $\rho_h v$ satisfies the estimate (2.3) whenever v does. For $h \in H_*$, using the definition of L_*, equivariance of the semiflow, and invariance of the isotropy subgroup H_* under conjugation with g_*, we calculate

$$\begin{aligned} L_*^\ell \rho_h v &= (\rho_{g_*}^{-1} \mathrm{D}\Phi_1(u_*))^\ell \rho_h v \\ &= (\rho_{g_*}^{-\ell} \rho_h \rho_{g_*}^\ell)(\rho_{g_*}^{-1} \mathrm{D}\Phi_1(\rho_{g_*}^{-\ell+1} \rho_h \rho_{g_*}^{\ell-1} u_*)) \cdot \ldots \cdot (\rho_{g_*}^{-1} \mathrm{D}\Phi_1(u_*)) v \\ &= \rho_{g_*}^{-\ell} \rho_h \rho_{g_*}^\ell L_*^\ell v \ . \end{aligned}$$

Take any v with $P_* v = 0$ and $h \in H_*$. Then

$$|L_*^\ell \rho_h v| \le \|\rho_{g_*}^{-\ell} \rho_h \rho_{g_*}^\ell\| \cdot |L_*^\ell v| \le |L_*^\ell v|$$

by the above calculation and isometry of the group action. Therefore, we conclude $P_* \rho_h v = 0$. In particular, $\mathcal{N}(P_*)$ is invariant under H_*.

It remains to show that $\mathcal{R}(P_*)$ is H_*-invariant. Given w such that $L_*^{-\ell} w$ exists for all $\ell > 0$ and

$$|L_*^{-\ell} w| \le C \frac{1}{\eta_{\mathrm{cu}}^\ell} |w| \tag{2.4}$$

for some C independent of ℓ, we conclude as before that $(\mathrm{id} - P_*) w = 0$. Next, take any $v \in E_*^{\mathrm{cu}}$ and $h \in H_*$. Since $v \in E_*^{\mathrm{cu}}$, there exists a $w \in E_*^{\mathrm{cu}}$ such that $L_* w = v$. Thus, since $L_*(\rho_{g_*} \rho_h \rho_{g_*}^{-1} w) = \rho_h v$, we have $L_*^{-1} \rho_h v = \rho_{g_*} \rho_h \rho_{g_*}^{-1} L_*^{-1} v$. Using the isometry of the group action, we obtain $|L_*^{-\ell} \rho_h v| = |L_*^{-\ell} v|$, and therefore $(\mathrm{id} - P_*) \rho_h v = 0$ by the discussion above. Hence, $P_* \rho_h v = \rho_h v = \rho_h P_* v$. □

Therefore, the splitting of Y into the center-unstable space E_*^{cu} and the stable space $E_*^{\mathrm{s}} = \mathcal{N}(P_*)$ is H_*-invariant. By Hypothesis (R)(iv), the space E_*^{cu} contains the tangent space $T_{u_*}(Gu_*)$ of the group orbit. Also, the vector field $\partial_t u_*(t)|_{t=0}$ lies in E_*^{cu}. Both subspaces $T_{u_*}(Gu_*)$ and $\mathrm{span}\{\partial_t u_*(t)|_{t=0}\}$ are invariant under the isotropy subgroup H_*. Note that these subspaces have trivial intersection since we excluded relative equilibria. We construct an H_*-invariant complement V_* of the sum of the aforementioned subspaces in E_*^{cu}.

Lemma 2.4. *There exists an H_*-invariant scalar product on E_*^{cu}.*

Proof. Since the representation ρ is isometric and H_* acts on E_*^{cu}, the image $\rho(H_*)$ of the isotropy group H_* under ρ is bounded in $\mathrm{GL}(E_*^{\mathrm{cu}})$. Hence, the closure $\mathrm{clos}(\rho(H_*)) \subset \mathrm{GL}(E_*^{\mathrm{cu}})$ of $\rho(H_*)$ is a compact group, and therefore admits a Haar measure. Using this Haar measure of $\rho(H_*)$, we endow E_*^{cu} with an H_*-invariant scalar product. □

Let V_* be the orthogonal complement in E_*^{cu} of the space

$$T_{u_*}\mathcal{O}_* = T_{u_*}(Gu_*) \oplus \mathrm{span}\left\{\frac{\mathrm{d}}{\mathrm{d}t}u_*(t)|_{t=0}\right\} \ ,$$

and denote the orthogonal projection onto V_* by P_{V_*}. In other words, we have the H_*-invariant splitting

$$E_*^{\mathrm{cu}} = T_{u_*}\mathcal{O}_* \oplus V_* \ . \tag{2.5}$$

Let $B_* := P_{V_*}L_*|_{V_*} \in \mathcal{L}(V_*)$. We emphasize that the spectrum of B_* determines bifurcations from the relative periodic orbit.

Next, we define the section

$$\mathcal{S} := \{\rho_g(u_* + v_*) + w_*;\ g \in G,\ v \in V_*,\ w_* \in \rho_g E_*^{\mathrm{s}} \text{ with } |v|, |w| < \delta\} \tag{2.6}$$

transverse to the time-orbit in u_*. In the next lemma, we show that $\mathcal{S}$ is smooth, and we construct the Poincaré-map $\Pi : \mathcal{S} \to \mathcal{S}$. It is here where the regularity hypothesis (R) is used.

Lemma 2.5. *The set $\mathcal{S}$ is a G-invariant and C^{k-1}-smooth hypersurface. It contains the group orbit Gu_* and is transverse to the semiflow. The corresponding Poincaré map $\Pi : \mathcal{S} \to \mathcal{S}$ is C^{k-1}-smooth, G-equivariant, and close to $\Phi_1|_{\mathcal{S}}$ in the C^0-topology. Moreover, $\mathrm{spec}(\rho_{g_*}^{-1}\mathrm{D}\Pi(u_*)) \subset \mathrm{spec}(L_*)$.*

Proof. By definition, $\mathcal{S}$ is G-invariant and contains Gu_*. Smoothness of $\mathcal{S}$ is a consequence of Hypothesis (R)(iii), and we refer to [26, Sect. 3.2] for the proof. The tangent space of $\mathcal{S}$ at u_* is $T_{u_*}(Gu_*) \oplus V_* \oplus E_*^{\mathrm{s}}$. By definition of V_*, the vector field at u_* is a complement to this tangent space.

It remains to verify the claim about the spectrum. The subspaces E_*^{s} and $T_{u_*}(Gu_*)$ of $T_{u_*}\mathcal{S}$ are invariant under the linearization L_*. Therefore, the spectrum of $\rho_{g_*}^{-1}\mathrm{D}\Pi(u_*)$ coincides with the spectrum of L_* on these subspaces. Recall that $B_* = P_{V_*}L_*|_{V_*}$, and let $B_*v = \lambda v$ for some $v \in V_*$. Using $P_{V_*}\partial_t u_*(0) = 0$, it is straightforward to show that $L_*(v + a\partial_t u_*(0)) = \lambda(v + a\partial_t u_*(0))$ for a suitable choice of a whenever $\lambda \neq 1$. This proves the lemma. □

Rather than investigating the semiflow, we shall concentrate on the Poincaré map Π. The relative periodic solution u_* of the semiflow is a *relative fixed point* of the corresponding Poincaré map Π, that is, $\Pi(u_*) = \rho_{g_*} u_*$. We now prove the existence of a center manifold near a relative fixed point of a G-equivariant map.

Theorem 2.6. *Suppose that $u_* \in \mathcal{S}$ is a relative fixed point of a G-equivariant, C^{k-1}-smooth map $\Pi : \mathcal{S} \to \mathcal{S}$. Assume that Hypotheses (S) and (R) are met. There exists then a G-invariant center manifold $M_\Pi^{\mathrm{cu}} \subset \mathcal{S}$ which is locally invariant under Π. If H_* is compact, the G-action on the manifold M_*^{cu} is proper.*

Proof. The proof is very similar to that given in [26] for relative equilibria. We will therefore only outline the proof and refer to [26] for the details.

First, we parametrize a neighborhood of Gu_* in the section $\mathcal{S}$ smoothly as in (2.6). The map Π is then modified for $v \in V_*$ outside a small H_*-invariant neighborhood U of the origin in V_* by adding a small positive multiple of the identity; see [26]. The modified map, which is still smooth and G-equivariant, is denoted by $\tilde{\Pi}$.

Next, consider the center-unstable and stable bundles

$$V^{\mathrm{cu}} = \{u = \rho_g(u_* + v);\ (g, v) \in G \times V_*,\ |v| < \delta\}$$

and $E^{\mathrm{s}} = \{\rho_g(u_* + w);\ (g, w) \in G \times E_*^{\mathrm{s}}\}$, respectively, along the group orbit Gu_*. The center manifold is sought as a graph of a map from the center-unstable into the stable bundle. Therefore, consider the closed metric space $\Sigma_\#$

$$\Sigma_\# = \{\sigma \in C^{0,1}(V^{\mathrm{cu}}, Y);\ \sigma(\rho_g(u_* + v)) \in \rho_g E_*^{\mathrm{s}},\ \ |\sigma(\cdot)| \le \delta,\ \ \mathrm{Lip}(\sigma) \le 1\}$$

of Lipschitz-continuous sections of the stable bundle equipped with the sup-norm.

Since the center manifold will be exponentially attracting, any other manifold nearby is attracted to it under forward iterations of $\tilde{\Pi}$. The idea of the graph transform is to define a map $\Pi_\#$ on $\Sigma_\#$ by mapping a section into its image under $\Pi_\#$, and seek the center manifold as a fixed point of $\Pi_\#$. More precisely, for some $\ell > 1$, define $\tilde{\sigma} = \Pi_\#(\sigma)$ by the condition

$$y + \tilde{\sigma}(y) \in \{\tilde{\Pi}^\ell(x + \sigma(x));\ x \in V^{\mathrm{cu}}\}$$

for each $y \in V^{\mathrm{cu}}$. The modification of the map Π takes care of the complication that the domain of the new section $\tilde{\sigma}$ may have shrunken under forward iteration of σ. On account of Hypothesis (S) and Lemma 2.5, the map $\tilde{\Pi}$ is normally hyperbolic. Hence, the graph transform $\Pi_\#$ is well-defined and a contraction provided the number $\ell \in \mathbb{N}$ is chosen sufficiently large; see, for instance, [8, 17]. The unique fixed point $\sigma_\#$ of $\Pi_\#$ defines the center manifold by

$$M_\Pi^{\mathrm{cu}} = \{(\mathrm{id} + \sigma_\#)(\rho_g(u_* + v));\ (g, v) \in G \times V_*,\ \ |v| < \delta\}\ . \qquad (2.7)$$

Again by [8, 17], M_Π^{cu} is exponentially attracting, locally invariant under $\tilde{\Pi}$, and of class C^{k-1}. Since Π and $\tilde{\Pi}$ coincide in the small neighborhood U of Gu_* in $\mathcal{S}$, the manifold M_Π^{cu} is also locally invariant under Π.

By G-equivariance of $\tilde{\Pi}$, the manifold $\rho_g M_\Pi^{\mathrm{cu}} \in \Sigma_\#$ is also invariant under $\tilde{\Pi}$ for any fixed $g \in G$. Therefore, by uniqueness, M_Π^{cu} is invariant under G. The G-action on M_Π^{cu} is smooth since the map $\sigma_\# : V^{\mathrm{cu}} \to Y$ is G-equivariant and the G-action on V^{cu} is smooth by assumption (R)(iii); see also [26, Sect. 3.4]. Finally, if H_* is compact, the G-action is proper on the group orbit Gu_* by Hypothesis (R)(ii). This remains true on M_Π^{cu} provided the V_*-component in the parametrization (2.7) is small enough; see again [26]. □

The following corollary, actually a by-product of the proof of Theorem 2.6, characterizes the structure of the manifold M_Π^{cu} and the Poincaré map $\Pi|_{M_\Pi^{\mathrm{cu}}}$. We recall that V_* is an H_*-invariant complement of $T_{u_*}(Gu_*) \oplus \mathrm{span}\{\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t(u_*)|_{t=0}\}$ in E_*^{cu}; see (2.5).

Corollary 2.7. *Assume in addition to the assumptions of Theorem 2.6 that H_* is compact. The center manifold M_Π^{cu} is then diffeomorphic to $G \times_{H_*} V_* := (G \times V_*)/\sim$ under the identification $(gh, v) = (g, \rho_h v)$ for $(g, v) \in G \times V_*$ and $h \in H_*$. The pull-back of the map $\Pi|_{M_\Pi^{\mathrm{cu}}}$ to $G \times V_*$ is*

$$\Pi(g, v) = \begin{pmatrix} g\Pi_G(v) \\ \Pi_N(v) \end{pmatrix} , \tag{2.8}$$

and (Π_G, Π_N) is H_*-equivariant

$$(\Pi_G, \Pi_N)(\rho_h v) = (\varphi_*(h)\Pi_G(v)\varphi_*(h)^{-1}, \rho_{\varphi_*(h)}\Pi_N(v)), \tag{2.9}$$

for all $h \in H_*$ and all $v \in V_*$, where $\varphi_*(h) = g_* h g_*^{-1}$.

Proof. It follows from the description (2.7) that the manifold M_Π^{cu} is diffeomorphic to $G \times_{H_*} V_*$. It is then possible to lift the map Π to the product $G \times V_*$ as in [9]. The structure of the pull-back of Π to $G \times V_*$ is a consequence of G-equivariance of Π. □

It has been shown in [9] that the vector field near relative equilibria is of skew-product form. Equation (2.8) establishes the same property near relative fixed points of equivariant maps.

Proof of Theorem 2.2. The Poincaré map Π defined near the relative periodic solution u_* of (2.1) is a G-equivariant map from the section $\mathcal{S}$ given in (2.6) into itself. By Theorem 2.6, there exists a G-invariant center-unstable manifold M_Π^{cu} of class C^{k-1} for the Poincaré map Π. We transport M_Π^{cu} with the semiflow Φ_t along the time direction. By local invariance of M_Π^{cu} under Π, the set

$$\{\Phi_t(u);\ u \in M_\Pi^{\mathrm{cu}},\ \ t \in [0, 3/2]\}$$

contains a smooth manifold M_*^{cu} in a possibly small, G-invariant, δ-neighborhood of $\mathcal{O}_*$. By construction, M_*^{cu} is locally invariant under the semiflow. Since M_Π^{cu} is G-invariant and attracting, so is M_*^{cu}.

It remains to prove that the action of G is proper on M_*^{cu} if H_* is compact. Let $g_n \in G$ and u_1, $u_2 \in M_*^{\mathrm{cu}}$ be such that $\rho_{g_n} u_1 \to u_2$. We have to show that the sequence $\{g_n\}$ has a convergent subsequence. With $j = 1, 2$, we find elements $\hat{u}_j \in M_{\Pi}^{\mathrm{cu}}$ and times $t_j \in [0, 3/2]$ with $u_j = \Phi_{t_j}(\hat{u}_j)$. Since Φ_t is a diffeomorphism in M_*^{cu}, we can assume that $t_1 = 0$ without loss of generality. Hence, $\rho_{g_n}\hat{u}_1 \to \Phi_{t_2}(\hat{u}_2)$. Since M_{Π}^{cu} is G-invariant, we have $t_2 = 0$. Furthermore, using that the G-action is proper on M_{Π}^{cu}, we conclude that $\{g_n\}$ has a convergent subsequence, and consequently the G-action is also proper on M_*^{cu}. □

2.3. The Skew-Product Flow on the Center Manifold

With Theorem 2.2 at hands, bifurcations from relative periodic orbits can be reduced to finite-dimensional smooth bifurcation problems. In order to analyze these bifurcations, the vector field on the manifold M_*^{cu} has to be computed. Therefore, we aim for a representation of the vector field in the space $G \times V_* \times \mathbb{R}$ corresponding to the directions along the group, the transverse directions in which bifurcations take place, and the time direction. However, the manifold M_*^{cu} might have a complicated topological structure; in general, it is not diffeomorphic to a direct product.

The main result in this section, Theorem 2.9, clarifies the geometric structure of the center manifold as a bundle. The manifold is diffeomorphic to the covering space $G \times V_* \times \mathbb{R}$ under an appropriate identification. It is important that the submersion describing this identification is defined in a uniform neighborhood of the possibly non-compact manifold $\mathcal{O}_*$. The vector field on M_*^{cu} is then lifted to the covering space. The lifted vector field inherits the G-equivariance from the original equation and has additional covering symmetries induced by the spatio-temporal symmetries of the relative periodic orbit.

In deriving the representation mentioned above, it is important to separate the effects of the operator $\mathrm{D}\Phi_1(u_*)$ and the group element g_*. First, we account for the bundle structure induced by the linearized map $\mathrm{D}\Phi_1(u_*)$. Recall the splitting (2.5), $E_*^{\mathrm{cu}} = T_{u_*}\mathcal{O}_* \oplus V_*$. The first subspace is mapped into itself by $\rho_{g_*}^{-1}\mathrm{D}\Phi_1(u_*)$. On the space V_*, we consider the matrix $B_* = P_{V_*}\rho_{g_*}^{-1}\mathrm{D}\Phi_1(u_*)|_{V_*} \in \mathcal{L}(V_*)$ which may induce a non-trivial bundle structure. For instance, an eigenvalue -1 of B_* may lead to a Möbius bundle. This bundle structure can be taken into account by following a basis in V_* along the time orbit of u_*. After one time unit, the resulting basis in $\rho_{g_*}V_*$ defines a map in $\mathcal{L}(V_*, \rho_{g_*}V_*)$ which we denote by $J(1)$. It describes the structure induced by the operator $\mathrm{D}\Phi_1(u_*)$. This matrix can be chosen in an H_*-equivariant fashion retaining some of the symmetry properties present in the system. The main point is that it is also an isometry; this property guarantees that the center manifold can be parametrized in a uniform neighborhood of $\mathcal{O}_*$. The composition $Q_* := \rho_{g_*}^{-1}J(1) \in \mathcal{L}(V_*)$ then encodes the entire bundle structure.

Therefore, consider the space $E_0 := E_*^{\mathrm{cu}}$. Let E_θ be the image of E_0 under $\mathrm{D}\Phi_\theta(u_*)$, that is, $E_\theta := \mathrm{D}\Phi_\theta(u_*)E_0$ with $\theta \in (-\delta, 1+\delta)$. The collection E_θ with $\theta \in (-\delta, 1+\delta)$ is a differentiable trivial vector bundle over $(-\delta, 1+\delta)$. Note that even if $g_* = \mathrm{id}$, we consider E_0 and E_1 as different spaces.

Lemma 2.8. *There exists an H_*-invariant splitting*

$$E_\theta = T_{\Phi_\theta(u_*)}\mathcal{O}_* \oplus V_\theta$$

such that $V_0 = V_*$. The associated projections $P_V(\theta)$ onto V_θ are H_*-equivariant, smooth, and satisfy $\rho_{g_*}P_V(\theta)\rho_{g_*}^{-1} = P_V(\theta+1)$ for $\theta \in (-\delta,\delta)$. Moreover, there exist isomorphisms $J(\theta) : V_* \to V_\theta$ which are smooth in θ and H_*-equivariant, such that $Q_* := \rho_{g_*}^{-1}J(1)$ is an isometry.

Proof. Note that

$$\{\theta\} \times E_0 \to \{\theta\} \times E_\theta, \quad (\theta, v_0) \mapsto (\theta, \mathrm{D}\Phi_\theta(u_*)v_0) \tag{2.10}$$

is an H_*-equivariant trivialization of the bundle $E = (E_\theta)_{\theta\in(-\delta,1+\delta)}$. We take the H_*-invariant scalar product $\langle u, v\rangle_0$ in E_0 which has been defined in Lemma 2.4. The splitting $E_*^{\mathrm{cu}} = T_{u_*}\mathcal{O}_* \oplus V_*$ is then H_*-invariant and orthogonal. Using the aforementioned trivialization (2.10), we may then choose H_*-invariant scalar products $\langle u, v\rangle_\theta$ in E_θ for $\theta \in (-\delta,\delta)$ which are smooth in θ. For $\theta \in (-\delta,\delta)$, we equip the spaces $E_{1+\theta}$ with the H_*-invariant scalar products

$$\langle u, v\rangle_{1+\theta} = \langle \rho_{g_*}^{-1}u, \rho_{g_*}^{-1}v\rangle_\theta \ . \tag{2.11}$$

Note that $g_*H_*g_*^{-1} = H_*$ since the isotropy group does not change along time orbits. Next, we connect the scalar products on E_δ and $E_{1-\delta}$ by a smooth, H_*-invariant family $\langle\cdot,\cdot\rangle_\theta$ of scalar products on E_θ for $\theta \in (\delta, 1-\delta)$ using again the H_*-equivariant trivialization (2.10). This can be accomplished exploiting the Haar measure of $\rho(H_*)$, see Lemma 2.4, and the fact that the set of positive definite, symmetric matrices is connected.

Using the smooth, H_*-invariant scalar product on the bundle, we define $P_V(\theta)$ as the orthogonal projection onto the complement of the tangent space $T_{\Phi_\theta(u_*)}\mathcal{O}_*$ in E_θ. The projections are smooth and H_*-equivariant since the tangent spaces and scalar products are smooth and H_*-invariant. For θ near zero, the scalar products on E_θ and $E_{1+\theta}$ are conjugated by g_*, see (2.11), and the tangent spaces $T_{\Phi_\theta(u_*)}\mathcal{O}_*$ are mapped into $T_{\Phi_{1+\theta}(u_*)}\mathcal{O}_*$ by ρ_{g_*}. Therefore, we have $\rho_{g_*}P_V(\theta)\rho_{g_*}^{-1} = P_V(\theta+1)$ for θ near zero.

The isometries $J(\theta)$ can now be constructed using the fact that $(V_\theta)_{\theta\in(-\delta,1+\delta)}$ is a Riemannian, H_*-invariant trivial subbundle of the bundle $(E_\theta)_{\theta\in(-\delta,1+\delta)}$ such that scalar product and H_*-action are compatible. We may choose $J(\theta)$ as a Riemannian, H_*-equivariant trivialization of this subbundle. □

Define $Q_* := \rho_{g_*}^{-1}J(1)$. The next theorem describes the skew-product structure.

Theorem 2.9. *Assume that Hypotheses (S) and (R) are satisfied and suppose that the isotropy group H_* is compact. The center manifold M_*^{cu} is then C^{k-1}-diffeomorphic to $(G \times V_* \times \mathbb{R})/\sim$ where the identification is defined by*

$$(gh, v, \theta) \sim (g, \rho_h v, \theta) \quad \textit{and} \quad (gg_*, Q_*v, \theta) \sim (g, v, \theta+1) \tag{2.12}$$

with $h \in H_*$ *and* $(g, v, \theta) \in G \times V_* \times \mathbb{R}$. *The differential equations on* M_*^{cu} *can be lifted to* $G \times V_* \times \mathbb{R}$ *such that*

$$\dot{g} = g f_G(v, \theta), \quad \dot{v} = f_N(v, \theta), \quad \dot{\theta} = f_\Theta(v, \theta) \ . \tag{2.13}$$

The vector field $(f_G, f_N, f_\Theta) \in \mathrm{alg}(G) \times V_* \times \mathbb{R}$ *is equivariant under the spatio-temporal symmetries of* u_*, *that is,*

$$(f_G, f_N, f_\Theta)(\rho_h v, \theta) = (h f_G(v, \theta) h^{-1}, \rho_h f_N(v, \theta), f_\Theta(v, \theta)) \tag{2.14}$$

for all $h \in H_*$ *and*

$$(f_G, f_N, f_\Theta)(v, \theta+1) = (g_* f_G(Q_* v, \theta) g_*^{-1}, Q_*^{-1} f_N(Q_* v, \theta), f_\Theta(Q_* v, \theta)) \ . \tag{2.15}$$

Moreover, $(f_G, f_N, f_\Theta)(0, \theta) = (0, 0, 1)$.

Note that the reduced differential equations (2.13) are of skew-product form. Indeed, the equations for (v, θ) decouple and can be solved independently of the equation on the group. Therefore, bifurcations are described entirely by the H_*-equivariant (v, θ)-equation. In particular, the equation for v describes the dynamics of the shape of the pattern, while the equation for θ determines the phase. On the other hand, drift along the group is determined by the g-equation where the bifurcating solutions act as an equivariant forcing. For non-compact groups, resonance phenomena in the g-equation may then lead to unbounded motion on the group. The skew-product structure is exploited in the applications, and we refer the reader to Sect. 7 for illustrative examples.

Proof. We have to construct an appropriate submersion from $G \times V_* \times \mathbb{R}$ onto the center manifold. Since the center manifold is given as an equivariant graph over the center-unstable bundle, it suffices to seek a submersion onto this bundle.

We choose a smooth function $\chi : (-\delta, 1+\delta) \to [0, 1]$ such that $\chi(\theta) = 0$ for $\theta \in (-\delta, \delta)$, $\chi(\theta) = 1$ for $\theta \in (1-\delta, 1+\delta)$, and $|\chi(\theta) - \theta| < 2\delta$ for all θ.

By Dunford-Taylor calculus, the operators $\rho_{g_*^{-1}} \mathrm{D}\Phi_1(\Phi_\theta(u_*))$ have spectral projections $P_*(\theta)$ with $P_*(0) = P_*$ which depend smoothly on θ. By Lemma 2.3, the projections are also H_*-equivariant. Furthermore, they satisfy $P_*(\theta+1) = \rho_{g_*} P_*(\theta) \rho_{g_*}^{-1}$ for all θ. We shall exploit these projections to extend the domain of definition of the projections $P_V(\theta)$ constructed in Lemma 2.8: The operators $P_V(\theta) P_*(\theta)$ are again projections defined on Y and retain all the properties described in Lemma 2.8. With a slight abuse of notation, we denote them again by $P_V(\theta)$.

The map τ from $G \times V_* \times (-\delta, 1+\delta)$ into the center-unstable bundle is then defined by

$$\tau(g, v, \theta) := \rho_g(\Phi_\theta(u_*) + P_V(\theta) J(\chi(\theta)) v) \ , \tag{2.16}$$

where $(g, v, \theta) \in G \times V_* \times (-\delta, 1+\delta)$. For $\theta \in (-\delta, \delta)$, we use the definition of χ and the properties of $P_V(\theta)$ and $J(\theta)$ described in Lemma 2.8, and obtain

$$\begin{aligned} \tau(g, v, 1+\theta) &= \rho_g(\Phi_{1+\theta}(u_*) + P_V(1+\theta) J(\chi(1+\theta)) v) \\ &= \rho_g(\rho_{g_*} \Phi_\theta(u_*) + \rho_{g_*} P_V(\theta) \rho_{g_*}^{-1} J(1) v) \end{aligned}$$

$$
\begin{aligned}
&= \rho_g \rho_{g_*}(\Phi_\theta(u_*) + P_V(\theta) Q_* v) \\
&= \rho_g \rho_{g_*}(\Phi_\theta(u_*) + P_V(\theta) J(\chi(\theta)) Q_* v) \\
&= \tau(g g_*, Q_* v, \theta) \ .
\end{aligned}
$$

Therefore, we may define

$$\tau(g, v, n+\theta) = \tau(g g_*^n, Q_*^n v, \theta) \tag{2.17}$$

for $\theta \in [0,1)$ and $n > 0$, and a similar expression for negative θ.

The derivative of τ is surjective and its kernel is given by $\mathrm{alg}(H_*) \times \{0\} \times \{0\}$. Indeed, it suffices to calculate the derivative of τ at $(\mathrm{id}, 0, 0)$. The kernel is induced by the following equivalence relation on $G \times V_* \times \mathbb{R}$. By H_*-equivariance of P_V and J, we have

$$\tau(gh, v, \theta) = \rho_g \rho_h(\Phi_\theta(u_*) + P_V(\theta) J(\chi(\theta)) v) = \tau(g, \rho_h v, \theta)$$

for any $(g, v, \theta) \in G \times V_* \times [0,1]$ and $h \in H_*$. Up to this equivalence relation, τ is a covering map where the covering symmetry is induced by the time-one shift (2.17). This proves that a uniform neighborhood of $\mathcal{O}_*$ in M_*^{cu} is diffeomorphic to $(G \times V_* \times \mathbb{R})/\sim$ under the equivalence relation mentioned in the theorem.

In the next step, we have to lift the differential equation from the center manifold to the covering space. We proceed here as in the proof of [9, Theorem 1.1]. In this reference, a manifold with a proper G-action was investigated in a neighborhood of a relative equilibrium with isotropy H_*. For $\theta \in [0,1)$, we lift the vector field as in the aforementioned reference. For points $\theta \notin [0,1)$, we then use the time shift $\theta \mapsto \theta + 1$ and conjugation by appropriate powers of ρ_{g_*} and Q_*. The relations (2.14) and (2.15) are consequences of G-equivariance and of the covering symmetries induced by H_* and the time shift $\theta \mapsto \theta + 1$, respectively. □

In the proof given above, we did not use the fact that M_*^{cu} is a center manifold. If M is a manifold with a smooth and proper action of a Lie group G, and $\dot{u} = f(u)$ is a G-equivariant vector field on M, then the same conclusions are true near relative periodic orbits $\mathcal{O}_*$ of the flow on M.

Motivated by the applications in Sect. 7, we will focus on several situations in which the bundle structure simplifies considerably. It is then possible to use the vector field on the relative periodic orbit in a more explicit way.

Lemma 2.10. *If $g_* = \exp(\xi_*)$ is in the centralizer of H_* for some $\xi_* \in \mathrm{alg}(G)$, then the following is true. The map $(g, v, \theta) \mapsto (g \exp(\xi_* \theta), v, \theta)$ transforms the vector field (2.13) on $G \times V_* \times \mathbb{R}$ into*

$$\dot{g} = g f_G(v, \theta), \quad \dot{v} = f_N(v, \theta), \quad \dot{\theta} = f_\Theta(v, \theta) \tag{2.18}$$

with $(f_G, f_N, f_\Theta)(0, \theta) = (\xi_*, 0, 1)$. In addition, the equivalence relations (2.14) and

$$(f_G, f_N, f_\Theta)(v, \theta+1) = (f_G(Q_* v, \theta), Q_*^{-1} f_N(Q_* v, \theta), f_\Theta(Q_* v, \theta)) \tag{2.19}$$

are met.

The proof is straightforward and will be omitted. In fact, it suffices that g_* is contained in the connected component of the identity in the centralizer of H_* in G. The description of the vector field in Lemma 2.10 is then still true. Indeed, in the expression for the map given in the lemma, we replace the homotopy $\exp(\xi_*\theta)$ by a path $g(\theta)$ which connects id and g_* in the centralizer of H_*.

Remark. By a similar argument, the relative periodic orbit $\mathcal{O}_*$ itself is diffeomorphic to $(G/H_* \times \mathbb{R})/\sim$ where points $(gH_*, \theta+1) \sim (gg_*H_*, \theta)$ are identified. If g_* lies in the connected component of the identity in the normalizer $N(H_*)$ of H_*, then $\mathcal{O}_*$ is diffeomorphic to $G/H_* \times S^1$. Note that it is the normalizer, and not the centralizer, which is relevant here since we only describe the structure of the manifold, and not the vector field on it.

Next, consider the case of trivial isotropy.

Lemma 2.11. *Assume that* $H_* = \{\mathrm{id}\}$ *and* $g_* = \exp(\xi_*)$ *for some* $\xi_* \in \mathrm{alg}(G)$. *If* $\det(B_*) > 0$, *then* M_*^{cu} *is diffeomorphic to the trivial bundle* $G \times V_* \times S^1$. *The vector field is as described in Lemma 2.10 with* $\theta \in S^1$ *and* $Q_* = \mathrm{id}$. *If* $\det(B_*) < 0$, M_*^{cu} *is covered twice by* $G \times V_* \times \mathbb{R}/\mathbb{Z}$. *The vector field lifted to the covering space is as described in Lemma 2.10 with* $\theta \in \mathbb{R}/2\mathbb{Z}$ *and* $Q_* \in \mathrm{O}(V_*)$ *with* $\det(Q_*) = -1$.

Proof. If $\det(B_*) > 0$ and $H_* = \{\mathrm{id}\}$, we have $Q_* = \mathrm{id}$. Therefore, the equivalence relations reduce to $(gg_*, v, \theta) \sim (g, v, \theta+1)$. Let $\xi_* \in \mathrm{alg}(G)$ such that $\exp(\xi_*) = g_*$. The map $\tilde{\tau}(g, v, \theta) = (g\exp(-\xi_*\theta), v, \theta)$ is then the required diffeomorphism which trivializes the bundle.

If $\det(B_*) < 0$, we describe M_*^{cu} as a bundle over $G \times S^1$ with an identification matrix Q_* in the fiber V_* which changes the orientation. The resulting non-orientable bundle over S^1 can be covered by a trivial bundle in the usual way. □

Finally, we focus on the situation where a Hopf bifurcation occurs in the transverse direction V_*.

Lemma 2.12. *Assume that* $g_* = \exp(\xi_*)$ *is in the centralizer of* H_* *for some* $\xi_* \in \mathrm{alg}(G)$. *Furthermore, suppose that the matrix* Q_* *is homotopic to the identity in* $\mathrm{O}(V_*)$ *in an* H_**-equivariant fashion. The center manifold is then diffeomorphic to* $(G \times V_* \times S^1)/\sim$ *under the identification*

$$(g, v, \theta) \sim (gh^{-1}, \rho_h v, \theta)$$

with $h \in H_*$. *The vector field on* $G \times V_* \times S^1$ *is as described in Lemma 2.10 with* $Q_* = \mathrm{id}$.

In particular, the assumption on Q_* *is met if* $\dim V_* = 2$ *and* $\mathrm{spec}(B_*) = \{\exp(\pm i\omega_*)\}$ *with* $\omega_* \neq 0 \bmod \pi$.

Proof. Since g_* is in the centralizer of H_*, Q_* commutes with elements $h \in H_*$. By construction, we can replace Q_* by id whenever Q_* is homotopic to the identity in $\mathrm{O}(V_*)$ in an H_*-equivariant fashion.

It remains to consider the last claim in the lemma. Since $\dim V_* = 2$ and B_* has two non-real eigenvalues, $H_* \subset \mathrm{SO}(2)$; otherwise, $B_* \neq \pm\mathrm{id}$ could not be H_*-equivariant. Furthermore, $Q_* \in \mathrm{SO}(2)$ since $\det(B_*) = 1$. Therefore, Q_* is homotopic in SO(2) to the identity in an H_*-equivariant fashion. □

For any subgroup K of G, we denote the connected component of the identity in K by K^0. Furthermore, $C(K)$ and $N(K)$ denote the centralizer and normalizer, respectively, of K in G. Suppose that the isotropy subgroup H_* is compact.

In the lemmata above, we have always required that $g_* = \exp(\xi_*)$ is in $C(H_*)$. This assumption is not optimal and can be relaxed considerably for many, even non-compact groups G including $\mathrm{SE}(N)$. For this class, arguing as in Field's work [11], we have the decomposition

$$N(H_*)^0 = C(H_*)^0 \cdot H_*^0 \ , \tag{2.20}$$

and, in addition, $g_*^\ell = \exp(\xi_*) \in N(H_*)^0$ for some $\ell \in \mathbb{N}$. We can then describe the flow on the center manifold in a way which is similar to that in [11].

3. Periodic Forcing

In the general set-up of Sect. 2, we consider (2.1) with a time-periodic right-hand side

$$u_t = -Au + F(u) + \mu F_{\mathrm{ext}}(t, u, \mu) \ , \tag{3.1}$$

where $u \in Y$; recall that $Y = X^\alpha$ and $\alpha \in [0, 1)$. The forcing F_{ext} is C^k from $\mathbb{R} \times Y \times \mathbb{R}$ to X for some $k \geq 3$. Suppose that F_{ext} is periodic in t with frequency Ω. We assume that the evolution operator $\Phi_{t,\tau}(u, \mu)$ on Y associated with (3.1) is G-equivariant.

Theorem 3.1. *Suppose that u_* is a relative 1-periodic solution of (3.1) for $\mu = 0$ which has compact isotropy H_*. Furthermore, suppose that Hypotheses (S) and (R) stated in Sect. 2.1 are satisfied with $V_* = \{0\}$, that is, $E_*^{\mathrm{cu}} = T_{u_*}\mathcal{O}_*$. There exists then a G-invariant center-manifold M_*^{cu}. The manifold M_*^{cu} is diffeomorphic to $(G/H_* \times \mathbb{R} \times S^1)/\sim$ under the equivalence relation $(gg_*, \theta, t) \sim (g, \theta{+}1, t)$. The vector field on M_*^{cu} is*

$$\dot{g} = g f_G(\theta, t, \mu), \quad \dot{\theta} = f_\Theta(\theta, t, \mu) \ , \tag{3.2}$$

where $(f_G, f_\Theta)(\theta, t, \mu)$ is periodic in t with frequency Ω for $\mu \neq 0$, while for $\mu = 0$ $(f_G, f_\Theta)(\theta, t, 0) = (f_G, f_\Theta)(\theta)$ does not depend on time. Moreover, (f_G, f_Θ) satisfies (2.14).

Proof. Consider the map $\Pi(u,\mu) := \Phi_{\frac{2\pi}{\Omega},0}(u,\mu)$. For $\mu = 0$, the orbit $\mathcal{O}_* = \{\rho_g \Phi_t(u_*,0);\ (g,t) \in G \times \mathbb{R}\}$ is a smooth, G-invariant, and normally hyperbolic manifold without boundary which is invariant under Π. As such, it persists under small perturbations; see [17]. Note that although $\mathcal{O}_*$ may not be compact, it is uniformly attracting since the contraction rates of $D\Pi(u)$ evaluated at $\rho_g \Phi_t(u_*,0)$ do not depend on $g \in G$ by G-equivariance and the isometric representation of G on Y. Hence, there is a G-invariant and Π-invariant manifold M_Π^{cu} for any small μ. The manifold $M_*^{\mathrm{cu}} := \cup_{0 \le t \le \frac{2\pi}{\Omega}} \Phi_{t,0}(M_\Pi^{\mathrm{cu}}, \mu)$ is then invariant under the time evolution $\Phi_{t,\tau}$. The structure of the vector field on M_*^{cu} follows as in Theorem 2.9. □

The theorem is also true if additional center-unstable directions are present. The main difficulty is that the domain of graphs may shrink when using the graph transform. The proof then requires a modification of the vector field which is similar to the procedure mentioned in the proof of Theorem 2.2; see also [26].

A similar reduction as described in Lemma 2.10 applies to periodic forcing provided $g_* = \exp(\xi_*)$ for some $\xi_* \in \mathrm{alg}(G)$ is in the centralizer of H_* in G. If, in addition, $H_* = \{\mathrm{id}\}$ is trivial, then the manifold M_Π^{cu} is diffeomorphic to

$$\{\rho_g \rho_{\exp(-\xi_*\theta)} \Phi_\theta(u_*,0);\ (g,\theta) \in G \times S^1\} = G \times S^1 \ .$$

The manifold M_*^{cu} is diffeomorphic to $G \times S^1 \times S^1$. For applications, we refer to Sect. 8 below.

Theorem 3.1 remains true if relative equilibria instead of relative periodic orbits are considered. Here, u_* is a relative equilibrium if $\Phi_t(u_*,0) = \rho_{\exp(\xi_* t)} u_* \in G u_*$ for all $t \in \mathbb{R}$. We then have $\Phi_1(u_*,0) = \rho_{\exp(\xi_*)} u_* = \rho_{g_*} u_*$.

Theorem 3.2. *Suppose u_* is a relative equilibrium of (3.1) for $\mu = 0$ with compact isotropy H_*. Assume Hypotheses (S) and (R) are satisfied for $\Phi_1(u_*,0)$. There exists then a G-invariant center-manifold M_*^{cu} diffeomorphic to $G/H_* \times S^1$, and the lifted vector field on $G \times S^1$ is*

$$\dot{g} = g f_G(t,\mu) \ , \tag{3.3}$$

where $f_G(t,\mu)$ is periodic in t with frequency Ω for $\mu \neq 0$, and $f_G(t,0) = \xi_*$ is independent of time. Moreover, $f_G(t,\mu) = h f_G(t,\mu) h^{-1}$.

4. Strongly Continuous Actions of Lie Groups on Banach Spaces

In this section, we prove regularity properties of linear representations of finite-dimensional Lie groups on Banach spaces. We summarize the results in Theorem 4.5 at the end of this section. Consider a linear representation ρ of a finite-dimensional Lie group G on a Banach space Y_0. We assume that this action is strongly continuous, that is, the map

$$G \times Y_0 \to Y_0, \quad (g,u) \mapsto \rho(g)u$$

is continuous. We mention that in this section we do *not* assume that the representation is isometric.

By strong continuity of ρ and semigroup theory, the generator of the one-parameter group $\rho(\exp(\xi t))$ for $\xi \in \mathrm{alg}(G)$ with $t \in \mathbb{R}$ is a closed operator in Y_0 which we denote by $\kappa(\xi)$. The domain of $\kappa(\xi)$

$$D(\xi) := \{u \in Y_0;\ \kappa(\xi)u := \lim_{t\to 0} \frac{1}{t}(\rho(\exp(\xi t))u - u) \text{ exists}\} \tag{4.1}$$

is dense in Y_0 for any $\xi \in \mathrm{alg}(G)$.

The first result of this section is that the intersection of all domains $D(\xi)$ over $\xi \in \mathrm{alg}(G)$ is also dense in Y_0. Dancer [6] proved this result for compact groups G using the Haar measure associated with G. Intuitively, however, the result should not depend on the global group structure but only on the Lie algebra $\mathrm{alg}(G)$. Indeed, the generators $\kappa(\xi)$ are related to the Lie algebra; the Lie algebra, however, may be the same for a compact and a non-compact group. The proof given here reflects this reasoning as only the local group structure in a neighborhood of the identity in G is used. It also seems to be more elementary than the one given in [6] for the particular case of compact groups.

Lemma 4.1. *The intersection* $\bigcap_{\xi\in\mathrm{alg}(G)} D(\xi)$ *is dense in* Y_0.

Proof. The proof is inspired by the treatment of semigroups in the textbook [24]. Using a local chart, we equip the Lie algebra $\mathrm{alg}(G) = T_{\mathrm{id}}G$ of G with a scalar product and the Lebesgue measure $\mathrm{d}\eta$ with $\eta \in \mathrm{alg}(G)$. For $r > 0$, define

$$M_r u = \frac{1}{|B_r(0)|} \int_{B_r(0)} \rho(\exp(\eta))u \,\mathrm{d}\eta \tag{4.2}$$

for $u \in Y_0$, where $B_r(0)$ is the ball with radius r and Lebesgue-volume $|B_r(0)|$ in $\mathrm{alg}(G)$. Note that the integrand is continuous in η by strong continuity of the G-action, and thus the integral is well-defined. It is straightforward to see that $\lim_{r\to 0} M_r u = u$. Therefore, it suffices to show that $M_r u \in D(\xi)$ for fixed $r > 0$, $u \in Y_0$, and $\xi \in \mathrm{alg}(G)$, that is,

$$\kappa(\xi)M_r u = \lim_{t\to 0} \frac{1}{t}(\rho(\exp(\xi t))M_r u - M_r u) \tag{4.3}$$

exists. For r small enough, the map

$$\phi_t : \mathrm{alg}(G) \to \mathrm{alg}(G), \quad \eta \mapsto \exp^{-1}(\exp(\xi t)\exp(\eta))$$

is a diffeomorphism from $B_r(0)$ into some neighborhood of $\eta = 0$ in $\mathrm{alg}(G)$. We fix $r > 0$ and write $B = B_r(0)$. Exploiting continuity and the transformation rule for integrals on $\mathbb{R}^n$, we have

$$\begin{aligned}
|B|\,\rho(\exp(\xi t))M_r u &= \rho(\exp(\xi t)) \int_{\eta\in B} \rho(\exp(\eta))u \,\mathrm{d}\eta \\
&= \int_{\eta\in B} \rho(\exp(\xi t)\exp(\eta))u \,\mathrm{d}\eta \\
&= \int_{\tilde\eta\in\phi_t(B)} \rho(\exp(\tilde\eta))u \det(\mathrm{D}\phi_{-t}(\tilde\eta))\mathrm{d}\tilde\eta \ .
\end{aligned}$$

Using this expression, we obtain

$$\begin{aligned}|B|\,(\rho(\exp(\xi t))-\mathrm{id})M_r u &= \int_{\phi_t(B)} \rho(\exp(\eta))u \det(\mathrm{D}\phi_{-t}(\eta))\,\mathrm{d}\eta \\ &\quad - \int_B \rho(\exp(\eta))u\,\mathrm{d}\eta \\ &= \int_{\phi_t(B)} \rho(\exp(\eta))u(\det(\mathrm{D}\phi_{-t}(\eta))-1)\,\mathrm{d}\eta \\ &\quad + \Big(\int_{\phi_t(B)} \rho(\exp(\eta))u\,\mathrm{d}\eta - \int_B \rho(\exp(\eta))u\,\mathrm{d}\eta\Big) \\ &=: I_1 + I_2 \ .\end{aligned}$$

It is straightforward to see that the integral $\frac{1}{t}I_1$ converges as $t \to 0$. Indeed, $\mathrm{D}\phi_0(\eta) = \mathrm{id}$ and $\mathrm{D}\phi_t(\eta)$ is smooth in t and η. Therefore, the limit $\lim_{t\to 0}\frac{1}{t}I_1$ exists.

It remains to show that $\lim_{t\to 0}\frac{1}{t}I_2$ exists. For any smooth function $f : \mathbb{R}^n \to \mathbb{R}$, we have

$$\int_{\phi_t(B)} f(\eta)\,\mathrm{d}\eta - \int_B f(\eta)\,\mathrm{d}\eta = \int_0^t \int_{\partial\phi_\tau(B)} f(\eta) n(\tau,\eta)\,\mathrm{d}\eta\,\mathrm{d}\tau \ , \qquad (4.4)$$

where $n(\tau,\eta)$ is the τ-component of the outer unit normal of

$$T_{(\tau,\eta)}\Big(\cup_{s\in[0,1]}\partial\phi_{\mathrm{s}}(B)\Big) \ .$$

Indeed, equality (4.4) is a consequence of Gauss's formula applied to the vector field $(f(\eta),0)$ in the domain $\cup_{s\in(0,1)}\phi_{\mathrm{s}}(B)$ since $\mathrm{div}(f(\eta),0) = \partial_\tau f(\eta) = 0$. By continuity of (4.4) with respect to the C^0-convergence in f, the formula is also true for continuous functions f. Finally, testing with functionals in Y_0^*, we see that (4.4) holds for $f \in C^0(\mathbb{R}^n, Y_0)$. To complete the argument, we apply (4.4) with $f(\eta) = \rho(\exp(\eta))u$. Observe that the integrand $n(\tau,\eta)\rho(\exp(\eta))u$ is continuous. Therefore,

$$\int_{\partial\phi_\tau(B)} n(\tau,\eta)\,\rho(\exp(\eta))u\,\mathrm{d}\eta$$

is continuous in τ as the domains varies smoothly. Hence, $\lim_{t\to 0}\frac{1}{t}I_2$ exists and is given by

$$\int_{\partial\phi_0(B)} n(0,\eta)\,\rho(\exp(\eta))u\,\mathrm{d}\eta = \int_{\partial B} \langle\nu(\eta),\xi\rangle\,\rho(\exp(\eta))u\,\mathrm{d}\eta \ ,$$

where ν is the outer unit normal of $\partial B \subset \mathrm{alg}(G)$. □

The next lemma shows that κ as defined in (4.3) is linear.

Lemma 4.2. *For any fixed* $u \in \cap_{\xi\in\mathrm{alg}(G)} D(\xi)$, *the map* $\mathrm{alg}(G) \to Y_0$, $\xi \mapsto \kappa(\xi)u$ *is linear in* ξ. *Furthermore, we have* $\kappa(\xi)\rho(g)u = \rho(g)\kappa(\mathrm{Ad}_g^{-1}(\xi))u$ *for any* $g \in G$.

Proof. The relations $\kappa(t\xi) = t\kappa(\xi)$ for $t \in \mathbb{R}$ and $\kappa(\xi)\rho(g) = \rho(g)\kappa(\mathrm{Ad}_g^{-1}(\xi))$ follow from the definition of κ. It suffices to prove that

$$\kappa(\xi_1 + \xi_2) = \kappa(\xi_1) + \kappa(\xi_2) \tag{4.5}$$

for all $\xi_1, \xi_2 \in \mathrm{alg}(G)$. Furthermore, it is sufficient to prove the identity (4.5) evaluated at elements $M_r u$ since these are dense in Y_0 and the operators are closed. Now, we can proceed as in the proof of the preceding lemma; instead of multiplying the element $M_r u$ by $\rho(\exp(\xi t)) - \mathrm{id}$, we multiply by $\rho(\exp((\xi_1 + \xi_2)t)) - \rho(\exp(\xi_1 t)) - \rho(\exp(\xi_2 t)) + \mathrm{id}$. We omit the details. □

We define

$$Y_1 := \cap_{\xi \in \mathrm{alg}(G)} D(\xi) \tag{4.6}$$

with norm $|u|_{Y_1} := |u| + \sup_{j=1,..,\dim \mathrm{alg}(G)} |\kappa(\xi_j)u|$ where ξ_j is a fixed basis of $\mathrm{alg}(G)$.

Lemma 4.3. *The space Y_1 is closed with respect to $|\cdot|_{Y_1}$, and hence a Banach space.*

The proof is straightforward using $\xi = \sum a_j \xi_j$ and closedness of ξ and ξ_j.

We remark that it is also possible to prove that $\kappa([\xi_1, \xi_2]) = [\kappa(\xi_1), \kappa(\xi_2)]$, that is, κ preserves the Lie structure of $\mathrm{alg}(G)$.

Lemma 4.4. *The representation κ is continuous, that is, the map $\mathrm{alg}(G) \times Y_1 \to Y_0$, $(\xi, u) \mapsto \kappa(\xi)u$, is continuous. Furthermore, the group G acts strongly continuously on Y_1, and $\rho(g)u$ is continuously differentiable as a function from G into Y_0 for any $u \in Y_1$.*

Proof. The first claim follows from the principle of uniform boundedness since the map $(\xi, u) \mapsto \kappa(\xi)u$ is linear and uniformly bounded for ξ bounded in $\mathrm{alg}(G)$.

Next, note that Y_1 is invariant under G since

$$\kappa(\xi)\rho(g)u = \rho(g)\kappa(\mathrm{Ad}_g^{-1}(\xi))u \in Y_0$$

by Lemma 4.2. For any two sequences $g_n \to g$ and $u_n \to u$, we have

$$\kappa(\mathrm{Ad}_{g_n}^{-1}(\xi))u_n \to \kappa(\mathrm{Ad}_g^{-1}(\xi))u$$

by continuity of κ proved above. Therefore, by strong continuity of the G-action on Y_0,

$$\kappa(\xi)\rho(g_n)u_n = \rho(g_n)\kappa(\mathrm{Ad}_{g_n}^{-1}(\xi))u_n \to \rho(g)\kappa(\mathrm{Ad}_g^{-1}(\xi))u$$

for $g_n \to g$, and the action on Y_1 is strongly continuous.

Finally, consider $g \mapsto \rho(g)u \in Y_0$. The partial derivative of $\rho(g)u$ in the ξ-direction evaluated at $\hat{g}$ is given by $\rho(\hat{g})\kappa(\mathrm{Ad}_{\hat{g}}^{-1}\xi)u$. Continuity of this expression with respect to $\hat{g}$ follows as above. □

We summarize the main results in the following theorem.

Theorem 4.5. *The space $Y_1 = \cap_{\xi \in \mathrm{alg}(G)} D(\xi)$ is dense in Y_0, and is a Banach space with norm*

$$|u|_{Y_1} = |u| + \sup\{|\kappa(\xi)u|;\ \xi \in \mathrm{alg}(G), |\xi| = 1\} \ .$$

Furthermore, the group G acts strongly continuously on Y_1 and the map $g \longrightarrow \rho(g)u \in Y_0$ is C^1 in g for any $u \in Y_1$.

For $j > 1$, define

$$Y_j := \{u \in Y_{j-1};\ \kappa(\xi)u \in Y_{j-1} \text{ for any } \xi \in \mathrm{alg}(G)\} \tag{4.7}$$

equipped with the graph norms $|\cdot|_{Y_j}$ defined by

$$|u|_{Y_j} = |u|_{Y_{j-1}} + \sup_{\xi \in \mathrm{alg}(G), |\xi|=1} |\kappa(\xi)u|_{Y_{j-1}} \ .$$

By induction, applying the aforementioned results to the spaces Y_j, we see that Y_j is dense in Y_0.

5. The Regularity Hypothesis

Here, we prove that the smoothness assumption (R) is essentially implied by the spectral hypothesis (S). Suppose that G acts on some Banach space Y via the isometric representation ρ_g. Let $|\cdot|$ and $\|\cdot\|$ denote the norms on Y and $\mathcal{L}(Y)$, respectively. Assume that $u_* \in Y$ is a relative fixed point of a G-equivariant, C^k-smooth map $\Phi : Y \longrightarrow Y$, that is, $\Phi(u_*) = \rho_{g_*} u_*$ for some $g_* \in G$. Let $Y_0 \subset Y$ be the largest subspace in Y in which G acts strongly continuously. We assume that the G-action on the Banach space Y is weakly continuous.

Hypothesis (W)

(i) If $\rho_g u \longrightarrow w$ as $g \longrightarrow \mathrm{id}$ for some $w \in Y$, then $u = w$.
(ii) The adjoint representation Ad_{g_*} on $\mathrm{alg}(G)$ has all its spectrum on the unit circle.

Note that Hypothesis (W)(ii) is on the group rather than on the representation. The following theorem is the main result of this section.

Theorem 5.1. *Suppose that $u_* \in Y_0$. Moreover, we assume that Hypotheses (S) and (W) are met. Then, Hypotheses (R)(i), (iii), and (iv) are satisfied.*

The theorem will be a consequence of Lemmata 5.4, 5.6, and 5.7 below. We start with the proof that $E_*^{\mathrm{cu}} \subset Y_0$.

Lemma 5.2. *If $u_* \in Y_0$, and Hypotheses (S) and (W)(i) are met, then $E_*^{\mathrm{cu}} \subset Y_0$. In particular, $\rho_g P_*$ is norm-continuous in $g \in G$.*

Proof. Assume first that the spaces $\rho_g E_*^{\mathrm{cu}}$ are continuous in g. For any $v \in E_*^{\mathrm{cu}}$ and any sequence $g_n \to \mathrm{id}$, the bounded sequence $\rho_{g_n} v$ converges to some element $w \in E_*^{\mathrm{cu}}$ possibly after choosing a subsequence. Indeed, the spaces $\rho_g E_*^{\mathrm{cu}}$ are continuous and E_*^{cu} is finite-dimensional. From Hypothesis (W)(i) we conclude that $w = v$. The same argument shows norm-continuity of $\rho_g P_*$.

It suffices therefore to prove that the spaces $\rho_g E_*^{\mathrm{cu}}$ are continuous as $g \to \mathrm{id}$ in G. We argue by contradiction. Since E_*^{cu} has finite dimension, there is then a sequence $g_n \to \mathrm{id}$ and an element $v_0 \in E_*^{\mathrm{cu}}$ with $|v_0| = 1$ such that $|(\mathrm{id} - P_*)\rho_{g_n} v_0| \geq \delta > 0$ for all n. For convenience, we use the notation $L_* = \rho_{g_*}^{-1} \mathrm{D}\Phi(u_*)$ and $L(u) = \rho_{g_*}^{-1} \mathrm{D}\Phi(u)$. On account of the spectral hypothesis (S), there are constants C_1, C_2, and $\eta < 1$ such that

$$|L_*^\ell (\mathrm{id} - P_*) v| \leq C_1 \eta^\ell |(\mathrm{id} - P_*) v|, \quad |L_*^\ell P_* v| \geq C_2 \ell^{-m} |P_* v|$$

for some $k > 0$ and any $\ell \in \mathbb{N}$ and $v \in Y$. Indeed, vectors in the center-unstable eigenspace can decay at most algebraically in ℓ. In particular, choosing ℓ large enough, there are then numbers $\eta_{\mathrm{s}} < 1$ and η_{cu} such that

$$\|L_*^\ell|_{E_*^{\mathrm{s}}}\| \leq \eta_{\mathrm{s}}, \quad \|(L_*^\ell|_{E_*^{\mathrm{cu}}})^{-1}\| \leq \frac{1}{\eta_{\mathrm{cu}}} \tag{5.1}$$

and

$$\delta \eta_{\mathrm{cu}} > (1 + \|P_*\|) \eta_{\mathrm{s}} \ . \tag{5.2}$$

In order to keep notation simpler, we assume that (5.1) and (5.2) are met with $\ell = 1$; otherwise, replace the map Φ by Φ^ℓ. Since $\mathrm{D}\Phi(\rho_g u_*) = \rho_g \mathrm{D}\Phi(u_*) \rho_g^{-1}$ by equivariance, we have

$$L(\rho_{g_*} \rho_g \rho_{g_*}^{-1} u_*) = \rho_g L_* \rho_{g_*} \rho_g^{-1} \rho_{g_*}^{-1} \tag{5.3}$$

for any $g \in G$. In particular,

$$|L(\rho_{g_*} \rho_{g_n} \rho_{g_*}^{-1} u_*)^{-1} \rho_{g_n} v_0| \leq \frac{1}{\eta_{\mathrm{cu}}} \ . \tag{5.4}$$

Suppose that there are elements v and w such that $L(\rho_g u_*) w = v$. We claim that

$$|w| = |L(\rho_g u_*)^{-1} v| \geq \frac{1}{\eta_{\mathrm{s}}} |(\mathrm{id} - P_*) v| - \frac{1}{\eta_{\mathrm{cu}}} |P_* v| + \mathrm{o}(1)|v| \tag{5.5}$$

for all g sufficiently close to $\mathrm{id} \in G$. Indeed, since $\rho_g u_*$ is continuous in g, $L(\rho_g u_*) w = (L_* + \mathrm{o}(1)) w = v$. It is then straightforward to prove (5.5) by projecting the expression $L(\rho_g u_*) w = v$ into center-unstable and stable eigenspaces, and estimating the resulting terms using (5.1).

Due to (5.3), for any n, there exists a w such that $L(\rho_{g_*} \rho_{g_n} \rho_{g_*}^{-1} u_*) w = \rho_{g_n} v_0$. Hence, by (5.5),

$$|L(\rho_{g_*} \rho_{g_n} \rho_{g_*}^{-1} u_*)^{-1} \rho_{g_n} v_0| \geq \frac{1}{\eta_{\mathrm{s}}} |(\mathrm{id} - P_*) \rho_{g_n} v_0| - \frac{1}{\eta_{\mathrm{cu}}} |P_* \rho_{g_n} v_0| + \mathrm{o}(1)|v_0| \ .$$

Using (5.4) and $|(\mathrm{id} - P_*)\rho_{g_n} v_0| \geq \delta$, we obtain

$$\frac{1}{\eta_{\mathrm{cu}}} \geq \frac{1}{\eta_{\mathrm{s}}}\delta - \frac{1}{\eta_{\mathrm{cu}}}\|P_*\| + \mathrm{o}(1) \ ,$$

which contradicts (5.2) for sufficiently large n. □

Let $Y^\star$ be the dual space to Y and define

$$Z_0^\star := \{y^\star \in Y^\star;\ \rho_g^\star y^\star \text{ is } C^0 \text{ in } g\} \ ,$$

where $\rho_g^\star$ denotes the adjoint operator of ρ_g. For $j > 1$, we define $Z_j^\star$ with norm $|\cdot|_{Z_j^\star}$ for the adjoint group action as in (4.7) with Y_0 replaced by $Z_0^\star$.

Lemma 5.3. *Under the assumptions of Lemma 5.2, the adjoint projection $P_*^\star$ maps $Y^\star$ into $Z_0^\star$.*

Proof. Arguing for the adjoint group action as in the proof of Lemma 5.2, we see that $\rho_g^\star P_*^\star Y^\star \to P_*^\star Y^\star$ as $g \to \mathrm{id}$. Since Hypothesis (W)(i) is not necessarily true in the dual space, we still have to prove pointwise convergence, that is, $\rho_g^\star P_*^\star y^\star \to P_*^\star y^\star$ for $g \to \mathrm{id}$. We argue by contradiction. Since the space $P_*^\star Y^\star$ is finite-dimensional, $\rho_{g_n}^\star P_*^\star y^\star \to P_*^\star z^\star$ for some $z^\star$ and some subsequence $g_n \to \mathrm{id}$. Therefore, $(\mathrm{id} - P_*^\star)\rho_{g_n}^\star P_*^\star y^\star \to 0$. Moreover, $\langle P_*^\star \rho_{g_n}^\star P_*^\star y^\star, \cdot\rangle = \langle P_*^\star y^\star, \rho_{g_n} P_* \cdot\rangle$, and, by Lemma 5.2, $\rho_{g_n} P_*$ converges to P_*. Hence, we conclude $P_*^\star y^\star = P_*^\star z^\star$. □

Lemma 5.4. *Suppose that $u_* \in Y_0$. Moreover, we assume that Hypotheses (S) and (W)(i) are met. The following is then true. For any small $\epsilon > 0$, there is a projection $\hat{P}_*$ which is ϵ-close to P_* in $\mathcal{L}(Y)$ such that $\hat{P}_* \rho_g$ is C^1 in $g \in G$.*

Proof. Throughout, the indices i and j are in the range $\{1, \ldots, \dim E_*^{\mathrm{cu}}\}$. Since E_*^{cu} is finite-dimensional, there are bases e_i and $e_i^\star$ of $E_*^{\mathrm{cu}} = P_* Y$ and $P_*^\star Y^\star$, respectively, such that

$$P_* = \sum_{i=1}^{\dim E_*^{\mathrm{cu}}} \langle e_i^\star, \cdot\rangle e_i \ . \tag{5.6}$$

By Theorem 4.5, there are elements $\hat{e}_i \in Y_1$ which are close to e_i in the Y-norm for all i. By Lemma 5.3, $P_*^\star Y^\star \subset Z_0^\star$, so that by Theorem 4.5 we may also approximate the vectors $e_i^\star$ by elements $\hat{e}_i^\star \in Z_1^\star$ in the $Y^\star$-norm. Using an appropriate normalization, we can assume that $\langle \hat{e}_i^\star, \hat{e}_j\rangle = \delta_{ij}$. We then define

$$\hat{P}_* y := \sum_{i,j=1}^{\dim E_*^{\mathrm{cu}}} \langle \hat{e}_i^\star, y\rangle \hat{e}_j \ .$$

Since

$$\hat{P}_* \rho_g y := \sum_{i,j=1}^{\dim E_*^{\mathrm{cu}}} \langle \hat{e}_i^\star, \rho_g y\rangle \hat{e}_j = \sum_{i,j=1}^{\dim E_*^{\mathrm{cu}}} \langle \rho_g^\star \hat{e}_i^\star, y\rangle \hat{e}_j$$

and $\rho_g^\star \hat{e}_i^\star$ is C^1 in g in the $Y^\star$-norm, we conclude that $\hat{P}_* \rho_g$ is C^1 in g. □

Next, we prove that G acts C^1-smoothly on u_* provided $u_* \in Y_0$, that is, G acts continuously on u_*. Recall the definition (4.1) of the operators $\kappa(\xi)$ on Y_0 for $\xi \in \mathrm{alg}(G)$.

Lemma 5.5. *Under the assumptions of Theorem 5.1, $u_* \in Y_1$, that is, $\rho_g u_*$ is C^1 in $g \in G$.*

Proof. It suffices to show that $\kappa(\xi)u_*$ exists for any $\xi \in \mathrm{alg}(G)$ since the derivative of $\rho_g u_*$ in g is continuous, see Lemma 5.2. We use the notation $\Phi_*(u) = \rho_{g_*}^{-1}\Phi(u)$. Since Y_1 is dense in Y_0, there is an element $u_0 \in Y_1$ such that $\Phi_*^n(u_0) =: u_n$ converges to u_*. In other words, u_0 is in the intersection of the strong stable manifold of u_* and Y_1. Note that this intersection is non-empty; we may approximate the affine space $u_* + E_*^{\mathrm{cu}}$ by an affine subspace of Y_1 of the same dimension, and use transversality of the strong stable manifold and $u_* + E_*^{\mathrm{cu}}$ in Y. Now, take any element $\xi \in \mathrm{alg}(G)$. It suffices to prove that $\kappa(\xi)u_{n_k}$ converges to some element $w \in Y_0$ for some subsequence $n_k \to \infty$ since then $u_* \in D(\xi)$ and $\kappa(\xi)u_* = w$ by closedness of $\kappa(\xi)$.

First, choose a projection $\hat{P}_*$ close to P_* such that $\hat{P}_*\rho_{\exp \xi t}$ is differentiable in t; see Lemma 5.4. The operator

$$R_\xi := \frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{P}_*\rho_{\exp \xi t}\right)\Big|_{t=0} \in \mathcal{L}(Y)$$

is then well-defined and bounded. In particular, $R_\xi u_* \in Y$ exists and

$$R_\xi \Phi_*^n(u_0) \to R_\xi u_* \quad \text{as} \quad n \to \infty \ .$$

Since $R_\xi v = \hat{P}_*\kappa(\xi)v$ for any $v \in D(\xi)$, we have $\hat{P}_*\kappa(\xi)\Phi_*^n(u_0) \to R_\xi u_*$ as $n \to \infty$, and we conclude that $\hat{P}_*\kappa(\xi)\Phi_*^n(u_0)$ is bounded uniformly in n.

In the second step, we use

$$\kappa(\xi)\Phi_*^n(u_0) = \mathrm{D}(\Phi_*^n)(u_0)\kappa(\mathrm{Ad}_{g_*^n}^{-1}\xi)u_0 \ .$$

We estimate the operators appearing on the right-hand side separately. It follows from the Roughness Theorem [15] for exponential dichotomies applied to $\mathrm{D}\Phi_*^n(u_*)$ that there are projections $P_n \in \mathcal{L}(Y)$ for $n \geq 0$ with $P_n \to P_*$ such that

$$\|\mathrm{D}(\Phi_*^n)(u_0)(\mathrm{id} - P_0)\| \leq C_1\eta^n \qquad \text{and} \qquad \mathrm{D}(\Phi_*^n)(u_0)P_0 = P_n\mathrm{D}(\Phi_*^n)(u_0) \quad (5.7)$$

for all $n \geq 0$, and some constants $C_1 > 0$ and $\eta < 1$ independent of n.

Next consider $\kappa(\mathrm{Ad}_{g_*^n}^{-1}\xi)u_0$. By Hypothesis (W)(ii), for any $\delta > 0$ there is a number C_2 such that

$$|\,\mathrm{Ad}_{g_*^n}^{-1}(\xi)| \leq C_2(1+\delta)^n|\xi| \ .$$

Moreover, the operator

$$T_0 : \mathrm{alg}(G) \to T_{u_0}(Gu_0) \quad \xi \mapsto \kappa(\xi)u_0$$

is onto and bounded. Thus, $\kappa(\mathrm{Ad}_{g_*^n}^{-1}\xi)u_0 = (T_0\,\mathrm{Ad}_{g_*^n}^{-1})(\xi)$, and we obtain

$$|\kappa(\mathrm{Ad}_{g_*^n}^{-1}\xi)u_0| \le \|T_0\|C_2(1+\delta)^n|\xi| \le C_3(1+\delta)^n \tag{5.8}$$

with $C_3 := \|T_0\|C_2|\xi|$. Therefore, using (5.7) and (5.8),

$$\begin{aligned}|(\mathrm{id}-P_n)\mathrm{D}(\Phi_*^n)(u_0)\kappa(\mathrm{Ad}_{g_*^n}^{-1}\xi)u_0| &\le C_1\eta^n|(\mathrm{id}-P_0)\kappa(\mathrm{Ad}_{g_*^n}^{-1}\xi)u_0| \\ &\le C_1C_3\|\mathrm{id}-P_0\|\eta^n(1+\delta)^n \to 0 \end{aligned}\tag{5.9}$$

for $n \to \infty$ provided we choose $\delta > 0$ sufficiently small.

Summarizing, we proved that $\hat{P}_*\kappa(\xi)\Phi_*^n(u_0)$ stays bounded and the expression $(\mathrm{id}-P_n)\kappa(\xi)\Phi_*^n(u_0)$ converges to zero as $n \to \infty$. Since $\|P_* - \hat{P}_*\|$ is small and $\|P_* - P_n\|$ tends to zero, the map $v \mapsto (\hat{P}_*v, (\mathrm{id}-P_n)v)$ is an isomorphism from Y onto $\mathcal{R}(\hat{P}_*) \times \mathcal{R}(\mathrm{id}-P_n)$ with uniformly bounded inverse. Hence, $\kappa(\xi)\Phi_*^n(u_0)$ is bounded as n tends to infinity. Using boundedness of $\kappa(\xi)\Phi_*^n(u_0)$, it follows that in fact $(\mathrm{id}-P_*)\kappa(\xi)\Phi_*^n(u_0) \to 0$ as $n \to \infty$. Therefore, $\mathrm{dist}(\kappa(\xi)\Phi_*^n(u_0), E_*^{\mathrm{cu}}) \to 0$ as $n \to \infty$, and, since E_*^{cu} is finite-dimensional, we have $\kappa(\xi)\Phi_*^{n_k}(u_0) \to w \in E_*^{\mathrm{cu}}$ for some subsequence $n_k \to \infty$ as $k \to \infty$. This proves the lemma. □

This lemma is the only result where we have used that the spectrum of the adjoint representation on $\mathrm{alg}(G)$ has its spectrum on the unit circle. Next, we prove that the group actually acts smoothly on u_* and E_*^{cu} provided $u_* \in Y_1$.

Lemma 5.6. *Under the assumptions of Theorem 5.1, Hypotheses (R)(i) and (iii) are satisfied. In other words, the relative fixed point u_* is contained in Y_k. Moreover, the operators $\rho_g P_*$ and $P_*\rho_g$ are C^{k-1} in g. In particular, $\rho_g(u_*+v)$ is C^{k-1}-smooth in $g \in G$ for any $v \in E_*^{\mathrm{cu}}$.*

Proof. By Lemma 5.5, $u_* \in Y_1$. Let $L_* = \rho_{g_*}^{-1}\mathrm{D}\Phi(u_*)$. Fix λ in the resolvent set of L_*. Differentiating

$$\exp(\xi t)\mathrm{D}\Phi(u_*) = \mathrm{D}\Phi(\exp(\xi t)u_*)\exp(\xi t)$$

with respect to t at $t = 0$, and multiplying by $\rho_{g_*}^{-1}$, we obtain

$$\kappa(\xi)(L_* - \lambda) = (L_* - \lambda)\kappa(\mathrm{Ad}_{g_*}\xi) + L \tag{5.10}$$

where $L = \rho_{g_*}^{-1}\mathrm{D}^2\Phi(u_*)[\kappa(\mathrm{Ad}_{g_*}\xi)u_*, \cdot] \in \mathcal{L}(Y_0)$. Multiplying both sides of (5.10) from the right and left with $(L_* - \lambda)^{-1}$, we see that the spectrum $\mathrm{spec}(L_*)$ of L_* considered as a map from Y_1 into itself is contained in the spectrum of L_* considered as map from Y_0 into itself. An analogous statement is true in the spaces Y_j with $j > 1$ whenever $u_* \in Y_j$. The projection P_* onto E_*^{cu} is given by

$$P_* = \frac{1}{2\pi\mathrm{i}}\oint_\Gamma(\lambda - L_*)^{-1}\,\mathrm{d}\lambda\ ,$$

where the closed curve Γ encloses precisely the center-unstable spectral set of $\mathrm{spec}(L_*)$ in the complex plane. In particular, P_* maps Y_j into itself provided $u_* \in Y_j$. Since Y_j is dense in Y_0 by Theorem 4.5, we conclude that $E_*^{\mathrm{cu}} \subset Y_j$

whenever $u_* \in Y_j$. Therefore, in particular, $\kappa(\xi)u_* \in Y_j$ for any $\xi \in \mathrm{alg}(G)$ since $\kappa(\xi)u_* \in E_*^{\mathrm{cu}}$. Hence, by induction, we see that $u_* \in Y_k$ and $E_*^{\mathrm{cu}} \subset Y_{k-1}$. The same arguments apply in the dual spaces. Using the expression (5.6), we see that the operators $\rho_g P_*$ and $P_* \rho_g$ are C^{k-1} in g. □

Finally, we have the following remark.

Lemma 5.7. *Hypothesis (R)(iv), that is $T_{u_*}(Gu_*) \subset E_*^{\mathrm{cu}}$, is an immediate consequence of (W)(ii) since $\rho_{g_*}^{-1} \mathrm{D}\Phi(u_*)\kappa(\xi)u_* = \kappa(\mathrm{Ad}_{g_*}\, \xi)u_*$ for $\xi \in \mathrm{alg}(G)$.*

6. SE(N)-Equivariant Reaction-Diffusion Systems

In this section, we show that reaction-diffusion equations on unbounded domains meet the basic hypotheses assumed in the sections above. It is also proved that the spectral hypothesis is satisfied if the relative periodic solution is localized, that is, converges to a stable homogeneous state as $|x| \to \infty$. Furthermore, if a relative equilibrium or relative periodic solution is not localized, that is, does not converge to a homogeneous state as $|x| \to \infty$, then the spectral hypothesis is not met.

Consider the following model for isotropic excitable media

$$u_t = d\Delta u + f(u) \qquad\qquad x \in \mathbb{R}^N \tag{6.1}$$

with $N = 2, 3$. The matrix d is diagonal with non-negative entries, and f is a smooth nonlinearity. The above equation is well-posed on the space $X = C^0_{\mathrm{unif}}(\mathbb{R}^N, \mathbb{R}^M)$ of bounded, uniformly continuous functions. In particular, it generates a smooth local semiflow on $Y = X$, see [15]. We denote solutions of (1.1) by $\Phi_t(u)$. In addition, (6.1) is equivariant with respect to the Euclidean group $\mathrm{SE}(N) = \mathrm{SO}(N) \dot{+} \mathbb{R}^N$ under the action $((R,S)u)(x) := u(R^{-1}(x - S))$ with $x \in \mathbb{R}^N$. The space $Y_0 = C^0_{\mathrm{eucl}}(\mathbb{R}^N)$ is defined as the largest subspace of $X = C^0_{\mathrm{unif}}(\mathbb{R}^N)$ on which the SE(N)-action is strongly continuous.

Suppose that u_* is a relative periodic solution of (6.1) with period T, that is, we have

$$\Phi_T(u_*) = (R_*, S_*)u_*$$

for some element $g_* := (R_*, S_*) \in \mathrm{SE}(N)$. Let $L_* = g_*^{-1}\mathrm{D}\Phi_T(u_*)$. Furthermore, assume that $\mathrm{SO}(N)u_*$ is continuous and the isotropy group H_* of u_* is compact.

Finally, suppose that $\{\lambda \in \mathrm{spec}((R_*, S_*)^{-1}\mathrm{D}\Phi_T(u_*));\ |\lambda| \geq 1\}$ consists of finitely many eigenvalues with finite multiplicity. We then have the following theorem.

Theorem 6.1. *In the above set-up, assume that Hypothesis (S) is met. Suppose furthermore that the rotations $\mathrm{SO}(N)$ act continuously on u_* and that u_* is not constant as a function of x. Hypothesis (R) is then also true, and the isotropy subgroup H_* of u_* is a compact subgroup of $\mathrm{SO}(N)$. In particular, the center-manifold theorem 2.2 and Theorem 2.9 on the skew-product structure apply.*

Proof. Since the action of the translations on $C^0_{\mathrm{unif}}(\mathbb{R}^N)$ is strongly continuous, the translational orbit of relative periodic points is always continuous. Therefore, by Lemma 6.2 below, the assumptions of Theorem 5.1 are satisfied. Thus, Hypotheses (R)(i), (iii), and (iv) are met as a consequence of Theorem 5.1. Finally, by [26, Lemma 4.1], Hypothesis (R)(ii) is satisfied and the isotropy is compact. There, we proved that if $u_* \in C^0_{\mathrm{unif}}(\mathbb{R}^N)$ meets Hypothesis (R)(i), that is, $(R,S)u_*$ is C^k in $(R,S) \in \mathrm{SE}(N)$, then Hypothesis (R)(ii) is also true and the isotropy is compact. In particular, the $\mathrm{SE}(N)$-orbit of u_* is embedded. □

6.1. Satisfaction of Hypothesis (W)

Lemma 6.2. *The* $\mathrm{SE}(N)$*-action defined above satisfies Hypothesis (W) on the space* C^0_{unif}.

Proof. It is straightforward to see that Hypothesis (W)(i) is met. Indeed, suppose that $(R_n, S_n)u \to w$ as $(R_n, S_n) \to (\mathrm{id}, 0)$ as $n \to \infty$, that is, $u(R_n^{-1}(x - S_n)) \to w(x)$ uniformly in x. However, for any fixed x, $u(R_n^{-1}(x-S_n)) \to u(x)$ as $n \to \infty$. Therefore, $w = u$.

It remains to verify Hypothesis (W)(ii), that is, that the spectrum of Ad_{g_*} on $\mathrm{alg}(G)$ is on the unit circle. This can be verified directly using the expression

$$(R,S)(r,s)(R,S)^{-1} = (RrR^{-1}, -RrR^{-1}S + Rs)$$

for the adjoint representation, see [9, Eqn. (4.3)]. Alternatively, we may use the fact that the Euclidean group $\mathrm{SE}(N)$ has an $\mathrm{SE}(N)$-invariant metric, namely the Killing form on $T\,\mathrm{SO}(N)$ and the Euclidean metric on $\mathbb{R}^N$. We then argue by contradiction. Suppose $\mathrm{Ad}_{g_*}\xi = \lambda\xi$ with $|\lambda| < 1$, for example. Hence, $\mathrm{Ad}_{g_*^n}\xi \to 0$ as $n \to \infty$, and therefore $g_*^n \exp(\xi t) g_*^{-n} \to \mathrm{id}$ for any fixed $t \geq 0$. By invariance of the metric, $\exp(\xi t) \to \mathrm{id}$, that is, $\xi = 0$. □

Note that the verification of (W)(ii) is not restricted to $\mathrm{SE}(N)$ but holds for any group with an invariant metric.

6.2. The Spectral Hypothesis (S)

If the relative periodic solution $\Phi_t(u_*)$ converges to zero uniformly as $|x| \to \infty$ for any $t \in [0,T]$, and the homogeneous state $u = 0$ is stable with respect to (6.1), then the spectral hypothesis (S) is satisfied for u_*.

This result is reminiscent of the situation for travelling waves on the real line, which are relative equilibria with respect to translations. If the asymptotic state is homogeneous and stable, the essential spectrum of the travelling wave is strictly contained in the left half-plane; see [15, Appendix to Sect. 5]. Here, the essential spectrum is defined as the complement in the spectrum of the set of isolated eigenvalues with finite multiplicity.

Lemma 6.3. *Suppose that the diffusion matrix d is non-singular. We assume that u_* is a relative periodic solution such that $u_*(t,x) \to 0$ uniformly as $|x| \to \infty$ for each t. Furthermore, assume that the spectrum of the operator $L_\infty := d\Delta + \mathrm{D}f(0)$ on C^0_{unif} satisfies $\mathrm{spec}(L_\infty) < -\beta < 0$. Under these conditions, Hypothesis (S) is met.*

Proof. We have

$$(R_*, S_*)^{-1}\mathrm{D}\Phi_T(u_*) = (R_*, S_*)^{-1}(\mathrm{e}^{L_\infty T} + K) \tag{6.2}$$

with

$$K := \int_0^T \mathrm{D}\Phi_{T-t}(\Phi_t(u_*))\Big(\mathrm{D}f(\Phi_t(u_*)) - \mathrm{D}f(0)\Big)\mathrm{e}^{L_\infty t}\,\mathrm{d}t \tag{6.3}$$

by the variation-of-constant formula. Note that L_∞ is sectorial with domain $C^2_{\mathrm{unif}}(\mathrm{I\!R}^N, \mathrm{I\!R}^M)$ since the diffusion matrix d is positive. Therefore, $\mathrm{spec}(\mathrm{e}^{L_\infty T})$ is contained in the circle with radius $\mathrm{e}^{-\beta T}$. Since (R_*, S_*) is an isometry and commutes with L_∞, we readily conclude that $\mathrm{spec}((R_*, S_*)^{-1}\mathrm{e}^{L_\infty T})$ is contained in the same circle. We claim that K is a compact operator in $\mathcal{L}(C^0_{\mathrm{unif}})$. Suppose for a moment that the claim is true. By [15, Theorem A.1], the essential spectra of $(R_*, S_*)^{-1}\mathrm{D}\Phi_T(u_*)$ and $(R_*, S_*)^{-1}\mathrm{e}^{L_\infty T}$ coincide, and the statement of the lemma is proved. It remains to show that K is compact. This follows from norm-continuity of the integrand in (6.3) and the fact that $\mathrm{D}f(\Phi_t(u_*)) - \mathrm{D}f(0)$ is compact for any t as an operator from C^2_{unif} to C^0_{unif}. We refer to [26, Lemma 5.4], see also [2, pp. 27-28], for the details. □

Note that Lemma 6.3 is also true in $L^2(\mathrm{I\!R}^N, \mathrm{I\!R}^M)$. The next lemma states that the spiral wave must be localized in space; otherwise Hypothesis (S) is never met since the essential spectrum contains part of the unit circle. We remark that the statement of the lemma is also true for relative equilibria.

Again, the result is not surprising. On the real line, that is, $x \in \mathrm{I\!R}$, it is well known that the essential spectrum of a travelling wave is determined by its asymptotic state with $|x| \to \infty$. However, the spectrum of periodic travelling waves on the real line consists entirely of essential spectrum, and therefore touches the imaginary axis since $\lambda = 0$ is always in the spectrum due to translational invariance; see [15].

Lemma 6.4. *Suppose that $\mathrm{SE}(N)$ acts smoothly on u_*. If $u_*|_{t=0}$ does not converge uniformly to a constant as $|x| \to \infty$, then the set $\{\lambda \in \mathrm{spec}(L_*);\ |\lambda| \geq 1\}$ is not isolated in $\mathrm{spec}(L_*)$. In other words, the essential spectrum of L_* contains elements on the unit circle. In particular, the spectral hypothesis (S) is not met.*

Proof. For relative equilibria, this lemma and the idea for its proof were communicated to us by Herbert Koch; see also [15, p. 139] for a similar idea.

Let $\chi(r)$ be a cut-off function defined for $r \geq 0$ such that

$$\chi(r) = \begin{cases} 1 & \text{for } r \geq 4, \\ 0 & \text{for } r \leq 1 \end{cases} .$$

Define $\chi_n(r) := \chi(\frac{r}{n})$ for $r \geq 0$. Note that we have

$$|\mathrm{D}\chi_n(r)| + |\mathrm{D}^2\chi_n(r)| \leq C\frac{1}{n}$$

for some constant $C > 0$.

Take any $\xi \in \mathrm{se}(N) = \mathrm{alg}(\mathrm{SE}(N))$. Let $u_*(t) = \Phi_t(u_*(0))$. By equivariance, $(\xi u_*(t))(x)$ satisfies

$$v_t = d\Delta v + \mathrm{D}f(u_*(t,x))v \ . \tag{6.4}$$

Let $v_n(t,x) := \chi_n(|x|)(\xi u_*(t))(x)$ and $v = v_n + w$. The function v satisfies (6.4) if and only if w satisfies

$$\begin{aligned} w_t &= d\Delta w + \mathrm{D}f(u_*(t,x))w + d\Delta v_n + \mathrm{D}f(u_*(t,x))v_n - \frac{\partial}{\partial t}v_n \\ &= d\Delta w + \mathrm{D}f(u_*(t,x))w + d\Big((\xi u_*)\Delta\chi_n + 2\nabla\chi_n \cdot \nabla(\xi u_*)\Big) \\ &= d\Delta w + \mathrm{D}f(u_*(t,x))w + \mathrm{O}(\frac{1}{n}) \end{aligned}$$

using the fact that ξu_* satisfies (6.4). Solving this equation with $w(0) = 0$, we obtain $w(t) = \mathrm{O}(\frac{1}{n})$ for $t \in [0,T]$ by Gronwall's Lemma. Thus,

$$\mathrm{D}\Phi_T(u_*(0))(\chi_n \xi u_*(0)) = \chi_n \xi u_*(T) + \mathrm{O}(\frac{1}{n}) \ .$$

Since $u_*(T) = g_* u_*(0)$, we have

$$L_*\xi u_*(0) = g_*^{-1}\mathrm{D}\Phi_T(u_*(0))\xi u_*(0) = g_*^{-1}\xi u_*(T) = \mathrm{Ad}_{g_*}^{-1}(\xi)u_*(0) \ ,$$

and therefore

$$L_*(\chi_n\xi u_*(0)) = (g_*^{-1}\chi_n)\,\mathrm{Ad}_{g_*}^{-1}(\xi)u_*(0) + \mathrm{O}(\frac{1}{n}) = \chi_n\,\mathrm{Ad}_{g_*}^{-1}(\xi)u_*(0) + \mathrm{O}(\frac{1}{n}) \ .$$

Indeed, the first equality is true due to the definition of the $\mathrm{SE}(N)$-action. The second equality follows since $g_* = (R_*, S_*)$ is fixed,

$$|g_*^{-1}\chi_n(|x|)| = |\chi_n(|x - S_*|)| \leq |\mathrm{D}\chi_n|\,|S_*| \ ,$$

and $\mathrm{D}\chi_n = \mathrm{O}(\frac{1}{n})$.

Comparing the last two equations, we have

$$L_*\xi u_*(0) = \mathrm{Ad}_{g_*}^{-1}(\xi)u_*(0), \quad L_*(\chi_n\xi u_*(0)) = \chi_n\,\mathrm{Ad}_{g_*}^{-1}(\xi)u_*(0) + \mathrm{O}(\frac{1}{n}) \ . \tag{6.5}$$

Consider the spaces

$$E_* := \{\xi u_*(0);\ \xi \in \mathrm{se}(N)\} \quad \text{and} \quad E_n := \{\chi_n\xi u_*(0);\ \xi \in \mathrm{se}(N)\} \ .$$

If $u_*(x)$ does not tend to a constant for $|x| \to \infty$, then there are elements in E_n which do not converge strongly (in norm) to zero as $n \to \infty$. These elements also have an $\mathrm{O}(1)$-distance from the space E_*. On the set of such elements, the linearization acts up to order $\mathrm{O}(\frac{1}{n})$ as on the tangent space E_* of the group orbit of $u_*(0)$ as $n \to \infty$. Therefore, the essential spectrum of L_* has to include the spectrum of L_* restricted to E_*, and the lemma is proved. □

In fact, the following slightly more general result is true.

Lemma 6.5. *Suppose that* $\mathrm{SE}(N)$ *acts smoothly on* u_*. *Assume that* $\lambda_* \in S^1 \subset \mathbb{C}$ *is an eigenvalue of* L_* *on the unit circle with eigenfunction* $v_*(x)$. *If* $v_*(x)$ *does not converge to zero uniformly as* $|x| \to \infty$, *then* λ_* *is not isolated in* $\mathrm{spec}(L_*)$. *In other words, the essential spectrum of* L_* *contains the element* λ_*. *In particular, the spectral hypothesis (S) is not met.*

Proof. The proof is completely analogous to that of the previous lemma with $\xi u_*(t)$ replaced by $v_*(t,x) := (\mathrm{D}\Phi_t(u_*)v_*)(x)$, which again satisfies (6.4). We omit the details. □

7. Bifurcations from Planar Spiral Waves in Excitable Media

In this section, we concentrate on planar waves. Hopf bifurcations of patterns occurring in reaction-diffusion equations in three dimensions can be investigated with the same techniques. Indeed, the reduced equations for Hopf bifurcations and periodic forcing are identical, so that the results presented in Sect. 8 also apply to Hopf bifurcations in three dimensions. The choice of examples we made here is motivated by chemical and numerical experiments. There, Hopf bifurcations from modulated waves have not yet been observed in three-dimensional media but similar phenomena have been produced through periodic forcing.

7.1. Hopf Bifurcation of Planar Meandering Spirals

In numerical simulations of reaction-diffusion systems in the plane, modulated waves with three frequencies have been observed in [23]. They may arise via a Hopf bifurcation from a meandering spiral to a relative invariant torus. Consider the reaction-diffusion system (1.1)

$$u_t = d\Delta u + f(u,\mu) \qquad\qquad x \in \mathbb{R}^2$$

with $N = 2$. Solutions of this equation are denoted by $\Phi_t(u,\mu)$. The semiflow $\Phi_t(u,\mu)$ is equivariant with respect to the group of rotations and translations of the plane $\mathrm{SE}(2)$. We write elements $(R,S) \in \mathrm{SE}(2)$ in the form $(R,S) = (\phi, a) \in S^1 \times \mathbb{C}$ where $a \in \mathbb{C} \cong \mathbb{R}^2$ is a translation and ϕ denotes the rotation around zero by the angle ϕ. Suppose that u_* is a meandering spiral wave of the above equation for $\mu = \mu_*$, that is, a relative periodic solution satisfying $(\phi_*, 0)u_* = \Phi_T(u_*, \mu_*)$. We assume that rotations act continuously on u_*.

Since we are mainly interested in Hopf bifurcations, we assume that the linearization about u_* has a complex conjugated pair of eigenvalues on the unit circle besides the eigenvalues enforced by symmetry. The rest of the spectrum should be strictly contained inside the unit circle. In other words, counting multiplicity, the spectrum of the operator $(\phi_*, 0)^{-1}\mathrm{D}\Phi_T(u_*)$ on the unit circle consists of the eigenvalues $\exp(\pm \mathrm{i}\omega_{\mathrm{H}} T)$, $\omega_{\mathrm{H}} \neq 0 \bmod \pi$, and the eigenvalues on the tangent space $\mathrm{span}\{\partial_{x_1} u_*, \partial_{x_2} u_*, \partial_\phi u_*, \partial_t u_*\}$ of the relative periodic orbit $\mathcal{O}_*$.

By Theorems 2.2, 2.9, and 6.1, and Lemma 2.12, the following is then true. The isotropy subgroup $H_* = \{(\phi, a) \in \mathrm{SE}(2);\ (\phi, a)u_* = u_*\}$ is either S^1 or $\mathbb{Z}_\ell$ for some ℓ. Furthermore, a center manifold exists. In particular, the essential dynamics near the relative periodic orbit can be reduced to an ODE on $\mathrm{SE}(2) \times V_* \times S^1$. The vector field on $\mathrm{SE}(2) \times V_* \times S^1$ is given by

$$\begin{aligned} \dot\phi &= f_1(v, \theta, \mu), \qquad & \dot a &= \mathrm{e}^{\mathrm{i}\phi} f_2(v, \theta, \mu), \\ \dot v &= f_N(v, \theta, \mu), \qquad & \dot\theta &= f_\Theta(v, \theta, \mu)\ . \end{aligned} \tag{7.1}$$

Here, the coordinates $(\phi, a) \in \mathrm{SE}(2)$ relate to the group orbit: $a \in \mathbb{C}$ and $\phi \in SO(2)$ correspond to the position of the spiral tip and the rotation angle of the spiral, respectively. The variable $\theta \in S^1$ is the time-phase of the meandering spiral and $v \in V_*$ is contained in the eigenspace associated with the Hopf eigenvalues. Hence, θ measures changes of the shape due to the time-dependence of the relative periodic orbit, while bifurcations will occur within the space V_*. We may assume that $f_1(0, \theta, \mu_*) = \omega_*$ where ω_* corresponds to the rotation frequency of the meandering spiral wave. In other words, $T\omega_* = \phi_*$ modulo 2π. The function $(f_1, f_2)(v, \theta, \mu)$ has values in the Lie algebra $\mathbb{R} \times \mathbb{C}$. Note that the equations for (v, θ) decouple from the equations on the group SE(2) due to the skew-product nature of the flow.

In order to simplify the discussion of the reduced equation (7.1) and according to the numerical observations [23], we assume that $\phi_* \neq 0$ and that the isotropy H_* of the meandering spiral is trivial. To avoid strong resonances, we assume that $\mathrm{e}^{\mathrm{i}n\omega_{\mathrm{H}}T} \neq 1$ for $n = 1, \dots, 4$. Furthermore, we assume that the eigenvalues $\mathrm{e}^{\mathrm{i}\omega_{\mathrm{H}}T}$ and $\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{H}}T}$ cross the unit circle with non-zero speed as μ changes. We may then apply the Hopf-bifurcation theorem for maps, see, for instance, [18], and obtain an invariant torus for the last two equations in (7.1). On this torus, frequency locking may occur for $\mu > \mu_*$, say; see [14, 18].

We discuss the full system (7.1) next. The aforementioned invariant torus for the (v, θ)-system corresponds to a relative invariant torus of (7.1). Let $(v(t), \theta(t))$ be a solution on the torus. The corresponding solution $(a(t), \phi(t))$ of the first two equations in (7.1) may then be unbounded. In particular, linear drift occurs if the rotation frequency ω_* satisfies

$$\omega_* = m\omega_{\mathrm{H}} + n\omega_T\ , \tag{7.2}$$

where $m, n \in \mathbb{Z}$, and ω_{H}, $\omega_T = \frac{2\pi}{T}$ are the frequencies on the relative invariant torus.

Indeed, integrating the first equation in (7.1), we obtain to leading order in μ

$$a(t) = a(0) + \int_0^t \mathrm{e}^{\mathrm{i}\omega_*\tau}\, f_2(v(\tau), \theta(\tau), \mu_*)\, \mathrm{d}\tau\ . \tag{7.3}$$

The function $f_2(v(t), \theta(t), \mu_*)$ is quasi-periodic with frequencies ω_{H} and ω_T. We expand f_2 into a Fourier series

$$f_2(v(t), \theta(t), \mu_*) = \sum_{k,\ell} \sigma_{k\ell}\, \mathrm{e}^{\mathrm{i}(\omega_{\mathrm{H}}k + \omega_T\ell)t}\ .$$

Substituting this expansion into (7.3), we see that linear drift $\lim_{t\to\infty}\frac{1}{t}a(t) = \sigma_{mn}$ occurs if (7.2) is satisfied. Indeed, the remainder term $a(t)-\sigma_{mn}t$ is periodic in t if the fraction $\omega_{\mathrm{H}}/\omega_T$ is rational. If the fraction is irrational, the above mentioned remainder term still grows only sub-linearly in t since

$$\mathrm{e}^{\mathrm{i}\omega_*\tau} f_2(v(\tau), \theta(\tau), \mu_*) - \sigma_{mn}$$

is quasi-periodic in τ and its constant term vanishes. Therefore, the mean value

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t (\mathrm{e}^{\mathrm{i}\omega_*\tau} f_2(v(\tau), \theta(\tau), \mu_*) - \sigma_{mn})\,\mathrm{d}\tau = 0$$

is zero.

A similar situation occurs for periodic forcing of meandering spiral waves. In this case, there are actually experimental results; see Sect. 8.2 for a discussion. If the solution $(a(t), \phi(t))$ of the first two equations in (7.1) is unbounded, the associated spiral waves are called *generalized drifting waves* since they may not be periodic but quasi-periodic in an appropriate moving frame.

7.2. Spiral Waves on Cylindrical Surfaces

Next, we consider spiral waves on cylindrical surfaces. Suppose that the reaction-diffusion system on the cylinder $C = S^1 \times \mathbb{R}$ is equivariant under the group $G := \mathrm{SO}(2) \times \mathbb{R}$. Note that this group is abelian. Suppose that the aforementioned spectral hypothesis is satisfied. By Theorems 2.2 and 5.1, the equivariant center-manifold reduction applies.

Relative equilibria move along helical curves without oscillating, that is, they satisfy $(\phi(t), a(t)) = (\phi_0+\omega_* t, a_0+s_* t)$. Typically, $\omega_* \neq 0$ except when the spiral has an additional reflection symmetry inside each cross-section. Relative equilibria may undergo a Hopf bifurcation to a relative periodic orbit. In numerical simulations, Steinbock [31] actually observed a spiral wave in the Belousov-Zhabotinsky system on a cylinder which appears to be a stable relative periodic orbit $u_* \in C^0_{\mathrm{unif}}(C, \mathbb{R}^M)$. After an appropriate time rescaling, the reduced vector field on the center manifold is given by

$$\dot{\phi} = f_1(t), \qquad \dot{a} = f_2(t) \ .$$

Here, $f_1(t)$ and $f_2(t)$ are periodic in t. The variables $\phi \in \mathrm{SO}(2)$ and $a \in \mathbb{R}$ correspond to the coordinates of the spiral tip on the cylindrical surface.

The relative periodic orbit typically oscillates around a helical curve. Indeed, if T denotes the period of u_*, we generically have $(\phi(T), a(T)) \neq 0$. This explains the numerical observation in [31] that the tip of the spiral wave follows helical loops along the cylinder.

We remark that further secondary Hopf bifurcations of the meandering spirals on the cylinder do not lead to more complicated dynamical phenomena. Indeed, the helical curve still depends continuously on parameters, while the oscillations around it depend on two frequencies.

8. Periodic Forcing of Spiral Waves

In this section, we consider periodically-forced reaction-diffusion systems

$$u_t = d\Delta u + f(u) + \mu f_{\text{ext}}(t,u) \qquad x \in \mathbb{R}^N,\ N = 2,3 \tag{8.1}$$

on $X = C^0_{\text{unif}}(\mathbb{R}^N)$ with $N = 2,3$. The forcing term $f_{\text{ext}}(t,u)$ has frequency Ω in t. Let $u_* \in X$ be a relative equilibrium or a relative periodic solution of (8.1) for $\mu = 0$. Suppose that $u_*(x)$ is not a constant function in x.

We assume that $\mathrm{SO}(N)$ acts continuously on u_*. Whether this assumption is satisfied or not depends strongly on the shape of the pattern; see Sect. 8.3 below. If this hypothesis is not met, then certain restrictions are imposed on the time evolution of the pattern. Indeed, suppose that the one-parameter family $\exp(r_*\tau)$ of rotations acts discontinuously on the function u_*, that is,

$$|u_* - \exp(r_*\tau)u_*|_{C^0} \geq \delta > 0$$

for small $\tau \neq 0$. As a result, the associated solution $\Phi_t(u_*)$ of (8.1) for $\mu = 0$ and t small also stays away from the rotated patterns $\exp(r_*\tau)u_*$. Hence, the time evolution of u_* cannot involve rotations of the pattern about the axis determined by r_*. In this situation, we may then consider the largest subgroup of $\mathrm{SO}(N)$ which acts continuously on u_*. We obtain a lower-dimensional center-manifold, which does not contain the functions $\exp(r_*\tau)u_*$ for small $\tau \neq 0$.

Next, we assume that the operator $(R_*, S_*)^{-1}\mathrm{D}\Phi_T(u_*)$ has finitely many eigenvalues on the unit circle, while the rest of the spectrum is strictly contained in the unit circle. Moreover, the center-unstable eigenspace associated with the eigenvalues on the unit circle coincides with the tangent space of the group orbit of u_*, plus the time-direction if u_* is a relative periodic solution.

For any small μ, there exists then an $\mathrm{SE}(N)$-invariant center manifold M^{cu}_* of (8.1). For relative periodic orbits, the center manifold can be described using the variables $((R,S),\theta) \in \mathrm{SE}(N) \times S^1$ and the time $t \in \mathbb{R}$. We refer to Theorems 3.1 and 3.2 for more details. The vector field on the center manifold is then given by

$$\dot{R} = R f_1(t,\theta,\mu), \qquad \dot{S} = R f_2(t,\theta,\mu), \qquad \dot{\theta} = f_\Theta(t,\theta,\mu)\ . \tag{8.2}$$

For $\mu = 0$, the function

$$(f_1, f_2, f_\Theta)(t,\theta,0) = (f_1, f_2, f_\Theta)(\theta,0)$$

does not depend on the time variable t. Note that the equation for θ decouples from the equations on the group $\mathrm{SE}(N)$. Equation (7.1) is equivariant with respect to the isotropy subgroup H_*, that is,

$$f_1(t,\theta,\mu) = h f_1(t,\theta,\mu)h^{-1}, \qquad f_2(t,\theta,\mu) = h f_2(t,\theta,\mu)h^{-1} \tag{8.3}$$

for any $h \in H_*$. These restrictions imposed by the isotropy subgroup can prevent spirals or scroll waves from drifting in certain directions.

8.1. Periodic Forcing of Rigidly-Rotating Spiral Waves

Periodic forcing of rigidly-rotating spiral waves leading to meandering and drifting spiral waves is described by the vector field

$$\dot{\phi} = f_1(t, \mu), \qquad \dot{a} = \mathrm{e}^{\mathrm{i}\phi} f_2(t, \mu) \ , \tag{8.4}$$

where $f_1(t, \mu)$ and $f_2(t, \mu)$ are time-independent for $\mu = 0$ and time-periodic with frequency Ω otherwise. Using a Fourier-series argument, see [20], it can then be shown that a path of drifting spiral waves emanates from the point $(0, \Omega) = (0, \frac{1}{n}\omega_*)$ in the (μ, Ω)-plane whenever the rotation frequency ω_* of the rigidly-rotating spiral wave is a multiple of Ω. We refer to [33] for a different approach to this phenomenon. In experiments [21, 34], drifting spirals have been observed for the resonances $\omega_* = \Omega$ and $\omega_* = 2\Omega$.

Note that the rotation orbit $\mathrm{SO}(2)u_*$ of the rigidly-rotating spiral u_* is automatically smooth since it is equal to the time-orbit of u_*. Thus, the center-manifold reduction applies whenever the Hypothesis (S) on the spectrum is satisfied.

8.2. Periodic Forcing of Meandering Spiral Waves

Planar meandering spiral waves u_* of an SE(2)-equivariant system satisfy

$$\Phi_T(u_*) = (\phi_*, 0)u_* \ .$$

Here $\omega_* = \frac{\phi_*}{T}$ is the non-zero rotation frequency of the meander. The reduced equations are given by

$$\dot{\phi} = f_1(t, \theta, \mu), \qquad \dot{a} = \mathrm{e}^{\mathrm{i}\phi} f_2(t, \theta, \mu), \qquad \dot{\theta} = f_\Theta(t, \theta, \mu)$$

where $\phi \in S^1$, $a \in \mathbb{C}$, and $\theta \in S^1$. We can choose coordinates such that $f_1(t, \theta, 0) = \omega_*$, $f_2(t, \theta, 0) = 0$, and $f_\Theta(t, \theta, 0) = 1$.

Note the equation for $\theta \in S^1$ decouples. Since this equation is periodically forced, frequency locking may occur. Let $\theta(t)$ be a solution. We then solve the equations for (ϕ, a) and denote the solutions by $(\phi(t), a(t))$.

If $a(t)$ is unbounded, the spiral is drifting. As mentioned in Sect. 7.1, such spirals are called generalized drifting solutions in the literature [34, 29], see also Fig. 1. The analysis is very similar to the one presented in Sect. 7.1, where we considered secondary Hopf bifurcation from meandering spiral waves.

Rescaling time, we obtain that $f_1(t, \theta, \mu) = \omega_*$ for all θ and t, as long as μ is small. Denote the second frequency of $\theta(t)$ by ω. Generically, $a(t)$ then grows linearly provided $\omega_* = n\omega + m\Omega$ for some $n, m \in \mathbb{Z}$. We omit the details.

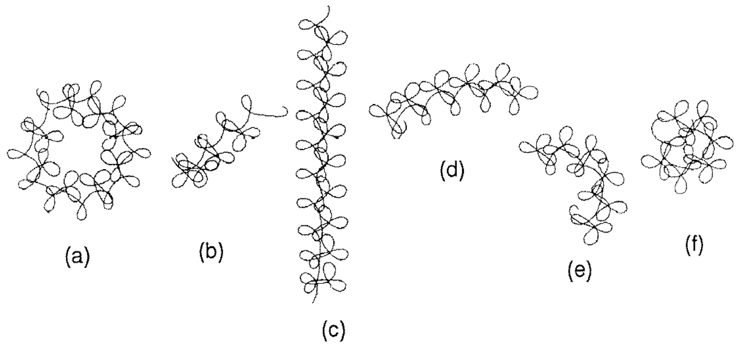

Fig. 1. Motions of spiral tips in the Oregonator-model of the Belousov-Zhabotinsky reaction with periodically-forced excitability for different values of the forcing period; courtesy of [29]. A generalized drifting spiral appears in (c).

8.3. Periodic Forcing of Helices

Helices have frequently been observed in reaction-diffusion systems in three dimensions: For instance, Henze et al. [16] observed stable helical waves in numerical simulations of the Oregonator model. Helices have a spiral wave in each horizontal plane such that the cores of these spirals are aligned along a helical filament.

For a helix, the rotation orbit $\mathrm{SO}(3)u_*$ might be discontinuous. Though rotations around the axis of the core helix act continuously, rotations around other axes may lead to large deviations from u_* in the C^0_{unif}-norm. Motivated by the observation that the rotations which do not fix the horizontal plane act discontinuously on the spatial pattern, we restrict the following discussion to vertically periodic patterns. The symmetry group is then given by $G = \mathrm{SE}(2) \times S^1$ rather than SE(3). Besides continuity of the group action, there is yet another reason for restricting to vertically periodic patterns: Lemma 6.4 states that the spectral hypothesis can only be satisfied if the underlying relative periodic solution is localized in space. Certainly, helices are not localized in the vertical direction. Restricting to the function space of vertically-periodic functions, however, allows that the essential spectrum may indeed be bounded away from the unit circle provided the spirals in each horizontal plane are localized.

We write the group elements of $\mathrm{SE}(2) \times S^1$ as (ϕ, a, ψ) with $(\phi, a) \in \mathrm{SE}(2)$ and $\psi \in S^1$. Near a periodically-forced helical wave, the reduced equations read

$$\dot{\phi} = f_1(t, \mu), \qquad \dot{a} = \mathrm{e}^{\mathrm{i}\phi} f_2(t, \mu), \qquad \dot{\psi} = f_3(t, \mu) \ , \tag{8.5}$$

where $a \in \mathbb{C}$ are the translations in the horizontal plane, $\phi \in S^1$ describes rotations in this plane, and $\psi \in S^1$ is the shift along the vertical axis due to vertical periodicity. We have $f_1(t, \mu)$, $f_3(t, \mu) \in \mathbb{R}$ and $f_2(t, \mu) \in \mathbb{C}$.

First, we set $\mu = 0$ and consider the unperturbed helix. Equation (8.5) is then autonomous. Typically, as far as symmetry is concerned, a helical wave is a relative equilibrium which rotates around its axis and drifts along the axis of rotation. Therefore, a helical wave u_* satisfies $\Phi_t(u_*) = (\omega_* t, 0, s_* t)u_*$. In particular, we have $f_1(t,0) = \omega_*$, $f_2(t,0) = 0$, and $f_3(t,0) = s_*$.

Next, let $\mu \neq 0$. The spiral waves in the horizontal planes along the helical filament will start to meander. Indeed, note that the first two equations in (8.5) decouple and are precisely the differential equations (8.4) for the motion along group orbits near rigidly-rotating spirals. Therefore, drift in the horizontal direction occurs if the rotation frequency ω_* is a multiple of the external frequency Ω. In addition, there is a small periodic perturbation added to the linear drift term along the vertical axis.

8.4. Periodic Forcing of Twisted Scroll Waves

A twisted scroll wave is similar to a helix with the only difference that the core filaments are aligned on a straight line rather than on a helix. For the reasons mentioned in Sect. 8.3, we restrict to functions which are spatially periodic in the vertical direction. The relevant symmetry group is then $G = \mathrm{SE}(2) \times S^1$. Its elements (ϕ, a, ψ) are rotations $\phi \in S^1$ and translations $a \in \mathbb{C}$ in the horizontal plane as well as vertical translations ψ with $\psi \in S^1$ due to vertical periodicity. Furthermore, by definition, a twisted scroll wave u_* has nontrivial isotropy S^1 given by $(\psi, 0, \psi)u_* = u_*$ for all $\psi \in S^1$. In other words, shifting the scroll wave along the vertical axis and rotating at the same time with the same speed in the horizontal plane does not change the pattern.

Without periodic forcing, twisted scroll waves u_* are rotating waves satisfying $\Phi_t(u_*) = (\omega_* t, 0, 0)u_*$. The reduced equations

$$\dot{\phi} = f_1(t,\mu), \qquad \dot{a} = \mathrm{e}^{\mathrm{i}\phi} f_2(t,\mu), \qquad \dot{\psi} = f_3(t,\mu)$$

coincide with the equations (8.5) for helices. However, the action of the isotropy group enforces

$$f_2(t,\mu) = \mathrm{e}^{\mathrm{i}\psi} f_2(t,\mu) \ ,$$

that is, $f_2(t,\mu) = 0$ for all μ and t. Therefore,

$$\dot{\phi} = f_1(t,\mu), \qquad \dot{a} = 0, \qquad \dot{\psi} = f_3(t,\mu) \ .$$

Under periodic forcing, the spiral waves in the horizontal planes begin to meander. On account of the isotropy group S^1, unbounded drift in the horizontal plane cannot occur since $a(t) = a_0$ is constant.

8.5. Periodic Forcing of Twisted Scroll Rings

Other patterns in reaction-diffusion systems in three dimensions are twisted scroll rings which have been studied numerically in [7, 22]. They rotate around the x_3-axis, say, while drifting with constant speed along the vertical axis. The spatial pattern typically resembles a one-parameter family of spirals with cores aligned along a circle. The spiral patterns occur in the bundle of planes normal to the core circle. Furthermore, the spirals have a phase difference along the family of normal planes; see Fig. 2. For *ℓ-twisted scroll rings*, this phase difference is ℓ-times the difference in angle between the core points on the unit circle.

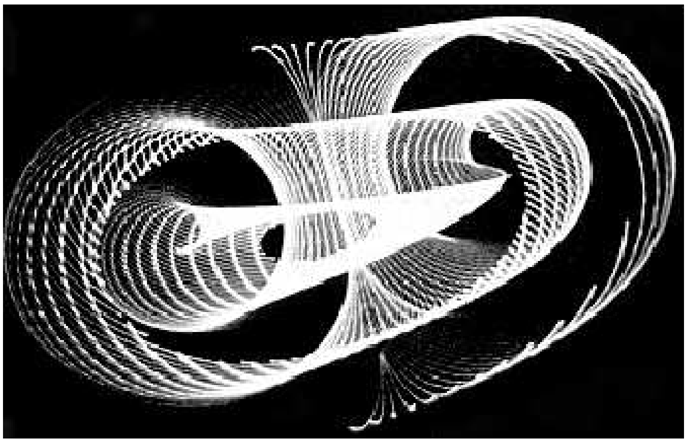

Fig. 2. A twisted scroll ring; reprinted from Fig. 13b of [32][1].

Without periodic forcing, an ℓ-twisted scroll ring u_* is a relative equilibrium with spatial isotropy $H_* = \mathbb{Z}_\ell$. Its time evolution is given by $\Phi_t(u_*) = \exp(\xi_* t)u_*$ for an element $\xi_* = (r_*, s_*) \in \mathrm{se}(3)$ where s_* lies in the fixed-point space of $\mathbb{Z}_\ell$; see [9]. We assume that the group orbit of u_* is continuous.

In passing, we remark that relative periodic orbits satisfying

$$\Phi_T(u_*) = (R_*, S_*)u_*$$

in three dimensions drift in a direction orthogonal to the axis of rotation. More precisely, among the elements in the group orbit $\mathrm{SE}(3)u_*$, there is one, say $\hat{u}$, such that $\Phi_T(\hat{u}) = (\hat{R}, \hat{S})\hat{u}$ and $\hat{R}\hat{S} = \hat{S}$. Indeed, let $S_* = \hat{S} + S_1$ such that $\hat{S} \in \mathcal{N}(\mathrm{id} - R_*)$ and $S_1 \perp \mathcal{N}(\mathrm{id} - R_*)$. Moreover, let $S_2 = (\mathrm{id} - R_*)^+ S_*$. Here, L^+ denotes the Moore-Penrose pseudo-inverse of the matrix L, that is, $L^+|_{\mathcal{R}(L)} = (L|_{\mathcal{N}(L)^\perp})^{-1}$ and $L^+|_{\mathcal{R}(L)^\perp} = 0$. Rather than investigating u_*, we may focus on $\hat{u} = (\mathrm{id}, -S_2)u_*$. We obtain

$$\begin{aligned} \Phi_T(\hat{u}) &= (\mathrm{id}, -S_2)(R_*, S_*)u_* = (\mathrm{id}, -S_2)(R_*, S_*)(\mathrm{id}, S_2)\hat{u} \\ &= (R_*, S_* + (R_* - \mathrm{id})S_2)\hat{u} = (R_*, \hat{S})\hat{u} \ . \end{aligned}$$

Therefore, without loss of generality, we assume that $R_* S_* = S_*$.

[1] With kind permission from Elsevier Science, Sara Burgerhartstraat 25, 1055 KV Amsterdam, The Netherlands

Note that (R_*, S_*) lies in the centralizer of $\mathbb{Z}_\ell = H_*$ by the above consideration. This is useful when studying Hopf bifurcations from twisted scroll rings since, under this condition, Lemma 2.12 ensures that the center manifold is diffeomorphic to the trivial product $\mathrm{SE}(3) \times S^1 \times V_*$ with $V_* = \mathbb{R}^2$.

The reduced equations are given by

$$\dot{R} = R f_1(t, \mu), \qquad \dot{S} = R f_2(t, \mu) \ ,$$

where $f_1(t, 0) = r_*$ and $f_2(t, 0) = s_*$. For ℓ-twisted scroll rings, we have

$$f_2(t, \mu) = R f_2(t, \mu)$$

for all $(R, 0) \in \mathbb{Z}_\ell$ in the isotropy group H_*.

For $\mu \neq 0$, that is, under periodic forcing, the spirals in the vertical planes start to meander. In the case of ℓ-twisted scroll rings with $\ell > 1$, drift is only possible along the symmetry axis of the scroll ring since then $f_2(t, \mu) \in \mathrm{span}\{s_*\}$.

Simply-twisted scroll rings typically drift in a direction different from the x_3-direction provided the group orbit is continuous, see [1]. The direction of drift generically varies in μ, regardless of resonances in the periodic forcing. If rotations around axes different from the vertical x_3-axis act discontinuously, the scroll ring generically drifts along the vertical axis. Additional slow horizontal drift occurs only at resonances, that is, when the rotation frequency ω_* is a multiple of the external frequency. Indeed, if rotations around axes different from the x_3-axis act discontinuously, the pattern cannot reach these rotated states in a small amount of time regardless how close the rotation is to the identity. Hence, the filament of the scroll ring is restricted to the vertical axis. Mathematically, we have to remove the corresponding rotations from the symmetry group SE(3) and obtain a lower-dimensional center manifold which is smooth and attracting. Note that, in function space, the patterns rotated around an axis different from the vertical axis are then not close to the center manifold.

9. Conclusions

In this article, we developed an equivariant center-manifold reduction near relative periodic orbits. The underlying symmetry group G is possibly non-compact and may act discontinuously. The flow on the center manifold is identified with an equivariant vector field of skew-product type on the product $G \times V_* \times \mathbb{R}$ under an equivalence relation involving the spatio-temporal symmetries of the relative periodic orbit. Here, V_* is some finite-dimensional vector space.

In particular, using only a priori known symmetries of patterns arising in chemical or physical systems, we can systematically derive equations-of-motion which govern the dynamical behavior and bifurcations of patterns.

Finally, we applied this method to several kinds of waves which were observed in experiments and numerical simulations. We assumed that the underlying chemical systems can be modeled by reaction-diffusion systems posed on unbounded domains such as the plane or the three-dimensional space. Our approach then applies, and the arising phenomena such as meandering and drifting of spiral waves can be explained using the Euclidean symmetry group as suggested first by Barkley.

It remains to discuss the validity of the aforementioned modeling assumption. There are three aspects involved. Firstly, neither experiments nor numerical simulations are posed on unbounded domains. However, both suggest that the boundaries are actually not important at all. The indications are that spiral waves behave dynamically as if there were no boundaries. Secondly, considering symmetry as a modeling parameter, it seems impossible to explain, for instance, drifting by using compact symmetries induced by bounded domains. Taking translations into account, we have to consider unbounded domains as a consequence of the Euclidean symmetry group. Thirdly, mathematically, the implications are that in order to apply the center-manifold reduction on unbounded domains, the underlying spirals must be localized. However, the spirals observed in experiments appear to be non-localized. This seems to be the only objection as to whether the results presented here actually apply to real-life chemical systems. Note that the predictions from center-manifold reduction are in excellent agreement with experiments and numerical simulations.

Clearly, spirals observed in experiments arise as parts of much more complicated patterns; they never occur as single patterns but only together with other spiral waves, target patterns, and travelling waves, all of which are not isolated. From that point-of-view, the analysis presented here is only a very small step towards an understanding of spirals.

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