Instability of Radially-Symmetric Spikes in Systems with a Conserved Quantity

Alin Pogan and Arnd Scheel
University of Minnesota
School of Mathematics
206 Church St. S.E.
Minneapolis, MN 55455, USA

Dedicated to George Sell on the occasion of his 70th birthday.

Abstract

We show that radially symmetric spikes are unstable in a class of reaction-diffusion equations coupled to a conservation law. [1]

1 Introduction

We consider a class of spatially extended systems that are governed by a scalar reaction-diffusion equation, coupled to a conservation law,

$$\begin{cases} u_t = \nabla \cdot [a(u,v)\nabla u + b(u,v)\nabla v], \\ v_t = \Delta v + f(u,v), \end{cases} \quad t \geq 0,\ x \in \mathbb{R}^k. \tag{1.1}$$

Here, the functions a, b, and f are of class $C^3(\mathbb{R}^2, \mathbb{R})$. In order to ensure well-posedness on appropriate function spaces, we also assume that $a(u,v) \geq a_0 > 0$ for all $(u,v) \in \mathbb{R}^2$. Equations of the type (1.1) arise in many physical, biological, and chemical applications. We mention the Keller-Segel model for chemotaxis [6], phase-field models for undercooled liquids [1], and chemical reactions in closed reactors, with stoichiometric conservation laws for chemical species. We refer to [13] for a somewhat more extensive review of the literature and specific examples. Many of those systems are known to exhibit patterns in large or unbounded domains. The simplest examples in one space-dimension, $k = 1$, are layers (or interfaces), and possible bound states formed between pairs of such layers. The simplest higher-dimensional patterns are radially symmetric, localized and time-independent patterns $(u,v)(t,x) = (u^*, v^*)(|x|)$. We refer to such solutions as spikes.

In previous work [13], we showed that such localized patterns are always unstable for the dynamics of (1.1), under quite mild, generic assumptions, in one space-dimension. Our goal here is to extend such instability results to higher space-dimensions. In this introduction, we first briefly characterize the types of radially symmetric solutions that we are interested in, and

[1] **AMS MSC 2000 Mathematics Subject Classification:** 37L15
Keywords: stability, essential spectrum, spikes, conservation law

then state our main results, which provides an instability statement for a large class of spike solutions. Our emphasis is on phenomenological assumptions, related to the spike solution, rather than assumptions on specific shapes and monotonicity properties of a, b, and f.

The following two assumptions characterize spikes as exponentially localized with a stable background.

(rs1) We assume that spikes are nonconstant, exponentially localized, that is

$$|(u^* - u^\infty, v^* - v^\infty)(x)| \le C\mathrm{e}^{-\eta|x|}, \quad \text{for all} \quad x \in \mathbb{R}^k, \quad (u^*, v^*) \not\equiv (u^\infty, v^\infty),$$

for some constants $u^\infty, v^\infty \in \mathbb{R}$, and $C, \eta > 0$.

(rs2) Spikes are asymptotic to constant states that are stable for the pure kinetics,

$$u' = 0 \qquad v' = f(u, v),$$

that is, we assume $f_v(u^\infty, v^\infty) < 0$.

We will now outline how to construct spikes that satisfy (rs1)–(rs2); for a more detailed discussion, see Section 2. One readily checks that radially symmetric spikes satisfy the system

$$\begin{cases} \left[r^{k-1}\left(a(u,v)u_r + b(u,v)v_r\right)\right]_r = 0, \\ v_{rr} + \frac{k-1}{r}v_r + f(u,v) = 0. \end{cases} \tag{1.2}$$

where $r := |x|$ is the radial variable. The first equation in the above system can be integrated and viewed as a differential equation for u in terms of v,

$$\frac{\mathrm{d}u}{\mathrm{d}v} = -\frac{b(u,v)}{a(u,v)}. \tag{1.3}$$

Solving this differential equation with appropriate initial conditions, one obtains a solution $u^* = \Phi(v^*)$. Substituting this solution into the second equation of the system (1.2) we obtain the equation for v^*,

$$v_{rr} + \frac{k-1}{r}v_r + f(\Phi(v), v) = 0. \tag{1.4}$$

This equation is nothing else than the equation for radially symmetric solutions to

$$\Delta v + f(\Phi(v), v) = 0. \tag{1.5}$$

Radially symmetric solutions to such a stationary nonlinear Schrödinger equation have been studied extensively in the literature, using a variety of techniques, for instance shooting or variational methods.

Different from the one-dimensional case, even positive solutions of (1.4) that decay to zero need not be unique and may bifurcate as problem parameters are varied. We will focus here on the simplest case, where the linearization of (1.4) is invertible and possesses an odd Morse index. More precisely, consider the linearization $\mathcal{K}$ of (1.5) at v^* as a self-adjoint operator on the closed subspace of $L^2(\mathbb{R}^k)$ consisting of radially symmetric functions. We will assume the following two conditions.

(rs3) The kernel of $\mathcal{K}$ in $L^2_{\rm rad}(\mathbb{R}^k)$ is trivial.

(rs4) The operator $\mathcal{K}$ on $L^2_{\rm rad}(\mathbb{R}^k)$ has an odd number of positive eigenvalues.

Condition (rs3) is typical in the sense that it is violated only for exceptional values of parameters; see for instance [2, §5]. It is also known to hold for several specific examples, see [5, 8, 12].

We note that in a typical situation the steady-states have Morse index one, they are in fact Mountain-Pass type extrema for the functional associated with (1.5). In this case, (rs4) is satisfied and $\mathcal{K}$ has exactly one positive eigenvalue.

We are now ready to state our main result.

Theorem 1.1. *Suppose (1.1) possesses an exponentially localized, radially symmetric spike solution (u^*, v^*) satisfying (rs1)–(rs4). Then (u^*, v^*) is unstable as an equilibrium to* (1.1)*, considered as an evolution equation on the space of bounded uniformly continuous functions $BUC(\mathbb{R}, \mathbb{C}^2)$ or on $BUC_{\rm rad}(\mathbb{R}, \mathbb{C}^2)$.*

The formal linearization of (1.1) along the spike (u^*, v^*) is the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{L}\begin{pmatrix} u \\ v \end{pmatrix}, \tag{1.6}$$

where

$$\mathcal{L} = \begin{bmatrix} \nabla\cdot(a^*\nabla + a^*_u\nabla u^* + b^*_u\nabla v^*) & \nabla\cdot(b^*\nabla + a^*_v\nabla u^* + b^*_v\nabla v^*) \\ f^*_u & \Delta + f^*_v \end{bmatrix}. \tag{1.7}$$

Here " $*$ " next to any of the functions a, b, f and their partial derivatives represents the composition of the respective function with the spike (u^*, v^*).

We can view the differential expression $\mathcal{L}$ as a densely defined, closed operator on various function spaces such as $BUC(\mathbb{R}^k, \mathbb{C}^2)$, $L^2(\mathbb{R}^k, \mathbb{C}^2)$ or the exponentially weighted $L^2_\eta(\mathbb{R}^k, \mathbb{C}^2)$ spaces that we define below. One can check that on these spaces $\mathcal{L}$ generates an analytic semigroup, see for instance [7].

We will show in this paper that the spectrum of $\mathcal{L}$ intersects $\operatorname{Re}\lambda > 0$. This readily implies that the spectral radius of the semigroup generated by $\mathcal{L}$ is larger than 1, and, using a result of Henry [3, Thm 5.1.5], that the spike is actually unstable for the nonlinear evolution. We refer to [13, §3] for more details on how spectral instability implies nonlinear instability in this context.

The remainder of the paper is organized as follows. We show that spikes naturally come in families parametrized by the asymptotic state, Section 2. We then recall some results on the essential spectrum of $\mathcal{L}$ in Section 3. The heart of our analysis is contained in Section 4, where we show that there exists at least one real unstable eigenvalue for $\mathcal{L}$ whenever the essential spectrum is stable. The main idea is similar to the construction in [13]. We perform a homotopy to a system where the linearization is known to exhibit an odd number of unstable eigenvalues and show that, during the homotopy, eigenvalues may not cross the origin. The crucial difficulty is the presence of essential spectrum at the origin, which makes it difficult to control multiplicities of eigenvalues. We therefore monitor eigenvalues near the origin by using a carefully crafted Lyapunov-Schmidt type reduction procedure, mimicking the extension of Evans functions at the essential spectrum. The procedure here is somewhat more subtle than in [13] since one would not expect Evans functions to be analytic in the presence of terms with decay $1/r$, generated by the Laplacian in higher space-dimension [15].

Notations: We collect some notation that we will use throughout this paper. For an operator T on a Hilbert space X we use T^*, $\mathrm{dom}(T)$, $\ker T$, $\mathrm{im}\, T$, $\sigma(T)$, $\rho(T)$ and $T_{|Y}$ to denote the adjoint, domain, kernel, range, spectrum, resolvent set and the restriction of T on a subspace Y of X. If $g : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function, e.g., a, b, f or one of its partial derivatives, we write $g^* := g(u^*, v^*)$ and $g^\infty := g(u^\infty, v^\infty)$. If u is a radially symmetric function, we write $u(r) = u(x)$ if $|x| = r$, slightly abusing the notation. We denote by $L^2_{\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^m)$ the space of radially symmetric functions that belong to $L^2(\mathbb{R}^k, \mathbb{C}^m)$. For $\eta \in \mathbb{R}$, $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^m)$ denotes the weighted space of vector-valued functions defined via the weighted L^2-norm

$$\|\underline{w}\|_\eta^2 = \int_{\mathbb{R}^k} |\underline{w}(x) \mathrm{e}^{\eta|x|}|^2 \mathrm{d}x.$$

Acknowledgment. The authors gratefully acknowledges support by the National Science Foundation under grant NSF-DMS-0806614.

2 Families of spikes

In this section we show that the existence of a spike satisfying conditions (rs1)–(rs4) implies the existence of a family of radially symmetric spikes. Recall that a radially symmetric spike (u^*, v^*) satisfies the system (1.2)

$$\begin{cases} \left[r^{k-1}\left(a(u,v)u_r + b(u,v)v_r\right)\right]_r = 0, \\ v_{rr} + \frac{k-1}{r} v_r + f(u,v) = 0. \end{cases}$$

Integrating the first equation we obtain that the spike (u^*, v^*) satisfies the equation $a(u,v)u_r + b(u,v)v_r = mr^{1-k}$ for some $m \in \mathbb{R}$. Since u^* and v^* converge exponentially as $r \to \infty$, we obtain that $v^*_r \to 0$ exponentially as $r \to \infty$ by standard ODE arguments. Therefore, u^* satisfies an equation of the form $u_r = mr^{1-k} + \mathcal{O}(e^{-\varepsilon r})$ for some $\varepsilon > 0$. Since, by (rs1), u^* converges exponentially as $r \to \infty$, we conclude that $m = 0$, which implies that a spike necessarily satisfies the ODE (1.3),

$$\frac{\mathrm{d}u}{\mathrm{d}v} = -\frac{b(u,v)}{a(u,v)}, \quad u(v_0) = u_0.$$

Since a spike is a bounded solution, we may assume without loss of generality that b is bounded. Moreover, a is bounded away from zero, so that this ODE possesses a global, smooth solution. We denote the solution to initial conditions u_0 at $v = v_0$ by

$$u(v) = \Phi(v, v_0; u_0), \qquad \Phi(v_0, v_0; u_0) = u_0. \tag{2.1}$$

For our particular spike (u^*, v^*), we note that $u^* = \Phi(v^*, v^\infty; u^\infty) =: \varphi_0(v^*)$. Hence, v^* satisfies the equation

$$v_{rr} + \frac{k-1}{r} v_r + H(v) = 0, \qquad H(v) := f(\varphi_0(v), v). \tag{2.2}$$

We recall that according to conditions (rs3)-(rs4), $\mathcal{K} = \partial_r^2 + \frac{k-1}{r}\partial_r + H'(v^*)$ has trivial kernel and the number of its positive eigenvalues is odd. Condition (rs1) asserts that $v^*(r) \to v^\infty$ exponentially for $r \to \infty$, which implies that $H'(v^\infty) < 0$. We will refer to this condition later on as

ODE-Hyperbolicity:

$$H'(v^\infty) = f_v^\infty - \frac{b^\infty}{a^\infty} f_u^\infty < 0. \tag{2.3}$$

In the next lemma, we prove the existence of a smooth family of spikes.

Lemma 2.1. *Under the assumptions of Theorem 1.1, there is $\varepsilon > 0$ and a family of spikes $(u^*(\cdot, \mu), v^*(\cdot, \mu))$ for $\mu \in (-\varepsilon, \varepsilon)$, such that*

(i) the asymptotic values $(u^\infty(\mu), v^\infty(\mu))$ are smooth functions and $0 \neq \partial_\mu u^\infty(\mu)$;

(ii) the spikes $(u^*(\cdot, \mu) - u^\infty(\mu), v^*(\cdot, \mu) - v^\infty(\mu))$ are given as smooth maps from $(-\varepsilon, \varepsilon)$ into $H^2_{\mathrm{rad}}(\mathbb{R}^k, \mathbb{R}^2)$; moreover, $(u^*(\cdot, 0), v^*(\cdot, 0)) = (u^*(\cdot), v^*(\cdot))$.

Proof. The proof, in most of its parts, is similar to the proof of [13, Lem. 2.1]. For completeness we give the main arguments. We first construct a family of asymptotic states that satisfy (i) by solving $f(u, v) = 0$ locally near (u^∞, v^∞) with the implicit function theorem, using (rs2), and denote the solution by $(u^\infty(\mu), v^\infty(\mu))$, $\mu \in (-\varepsilon, \varepsilon)$, with

$$\partial_\mu(u^\infty(0), v^\infty(0)) = (-f_v^\infty, f_u^\infty). \tag{2.4}$$

We now define

$$\Phi^\mu : \mathbb{R} \to \mathbb{R}, \qquad \Phi^\mu(v) = \Phi(v, v^\infty(\mu); u^\infty(\mu))$$

and

$$\tilde{H} : \mathbb{R} \times (-\varepsilon, \varepsilon) \to \mathbb{R}, \qquad \tilde{H}(v, \mu) = f(\Phi^\mu(v), v).$$

Note that $\tilde{H}(v^\infty(\mu), \mu) = 0$. Next, we seek radially symmetric spikes that are solutions to

$$v_{rr} + \frac{k-1}{r} v_r + \tilde{H}(v, \mu) = 0, \tag{2.5}$$

with $u = \Phi^\mu(v)$. Using ODE-Hyperbolicity (2.3) again, we infer $\tilde{H}_v(v^\infty, 0) = H'(v^\infty) < 0$. It follows that $v^\infty(\mu)$ is the locally unique equilibrium to (2.5), and v^∞ is hyperbolic.

To solve equation (2.5) for $\mu \approx 0$ we make the change of variables $w = v - v^\infty(\mu)$, which yields a nonlinear equation

$$G(w, \mu) = w_{rr} + \frac{k-1}{r} w_r + \tilde{H}(w(\cdot) + v^\infty(\mu), \mu) = 0, \tag{2.6}$$

where the nonlinearity vanishes at the origin so that $G : H^2_{\mathrm{rad}}(\mathbb{R}^k) \times (-\varepsilon, \varepsilon) \to L^2_{\mathrm{rad}}(\mathbb{R}^k)$ is a smooth map. Since v^* satisfies (2.2), it follows that $G(v^* - v^\infty, 0) = v^*_{rr} + \frac{k-1}{r} v^*_r + H(v^*) = 0$. The w-derivative of G is given by:

$$\big(G_w(v^* - v^\infty, 0)\big)w = w_{rr} + \frac{k-1}{r} w + \tilde{H}_v(v^*(\cdot), 0)w = \mathcal{K}w.$$

Next, we will show that $\mathcal{K}$ is invertible. Since the kernel of $\mathcal{K}$ is assumed to be trivial, (rs3), it is sufficient to show that $\mathcal{K}$ is Fredholm of index zero. To see this, we first consider $\mathcal{K}$ as an operator from $H^2_{\mathrm{rad}}(\mathbb{R}^k)$ into $L^2_{\mathrm{rad}}(\mathbb{R}^k)$. Since the spike (u^*, v^*) is exponentially localized, $v^*(r) \to v^\infty$, *exponentially* for $r \to \infty$. Thus, the operator $\mathcal{K}$ is a relatively compact (actually, even in the Schatten-von Neumann B_p ideal, for the right choice of p, see [17, Thm. 4.1]) perturbation of $\mathcal{K}_\infty = \Delta + H'(v_\infty)$. By ODE-Hyperbolicity (2.3), we have $H'(v_\infty) < 0$ so that $\mathcal{K}_\infty$ is invertible and $\mathcal{K}$ is Fredholm with index 0.

Using the Implicit Function Theorem, we now find a local smooth solution $w(\cdot, \mu) \in H^2_{\mathrm{rad}}(\mathbb{R}^k)$. One readily concludes that $w(r; \mu) \to 0$ for $r \to \infty$, which gives the asymptotics of the spike solution $v = w + v^\infty$ as claimed. ∎

We note that one can verify that all spikes in the family satisfy the conditions (rs1)–(rs4).

3 Essential spectrum

In this section we compute the essential spectrum and show that it is stable, whenever the asymptotic equilibrium is stable.

First, we define the limiting operator $\mathcal{L}_\infty$ through

$$\mathcal{L}_\infty = \begin{bmatrix} a^\infty \Delta & b^\infty \Delta \\ f^\infty_u & \Delta + f^\infty_v \end{bmatrix}. \tag{3.1}$$

Just like the operator $\mathcal{L}$, we will consider the operator $\mathcal{L}_\infty$ on various function spaces, or merely as a differential expression, slightly abusing notation. We recall the definition of essential spectrum that we use in this paper. For a given choice of function space, we say λ is in the essential spectrum $\sigma_{\mathrm{ess}}(\mathcal{L})$ if $\mathcal{L} - \lambda$ is not Fredholm index zero. We refer to the complement of the essential spectrum in the spectrum as the point spectrum $\sigma_{\mathrm{point}}(\mathcal{L})$.

One can compute the essential spectrum using arguments similar to [3, §5, Appendix]. In the next proposition we collect some results on the essential spectrum. For the proof we refer to [13, Sec. 4].

Proposition 3.1. *Under the assumptions of Theorem 1.1, the following hold true:*

(i) The essential spectra of the operators $\mathcal{L}$ and $\mathcal{L}_\infty$ coincide, and are equal for the choices of function space $X = L^2(\mathbb{R}^k, \mathbb{C}^2)$ and $X = BUC(\mathbb{R}^k, \mathbb{C}^2)$.

(ii) The essential spectrum of $\mathcal{L}$ is given by

$$\sigma_{\mathrm{ess}}(\mathcal{L}) = \Big\{ \lambda_\pm(\xi) : \xi \in \mathbb{R}^k \Big\}, \quad \textit{where} \quad \lambda_\pm(\xi) = \frac{\mathrm{tr}(\xi) \pm \sqrt{\mathrm{tr}(\xi)^2 - 4\mathrm{det}(\xi)}}{2},$$

with

$$\operatorname{tr}(\xi) = -(a^\infty + 1)|\xi|^2 + f_v^\infty, \qquad \det(\xi) = a^\infty|\xi|^4 + (f_u^\infty b^\infty - f_v^\infty a^\infty)|\xi|^2.$$

(iii) The essential spectral radius of $\mathrm{e}^{\mathcal{L}}$ *is larger than 1 if* $f_v^\infty > 0$.

(iv) The essential spectral radius of $\mathrm{e}^{\mathcal{L}}$ *is 1 if* $f_v^\infty < 0$.

On the radially symmetric subspace, the differential expression for $\mathcal{L}$ is

$$\mathcal{L}_{\text{rad}} = \begin{bmatrix} \frac{1}{r^{k-1}}\partial_r\left[r^{k-1}\left(a^*\partial_r + l_1\right)\right] & \frac{1}{r^{k-1}}\partial_r\left[r^{k-1}\left(b^*\partial_r + l_2\right)\right] \\ f_u^*(r) & \frac{1}{r^{k-1}}\partial_r(r^{k-1}\partial_r) + f_v^*(r) \end{bmatrix}, \tag{3.2}$$

where

$$l_1 = a_u^* u_r^* + b_u^* v_r^*, \qquad \text{and } l_2 = a_v^* u_r^* + b_v^* v_r^*. \tag{3.3}$$

Lemma 3.2. *Under the assumptions of Theorem 1.1, the essential spectrum of* $\mathcal{L}$ *and* $\mathcal{L}_{\text{rad}}$ *coincide.*

Proof. Using the same compact perturbation argument given in the proof of Proposition 3.1(i), see also [13, Prop. 4.1], one can show that the essential spectrum of $\mathcal{L}_{\text{rad}}$ and of $\mathcal{L}_{\text{rad}}^\infty$, the restriction of $\mathcal{L}_\infty$ to the set of radially symmetric functions, coincide. From Proposition 3.1(i) it follows that to finish the proof of lemma it is enough to show that $\sigma_{\text{ess}}(\mathcal{L}_{\text{rad}}^\infty) = \sigma_{\text{ess}}(\mathcal{L}_\infty)$.

Simply restricting Fredholm properties to the closed radially symmetric subspace, we find that $\sigma_{\text{ess}}(\mathcal{L}_{\text{rad}}^\infty) \subseteq \sigma_{\text{ess}}(\mathcal{L}_\infty)$. Let $\xi \in \mathbb{R}^k$ and $\lambda \in \{\lambda_-(\xi), \lambda_+(\xi)\}$. From the definition of $\lambda_\pm$ we infer that there is vector $z \neq 0$ such that

$$\left(-D_\infty|\xi|^2 + N_\infty - \lambda\right) z = 0,$$

where

$$D_\infty = \begin{bmatrix} a^\infty & b^\infty \\ 0 & 1 \end{bmatrix}, \; N_\infty = \begin{bmatrix} 0 & 0 \\ f_u^\infty & f_v^\infty \end{bmatrix}. \tag{3.4}$$

Let $Z = L^2_{\text{rad}}(\mathbb{R}^k) \otimes z = \{f \otimes z : f \in L^2_{\text{rad}}(\mathbb{R}^k)\}$ and let $\mathcal{L}_Z^\infty$ be the restriction of $\mathcal{L}_{\text{rad}}^\infty$ to $Z \cap H^2_{\text{rad}}(\mathbb{R}^k)$. One can easily check that

$$D_\infty^{-1}(\mathcal{L}_Z^\infty - \lambda)(f \otimes z) = \Delta_{\text{rad}} f \otimes z + D_\infty^{-1}(N_\infty - \lambda) f \otimes z = (\Delta_{\text{rad}} + |\xi|^2) f \otimes z.$$

Since $\sigma_{\text{ess}}(\Delta_{\text{rad}}) = (-\infty, 0]$ (see for instance [4, Thm.2]), we have that $\Delta_{\text{rad}} + |\xi|^2$ is not Fredholm, which implies that $D_\infty^{-1}(\mathcal{L}_Z^\infty - \lambda)$ is not Fredholm, as an operator from $Z \cap H^2_{\text{rad}}(\mathbb{R}^k)$ to Z. Thus, $\mathcal{L}_{\text{rad}}^\infty - \lambda$ is not Fredholm. Hence, $\sigma_{\text{ess}}(\mathcal{L}_{\text{rad}}^\infty) = \sigma_{\text{ess}}(\mathcal{L}_\infty)$, proving the lemma. ∎

4 Tracing the point spectrum

This section presents the core of our arguments which yield the existence of an unstable eigenvalue provided that the essential spectrum is stable. We first construct a homotopy of our equation to a simpler equation, Section 4.1 and show that spikes are unstable at the end of the homotopy, Section 4.2. We discuss the kernel of the linearization, Fredholm properties in weighted spaces, and far-field asymptotics of eigenfunctions in Sections 4.3–4.5. The crucial step is carried out in Section 4.6, where we control for eigenvalues in a neighborhood of $\lambda = 0$. The discussion in Sections 4.3–4.6 is valid during the entire homotopy and will allow us to prove our main result in Section 4.7.

4.1 Homotopy

In this section we make use of the homotopy constructed in [13, Sec. 5.1] in order to relate (1.1) to a "simpler" system. For this simpler system, we can easily compute the point spectrum. The homotopy is constructed so that it does not modify the structure of the spikes and does not change the stability of $\sigma_{\text{ess}}(\mathcal{L})$. To be precise, we introduce the homotopy parameter $\tau \in [0,1]$ and consider the system

$$\begin{cases} u_t = \nabla \cdot \left[a_\tau(u,v)\nabla u + b_\tau(u,v)\nabla v\right], \\ v_t = \Delta v + \tilde{f}(u,v,\tau), \end{cases} \quad t \geq 0,\ x \in \mathbb{R}^k, \tag{4.1}$$

The functions $a_\tau, b_\tau : \mathbb{R}^2 \to \mathbb{R}$ and $\tilde{f} : \mathbb{R}^2 \times [0,1] \to \mathbb{R}$ are defined by

$$\begin{aligned} a_\tau(u,v) &= (1-\tau)a(u,v) + \tau, \quad b_\tau(u,v) = (1-\tau)b(u,v), \\ \tilde{f}(u,v,\tau) &= f(u,v) - f(\varphi_\tau(v),v) + f(\varphi_0(v),v), \end{aligned}$$

where φ_τ is the solution of the Cauchy problem

$$\frac{du}{dv} = -\frac{b_\tau(u,v)}{a_\tau(u,v)}, \quad u(v^\infty) = u^\infty. \tag{4.2}$$

We collect some aspects of this homotopy in the following remark.

Remark 4.1. The homotopy satisfies the following properties.

(i) The homotopy originates at our equation (1.1), $\tilde{f}(u,v,0) = f(u,v)$;

(ii) $H_\tau(v) := \tilde{f}(\varphi_\tau(v),v,\tau) = H(v)$;

(iii) If we define $u_\tau^* := \varphi_\tau(v^*)$ and $v_\tau^* := v^*$ then (u_τ^*, v_τ^*) is a spike for (4.1) satisfying conditions (rs1)–(rs4);

(iv) The background states for the system (4.1), $\lim_{|x|\to\pm\infty} u_\tau^*(x) = u^\infty$ and $\lim_{|x|\to\pm\infty} v_\tau^*(x) = v^\infty$, do not depend on τ.

The linearization of (4.1) along the spike (u_τ^*, v_τ^*) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{L}_\tau \begin{pmatrix} u \\ v \end{pmatrix}, \tag{4.3}$$

where $\mathcal{L}_\tau$ is defined by replacing a, b and f in the definition (1.7) of $\mathcal{L}$ by a_τ, b_τ and $\tilde{f}(\cdot,\cdot,\tau)$, respectively.

Lemma 4.2. *If $f_v^\infty < 0$ the essential spectrum of $\mathcal{L}_\tau$ is stable.*

Proof. From Proposition 3.1(iv) it follows that it suffices to show that $\tilde{f}_v^\infty < 0$. The latter was proved in [13, Lem. 5.2]. ∎

4.2 Instability at $\tau = 1$

At the end of the homotopy, the system possesses a lower triangular structure and we can readily infer instability.

Lemma 4.3. *If $f_v^\infty < 0$, then the spikes in the system (4.1) with $\tau = 1$ are unstable. Moreover, $\mathcal{L}_1$ has an odd number of positive eigenvalues.*

Proof. We note that if $\tau = 1$ the operator $\mathcal{L}_1$ has lower triangular block structure

$$\mathcal{L}_1 = \begin{bmatrix} \Delta & 0 \\ f_u^* & \mathcal{R}_1 \end{bmatrix},$$

where $\mathcal{R}_1 = \Delta + H_1'(v^*) = \Delta + H'(v^*)$, using Remark 4.1(ii) . Therefore, the spectrum of $\mathcal{L}_1$ is the union of the spectra of Δ and $\mathcal{R}_1$. We note that the restriction of $\mathcal{R}_1$ to the space of radially symmetric functions is $\mathcal{K}$, which according to (rs4) has an odd number of unstable eigenvalues. Hence, $\mathcal{L}_1$ has an odd number of unstable eigenvalue, proving the lemma. ∎

4.3 The kernel of $\mathcal{L}_{\mathrm{rad}}$

From our assumptions on a, b, f, one can see that the kernel of $\mathcal{L}_{\mathrm{rad}}$ (and of $\mathcal{L}_\tau$ for all τ) consists of smooth functions for all functions spaces in consideration here. In fact, functions in the kernel solve the system of ODEs

$$a^* u_r + (a_u^* u_r^* + b_u^* v_r^*)u + b^* v_r + (a_v^* u_r^* + b_v^* v_r^*)v = 0; \tag{4.4}$$

$$v_{rr} + \frac{k-1}{r} v_r + f_u^* u + f_v^* v = 0. \tag{4.5}$$

To solve the system (4.4)–(4.5), we first solve the first equation for u in terms of v.

Lemma 4.4. *If $u, v \in BUC(\mathbb{R}_+)$ satisfy equation (4.4) then $u = \alpha(\partial_\mu u^*_{|\mu=0} + \frac{b^*}{a^*}\partial_\mu v^*_{|\mu=0}) - \frac{b^*}{a^*}v$, for some constant $\alpha \in \mathbb{C}$. Here, $(u^*(\cdot,\mu), v^*(\cdot,\mu))$ refers to the family of spikes as constructed in Lemma 2.1.*

Proof. The proof is a straightforward adaptation of the argument given in [13, Lem. 5.4] and will be omitted here. ∎

Lemma 4.5. *A pair (u,v) belongs to the kernel of $\mathcal{L}$ in $BUC(\mathbb{R},\mathbb{C}^2)$ if and only if for some $\alpha\in\mathbb{C}$,*

$$u=\alpha\partial_\mu u^*_{|\mu=0}\quad v=\alpha\partial_\mu v^*_{|\mu=0}. \tag{4.6}$$

Proof. From Lemma 4.4 we have that $u=\alpha(\partial_\mu u^*_{|\mu=0}+\frac{b^*}{a^*}\partial_\mu v^*_{|\mu=0})-\frac{b^*}{a^*}v$, for some constant $\alpha\in\mathbb{C}$. Substituting this expression into equation (4.5) we obtain that a function v from the kernel of $\mathcal{L}$ in BUC satisfies the following equation

$$v_{rr}+\frac{k-1}{r}v_r+(f^*_v-f^*_u\frac{b^*}{a^*})v=-\alpha(f^*_u\partial_\mu u^*_{|\mu=0}+\frac{b^*}{a^*}\partial_\mu v^*_{|\mu=0}). \tag{4.7}$$

Once again using the fact that $(u^*(\cdot,\mu),v^*(\cdot,\mu))$ is a family of spikes, that $v^*(\cdot,\mu)$ satisfies the equation

$$v^*_{rr}(r,\mu)+\frac{k-1}{r}v^*_r(r,\mu)+f(u^*(r,\mu),v^*(r,\mu))=0.$$

Differentiating with respect to μ in this equation and setting $\mu=0$, we infer that $\alpha\partial_\mu v^*_{|\mu=0}$ is a particular solution of equation (4.7). Hence, the general solution of equation (4.7) is of the form $v=\alpha\partial_\mu v^*_{|\mu=0}+\tilde v$, where $\tilde v$ is a solution of the equation

$$\tilde v_{rr}+\frac{k-1}{r}\tilde v_r+(f^*_v-f^*_u\frac{b^*}{a^*})\tilde v=0, \tag{4.8}$$

which is equivalent to $\mathcal{K}\tilde v=0$. From (rs3) we obtain that $\tilde v=0$, proving the lemma. ∎

4.4 Fredholm properties of $\mathcal{L}_{\mathrm{rad}}$ on exponentially weighted spaces

We will set up our perturbation problem using exponentially weighted spaces, introduced at the end of Section 1. It turns out that the linearized operator is Fredholm in spaces with small nonzero exponential weight, so that one can attempt to use regular Fredholm perturbation theory for eigenvalues.

Lemma 4.6. *There exists $\eta^*>0$ such that the operator $\mathcal{L}_{\mathrm{rad}}$ is Fredholm with index -1 in $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k,\mathbb{C}^2)$ for all $\eta\in(0,\eta^*)$.*

Proof. The proof is a somewhat non-standard generalization of Palmer's theorem; see [10, 11], and [16] for generalizations and applications to perturbation theory. The details of the proof will be presented elsewhere [14]. ∎

Next, we consider the adjoint of $\mathcal{L}_{\mathrm{rad}}$ with respect to the non-weighted L^2-scalar product, so that $\mathcal{L}^*_{\mathrm{rad}}$ is a closed operator on $L^2_{-\eta}(\mathbb{R},\mathbb{C}^2)$.

Lemma 4.7. *The kernel of $\mathcal{L}^*_{\mathrm{rad}}$ in $L^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k,\mathbb{C}^2)$ is spanned by the constant vector-valued function $(1,0)^{\mathrm{T}}$.*

Proof. From Lemma 4.5 it follows that the kernel of $\mathcal{L}_{\text{rad}}$ on $L^2_{\eta,\text{rad}}(\mathbb{R}^k,\mathbb{C}^2)$ is trivial, hence its cokernel is one-dimensional by Lemma 4.6. We conclude that the kernel of $\mathcal{L}^*_{\text{rad}}$ in $L^2_{-\eta,\text{rad}}(\mathbb{R}^k,\mathbb{C}^2)$ is one-dimensional. A short explicit calculation shows that the vector $(1,0)^{\mathrm{T}}$ belongs to the kernel of $\mathcal{L}^*_{\text{rad}}$, which proves the lemma. ∎

4.5 Asymptotics of eigenfunctions

We recall that $\mathcal{L}^\infty_{\text{rad}}$ is the restriction of $\mathcal{L}_\infty$ to the set of radially symmetric functions. The eigenvalue problem associated with the operator $\mathcal{L}^\infty_{\text{rad}}$, is given by

$$\begin{cases} a^\infty(u_{rr} + \frac{k-1}{r}u_r) + b^\infty(v_{rr} + \frac{k-1}{r}v_r) = \lambda u \\ v_{rr} + \frac{k-1}{r}v_r + f^\infty_u u + f^\infty_v v = \lambda v, \end{cases} \tag{4.9}$$

This system can be rewritten in the form

$$D_\infty\left(\partial_{rr} + \frac{k-1}{r}\partial_r\right)\begin{pmatrix} u \\ v \end{pmatrix} + \left(N_\infty - \lambda\right)\begin{pmatrix} u \\ v \end{pmatrix} = 0, \tag{4.10}$$

where D_∞ and N_∞ were defined in (3.4). We therefore define the linear dispersion relation

$$\Lambda(\lambda,\nu) = \begin{bmatrix} a^\infty\nu^2 - \lambda & b^\infty\nu^2 \\ f^\infty_u & \nu^2 + f^\infty_v - \lambda \end{bmatrix} \quad \text{and} \quad d(\lambda,\nu) = \det\Lambda(\lambda,\nu).$$

Remark 4.8. Next, we collect some results proved in [13, Lem. 5.10].

(i) In case $\operatorname{Re}\lambda > 0$, there are two roots ν_j, $j = 1,2$ of the equation $d(\lambda,\nu) = 0$ with $\operatorname{Re}\nu_j > 0$.

(ii) Setting $\lambda = \gamma^2$, the ν_j, $j = 1,2$, can be considered as analytic functions of γ with expansion

$$\nu_1(\gamma) = \sqrt{\frac{f^\infty_v}{a^\infty f^\infty_v - b^\infty f^\infty_u}}\gamma + \mathcal{O}(\gamma^2) \quad \nu_2(\gamma) = \sqrt{-H'(v^\infty)} + \mathcal{O}(\gamma^2),$$

where H was defined in (2.2), and $H'(v^\infty) < 0$ is guaranteed by (2.3).

We can now state the main result of this subsection.

Lemma 4.9. *The solutions to (4.9) for λ in a complex neighborhood of the origin can be characterized as follows.*

(i) In case $\operatorname{Re}\lambda > 0$, *the solutions of the system* (4.9) *are of the form*

$$(u(r),v(r))^{\mathrm{T}} = C_1 r^{1-k/2}K_{k/2-1}(\nu r) + C_2 r^{1-k/2}I_{k/2-1}(\nu r),$$

where, C_j, $j = 1,2$ are vectors in the kernel of $\Lambda(\lambda,\nu_j)$ and $\nu = \nu_j$, $j = 1,2$.[2]

[2]Here, I_α and K_α are the modified Bessel functions satisfying the equation $r^2h'' + rh - (r^2+\alpha^2)h = 0$, such that K_α is bounded at ∞, I_α is bounded at 0, see [9].

(ii) *Setting* $\lambda = \gamma^2$, *the solution* $r^{1-k/2}K_{k/2-1}(\nu_1 r)$ *is bounded at* $+\infty$ *and the constant* C_1 *can be chosen as an analytic function* $C_1 = \alpha(\gamma)$ *with expansion*

$$\alpha(\gamma) = (-f_v^\infty, f_u^\infty)^{\mathrm{T}} + \mathrm{O}(\gamma)$$

such that the function defined by $\alpha(\gamma)r^{1-k/2}K_{k/2-1}(\nu_1(\gamma)r)$ *satisfies* (4.9).

Proof. The proof of (i) follows immediately from the definition of the modified Bessel functions and Remark 4.8. To prove (ii) one can argue in a similar way to the proof of [13, Lem. 5.10(ii)]. ∎

4.6 The eigenvalue problem near 0

In this section we discuss the eigenvalue problem $\mathcal{L}_{\mathrm{rad}}u = \lambda u$, near $\lambda = 0$ using Lyapunov-Schmidt reduction. The following proposition states that the eigenvalue problem can be reduced to finding the roots of a single scalar function.

Proposition 4.10. *Under the assumptions of Theorem 1.1, there exists* $\delta > 0$ *and function* $E : [0, \delta] \to \mathbb{C}$, *such that for any* $\gamma > 0$,

$$\gamma^2 \in \sigma_{\mathrm{point}}(\mathcal{L}) \quad \textit{if and only if} \quad E(\gamma) = 0. \tag{4.11}$$

Moreover, we have that

(i) *if* $k = 2$ *the function* E *is continuous on* $[0, \delta]$, *differentiable on* $(0, \delta]$, $E(0) = 0$ *and*

$$E(\gamma) = \frac{a^\infty f_v^\infty - b^\infty f_u^\infty}{\ln\gamma} + \frac{1}{\ln\gamma}\mathcal{O}(\gamma^2) + \mathcal{O}(\gamma^2).$$

It can be extended analytically to $B(0, \delta) \setminus [-\delta, 0]$;

(ii) *if* $k \geq 3$ *the function* E *can be extended analytically to* $B(0, \delta)$ *and* $E(0) \neq 0$.

Step 1: The ansatz. We are interested in solutions to

$$\left(\mathcal{L} - \gamma^2\right)\begin{pmatrix} u \\ v \end{pmatrix} = 0, \tag{4.12}$$

for $\gamma \sim 0$. We use the ansatz

$$(u, v)^{\mathrm{T}} = w + \beta\alpha(\gamma)h_k(\gamma), \tag{4.13}$$

where $w \in L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^2)$ and $\beta \in \mathbb{C}$, $\alpha(\cdot)$ are defined in Remark 4.8. The function h_k is defined as

$$[h_k(\gamma)](r) = \chi(r)j_k(\gamma)r^{1-k/2}K_{k/2-1}(\nu_1(\gamma)r) \tag{4.14}$$

for $r > 0$, $\gamma \in B(0, \delta) \setminus [-\delta, 0]$ and $\chi \in C^\infty(\mathbb{R}_+)$ is a smooth function satisfying $\chi(r) = 0$ for all $r \leq 1$ and $\chi(r) = 1$ for all $r \geq 2$. The function j_k is defined as follows: $j_2(\gamma) = (\ln\gamma)^{-1}$ and $j_k(\gamma) = \gamma^{k-2}(\nu_1(\gamma))^{1-k/2}$ for $k \geq 3$. Again, $\nu_1(\cdot)$ is as in Remark 4.8. In the sequel we will show that with this choice of j_k, the function h_k can be extended smoothly up to $\gamma = 0$ in an appropriate sense. Clearly, a function $(u, v)^{\mathrm{T}}$ of the form (4.13) that solves the eigenvalue equation (4.12) for $\gamma > 0$ belongs to the kernel of $\mathcal{L}_{\mathrm{rad}}$. We will see in Step 5 that any eigenfunction is actually of the form (4.13).

Summarizing, the ansatz allows us to consider the eigenvalue problem in the smaller space $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^2)$, only, instead of $L^2_{\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^2)$, at the expense of adding a free parameter β.

Since spikes are exponentially localized, we obtain that

$$\begin{aligned} w_0 :=& (\partial_\mu u^*_{|\mu=0} - \partial_\mu u^\infty(0), \partial_\mu v^*_{|\mu=0} - \partial_\mu v^\infty(0))^{\mathrm{T}} \\ =& (\partial_\mu u^*_{|\mu=0}, \partial_\mu v^*_{|\mu=0})^{\mathrm{T}} - \alpha(0) \in L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^2). \end{aligned} \tag{4.15}$$

Step 2: Setup of the bifurcation problem. As shown in Section 4.4 , we can choose $\eta > 0$ small enough, but fixed, so that $\mathcal{L}_{\mathrm{rad}}$ is Fredholm on $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^2)$ and $\ker \mathcal{L}^*_{\mathrm{rad}} = \mathrm{Span}\{(1, 0)^{\mathrm{T}}\}$ on $L^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^2)$. Here $\mathcal{L}^*_{\mathrm{rad}}$ refers to the L^2 adjoint. Thus, we have the following characterization of the image:

$$\mathrm{im}\, \mathcal{L}_{\mathrm{rad}} = \Big\{ (u, v)^{\mathrm{T}} \in L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^2) : \int_{\mathbb{R}} u(r) r^{k-1} \mathrm{d}r = 0 \Big\}.$$

It follows that equation (4.12) with ansatz (4.13) is equivalent to the following system,

$$\left\{ \begin{array}{l} F(w, \beta, \gamma) = 0 \\ \Big\langle \Big(\mathcal{L}_{\mathrm{rad}} - \gamma^2 \Big) \Big(w + \beta \alpha(\gamma) h_k(\gamma) \Big), (1, 0)^{\mathrm{T}} \Big\rangle_{L^2(0,\infty; r^{k-1} dr)} = 0. \end{array} \right. \tag{4.16}$$

Here, the function $F : H^2_{\eta,\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^2) \times \mathbb{C}^2 \to \mathrm{im}\, \mathcal{L}_{\mathrm{rad}}$ is defined by

$$F(w, \beta, \gamma) = P_0 \Big(\mathcal{L}_{\mathrm{rad}} - \gamma^2 \Big) \Big(w + \beta \alpha(\gamma) h_k(\gamma) \Big)$$

and P_0 is the orthogonal projection in $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k, \mathbb{C}^2)$ onto $\mathrm{im}\, \mathcal{L}_{\mathrm{rad}}$. We view (4.16) as an equation $\mathcal{F}(w, \beta, \gamma) = 0$ for the variables $(w, \beta, \gamma) \in H^2_{\eta,\mathrm{rad}}(\mathbb{R}, \mathbb{C}^2) \times \mathbb{C}^2$, with values $\mathcal{F} \in \mathrm{im}\, \mathcal{L}_{\mathrm{rad}} \times \mathbb{C}$.

Step 3: Smoothness of the bifurcation problem. For our perturbation analysis, we will rely on expansions of F. For this it is essential to establish smoothness properties. We will see that we can extend the function

$$\mathcal{F}_0 : \Omega_k(\delta) \to \left(\mathcal{L}_{\mathrm{rad}} - \gamma^2 \right) (\alpha(\gamma) h_k(\gamma)) \in L^2_\eta(\mathbb{R}, \mathbb{C}^2)$$

to a domain $\Omega_k(\delta)$,

$$\begin{aligned} \Omega_2(\delta) =& B(0, \delta) \setminus [0, \delta], \\ \Omega_k(\delta) =& B(0, \delta), \text{ for } k \geq 3. \end{aligned}$$

which is in fact given by a domain of analyticity of h_k; see Lemma A.4.

Now, let $\psi, \phi_0 \in C^\infty(\mathbb{R}_+)$ be smooth functions satisfying $\psi(r) = 1$ for all $r \in [1, 2]$ and $\psi(r) = 0$ for all $r \geq 3$ and all $r \in [0, \frac{1}{2}]$. We choose the function ψ_0 such that it satisfies the conditions

$\psi_0(r) = 0$ for all $r \in [0, \frac{1}{2}]$ and $\psi_0(r) = 1$ for all $r \geq 1$. We note that $\psi_0\chi = \chi$, and thus, from (4.14) we obtain that

$$\psi_0 h_k(\gamma) = h_k(\gamma) \quad \text{for all} \quad \gamma \in \Omega_k(\delta). \tag{4.17}$$

Next, we will show that

$$\left(\mathcal{L}^\infty_{\text{rad}} - \gamma^2\right)(\alpha(\gamma)h_k(\gamma)) = \chi_{[1,2]}\left(\mathcal{L}^\infty_{\text{rad}} - \gamma^2\right)(\alpha(\gamma)\psi h_k(\gamma)), \tag{4.18}$$

where $\chi_{[1,2]}$ is the characteristic function of the interval $[1, 2]$. Since from Lemma 4.9(ii) we know that $\alpha(\gamma)[h_k(\gamma)](r)$ satisfies (4.9) and $\chi(r) = 1$ for all $r \geq 2$ and $\chi(r) = 0$ for all $r \in [0, 1]$, we obtain that

$$\left(\mathcal{L}^\infty_{\text{rad}} - \gamma^2\right)(\alpha(\gamma)h_k(\gamma))(r) = 0 \quad \text{for all} \quad r \in [0, 1] \quad \text{and all} \quad r \geq 2. \tag{4.19}$$

Also, since $\mathcal{L}^\infty_{\text{rad}}$ is a differential operator and $\psi(r) = 1$ and $\psi'(r) = \psi''(r) = 0$ for all $r \in [1, 2]$ one can verify that

$$\chi_{[1,2]}\mathcal{L}^\infty_{\text{rad}}(\psi u) = \chi_{[1,2]}\mathcal{L}^\infty_{\text{rad}}u \quad \text{for all} \quad u \in H^2_{\text{loc}}(\mathbb{R}_+, \mathbb{C}^2).$$

This, together with (4.19), proves (4.18) which together with (4.17) implies the representation for $\mathcal{F}_0$

$$\mathcal{F}_0(\gamma) = \left(\mathcal{L}_{\text{rad}} - \mathcal{L}^\infty_{\text{rad}}\right)(\psi_0\alpha(\gamma)h_k(\gamma)) + \chi_{[1,2]}\left(\mathcal{L}^\infty_{\text{rad}} - \gamma^2\right)(\alpha(\gamma)\psi h_k(\gamma)). \tag{4.20}$$

Since $\mathcal{L} - \mathcal{L}_\infty$ is a second order differential operator whose matrix-valued coefficients decay exponentially at ∞ and ψ_0 is a bounded C^∞ function and $\text{supp}(\psi_0) \subseteq [\frac{1}{2}, \infty)$, we have that the linear operator defined by $\underline{w} \to (\mathcal{L} - \mathcal{L}_\infty)(\psi_0\underline{w})$ is bounded from $H^2_{-\eta,\text{rad}}(\mathbb{R}, \mathbb{C}^2)$ to $L^2_{\eta,\text{rad}}(\mathbb{R}, \mathbb{C}^2)$. Moreover, since $\psi \in C^\infty(\mathbb{R}_+)$ has compact support, $\text{supp}(\psi_0) \subseteq [\frac{1}{2}, 3]$, the operator of multiplication by ψ is bounded from $H^2_{-\eta,\text{rad}}(\mathbb{R}^k)$ to $H^2_{\eta,\text{rad}}(\mathbb{R}^k)$. Recall that ν_1 is analytic on $B(0, \delta)$ by Remark 4.8.

It remains to investigate smoothness of h_k. Lemma A.4 in the appendix states that h_k is analytic from $\Omega_k(\delta)$ to $H^2_{-\eta,\text{rad}}(\mathbb{R}^k)$. Moreover, in the case $k = 2$, the limit $\lim\limits_{\gamma \to 0, \gamma > 0} \mathcal{F}_0(\gamma) = \mathcal{L}_{\text{rad}}(\alpha(0)h^0_k)$ exists in $L^2_{\eta,\text{rad}}(\mathbb{R}^k)$. Now, using the representation (4.20), we conclude that $\mathcal{F}_0$ is well-defined and analytic on $\Omega_k(\delta)$, and continuous on $[0, \delta]$ in the case $k = 2$.

Step 4: Construction of the Evans function. From the definition of $\underline{w}_0$ in (4.15) it follows that $F(w_0, 1, 0) = 0$. Since $\mathcal{F}$ is bounded linear in w and β, and since $\mathcal{F}_0$ is analytic on $\Omega_k(\delta)$, we conclude that $\mathcal{F}$ is analytic on $H^2_{\eta,\text{rad}}(\mathbb{R}^k) \times \mathbb{C} \times \Omega_k(\delta)$. Differentiating F in w, we obtain that

$$F_w(w_0, 1, 0) = P_0\mathcal{L}_{\text{rad}}.$$

Since $\mathcal{L}_{\text{rad}}$ is Fredholm of index -1 with trivial kernel, and P_0 projects onto its range, we infer that the linearization in w, $F_w(w_0, 1, 0)$, is boundedly invertible. From the Implicit Function Theorem it follows that we can solve locally the first equation in (4.16), and find a unique smooth solution

$$w^* : B(1, \delta) \times \Omega_k(\delta) \subset \mathbb{C}^2 \to H^2_{\eta,\text{rad}}(\mathbb{R}^k, \mathbb{C}^2), \quad w^*(1, 0) = w_0,$$

so that locally

$$F(w,\beta,\gamma)=0 \Longleftrightarrow w=w^*(\beta,\gamma).$$

From this local uniqueness we conclude that $w^*(\beta,\gamma)=\beta w^*(1,\gamma)$ for all $(\beta,\gamma)\in B(1,\delta)\times\Omega_k(\delta)$, so that we may restrict to $\beta=1$ in the sequel. By continuity of the solution, $w^*(1,\gamma)+\alpha(\gamma)h(\gamma)\neq 0$ for small γ.

Substituting w^* into the second equation of (4.16) completes the Lyapunov-Schmidt reduction and gives us a bifurcation equation $E(\gamma)=0$, where $E:\Omega_k(\delta)\to\mathbb{C}$ is defined as

$$E(\gamma)=\left\langle\left(\mathcal{L}_{\mathrm{rad}}-\gamma^2\right)\left(w^*(1,\gamma)+\alpha(\gamma)h_k(\gamma)\right),(1,0)^{\mathrm{T}}\right\rangle_{L^2}. \tag{4.21}$$

Now, w^* is an analytic function on $\Omega_k(\delta)$, continuous on $[0,\delta]$ for $k=2$ by the implicit function theorem. Also, the map $\mathcal{F}_0:\gamma\to\left(\mathcal{L}-\gamma^2\right)(\alpha(\gamma)h_k(\gamma))$ is an analytic functions on $\Omega_k(\delta)$ and continuous on $[0,\delta]$ by Step 3. We can conclude that E is analytic and continuous on $\gamma\in[0,\delta]$ in the case $k=2$.

Step 5: Invertibility for $E(\gamma)\neq 0$. In this subsection we show that the eigenfunctions are necessarily of the form described in the ansatz (4.13) which is equivalent to prove that $\mathcal{L}_{\mathrm{rad}}-\lambda$ is invertible for $\operatorname{Re}\lambda>0$ whenever $E(\gamma)\neq 0$, where $\lambda=\gamma^2$. Consider the system

$$\left(\mathcal{L}_{\mathrm{rad}}-\gamma^2\right)\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}g_1\\g_2\end{pmatrix}, \tag{4.22}$$

for right-hand sides $g_1,g_2\in L^2_{\mathrm{rad}}(\mathbb{R}^k,\mathbb{C}^2)$. Since γ^2 is not in the essential spectrum for $\operatorname{Re}\lambda>0$ we have that $\mathcal{L}_{\mathrm{rad}}-\gamma^2$ is Fredholm of index 0. It is therefore sufficient to solve this equation for right-hand sides g_j, $j=1,2$ in a dense subset of $L^2_{\mathrm{rad}}(\mathbb{R}^k,\mathbb{C}^2)$, for example $L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k,\mathbb{C}^2)$. Seeking solutions of (4.22) in the form of our ansatz (4.13), which for $\operatorname{Re}\lambda>0$ clearly provides us with L^2-functions, we obtain the system

$$\left\{\begin{array}{l} F(w,\beta,\gamma)=P_0(g_1,g_2)^{\mathrm{T}}\\ \left\langle\left(\mathcal{L}_{\mathrm{rad}}-\gamma^2\right)\left(w+\beta\alpha(\gamma)h_k(\gamma)\right),(1,0)^{\mathrm{T}}\right\rangle_{L^2}=\left\langle 1,g_1\right\rangle_{L^2}.\end{array}\right. \tag{4.23}$$

This is a linear system in w and β, and the joint linearization is Fredholm of index zero, since the Fredholm index of $\mathcal{L}_{\mathrm{rad}}$ is -1. Thus, we can solve this equation with bounded inverse provided that there is no kernel, which is equivalent to $E(\gamma)\neq 0$.

Step 6: Estimating $E(\gamma)$ for $\gamma>0$. We will split the argument for the two cases that are significantly different, that is , when $k=2$ or $k\geq 3$.

The case $k=2$. Since $w^*(1,\gamma)\in H^2_{\eta,\mathrm{rad}}(\mathbb{R}^2,\mathbb{C}^2)$ we have that

$$E(\gamma)=\left\langle\mathcal{L}_{\mathrm{rad}}\left(\alpha(\gamma)h_2(\gamma)\right),(1,0)^{\mathrm{T}}\right\rangle_{L^2}+\frac{1}{\ln\gamma}\mathcal{O}(\gamma^2).$$

Recall the definition of the functions $l_j(r)$ from (3.3). Since $l_j(r)\to 0$ exponentially, as $r\to\infty$, $j=1,2$, one has that

$$\int_0^\infty\frac{1}{r}\partial_r[r(l_j(r)v(r))]rdr=0$$

for any function $v \in H^2_{-\eta,\mathrm{rad}}(\mathbb{R}^2)$ whose support does not include $r=0$. This implies that

$$E(\gamma) = \int_0^\infty \frac{1}{r}\partial_r\Big(r((b^*(r)f_u^\infty - a^*(r)f_v^\infty)[h_2(\gamma)]'(r)\Big) r dr. \tag{4.24}$$

We expand $h_2(\gamma)$, next. Therefore recall that the modified Bessel function can be written in the form

$$K_0(z) = (\ln z) f_0(z) + g_0(z), \qquad \text{for all } z \in \mathbb{C}\setminus\mathbb{R}_-, \tag{4.25}$$

with two entire function f_0 and g_0.

Since ν_1 is analytic, $\nu_1(0)=0$ and $\nu_1'(0)>0$ the function $\rho: B(0,\delta)\to\mathbb{C}$ defined by $\rho(\gamma) = \ln\left(\frac{\nu_1(\gamma)}{\gamma}\right)$ is analytic, for some small $\delta>0$. Next we note that

$$\ln(\nu_1(\gamma) r) = \ln\gamma + \rho(\gamma) + \ln r \quad \text{for all} \quad \gamma \in B(0,\delta), r>0. \tag{4.26}$$

From equation (4.26) and since the functions f_0 and g_0 are entire functions, we can expand

$$[h_2(\gamma)](r) = \frac{\chi(r)}{\ln\gamma}\left(g_0(0) - \ln\gamma - \rho(\gamma) + \mathcal{O}(\gamma^2) + \mathcal{O}(\gamma^2)\ln\gamma\right),$$

and

$$\begin{aligned}
[h_2(\gamma)]'(r) =& \frac{\chi'(r)}{\ln\gamma}\left(g_0(0) - \ln\gamma - \rho(\gamma) + \mathcal{O}(\gamma^2) + \mathcal{O}(\gamma^2)\ln\gamma\right) \\
&+ \frac{\chi(r)}{\ln\gamma}(-r^{-1} + \mathcal{O}(\gamma^2) + \mathcal{O}(\gamma^2)\ln\gamma).
\end{aligned}$$

Substituting these expansions for h_2 into (4.24), we arrive at the expansion for E

$$E(\gamma) = \frac{a^\infty f_v^\infty - b^\infty f_u^\infty}{\ln\gamma} + \frac{1}{\ln\gamma}\mathcal{O}(\gamma^2) + \mathcal{O}(\gamma^2). \tag{4.27}$$

The case $k \geq 3$. By Lemma A.4, the limit $\lim\limits_{\gamma\to 0,\gamma>0} h_k)(r) = c_k\chi(r)r^{2-k}$ exists for any fixed r, and convergence holds in the $H^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$-norm. Moreover, we have $c_k \neq 0$. Using the definition of the function E, we therefore obtain

$$\begin{aligned}
E(0) &= \left\langle \mathcal{L}_{\mathrm{rad}}(\alpha(0)h_k(0)), (1,0)^{\mathrm{T}} \right\rangle_{L^2} \\
&= \int_0^\infty \frac{1}{r^{k-1}}\partial_r\Big(r^{k-1}((b^*(r)f_u^\infty - a^*(r)f_v^\infty)c_k\partial_r(\chi(r)r^{2-k})\Big) r^{k-1} dr \\
&\quad + \int_0^\infty \frac{1}{r^{k-1}}\partial_r\Big(r^{k-1}(l_1(r)+l_2(r))\chi(r)r^{2-k}\Big) r^{k-1} dr \\
&= (k-2)c_k(a^\infty f_v^\infty - b^\infty f_u^\infty),
\end{aligned}$$

where we used $l_j(r)\to 0$ as $r\to\infty$, exponentially, $j=1,2$, in the last equality.

4.7 Proof of Theorem 1.1

Proposition 4.10 implies that for any $\tau \in [0,1]$, there exists $\delta_\tau > 0$ and an analytic function $E_\tau : \Omega_k(\delta_\tau) \to \mathbb{C}$ that detects the eigenvalues of $\mathcal{L}_{\tau,\mathrm{rad}}$ according to (4.11). Since E is smooth in τ, there is a constant c^* independent of τ such that $|E_\tau(\gamma)| \geq \frac{c^*}{\ln \gamma}$, uniformly in τ, so that we can exclude eigenvalues of $\mathcal{L}_{\tau,\mathrm{rad}}$ in $(0,\delta^*)$. In addition, equation (1.6) is well-posed for all τ, which implies $\sup \mathrm{Re}\, \sigma(\mathcal{L}_{\tau,\mathrm{rad}}) < \infty$. Thus, the number of real unstable eigenvalues of $\mathcal{L}_{\mathrm{rad}}$ is finite,

$$N(\tau) = \#\{\lambda(\tau) \in \sigma_{\mathrm{point}}(\mathcal{L}_\tau) : \lambda(\tau) > 0\} < \infty,$$

for all $\tau \in [0,1]$. This fact allows us to define a parity index as follows,

$$i_{\mathrm{p}}(\tau) = (-1)^{N(\tau)}. \tag{4.28}$$

Since eigenvalues are uniformly bounded away from 0 and ∞ on the positive real axis, they can only leave the positive axis in complex pairs. Therefore, i_{p} is constant, independent of τ. Also, by Lemma 4.3 $i_p(1) = -1$, so that $N(\tau) \neq 0$. This proves the linear instability of spikes and Theorem 1.1.

A Appendix

In this appendix we discuss the analyticity of h_k and the possibility of extending the function analytically to a neighborhood of 0. We start with two abstract lemmas.

Lemma A.1. *Given $\eta > 0$, $p : \mathbb{R}_+ \to \mathbb{R}$ a C^∞ function, and f an entire function such that*

(i) $p(r) = 0$ for all $r \in [0,1]$;

(ii) $|p(r)| \leq cr^m$ for all $r > 0$, for some constants $c > 0$ and $m \in \mathbb{N}$;

(iii) $|f(z)| \leq ce^{\omega|z|}$ for all $z \in \mathbb{C}$, for some constants $c > 0$ and $\omega \in \mathbb{R}_+$.

Then, there exists $\delta > 0$ such that the function $F_p : B(0,\delta) \to L^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$, given by $[F_p(\gamma)](r) = p(r)f(\nu_1(\gamma)r)$, is well-defined and analytic on $B(0,\delta)$.

Proof. Since ν_1 is analytic and $\nu_1(0) = 0$ we can choose $\delta > 0$ such that $\omega|\nu_1(\gamma)| \leq \eta/2$. Using the hypothesis (i)–(iii) we estimate

$$|[F_p(\gamma)](r)| \leq c|p(r)|e^{\omega|\nu_1(\gamma)r|} \leq c\chi_{[1,\infty)}(r) r^m e^{\eta/2r} \quad \text{for all} \quad \gamma \in B(0,\delta), r > 0. \tag{A.1}$$

It follows that $F_p(\gamma) \in L^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$ for all $\gamma \in B(0,\delta)$ and so, F_p is well-defined. From Lebesgue's Dominated Convergence Theorem and the estimate (A.1) we infer that F_p is continuous. Next, we prove that F_p is weakly analytic, that is, the function defined by $F_p^v(\gamma) := \langle F_p(\gamma), v\rangle_{L^2}$ is analytic for all $v \in L^2_{\eta,\mathrm{rad}}(\mathbb{R}^k)$, which will then imply that F_p is analytic. To check this, we integrate F_p^v on the boundary of a rectangle $\mathrm{R} \subset B(0,\delta)$ using Fubini's Theorem. Since ν_1 and f are analytic we have

$$\oint_{\partial \mathrm{R}} F_p^v(\gamma) d\gamma = \int_0^\infty \left(\oint_{\partial \mathrm{R}} \left(p(r) f(\nu_1(\gamma) r)\right) d\gamma \right) v(r) r^{k-1} dr = 0.$$

■

Lemma A.2. *Given $\eta > 0$, $p : \mathbb{R}_+ \to \mathbb{R}$ a C^∞ function, and f an entire function such that*

(i) $p(r) = 0$ *for all* $r \in [0,1]$*;*

(ii) $\max(|p(r)|, |p'(r)|, |p''(r)|) \le c r^m$ *for all* $r > 0$*, for some constants* $c > 0$ *and* $m \in \mathbb{N}$*;*

(iii) $|f(z)| \le c e^{\omega |z|}$ *for all* $z \in \mathbb{C}$*, for some constants* $c > 0$ *and* $\omega \in \mathbb{R}_+$*.*

Then, there exists $\delta > 0$ such that the function $F_p : B(0,\delta) \to H^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$, defined in Lemma A.1, is well-defined and analytic on $B(0,\delta)$.

Proof. First we note that using standard complex analysis arguments one can show that the functions f' and f'' are entire functions and they satisfy condition (iii) from Lemma A.1. Also, we note that by our assumption the functions p, p' and p'' satisfy conditions (i)–(ii) from Lemma A.1. We have

$$[F_p(\gamma)]'(r) = p'(r) f(\nu_1(\gamma) r) + \nu_1(\gamma) p(r) f'(\nu_1(\gamma) r)$$

$$[F_p(\gamma)]''(r) = p''(r) f(\nu_1(\gamma) r) + 2\nu_1(\gamma) p'(r) f'(\nu_1(\gamma) r) + \nu_1(\gamma)^2 p(r) f''(\nu_1(\gamma) r),$$

for all $\gamma \in B(0,\delta)$ and all $r > 0$. Now, using analyticity of ν_1, we conclude from Lemma A.1 that the maps $\gamma \to [F_p(\gamma)]^{(j)} : B(0,\delta) \to L^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$ are well-defined and analytic. This proves the lemma. ■

Applying these two lemmas to our particular situation we obtain the following result.

Lemma A.3. *For each $\eta > 0$ there exists $\delta > 0$ such that*

(i) There exists $F_2, G_2 : B(0,\delta) \to H^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$ two analytic functions such that

$$h_2(\gamma) = \frac{1}{\ln \gamma} F_2(\gamma) + G_2(\gamma) \quad \textit{for all} \quad \gamma \in B(0,\delta) \setminus [0,\delta]; \tag{A.2}$$

(ii) For $k \ge 3$, the function h_k can be extended as an analytic function from $B(0,\delta)$ into $H^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$.

Proof. (i) Recall the decomposition of K_0, the definitions of f_0 and g_0 (4.25), and the expansion of $\ln(\nu_1(\gamma) r)$, (4.26).

Using this representation and the definition of h_2 in (4.14) we calculate

$$[h_2(\gamma)](r) = \frac{1}{\ln \gamma} \chi(r) K_0(\nu_1(\gamma) r) = \frac{1}{\ln \gamma} \chi(r) \left[(\ln \gamma + \rho(\gamma) + \ln r) f_0(\nu_1(\gamma) r) + g_0(\nu_1(\gamma) r) \right]$$

$$= \frac{1}{\ln\gamma}\left[\rho(\gamma)\chi(r)f_0(\nu_1(\gamma)r) + \chi(r)g_0(\nu_1(\gamma)r) + \chi(r)(\ln r)f_0(\nu_1(\gamma)r)\right] + \chi(r)f_0(\nu_1(\gamma)r).$$

We note that the functions $p(r) = \chi(r)$ and $q(r) = \chi(r)\ln r$ satisfy conditions (i)–(ii) from Lemma A.2, the functions f_0 and g_0 have exponential growth, that is they satisfy condition (iii) from Lemma A.2. Since, in addition, ρ is analytic, we obtain that there is a $\delta > 0$ such that the functions $F_2, G_2 : B(0,\delta) \to H^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$ defined by

$$[F_2(\gamma)](r) = \chi(r)\left[(\rho(\gamma) + \ln r)f_0(\nu_1(\gamma)r) + g_0(\nu_1(\gamma)r)\right], \quad [G_2(\gamma)](r) = \chi(r)f_0(\nu_1(\gamma)r),$$

are analytic, proving (i).

(ii) The proof of (ii) is similar to the proof of (i). Indeed, if $k \geq 3$, then there are two entire functions $f_{k/2}$ and $g_{k/2}$ such that $K_{k/2-1}(z) = z^{k/2-1}f_{k/2}(z) + z^{1-k/2}g_{k/2}(z)$. From the definition of h_k in (4.14) we calculate

$$\begin{aligned}
[h_k(\gamma)](r) &= \gamma^{k-2}\chi(r)(\nu_1(\gamma)r)^{1-k/2}K_{k/2-1}(\nu_1(\gamma)r)\\
&= \gamma^{k-2}\chi(r)\left[f_{k/2}(\nu_1(\gamma)r) + (\nu_1(\gamma)r)^{2-k}g_{k/2}(\nu_1(\gamma)r)\right]\\
&= \gamma^{k-2}\chi(r)f_{k/2}(\nu_1(\gamma)r) + \left(\frac{\gamma}{\nu_1(\gamma)}\right)^{k-2}\chi(r)r^{2-k}g_{k/2}(\nu_1(\gamma)r)\\
&= \gamma^{k-2}\chi(r)f_{k/2}(\nu_1(\gamma)r) + e^{-(k-2)\rho(\gamma)}\chi(r)r^{2-k}g_{k/2}(\nu_1(\gamma)r)
\end{aligned}$$

Again, we note $p(r) = \chi(r)$ and $q(r) = \chi(r)r^{2-k}$ satisfy conditions (i)–(ii) from Lemma A.2, the functions $f_{k/2}$ and $g_{k/2}$ satisfy condition (iii) from Lemma A.2. Since ρ is analytic, from Lemma A.2 we conclude that h_k can be extended as an analytic function from $B(0,\delta)$ into $H^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$. ∎

We collect the main conclusions of this appendix in the following lemma:

Lemma A.4. *Let $\Omega_2(\delta) = B(0,\delta) \setminus [0,\delta]$ and $\Omega_k(\delta) = B(0,\delta)$ for $k \geq 3$. The function $h_k : \Omega_k(\delta) \to H^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$ is well-defined and analytic. For all k, the limit $\lim_{\gamma\to 0,\gamma>0} h_k = h^0_k$ exists in $H^2_{-\eta,\mathrm{rad}}(\mathbb{R}^k)$, and $h^0_k(r) = c_k\chi(r)r^{2-k}$, for some non-zero constant c_k.*

References

[1] G. Caginalp, *An analysis of a phase field model of a free boundary*, Arch. Ration. Mech. Anal. **92** (1986), 205–245.

[2] E. N. Dancer, *Real analyticity and non-degeneracy*, Math. Ann. **325** (2003), 369–392.

[3] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Lect. Notes Math. **840**, Springer–Verlag, New York, 1981.

[4] R. Hempel, A. Hinz, H. Kalf, *On the essential spectrum of Schrödinger operators with spherically symmetric potentials. With a comment by J. Weidmann*, Math. Ann. **277** (1987), 197–211.

[5] Y. Kabeya, K. Tanaka, *Uniqueness of positive radial solutions of semilinear elliptic equations in $\mathbb{R}^N$ and Seres's non-degeneracy condition*, Comm. Partial Diff. Eq. **24** (1999), no. 3-4, 563–598.

[6] E. Keller and L. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol. **26** (1970), 399–415.

[7] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Birkhäuser Verlag, Basel (1995).

[8] W. M. Ni, I. Takagi, *Locating the peaks of least-energy solutions to a semilinear Neumann problem*, Duke Math. J. **70** (1993), 247–281.

[9] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, 1974.

[10] K. J. Palmer, *Exponential dichotomies and transversal homoclinic points*, J. Diff. Eq. **55** (1984), 225–256.

[11] K. J. Palmer, *Exponential dichotomies and Fredholm operators*, Proc. Amer. Math. Soc. **104** (1988), 149–156.

[12] P. Polacik, *Morse indices and bifurcations of positive solutions of $\Delta u + f(u) = 0$ on $\mathbb{R}^N$*, Indiana Univ. Math. J. **50** (2001), 1407–1432.

[13] A. Pogan, A. Scheel, *Instability of Spikes in the Presence of Conservation Laws*, preprint.

[14] A. Pogan, A. Scheel, *Fredhom properties of radially symmetric, second order differential operators*, in preparation.

[15] B. Sandstede, A. Scheel *Evans function and blow-up methods in critical eigenvalue problems*, Discr. Cont. Dyn. Sys. **10** (2004), 941-964.

[16] B. Sandstede and A. Scheel, *Relative Morse indices, Fredholm indices, and group velocities,* Discr. Cont. Dynam. Syst., **20** (2008), 139–158.

[17] B. Simon, *Trace Ideals and Their Applications*, Math. Surv. Monographs, Vol. **120**, Amer. Math. Soc., Providence, RI, 2005.