Lattice differential equations embedded into reaction-diffusion systems

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Abstract

We show that lattice dynamical systems naturally arise on infinite-dimensional invariant manifolds of reaction-diffusion equations with spatially periodic diffusive fluxes. The result connects wave pinning phenomena in lattice differential equations and in reaction-diffusion equations in inhomogeneous media. The proof is based on a careful singular perturbation analysis of the linear part, where the infinite-dimensional manifold corresponds to an infinite-dimensional center eigenspace.

1 Introduction

The goal of this article is to connect the dynamics of lattice differential equations with the dynamics of reaction-diffusion systems. From a technical point of view, lattice differential equations are ordinary differential equations, typically posed on Banach spaces such as ℓ^∞. A simple prototype is the set of scalar, diffusively coupled ODEs

$$\frac{\mathrm{d}}{\mathrm{d}t}u_j = d\,(u_{j+1} - 2u_j + u_{j-1}) + f(u_j), \quad u_j \in \mathbb{R},\ j \in \mathbb{Z}, \tag{1.1}$$

where $d \in \mathbb{R}$, and f is a smooth nonlinearity. The initial value problem to (1.1) is well-posed in forward *and* backward time $t \in \mathbb{R}$, with initial condition $(u_j)_{j\in\mathbb{Z}} \in \ell^\infty$, and norm $|u|_{\ell^\infty} = \sup_j |u_j|$. If $f(0) = 0$, (1.1) is also well-posed on ℓ^2, with norm $|u|^2_{\ell^2} = \sum_j |u_j|^2$.
We also consider reaction-diffusion systems posed on the real line,

$$\partial_t u = u_{xx} + f(u), \quad x \in \mathbb{R}. \tag{1.2}$$

The initial-value problem to (1.2) is well-posed in forward time, $t \geq 0$ on the space of uniformly integrable functions H^1_u, defined as the closure of C^∞_0 with respect to the norm

$$|u|_{H^1_\mathrm{u}} = |u|_{L^2_\mathrm{u}} + |u_x|_{L^2_\mathrm{u}}, \quad |u|_{L^2_\mathrm{u}} = \sup_j |u(x)|_{L^2(2\pi j, 2\pi(j+1))}. \tag{1.3}$$

When $f(0) = 0$, (1.2) is well-posed on H^1, too.
While lattice differential equations and reaction-diffusion systems appear to be very different on a technical level, they share many common phenomena. For instance, with a bistable nonlinearity

$f(u) = u(1-u)(u-b)$, $0 < b < 1$, both exhibit stable traveling fronts $u_*(j-ct)$, or $u_*(x-ct)$, respectively, between the metastable states $u = 0$ and $u = 1$.

A phenomenological difference between the two problems is the pinning of fronts: in the lattice differential equation, there exists a nontrivial interval $b \in (\frac{1}{2} - b_{\mathrm{up}}, \frac{1}{2} + b_{\mathrm{up}})$, where $c = 0$, the front is pinned, it does not move. At b_{up}, a saddle-node bifurcation of pinned fronts leads to unpinning, the existence of propagating fronts. In the reaction-diffusion system, the pinning regime collapses, $c = 0$ for $b = 1/2$, only. One reason for this is the different symmetry of both problems: the lattice differential equation is invariant under discrete translations, the reaction diffusion system allows for continuous translations. In the steady-state problem, pinned fronts are heteroclinic orbits, which are typically transverse, hence robust in the spatially discrete set-up, but non-transverse in the spatially continuous setup due to the trivial kernel generated by translations.

Phenomena are very similar, when reaction-diffusion systems with spatially periodic coefficients are compared to lattice differential equations. Take for instance a periodically modulated flux $a(x) = a(x + 2\pi)$ in

$$\partial_t u = \partial_x(a(x)\partial_x u) + f(u), \quad x \in \mathbb{R}. \tag{1.4}$$

Stationary solutions $c = 0$ solve a time-periodic ODE. Heteroclinic orbits are now typically transverse, so that we expect pinned fronts for an interval of values of b.

Going beyond pure front propagation and pinning, one can ask more challenging questions, such as for the interaction of fronts, the collision of fronts and coarsening, or, more generally, dynamics on the attractor. Our main result relates the dynamics of lattice differential equations (1.1) and reaction-diffusion systems (1.4) in a specific case of fluxes

$$a(x) = \frac{1}{\varepsilon}, \text{ for } x \in (0, 2\pi - \varepsilon), \qquad a(x) = \varepsilon, \text{ for } x \in (2\pi - \varepsilon, 2\pi). \tag{1.5}$$

Since a is not uniformly bounded away from 0, regularization properties are not uniform in ε. We therefore need to adapt the norm in H^1_{u} to

$$|u|_{H^1_{\mathrm{u},\varepsilon}} = |u|_{L^2_{\mathrm{u}}} + |a(x)^{1/2} u_x|_{L^2_{\mathrm{u}}}. \tag{1.6}$$

We will see below that (1.4) generates a smooth semiflow on $H^1_{\mathrm{u},\varepsilon}$, and the embedding $H^1_{\mathrm{u},\varepsilon} \to BC^0_{\mathrm{u}}$, the bounded and uniformly continuous functions, is uniformly bounded in ε.

In order to relate the dynamics on discrete physical space $j \in \mathbb{Z}$ and $x \in \mathbb{R}$, we consider the natural embedding

$$\iota : u(x)_{x \in \mathbb{R}} \mapsto (u_\ell)_{\ell \in \mathbb{Z}} = u(2\pi\ell).$$

This embedding defines a continuous, uniformly bounded map $\iota : H^1_{\mathrm{u},\varepsilon} \to \ell^\infty$. Note that ι is onto but possesses a large kernel.

For our main result, we also assume that f is globally Lipschitz continuous with Lipschitz constant L. We think of possibly modifying f for large u, whenever f is not globally Lipschitz but dissipative, $f(u) \cdot u < 0$ for $|u| > R$ for some sufficiently large R. For those, dissipative f, there exists a bounded attractor $\mathcal{A}_\varepsilon$, with $|u|_{H^1_{\mathrm{u},\varepsilon}} < C$ for all $u \in \mathcal{A}_\varepsilon$. Since all the long-time dynamics is captured by the dynamics near the attractor, we can modify f outside of a large ball and still capture the correct long-time dynamics. In the following, "smooth" refers to a function of class C^k for some k arbitrarily large but finite.

Theorem 1. *Consider the semiflow $\Phi_\varepsilon(t) : H^1_{\mathrm{u},\varepsilon} \to H^1_{\mathrm{u},\varepsilon}$ to the reaction-diffusion system (1.4) with $a(x)$ given in (1.5) and f, f' globally Lipschitz. Let $\phi(t)$ denote the flow to the lattice differential equation (1.1) with $\mathrm{d} = 1/(2\pi)$. For ε sufficiently small, there exists a smooth manifold $\mathcal{M}_\varepsilon \subset H^1_{\mathrm{u},\varepsilon}$, diffeomorphic to ℓ^∞, with diffeomorphism given by the restriction $\iota\mid_{\mathcal{M}_\varepsilon}$. The manifold $\mathcal{M}_\varepsilon$ is invariant under the flow Φ_ε. Moreover, $\Phi_\varepsilon(t)$ is a smooth diffeomorphism on $\mathcal{M}_\varepsilon$, and thereby extends to a smooth flow. In the limit $\varepsilon \to 0$, we have*

$$\left|\iota\Phi_\varepsilon(t)\,(\iota\mid_{\mathcal{M}_\varepsilon})^{-1} - \phi(t)\right|_{C^k(\ell^\infty)} \to 0, \tag{1.7}$$

for all $t \in \mathbb{R}$, fixed.

The manifold $\mathcal{M}_\varepsilon$ can be viewed as an infinite-dimensional inertial manifold [5], or an infinite-dimensional slow manifold in the sense of Fenichel's singular perturbation theory [3].

There has been interest in the comparison of continuum equations (both homogeneous and inhomogeneous) and discrete equations dating back at least to the work of Cahn [2] through comparison of continuous and discrete potential energy functionals; see more recently [7]. We also mention the work of Keener [6] in which averaging and homogenization techniques are used to obtain predictions on the range of pinning for a problem somewhat similar to (1.4) with (1.5). The paper of Mallet-Paret [8] reviews recent developments in the analysis of the discrete problem (1.1), in particular for traveling wave solutions, while the review article of Xin [14] includes results on wave propagation for (1.4).

The remainder of this article is organized as follows. We first review the spectral theory of diffusion operators with periodic coefficients in Section 2. We then show the existence of the invariant manifold for the nonlinear system in Section 3, and expand the reduced flow in terms of ε. We conclude with a brief illustration of our results, where we investigate the stationary equation for both lattice differential equation and the reaction-diffusion system.

2 Linear theory

2.1 Bloch waves and the spectral family for periodic coefficients

We review results on spectral analysis of operators with periodic coefficients. Consider the operator $\mathcal{L}u = (a(x)u_x)_x$, with spatially periodic flux, $a(x) = a(x + 2\pi)$ for all $x \in \mathbb{R}$. We view $\mathcal{L}$ as a self-adjoint operator on $L^2(\mathbb{R})$.

Our key ingredient is the theory of Bloch waves, similar to Floquet analysis, and a generalization of Fourier analysis [11, XIII.16]. For $u \in L^2(\mathbb{R})$, we define the unitary transformation $\mathcal{T} : L^2(\mathbb{R}) \to L^2(S^1, L^2(0, 2\pi))$

$$w(x;\gamma) = (\mathcal{T}u)(x;\gamma) := \frac{1}{2\pi}\sum_{j\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}jx}\int_{y\in\mathbb{R}} \mathrm{e}^{-\mathrm{i}(\gamma+j)y}u(y)\mathrm{d}y, \quad \gamma \in S^1 = \mathbb{R}/\mathbb{Z}. \tag{2.1}$$

The inverse of $\mathcal{T}$ is

$$u(x) = (\mathcal{T}^{-1}w)(x) := \int_{\gamma\in S^1} \mathrm{e}^{\mathrm{i}\gamma x} w(x;\gamma)\mathrm{d}\gamma. \tag{2.2}$$

The operator $\mathcal{L}$ is conjugate to the direct product of operators $\mathcal{L}_\gamma = (\partial_x + \mathrm{i}\gamma)(a(x)(\partial_x + \mathrm{i}\gamma))$, equipped with periodic boundary conditions on $(0, 2\pi)$. We can formally write

$$\mathcal{L} = \mathcal{T}^{-1}\left(\int_\gamma^\oplus \mathcal{L}_\gamma \mathrm{d}\gamma\right)\mathcal{T};$$

see [11, XIII.16] for definitions and details on the direct integral in this equation. Each of the operators $\mathcal{L}_\gamma$ is self-adjoint with compact resolvent. The spectrum of $\mathcal{L}$ is given by the union of the spectra of the $\mathcal{L}_\gamma$ whose eigenvalues we denote by $\lambda_k(\gamma)$, $k = 0, 1, 2, \ldots$, with normalized eigenvectors $w_k(\gamma)$. In our particular example, we will find that $\lambda_0(\gamma)$ is simple and moreover

$$\min_\gamma \lambda_0(\gamma) - \sup_{k>1,\gamma} \mathrm{Re}\,\lambda_k(\gamma) = \delta_0 > 0; \tag{2.3}$$

see Section 2.2. As a consequence, $\lambda_0(\gamma)$ is analytic and periodic in γ. Moreover, the eigenvector $w_0(\gamma)$ solves

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathrm{i}\gamma\right)\left(a(x)\left(\frac{\mathrm{d}}{\mathrm{d}x} + \mathrm{i}\gamma\right)w_0\right) = \lambda_0(\gamma), \qquad w_0(0) = w_0(2\pi),\ w_0'(0) = w_0'(2\pi)$$

so that $u_0(x;\gamma) = w_0(x;\gamma)\mathrm{e}^{-\mathrm{i}\gamma x}$ solves

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(a(x)\frac{\mathrm{d}}{\mathrm{d}x}u_0\right) = \lambda_0(\gamma), \qquad \rho u_0(0) = u_0(2\pi),\ \rho u_0'(0) = u_0'(2\pi), \quad \rho = \mathrm{e}^{2\pi\mathrm{i}\gamma}. \tag{2.4}$$

In particular, u_0 solves a boundary-value problem which only depends on $\gamma \bmod 1$. Since u_0 is unique up to complex scalar multiples, we can write $u_0 = \tilde{u}_0/|\tilde{u}_0|_{L^2}$, where $\tilde{u}_0$ can be assumed to be analytic, and $\tilde{u}_0(x;1) = \sigma\tilde{u}_0(x;0)$, with $\sigma \neq 0$. Substituting $\tilde{u}_0 \mapsto \mathrm{e}^{\gamma\log\sigma}\tilde{u}_0$, we therefore find $\tilde{u}_0$ to be analytic and periodic in γ. In fact, $\tilde{u}_0$ immediately extends to the strip $\gamma \pm \mathrm{i}\eta$ for η small, with again analyticity and periodicity in γ.

Using the spectral resolution, or Dunford's integral, we find a projection on the direct sum of the eigenspaces associated with $\lambda_0(\gamma)$, which we refer to as the *center eigenspace* E^{c} of $\mathcal{L}$. The projection on this eigenspace is the direct product of the projections on the first eigenfunctions of the operators $\mathcal{L}_\gamma$,

$$P_\gamma w(x;\gamma) = \int_x \overline{w_0(y;\gamma)}w(y;\gamma)\mathrm{d}y w_0(x;\gamma),$$

where $\mathcal{L}_\gamma w_0(x;\gamma) = \lambda_0(\gamma)w_0(x;\gamma)$, and we assumed that $|w_0(\cdot;\gamma)|_{L^2} = 1$. In particular, any element in the center eigenspace can be written as a direct product

$$w(x;\gamma) = \hat{v}(\gamma)w_0(x;\gamma), \tag{2.5}$$

with $\hat{v} \in L^2(S^1)$. By the spectral resolution formula, the projection onto the center eigenspace is given through

$$[P^{\mathrm{c}}u](x) = \int_{\gamma=0}^1 \mathrm{e}^{\mathrm{i}\gamma x}w_0(x;\gamma)\left(\int_{y\in\mathbb{R}} \mathrm{e}^{-\mathrm{i}\gamma y}\overline{w_0(y;\gamma)}u(y)\mathrm{d}y\right)\mathrm{d}\gamma,$$

which simplifies to

$$[P^{\mathrm{c}}u](x)=\int_{y\in\mathbb{R}}G^{\mathrm{c}}(x,y)u(y)\mathrm{d}y,\qquad G^{\mathrm{c}}(x,y)=\int_{\gamma=0}^{1}\mathrm{e}^{\mathrm{i}\gamma(x-y)}w_0(x;\gamma)\overline{w_0(y;\gamma)}\mathrm{d}\gamma.\tag{2.6}$$

Note that G^{c} is 2π-periodic, $G^{\mathrm{c}}(x,y)=G^{\mathrm{c}}(x+2\pi,y+2\pi)$.

2.2 Spectral theory for a_ε

We show that in our specific example $a=a_\varepsilon(x)$, defined in (1.5), the first spectral gap as defined in (2.3) is of size ε^{-1}, and compute the approximations to eigenfunctions and the center projection.

Lemma 2.1 (Expansion for spectrum)**.** *The spectrum of $\mathcal{L}$ is contained in the intervals $(-2/\pi-\delta,0]\cup(-\infty,-1/\delta)$ for some δ with $\delta\to 0$ when $\varepsilon\to 0$. More precisely,*

$$\lambda_0(\gamma;\varepsilon)=(\cos(2\pi\gamma)-1)/\pi+\varepsilon\tilde{\lambda}(\gamma;\varepsilon),\qquad \textit{where } \gamma\in S^1=\mathbb{R}/\mathbb{Z},$$

and $\lambda_1(\gamma)\to-\infty$ as $\varepsilon\to 0$.

Proof. Step 1: ODE formulation
We rewrite the eigenvalue problem as two systems of differential equations

$$u_{1,x}=\varepsilon v_1,\quad v_{1,x}=\lambda u_1,\qquad \text{for } x\in[0,2\pi-\varepsilon],$$

and

$$u_{2,x}=\frac{1}{\varepsilon}v_2,\quad v_{2,x}=\lambda u_2,\qquad \text{for } x\in[2\pi-\varepsilon,2\pi],$$

periodically extended in x, with continuity of u,v across the jump points. Rescaling $y=2\pi x/(2\pi-\varepsilon)$ in the first and $y=(x-2\pi+\varepsilon)/\varepsilon$ in the second system, we are lead to the equivalent formulation

$$u_{1,y}=\frac{2\pi\varepsilon}{2\pi-\varepsilon}v_1,\quad v_{1,y}=\frac{2\pi}{2\pi-\varepsilon}\lambda u_1,\qquad \text{for } y\in[0,2\pi],\tag{2.7}$$

and

$$u_{2,y}=v_2,\quad v_{2,y}=\varepsilon\lambda u_2,\qquad \text{for } y\in[0,1].\tag{2.8}$$

Continuity and Floquet boundary conditions for $\mathcal{L}$ translate into

$$u_1(2\pi-\varepsilon)=u_2(0),\quad v_1(2\pi-\varepsilon)=v_2(0),\tag{2.9}$$

and

$$u_2(2\pi)=\mathrm{e}^{2\pi\mathrm{i}\gamma}u_1(0),\quad v_2(2\pi)=\mathrm{e}^{2\pi\mathrm{i}\gamma}v_1(0),\tag{2.10}$$

respectively.
The boundary-value problem (2.7)-(2.10) defines a closed unbounded operator A_λ^γ on $L^2([0,2\pi],\mathbb{C}^2)\times L^2([0,1],\mathbb{C}^2)$ with domain $H^1([0,2\pi],\mathbb{C}^2)\times H^1([0,1],\mathbb{C}^2)\cap\{(2.9)-(2.10)\text{ hold}\}$. In fact, A_λ^γ is readily seen to be Fredholm of index zero and smooth in ε. The spectrum of L_γ coincides with the

values of λ for which A_λ^γ is not invertible (this can be readily seen from Fredholm properties of both operators and a comparison of the kernels).
Our goal is to compute kernels for A_λ^γ using perturbation from $\varepsilon = 0$ and Lyapunov-Schmidt reduction.

Step 2: The singular limit spectrum

We evaluate at $\varepsilon = 0$,

$$u_{1,y} = 0, \quad v_{1,y} = \lambda u_1, \qquad \text{for } y \in [0, 2\pi], \tag{2.11}$$

and

$$u_{2,y} = v_2, \quad v_{2,y} = 0, \qquad \text{for } y \in [0, 1]. \tag{2.12}$$

The explicit solution readily shows that the flow maps the values $u = u_1$, $v = v_1$ at $y = 0$ to $u = (1 + 2\pi\lambda)u_1 + v_1$, $v = 2\pi\lambda u_1 + v_1$ at $y = 2\pi$. Solving the boundary conditions therefore amounts to the condition

$$0 = E_0(\lambda; \gamma) = \begin{vmatrix} 1 + 2\pi\lambda - \mathrm{e}^{2\pi\mathrm{i}\gamma} & 1 \\ 2\pi\lambda & 1 - \mathrm{e}^{2\pi\mathrm{i}\gamma} \end{vmatrix} = 2\mathrm{e}^{2\pi\mathrm{i}\gamma}(\cos(2\pi\gamma) - 1 - \pi\lambda)$$

In particular, $\lambda_0(\gamma) = (\cos(2\pi\gamma) - 1)/\pi$ is the sole eigenvalue, so that A_λ^γ is invertible for all $\lambda \neq \lambda(\gamma)$ at $\varepsilon = 0$. In particular, the norm of the resolvent is uniformly bounded for $\lambda \in [-1/\delta, -(2/\pi) - \delta]$, for any fixed $\delta > 0$.

Step 4: Lyapunov-Schmidt perturbation analysis

The ε-dependent perturbations will introduce perturbations so that eigenvalues are given as roots of $E(\lambda; \gamma, \varepsilon) = E_0(\lambda; \gamma) + \varepsilon E_1(\lambda; \gamma, \varepsilon)$. Since all roots are simple, we find that the spectrum in any large disk $|\lambda| \leq 1/\delta$ is given by

$$\lambda(\gamma; \varepsilon) = \frac{1}{\pi}(\cos(2\pi\gamma) - 1) + \varepsilon\lambda_1(\gamma; \varepsilon), \tag{2.13}$$

so that λ is smooth and 2π-periodic in γ. This proves the lemma. ∎

Corollary 2.2 (Expansions for Eigenfunctions)**.** *We have the following expansions for the eigenfunctions*

$$w_0(x; \gamma)\mathrm{e}^{\mathrm{i}\gamma x} = \frac{1}{\sqrt{2\pi}} \left\{ \begin{array}{l} 1, \ \textit{for } x \in [0, 2\pi - \varepsilon], \\ \frac{x - 2\pi + \varepsilon}{\varepsilon}\mathrm{e}^{2\pi\mathrm{i}\gamma}, \ \textit{for } x \in (2\pi - \varepsilon, 2\pi), \end{array} \right\} + \mathrm{O}(\varepsilon), \tag{2.14}$$

with error terms uniformly bounded in $x \in \mathbb{R}$.

We may also approximate the projection on the center eigenspace. The projection is given by the integral operator $G^{\mathrm{c}}(x, y)$ defined through (see (2.6))

$$G^{\mathrm{c}}(x, y) := \int_0^1 \mathrm{e}^{\mathrm{i}\gamma(x-y)} w_0(x; \gamma)\overline{w_0(y; \gamma)}\mathrm{d}\gamma.$$

Using the representation for w_0 in the form

$$w_0(x; \gamma) = \mathrm{e}^{-\mathrm{i}\gamma(x - 2\pi k)}\theta(x - 2\pi k) + \mathrm{O}(\varepsilon), \qquad 0 \leq x - 2\pi k < 2\pi,$$

with

$$\theta(\xi)=\left\{\begin{array}{ll}1, & \xi\le 2\pi-\varepsilon\\ \frac{\xi-2\pi+\varepsilon}{\varepsilon}\mathrm{e}^{2\pi\mathrm{i}\gamma}, & 2\pi-\varepsilon\le\xi,\end{array}\right.$$

we find that for $0\le x-2\pi k\le 2\pi$, and $0\le y-2\pi\ell\le 2\pi$,

$$G^{\mathrm{c}}(x,y)=G^{\mathrm{c}}_0(x,y)+\mathrm{O}(\varepsilon),\qquad G^{\mathrm{c}}_0(x,y)=\frac{1}{2\pi}\left\{\begin{array}{ll}1, & k=\ell,\\ 0, & k\neq\ell,\end{array}\right\}.$$

The error terms and derivatives are bounded in $L^1_xL^\infty_y+L^\infty_xL^1_y$, so that we have the following Corollary.

Corollary 2.3. *The projection on the center subspace P^{c} can be expanded as $P^{\mathrm{c}}=P^{\mathrm{c}}_0+\mathrm{O}(\varepsilon)$ in the space of linear bounded operators on L^2, where*

$$[P^{\mathrm{c}}_0u](x)=\int_y G^{\mathrm{c}}_0(x,y)u(y)\mathrm{d}y.$$

We denote the center eigenspace, given as the range of P^{c}, as E^{c}.

2.3 Higher regularity

The spectral results immediately carry over to spaces based on interpolation theory with the domain of definition of $\mathcal{L}$. The domain of definition of $\mathcal{L}$ is given by

$$X^1:=\mathcal{D}(\mathcal{L})=\{u\in H^1\mid\ a(x)u_x(x)\in H^1\}.$$

Using the spectral family, one can define $\mathcal{L}^\alpha$ for $\alpha\ge 0$, and $X^\alpha:=\mathcal{D}(\mathcal{L}^\alpha)$. We will use $X^{1/2}$, which can easily seen to be equivalent to H^1 using the bilinear form generated by $\mathcal{L}$ on X,

$$X^{1/2}:=\mathcal{D}(\mathcal{L}^{1/2})=\{u\in H^1\mid\ \sqrt{a(x)}u_x(x)\in L^2\}=H^1.$$

However, the graph norm induced on $X^{1/2}$ by $\mathcal{L}^{1/2}$ is not uniformly equivalent to the H^1-norm as $\varepsilon\to 0$. In the sequel, when we write $|\cdot|_{X^\alpha}$, we refer to the ε-dependent graph norms, not to an ε-independent H^s-norm.

Lemma 2.4. *There exists a constant $C>0$, independent of ε, such that*

$$|u|_{C^0}\le C|u|_{X^{1/2}}.$$

In particular, $X^{1/2}$ embeds into the continuous functions with uniform embedding constant. Moreover, any smooth function $f\in C^k(\mathbb{R})$ induces a smooth Nemitskii operator $\tilde f$ on $X^{1/2}$ such that norms of derivatives of $\tilde f$ are bounded in terms of derivatives of f on bounded subsets of $X^{1/2}$, uniformly in ε.

Proof. We only prove the embedding. The consequences for the Nemitskii operator are straightforward, following for example [1, 4]. The difficulty in proving the embedding is the fact that the norm in $X^{1/2}$ does not control the H^1-norm uniformly. However, if one changes the independent variable x to y by rescaling $y = x/\varepsilon$ in the intervals where $a(x) = \varepsilon$, one finds that the flux $\tilde{a}(y) \geq 1$ in the new variables, that is, the norm in $X^{1/2}$ does control the H^1-norm in the independent variable y. One therefore obtains supremum bounds in the y-variable by the standard embedding $H^1 \to C^0$, which immediately give supremum bounds in the x-variable. ∎

2.4 The lattice parametrization

It turns out that the center eigenspace E^{c} is naturally isomorphic to the space of functions on a lattice.

Lemma 2.5. *The map*

$$\iota : u \in E^{\mathrm{c}} \subset L^2 \mapsto (u_k)_{k\in\mathbb{Z}} := (u(2\pi k))_{k\in\mathbb{Z}} \in \ell^2,$$

is an isomorphism.

Proof. Since $\mathcal{L}$ is bounded on E^{c}, it follows that E^{c} is a closed subspace of X^α for any α. Lemma 2.4, then yields a uniformly bounded embedding $E^{\mathrm{c}} \subset C^0$. Explicitly, we have from Lemma 2.4

$$|u(2\pi j)|^2 \leq C \int_{2\pi j}^{2\pi(j+1)} \left(|a(x)u_x|^2 + |u|^2\right),$$

with C independent of ε, and, since

$$|u|^2_{X^{1/2}} = \sum_j \int_{2\pi j}^{2\pi(j+1)} \left(|a(x)u_x|^2 + |u|^2\right),$$

we find

$$|u_k|^2_{\ell^2} \leq C|u|^2_{X^{1/2}} = C|(1+\mathcal{L}^{1/2})u|^2_X \leq C'|u|^2_X$$

where we used boundedness of $\mathcal{L}$ in the last inequality. This shows that ι is a (uniformly) bounded linear operator on E^{c}. It remains to show that we can invert ι. Inspecting (2.2) and (2.5), one sees that $u(x) \in E^{\mathrm{c}}$ is given through the Fourier-type integral

$$u(x) = \int_0^1 \hat{u}(\gamma)\mathrm{e}^{\mathrm{i}\gamma x} w(x;\gamma)\mathrm{d}\gamma,$$

with $\hat{u}(\gamma)$ defined as the Bloch wave representation on the center eigenspace,

$$\hat{u}(\gamma) := \int_0^1 \mathrm{e}^{-\mathrm{i}\gamma}\overline{w_0(y;\gamma)}u(y)\mathrm{d}\gamma.$$

In particular, we have

$$u_k := u(2\pi k) = \int_0^1 \mathrm{e}^{2\pi\mathrm{i}k\gamma}\hat{u}(\gamma)w_0(0;\gamma)\mathrm{d}\gamma = \mathcal{F}\left(\hat{u}(\cdot)w_0(0;\cdot)\right), \tag{2.15}$$

where we introduced the Fourier transform on periodic functions

$$\mathcal{F}\left(v(\cdot)\right)_\ell := \int_0^1 \mathrm{e}^{2\pi\mathrm{i}\ell\gamma} v(\gamma)\mathrm{d}\gamma,$$

with inverse

$$v(\gamma) = \sum_\ell \mathcal{F}(v(\cdot))_\ell \mathrm{e}^{-2\pi\mathrm{i}\ell\gamma}.$$

Inverting the Fourier transform in (2.15), we can reconstruct u from u_k as follows. We have

$$\hat{u}(\gamma) = w_0(0;\gamma)^{-1} \sum_k \mathrm{e}^{-2\pi\mathrm{i}k\gamma} u_k,$$

so that

$$u(x) = \sum_k g^{\mathrm{c}}(x;k) u_k, \qquad g^{\mathrm{c}}(x;k) = \int_0^1 \frac{w_0(x;\gamma)}{w_0(0;\gamma)} \mathrm{e}^{\mathrm{i}\gamma(x-2\pi k)}\mathrm{d}\gamma.$$

We claim that $|g^{\mathrm{c}}(x;k)| \leq C\mathrm{e}^{-\eta|x-2\pi k|}$. This will then show that the map from the u_k to u is bounded. To prove the claim, we first note that

$$\frac{w_0(x;\gamma)}{w_0(0;\gamma)} \mathrm{e}^{\mathrm{i}\gamma(x-2\pi k)} = \frac{u_0(x-2\pi k;\gamma)}{u_0(0;\gamma)},$$

where $u_0(x;\gamma)$ was defined in (2.4), periodic in γ and analytic in a strip $\gamma \pm \mathrm{i}\eta$. We can therefore shift the integration over $\gamma \in [0,1]$ into the complex plain $\gamma \in [0,1] + \mathrm{i}\eta$,

$$\left|\int_0^1 \frac{w_0(x;\gamma)}{w_0(0;\gamma)} \mathrm{e}^{\mathrm{i}\gamma(x-2\pi k)}\mathrm{d}\gamma\right| = \left|\int_0^1 \frac{w_0(x;\gamma+\mathrm{i}\eta)}{w_0(0;\gamma+\mathrm{i}\eta)} \mathrm{e}^{(\mathrm{i}\gamma-\eta)(x-2\pi k)}\mathrm{d}\gamma\right| \leq C\mathrm{e}^{-\eta(x-2\pi k)},$$

which proves the claim since the sign of η was arbitrary. ∎

Remark 2.6. *In fact, all we used for this result is the fact that $w_0(0;\gamma) > 0$ for all $\gamma \in [0,1]$. One can go further and weaken this assumption by choosing a local average around $2\pi k$ instead of a pointwise evaluation at $2\pi k$ for ι.*

Using the parametrization by the map ι from Lemma 2.5, we find that

$$u_k^\gamma := \iota(w_0(x;\gamma)\mathrm{e}^{\mathrm{i}\gamma x})_k = \mathrm{e}^{2\pi\mathrm{i}\gamma k}.$$

In the lattice coordinates, we therefore have the linear operator $\mathcal{L}_0$ diagonalized in terms of the u_k^γ, that is

$$\mathcal{L}u_k^\gamma = \lambda(\gamma)u_k^\gamma.$$

If we set $e_j = \delta_{jk}$, the indicator function unit basis vector in the lattice coordinate space ℓ^2, and exploit that

$$e_0 = \int_0^1 \mathrm{e}^{2\pi\mathrm{i}\gamma k}\mathrm{d}\gamma,$$

we find

$$\mathcal{L}e_0 = \int_0^1 \lambda(\gamma)\mathrm{e}^{2\pi\mathrm{i}\gamma k}\mathrm{d}\gamma = \frac{1}{2\pi}(e_1 + e_{-1} - 2e_0) + \mathrm{O}(\varepsilon),$$

which in turn yields the representation

$$(\mathcal{L}v)_j = \frac{1}{2\pi}(v_{j+1} - 2v_j + v_{j-1}) + \varepsilon(R(\varepsilon)v)_j.$$

By the exponential decay estimates on g, the remainder terms are a discrete convolution, translation invariant,

$$(R(\varepsilon)v)_j = \sum_{j'} R_{j-j'}v_{j'}, \quad |R_k| \le C\mathrm{e}^{-c|k|/\varepsilon}.$$

2.5 Spectral theory in L^2_u

In order to include traveling waves, which are not necessarily in H^1 or L^2, we extend our functional analytic setup to uniformly integrable spaces. In order to invert (and thereby define) $\mathcal{L} - \mathrm{id}$ on L^2_u, we first decompose

$$g = \sum_j g_j \in L^2_\mathrm{u},\ g_j(x) = \chi_{[0,2\pi]}(x - 2\pi j)g(x),\ \text{so that } |g|_{L^2_\mathrm{u}} = \sup_j |g_j|_{L^2}.$$

Next, consider the exponentially weighted space L^2_η, with norm

$$|u|_{L^2_{\eta,j}} = |\cosh(\eta(\cdot - 2\pi j))u(\cdot)|_{L^2}.$$

Note that $|g_j|_{L^2_{\eta,j}} \le C|g|_{L^2_\mathrm{u}}$, where $C = 1 + \mathrm{O}(\eta)$, independent of j.
Next, we claim that we can invert $\mathcal{L} - \mathrm{id}$ on $L^2_{\eta,j}$ with uniform bounds in j and $\eta \sim 0$, small. We can define the operator $\mathcal{L}$ on $L^2_{\eta,j}$ via the associated quadratic form. Using the isomorphism $\psi_{\eta,j} : L^2_{\eta,j} \to L^2$, $u \mapsto \cosh(\eta(\cdot - 2\pi j))u$, we find that $\mathcal{L}$ on L^2_η is conjugate to

$$\psi^{-1}_{\eta,j}\mathcal{L}\psi_{\eta,j} = \mathcal{L} + \eta^2 a + \eta\tanh(\eta(\cdot - 2\pi j))((au)_x + au_x).$$

In particular, $(\psi^{-1}_{\eta,j}\mathcal{L}\psi_{\eta,j} - \mathcal{L})(\mathrm{id} - \mathcal{L})^{-1}$ is bounded and smooth in η. The resolvent identity then yields smooth dependence of the resolvent of $\mathcal{L}$ on η (when pulled back to L^2 with the isomorphism ψ). Moreover, the pointwise Greens function which defines the resolvent of $\mathcal{L}$ in L^2 also yields the resolvent in $L^2_{\eta,j}$, since the two resolvents coincide on $L^2 \subset L^2_{\eta,j}$. Because of the discrete shift symmetry, the resolvents are bounded uniformly in $j \in \mathbb{Z}$. Using the resolvent in $L^2_{\eta,j}$, we find bounds on the solution $u_j = (\mathrm{id} - \mathcal{L})^{-1}g_j$ of the form

$$|\cosh(\eta(\cdot - 2\pi j))u_j(\cdot)|_{L^2} \le C|g_j|_{L^2},$$

where C does not depend on j. We next define

$$u = (\mathrm{id} - \mathcal{L})^{-1}g := \sum_j u_j = \sum_j(\mathrm{id} - \mathcal{L})^{-1}g_j.$$

We find that by definition,

$$|u|_{L^2_{\rm u}} \le \sup |\chi_{[0,2\pi]}(\cdot - 2\pi\ell)u_j(\cdot)|_{L^2},$$

and

$$|\chi_{[0,2\pi]}(\cdot - 2\pi\ell)u_j(\cdot)|_{L^2} \le \sum_j \cosh^{-1}(2\pi\eta(j-\ell))C|g_j|_{L^2} \le \frac{C'}{\eta}|g|_{L^2_{\rm u}}.$$

Similar estimates hold for derivatives. As a consequence, the resolvent exists and is bounded in $L^2_{\rm u}$ whenever it exists and is bounded in L^2. Both resolvents coincide on $L^2 \subset L^2_{\rm u}$, so that the representation of the resolvent on $L^2_{\rm u}$ is given by the same Greens function. From the construction, the domain of definition is given by

$$\mathcal{D}(\mathcal{L}) = \{u \in L^2_{\rm u} \mid a(x)u_x \in H^1_{\rm u}\}$$

where $H^1_{\rm u}$ is defined as in (1.3). Interpolation theory then implies that the interpolation space $X_{\rm u}^{1/2}$ is given through

$$\mathcal{D}(\mathcal{L}) = \{u \in L^2_{\rm u} \mid a(x)^{1/2}u_x \in L^2_{\rm u}\},$$

with associated norm.
As a consequence of the resolvent characterization and the representation of spectra via bounded eigenfunctions, the spectra coincide in the spaces L^2 and $L^2_{\rm u}$. Using the representation of the spectral projections via Dunford's integral, we find that spectral projections on both spaces are given by the same kernel $G^{\rm c}(x,y)$.
Using the same technique, one can also extend the lattice parametrization to a bounded invertible map

$$\iota : u \in E^{\rm c} \subset L^2_{\rm u} \mapsto (u_k)_{k\in\mathbb{Z}} \in \ell^\infty.$$

2.6 Dichotomies

The spectral results immediately imply exponential decay estimates on the heat flow since the operators are sectorial in the spaces we are considering. We are interested in the behavior when ε tends to zero and the spectral gap is large.

Lemma 2.7. *Assume that $[\alpha_-, \alpha_+]$ is contained in the first gapand denote by $E^{\rm c}$ and $E^{\rm ss}$ the center eigenspace and its spectral complement. Then the heat equation solution operator possesses an exponential dichotomy: there exists a constant C such that*

$$\begin{aligned}
\left|{\rm e}^{\mathcal{L}t}|_{E^{\rm c}}\right|_{Y\to Y^{1/2}} &\le Ct^{-1/2}{\rm e}^{\alpha_+ t}, \qquad t \le 0\\
\left|{\rm e}^{\mathcal{L}t}|_{E^{\rm ss}}\right|_{Y\to Y^{1/2}} &\le Ct^{-1/2}{\rm e}^{\alpha_- t}, \qquad t \ge 0,
\end{aligned}$$

with $Y = X$ or $Y = X_{\rm u}$. In particular, C is uniformly bounded in ε for a choice $\alpha_- = -1/\delta$ and $\alpha_+ = -3/\pi$, and $\delta \to 0$ as $\varepsilon \to 0$.

Proof. The estimates are an immediate consequence of the spectral resolution in the self-adjoint case $X = L^2$. In case of $L^2_{\rm u}$, the estimates follow from the uniform resolvent estimates for the sectorial operators. ∎

3 Nonlinear theory — invariant manifolds

3.1 Invariant manifolds

We find invariant manifolds using the standard variation-of-constant formula for the unstable manifold

$$u(t) = \mathrm{e}^{\mathcal{L}t}u_0^{\mathrm{c}} + \int_0^t \mathrm{e}^{\mathcal{L}(t-\tau)}P^{\mathrm{c}}f(u(\tau))\mathrm{d}\tau + \int_{-\infty}^t \mathrm{e}^{\mathcal{L}(t-\tau)}P^{\mathrm{ss}}f(u(\tau))\mathrm{d}\tau. \qquad (3.1)$$

We solve this equation in the case of uniform spaces X_{u}, the L^2-case is similar.
We claim that the right-hand side defines a mapping on the space of functions

$$Z_\eta = \{u \in C^0(\mathbb{R}^-, X_{\mathrm{u}}^{1/2}) \mid |u|_\eta < \infty\}, \quad \|u\|_\eta = \sup_t \mathrm{e}^{\eta t}|u(t)|_{X_{\mathrm{u}}^{1/2}}, \quad \eta \in (\alpha_-, \alpha_+),$$

where $\alpha_\pm$ are chosen as in Lemma 2.7. Indeed, we can view f as a smooth, globally Lipschitz-continuous $\tilde{f}$ mapping from $X_{\mathrm{u}}^{1/2}$ into C^0; see Lemma 2.4. With the embedding $C^0 \to L^2_{\mathrm{u}}$, and using the estimates on the semigroups as maps from X_{u} into $X_{\mathrm{u}}^{1/2}$, we see that the right-hand side defines a globally Lipschitz continuous map on Z_η. The Lipschitz constant depends only on the Lipschitz constant of f, norms of embeddings, and the norms of the integral convolution operators. For the latter ones, these are estimated by the L^1 norms in τ of the integral kernels $\mathrm{e}^{\mathcal{L}\tau}P^{\mathrm{ss}}$ and $\mathrm{e}^{-\mathcal{L}\tau}P^{\mathrm{c}}$,

$$\int_0^\infty \left(\left|\mathrm{e}^{(\mathcal{L}+\eta)\tau}P^{\mathrm{ss}}\right|_{X_{\mathrm{u}}\to X_{\mathrm{u}}^{1/2}} + \left|\mathrm{e}^{-(\mathcal{L}+\eta)\tau}P^{\mathrm{c}}\right|_{X_{\mathrm{u}}\to X_{\mathrm{u}}^{1/2}} \right) \mathrm{d}\tau \le \tilde{C}\delta_0^{1/2}.$$

We choose $\eta = (\alpha_+ - \alpha_-)/2$, in the center of the spectral gap. The fixed point $u_*(t)$ is Lipschitz continuous in the parameter u_0^{c}. A fiber contraction argument [12, 10] shows that the dependence is actually $C^{k'}$ provided the spectral gap is large enough. The projection at time $t = 0$ yields the graph Ψ of the invariant manifold, $\Psi(u_0^{\mathrm{c}}) := P^{\mathrm{ss}}u_*(0; u_0^{\mathrm{c}})$. From the above discussion and the estimates of Section 2.2, we immediately conclude the following corollary.

Corollary 3.1. *In the explicit example, with $a(x)$ given by (1.5), we have the estimate on the representation of the invariant manifold as a graph,*

$$|\Psi|_{BC^1(E^{\mathrm{c}},E^{\mathrm{ss}})} \le C\varepsilon.$$

3.2 The reduced flow

We focus on our specific example of $a(x)$ given by (1.5) and derive the expansion of the flow on the invariant manifold in terms of ε.
Our goal in this section is to compute the reduced vector field. On the slow manifold, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}u^{\mathrm{c}} = \mathcal{L}u^{\mathrm{c}} + P^{\mathrm{c}}f(u^{\mathrm{c}} + \Psi(u^{\mathrm{c}})). \qquad (3.2)$$

Since $\Psi = \mathrm{O}(\varepsilon)$, we find to leading order the nonlinearity evaluated on the center eigenspace and then projected back onto the center eigenspace. We parametrize the center eigenspace by $u_j = u(2\pi j)$. The function $u(x)$ is then given by $u(x) = \sum g(x;k)u_k \in E^{\mathrm{c}}$, which is to leading order given by linear interpolation on intervals of length ε between step functions on intervals $[2\pi j, 2\pi(j+1)-\varepsilon]$. We next substitute into the nonlinearity and project. Now $f_{\text{eff}} = P^{\text{c}} f(u(x))$ is given by

$$f_j = f_{\text{eff}}(u(2\pi j)) = \int_y G^{\text{c}}(2\pi j; y) f(\sum_j g(y;j) u_j).$$

For $\varepsilon \to 0$, $f_j \to f(u_j)$, uniformly on $\bigcup_j [2\pi j + \delta, 2\pi(j+1) - \delta]$. Projection and evaluation shows that

$$f_j = f(u_j) + r(\varepsilon, (u)_{k-j}),$$

where the error terms are bounded, smooth in $u_k \in \ell^\infty$, and exponentially localized, $\partial_{u_m} r = \mathrm{O}(\mathrm{e}^{-\eta|m|})$ for some $\eta > 0$. This concludes the proof of Theorem 1.

Remark 3.2. *Most of our results can be obtained from the assumption of a large spectral gap, only, thus allowing for a much larger class of fluxes and more general inhomogeneities in the linear operator. In fact, the exponential separation and the strong localization of Greens functions and spectral projections are related: resolvents can be continues to exponentially weighted spaces* $L^2_{\eta,j}$ *as long as* $\eta|(\lambda - \mathcal{L})^{-1}|$ *is small. Since the norm of the resolvent is estimated by the distance to the spectrum in the self-adjoint case, we can let* $\eta \to \infty$ *when the gap becomes large,* $\delta_0 \to \infty$.

4 Pinned fronts — an illustration

We examine the existence of standing fronts in both the lattice differential equation and the reaction-diffusion system. For the reaction-diffusion system, steady-states solve

$$u' = \frac{1}{a(x)} v, \qquad v_x = -f(u),$$

with $a(x) = \varepsilon$ on $x \bmod 2\pi \in (2\pi - \varepsilon, 2\pi)$, and $a = 1/\varepsilon$ otherwise.
We want to expand the time-one map of this periodically forced flow, expanded in ε. On $0 \le x \le (2\pi - \varepsilon)$, we find

$$u_x = \varepsilon v, \qquad v_x = -f(u),$$

on $2\pi - \varepsilon \le x \le 2\pi$, we have

$$u_x = \frac{1}{\varepsilon} v, \qquad v_x = -f(u),$$

or, equivalently, on $0 \le y \le 1$,

$$u_y = v, \qquad v_x = -\varepsilon f(u),$$

We write u_0, v_0 for the values at $x = 0$, u_1, v_1 for the values at $x = 2\pi - \varepsilon$, and u_2, v_2 for the values at $x = 2\pi$. We immediately find

$$u_1 = u_0 + \mathrm{O}(\varepsilon),\ v_1 = v_0 - 2\pi f(u_0) + \mathrm{O}(\varepsilon), \qquad u_2 = u_1 + v_1 + \mathrm{O}(\varepsilon),\ v_2 = v_1 + \mathrm{O}(\varepsilon),$$

so that

$$u_2 = u_0 + v_0 - 2\pi f(u_0) + \mathrm{O}(\varepsilon), \qquad v_2 = v_0 - 2\pi f(u_0) + \mathrm{O}(\varepsilon).$$

Changing variables, $w = u - v$, we find

$$w_2 = u_0 + \mathrm{O}(\varepsilon), \qquad u_2 = 2u_0 - w_0 - 2\pi f(u_0) + \mathrm{O}(\varepsilon),$$

so that

$$w_4 - 2w_2 + w_0 + 2\pi f(w_2) = \mathrm{O}(\varepsilon).$$

In particular, at leading order after relabeling indices, we find the steady-state problem to the lattice differential equation

$$\dot{w} = \frac{1}{2\pi}\left(w_{n+1} - 2w_n + w_{n-1}\right) + f(w_n).$$

Pinned traveling waves are transverse heteroclinic orbits for this discrete-time dynamical system. They are hence robust under perturbations of size $\mathrm{O}(\varepsilon)$. Unpinning, and hence the transition towards propagating fronts occurs for a parameter value b_{up} in $f(u) = u(1-u)(u-b)$, where the intersection of stable and unstable manifolds of 0 and 1, respectively, is not transverse, with a quadratic tangency. This tangency unfolds in the parameter $b \sim b_{\mathrm{up}}$. Again, the unfolding of the tangency is robust as a singularity, so that we recover the same dynamics in the reaction-diffusion system with small diffusivities, for unpinning parameter values $b = b_{\mathrm{up}} + \mathrm{O}(\varepsilon)$.

We note that this result can also be inferred from our main theorem, where the unpinning corresponds to a saddle-node of hyperbolic equilibria in the lattice differential equation, which persists on the slow manifold of the reaction-diffusion system.

The unpinning as the transition towards traveling waves has been studied recently in lattice differential equations; see the review [8] and the references therein. Our main theorem lifts these results to a specific class of reaction-diffusion systems with periodic fluxes. We refer to [14, 13] for results on propagating fronts in reaction-diffusion systems.

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