BIFURCATION TO SPIRAL WAVES IN REACTION-DIFFUSION SYSTEMS

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Abstract. For a large class of reaction-diffusion systems on the plane, we show rigorously that m-armed spiral waves bifurcate from a homogeneous equilibrium when the latter undergoes a Hopf bifurcation. In particular, we construct a finite-dimensional manifold which contains the set of small rotating waves close to the homogeneous equilibrium. Examining the flow on this center-manifold in a very general example, we find different types of spiral waves, distinguished by their speed of rotation and their asymptotic shape at large distances of the tip. The relation to the special class of λ-ω systems and the validity of these systems as an approximation is discussed.

Key words. Spiral waves, Center-Manifolds, Ginzburg-Landau Equations, λ-ω Systems

AMS subject classifications. 35B32, 58F39, 35K57, 35J60

1. Introduction .

We study reaction-diffusion systems

$$U_t = D\triangle U + F(\lambda, U), \qquad U(x,t) \in \mathbb{R}^N, \lambda \in \mathbb{R}^p \tag{1.1}$$

on the plane $x \in \mathbb{R}^2$. The N-dimensional vector U typically describes a set of chemical concentrations and temperature, depending on time $t \in \mathbb{R}$ and the space variable x. The parameter λ is a p-dimensional control parameter which shall allow us to create instabilities of spatially homogeneous equilibria. We shall be interested in rotating wave solutions $U(t,x) = U(0, R_{ct}x)$, $c \neq 0$, where R_φ is the rotation in $\mathbb{R}^2$ around the origin by the angle $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$. Our analysis show that this class of solutions appears naturally via some type of Hopf bifurcation. Moreover the spatial structure resembles n-armed spiral waves.

Experimentally this type of spatio-temporal pattern has been observed frequently in chemical, biological, physiological and physical experiments (e.g. the Belousov-Zhabotinsky reaction, the catalysis on platinum surfaces, electro-chemical waves in the cortex of the brain, signaling patterns of the slime mold and the Rayleigh-Bénard convection). Nevertheless a rigorous treatment of existence and creation is still not available — for various reasons.

Spiral waves are typically observed in spatially extended oscillatory processes. Near Hopf bifurcation points the dynamics of these processes is approximated by Ginzburg-Landau equations or λ-ω-systems [1, 10]. This has been shown using formal asymptotic methods; see [1] for example. Recently a rigorous proof of the approximation property of Ginzburg-Landau equations has been given by Schneider; see [21]. The important property of the approximating equations is a decoupling of Fourier modes which was exploited by several authors in order to construct spiral wave solutions [1, 5, 7, 11], though the methods are still formal or do not cover the typical nonlinearities appearing close to bifurcation points.

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Of course a treatment of Hopf bifurcation using classical bifurcation methods with symmetry is not possible because neither center-manifolds exist nor Lyapunov-Schmidt reduction can be applied, due to presence of continuous spectrum. Another explanation is provided by the fact that the symmetry of the reaction-diffusion equation, the Euclidean symmetry, does not have bounded finite dimensional representations; see [12] for an approach to Ginzburg-Landau equations exploiting symmetry.

In spatially extended systems including only one unbounded spatial variable, typically cylindrical domains, the continuous spectrum can be avoided by restricting to steady state solutions and considering the unbounded spatial variable as a new time direction. This approach was introduced by Kirchgässner [9] and applied to various interesting problems in mechanics, fluid dynamics and physics; see for example [13, 17]. Unfortunately, considering systems with several unbounded directions, this method becomes less successful; see however [14].

We adopt the idea of spatial dynamics, now considering the radial direction in polar coordinates as a new time variable, in order to describe the spatial structure of rotating waves by a surprisingly finite-dimensional, non-autonomous ODE on a center-manifold. In particular any small bounded solution to this ODE corresponds to a rotating wave of the original reaction-diffusion system. This approach allows us to describe systematically the creation of rotating waves from homogeneous equilibria of reaction-diffusion systems.

The paper is organized as follows. In the next section we fix the abstract functional analytic setting in which we formulate a center-manifold reduction theorem. This theorem, our main result, is stated in § 3 and proved in the subsequent two sections. The key to the proof are exponential dichotomies which are proved to exist in our functional analytic framework in § 4. In § 6 we comment on a localization of our main theorem using cut-off procedures and we briefly discuss regularity of solutions in § 7. The last important abstract result is stated in § 8, where we formulate and prove the existence of a larger manifold, containing solutions which might be singular at the origin. This manifold is tangent to a subspace which is independent of time τ (alias the distance to the origin in $\mathbb{R}^2$) and therefore allows to derive explicit bifurcation equations. We conclude by applying our results to a model problem in the remaining sections.

2. The abstract setting. Introducing polar coordinates $x = (r\cos\varphi, r\sin\varphi)$, the equation for rotating wave solutions of (1.1) becomes

$$D(U'' + \frac{1}{r}U' + \frac{1}{r^2}U_{\varphi\varphi}) + F(\lambda, U) = cU_\varphi, \tag{2.1}$$

where $' = \frac{\partial}{\partial r}$, $\varphi \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$, $r \in (0,\infty)$ and c is again the speed of rotation.

We suppose that for an initial parameter value λ_0 we are given a spatially homogeneous equilibrium $U_0(\lambda_0)$ which solves $F(\lambda_0, U_0(\lambda_0)) = 0$. As we are merely interested in Hopf bifurcation we suppose that this solution can be continued in λ to a branch $U_0(\lambda)$ which we can assume without loss of generality to be the zero solution.

To sum up we suppose that $F(\lambda, 0) = 0$ and $D = D^* > 0$.

Note however that the first assumption is merely a simplification for clarity of the statements and the second assumption might be generalized slightly.

As a most elementary example, which is discussed in § 9, the reader might think of $D = id$ and $F(\lambda, U) = iU + \mathrm{O}(|U|^2)$, $U \in \mathbb{R}^2 \simeq \mathbb{C}$.

Next we multiply (2.1) by D^{-1}

$$U'' + \frac{1}{r}U' + \frac{1}{r^2}U_{\varphi\varphi} = -D^{-1}F(\lambda, U) + cD^{-1}U_\varphi \tag{2.2}$$

and linearize around $U = 0$

$$U'' + \frac{1}{r}U' + \frac{1}{r^2}U_{\varphi\varphi} = -D^{-1}F_U(\lambda, 0)U + cD^{-1}U_\varphi \tag{2.3}$$

We work in the function spaces $H^l(S^1, \mathbb{R}^N), l \geq 0$. Functions $u \in H^l$ may be represented by their Fourier series

$$u(\varphi) = \sum_{k \in \mathbb{Z}} u_k e^{ik\varphi}$$

with $u_{-k} = \bar{u}_k$ and $\|u\|_{H^l}^2 = \sum_{k\in\mathbb{Z}} |u_k|^2 k^{2l} < \infty$.

The operator $A = -\partial_{\varphi\varphi}$ is self-adjoint and positive in H^l with spectrum $\{k^2;\ k \in \mathbb{Z}\}$ and domain of definition H^{l+2}. The spectral information on our bifurcation problem is contained in the operator

$$\begin{aligned} B_{\lambda,c} : H^{l+1} \subset H^l &\to H^l \\ u(\cdot) &\mapsto -D^{-1}F_U(\lambda, 0)u(\cdot) + cD^{-1}u_\varphi(\cdot). \end{aligned}$$

As $c \neq 0$, the operator $B_{\lambda,c}$ can be considered as a bounded perturbation of the closed, antisymmetric, unbounded operator $C_c = cD^{-1}\partial_\varphi$ on H^l. Thereby $B_{\lambda,c}$ has point spectrum, and any strip $\{z \in \mathbb{C};\ |\operatorname{Im} z| < M\}$ contains only finitely many eigenvalues, each of finite multiplicity.

We denote by $\tilde{P}_+^c(\lambda, c)$ the spectral projection to the operator $B_{\lambda,c}$ on $(-\infty, 0] \subset \mathbb{C}$. Of course $\tilde{P}_+^c(\lambda, c)$ can be constructed via Dunford's integral.

We rewrite equation (2.2) as an equation in function space

$$u'' + \frac{1}{r}u' - \frac{1}{r^2}Au = -\tilde{F}(\lambda, u) + C_c u \tag{2.4}$$

and linearize around $u = 0$

$$u'' + \frac{1}{r}u' - \frac{1}{r^2}Au = B_{\lambda,c}u. \tag{2.5}$$

We suppose that $\tilde{F} \in C^K(\mathbb{R}^p \times H^{l+1/2}, H^l)$, $K \geq 1$ which can be achieved assuming $F \in C^K$ and either $l > 0$ such that $H^{l+1/2} \hookrightarrow C^0$, or assuming suitable growth conditions on F.

We say that u is a solution of (2.4) (or (2.5)) on an interval $I \subset (0, \infty)$ if

$$u \in C^0(\operatorname{clos} I, H^{l+1}) \cap C^1(\operatorname{clos} I, H^l) \cap C^2(\operatorname{int} I, H^l) \cap C^1(\operatorname{int} I, H^{l+1}) \cap C^0(\operatorname{int} I, H^{l+2})$$

and if (2.4) (or (2.5) respectively) is satisfied in H^l.

Furthermore we introduce a new time

$$\tau(r) = \begin{cases} \log r, & \text{if } r \leq \bar{r} \\ r, & \text{if } r \geq 2\bar{r} \end{cases}$$

defined by smooth, monotone interpolation on $(\bar{r}, 2\bar{r})$ such that $\tau \in C^\infty(\mathbb{R}^+, \mathbb{R})$. The exact value of the positive constant $\bar{r}$ is of no importance for the statement of the results.

For $r < \bar{r}$ the differential equation (2.4) becomes

$$u_{\tau\tau} - Au = -e^{2\tau}(\tilde{F}(\lambda, u) - C_c u).$$

Of course the equation (2.4) remains unchanged for $r \geq 2\bar{r}$.

We conclude this section emphasizing on the symmetry of equation (2.4). The rotation $\varphi \to \varphi + \bar{\varphi}$ on $S^1 \simeq \mathcal{SO}(2)$ acts on $H^l(S^1, \mathbb{R}^N)$ by shifting functions on the circle: $u(\varphi) \to u(\varphi - \bar{\varphi})$. Of course this symmetry is inherited from the Euclidean symmetry of the original problem. Note that translations are ruled out by the ansatz for equilibria in a fixed rotating coordinate system. Translates of any solution found with this ansatz would be periodic solutions in our coordinate system.

3. Main results. We write (2.4) as a first order differential equation

$$\frac{d}{d\tau}\underline{u}(\tau) = \mathcal{A}(\tau)\underline{u}(\tau) + \mathcal{G}(\underline{u}(\tau)), \qquad \underline{u} = (u, \frac{d}{d\tau}u(\tau)) \tag{3.1}$$

on the space $\underline{u} \in X = H^{l+1}(S^1, \mathbb{R}^N) \times H^l(S^1, \mathbb{R}^N)$. Here

$$\mathcal{A}(\tau) = \begin{pmatrix} 0 & 1 \\ A + e^{2\tau}B_{\lambda,c} & 0 \end{pmatrix}, \qquad \mathcal{G}(\underline{u}) = -e^{2\tau}\begin{pmatrix} 0 \\ \tilde{F}(\lambda, u) + B_{\lambda,c}u - C_c u \end{pmatrix},$$

if $\tau \leq \log \bar{r}$ and

$$\mathcal{A}(\tau) = \begin{pmatrix} 0 & 1 \\ \tau^{-2}A + B_{\lambda,c} & -\tau^{-1} \end{pmatrix}, \qquad \mathcal{G}(\underline{u}) = -\begin{pmatrix} 0 \\ \tilde{F}(\lambda, u) + B_{\lambda,c}u - C_c u \end{pmatrix},$$

if $\tau \geq 2\bar{r}$. For $\tau \in (\log \bar{r}, 2\bar{r})$, the exact form of the equation is not important, as solutions arise from solutions of (2.4) by a bounded diffeomorphic rescaling of time. The linearisation of (3.1) along $\underline{u} = 0$ is

$$\frac{d}{d\tau}\underline{u}(\tau) = \mathcal{A}(\tau)\underline{u}(\tau) \tag{3.2}$$

We let

$$Y_\delta = \{\underline{u} \in C^0(\mathbb{R}, X);\ ||u||_{Y_\delta} < \infty\} \quad \text{where } ||u||_{Y_\delta} := \sup_{\tau \in \mathbb{R}} e^{-\delta|\tau|}|\underline{u}(\tau)|_{X_\tau}$$

and, similarly,

$$Y_\delta^\pm = \{\underline{u} \in C^0(\overline{\mathbb{R}}^\pm, X);\ ||u||_{Y_\delta} < \infty\} \quad \text{where } ||u||_{Y_\delta^\pm} := \sup_{\tau \in \mathbb{R}^\pm} e^{-\delta|\tau|}|\underline{u}(\tau)|_{X_\tau}.$$

The norm in X_τ of $\underline{u}(\tau) = (u(\tau), v(\tau))$ is defined as

$$|\underline{u}(\tau)|_{X_\tau} := \begin{cases} \tau^{-1}|u|_{H^{l+1}} + |u|_{H^{l+1/2}} + |v|_{H^l}, & \text{if} \tau \geq 2\bar{r} \\ |u(\tau)|_{H^{l+1}} + |v(\tau)|_{H^l}, & \text{if} \tau < 2\bar{r} \end{cases}$$

Let us denote by $E^c(\tau)$ the (possibly empty) linear subspace of initial values of the linear equation (3.2) at time τ which give rise to Y_δ-bounded solutions. Of course E^c depends also on δ.

THEOREM 1. *Suppose the superposition operator $\tilde{F}$ to the nonlinearity F belongs to the class $C^K(\mathbb{R}^p \times H^{l+1/2}(S^1, \mathbb{R}^N), H^l(S^1, \mathbb{R}^N))$, $1 \leq K < \infty$. Suppose furthermore that for $\lambda = \lambda_0$ and $c = c_0$ we have $\tilde{P}_+^c(\lambda, c) \neq 0$.*

Then there are $\epsilon, \delta > 0$ such that, if

$$\mathrm{Lip}_u[\tilde{F} - \tilde{F}_u(\lambda, 0)] + |\lambda - \lambda_0| + |c - c_0| < \epsilon,$$

there exists a unique finite-dimensional C^K-center-manifold $\mathcal{M} \subset X \times \mathbb{R}$ which contains all solutions of (3.1) which are bounded in Y_δ. The manifold $\mathcal{M}$ is given as a graph over $\{E^c(\tau);\ \tau \in \mathbb{R}\}$ and depends smoothly on λ, c. In any section $\tau = \tau_0$, it is tangent to $E^c(\tau)$ at $\lambda = \lambda_0, c = c_0$.

Moreover we have

(i) *flow property: for any $\underline{u}_0 = \underline{u}(\tau_0, c, \lambda) \in \mathcal{M}$, there is a unique Y_δ-bounded solution $\underline{u}(\tau), \tau \in \mathbb{R}$ to (3.1) with $\underline{u}(\tau_0) = \underline{u}_0$.*

(ii) *invariance: this unique solution $\underline{u}(\tau)$ lies on $\mathcal{M}$ for all times τ and depends C^K on $\underline{u}_0, \tau, \lambda$ and c.*

(iii) *dimension: the dimension of $E^c(\tau)$ is $\dim R(\tilde{P}^c_+(\lambda_0, c_0)) + \dim \tilde{P}^c_+(\lambda_0, 0)\mathbb{R}^N$ where the second summand is the dimension of the range when restricted on the homogeneous N-dimensional subspace of H^l.*

(iv) *symmetry: the manifold $\mathcal{M}$ is invariant and the flow on $\mathcal{M}$ is equivariant under the diagonal action of $\mathcal{SO}(2)$ on $X = H^{l+1} \times H^l$.*

Remarks:

(i) Let us emphasize that the operator $D\Delta + \partial_\varphi$ has continuous spectrum close to the imaginary axis which makes a standard, finite-dimensional bifurcation approach to the dynamical reaction-diffusion problem impossible.

(ii) We will later give expansions for the spaces $E^c(\tau)$ at $\tau = \infty$ and describe how to obtain expansions for $\mathcal{M}$.

(iii) It is possible to treat the case of F depending on ∇u with the same methods. Indeed, both components of the gradient, u_r and $\frac{1}{r}u_\varphi$, are bounded with respect to $|(u, u_r)|_{X_\tau}$.

(iv) A slight generalization could be obtained by the use of interpolation spaces between X_τ and $D(\mathcal{A}(\tau))$. We avoided these additional technical difficulties for the sake of clarity.

(v) Making δ larger it is possible to allow singularities of the rotating waves at the origin, a phenomenon which is frequently attributed in the literature to spiral waves. The manifold will be larger if we allow for this type of solutions, but still finite-dimensional. However, the point in this work is, that even spiral wave like solutions without singularities at the tip are created via Hopf bifurcation.

4. The linearized equation. The key to a center-manifold theorem is the construction of exponential dichotomies for the linear equation. Background information on exponential dichotomies might be found in the textbook [2], in [15] or, in a non-evolutionary, elliptic context, in [16].

4.1. Bounded solutions for $\tau \to -\infty$. We construct a family of projections $P^{cu}_-(\tau)$ which project on the initial values of bounded solutions to (3.2) on $(-\infty, \tau]$. In a more general context this problem has been studied in [16]. The main theorems there (Theorem 1 and Theorem 3), applied to our setting, state the following:

LEMMA 2. *Under the conditions of Theorem 1, suppose $\tau_0, \tau_1, \tau \le 2\bar{r}$. Then there are families of evolution operators, smoothly depending on λ and c,*

$$\Phi^u_-(\tau, \tau_0) : X \to X, \quad \tau \le \tau_0$$
$$\Phi^s_-(\tau, \tau_0) : X \to X, \quad \tau \ge \tau_0$$

and constants $C > 0$, $\eta^u_- > \eta^s_- > 0$, such that

(i) $\Phi_-^{u/s}(\cdot,\tau_0)\underline{u}$ *is a solution of (3.2) for any* $\underline{u} \in X$,

(ii) $\Phi_-^{u/s}(\cdot,\cdot)\underline{u}$ *is continuous in* X,

(iii) $\Phi_-^u(\tau_0,\tau_0) + \Phi_-^s(\tau_0,\tau_0) = id$,

(iv) $\Phi_-^{u/s}(\tau,\tau_1)\Phi_-^{u/s}(\tau_1,\tau_0) = \Phi_-^{u/s}(\tau,\tau_0)$, $\Phi_-^{u/s}(\tau,\tau_1)\Phi_-^{s/u}(\tau_1,\tau_0) = 0$, *and*

(v) $|\Phi_-^u(\tau,\tau_0)|_{L(X,X)} \leq Ce^{-\eta_-^u(\tau-\tau_0)}$, $|\Phi_-^s(\tau,\tau_0)|_{L(X,X)} \leq Ce^{-\eta_-^s(\tau-\tau_0)}$, *and we can choose any* $\eta_-^u > 0$.

We define $P_-^{cu}(\tau) := \Phi_-^u(\tau,\tau)$.

We will later see how we can give a more explicit representation of the evolution operators Φ in terms of Bessel functions. This will also show why the uniqueness assumption from [16] is automatically satisfied in our context because the linear equation splits into an infinite product of ODE's, which are all uniquely solvable — in forward and in backward time.

4.2. Bounded solutions for $\tau \to +\infty$**.** The situation at $\tau = +\infty$ is considerably more difficult as $B_{\lambda,c}$ is no more τ-uniformly bounded with respect to $\tau^{-2}A$. It is due to our careful choice of norms in X_τ, that we still have an analogous result to Lemma 2.

LEMMA 3. *Under the conditions of Theorem 1, suppose* $\tau_0, \tau_1, \tau \geq 2\bar{r}$. *Then there are families of evolution operators, smoothly depending on* λ, c,

$$\Phi_+^u(\tau,\tau_0) : X_{\tau_0} \to X_\tau, \quad \tau \leq \tau_0$$
$$\Phi_+^s(\tau,\tau_0) : X_{\tau_0} \to X_\tau, \quad \tau \geq \tau_0$$

and constants $C > 0$, $\eta_+^u > \eta_+^s > 0$, *such that*

(i) $\Phi_+^{u/s}(\cdot,\tau_0)\underline{u}$ *is a solution of (3.2) for any* $\underline{u} \in X$,

(ii) $\Phi_+^{u/s}(\cdot,\cdot)\underline{u}$ *is continuous in* X,

(iii) $\Phi_+^u(\tau_0,\tau_0) + \Phi_+^s(\tau_0,\tau_0) = id$,

(iv) $\Phi_+^{u/s}(\tau,\tau_1)\Phi_+^{u/s}(\tau_1,\tau_0) = \Phi_+^{u/s}(\tau,\tau_0)$, $\Phi_+^{u/s}(\tau,\tau_1)\Phi_+^{s/u}(\tau_1,\tau_0) = 0$, *and*

(v) $|\Phi_+^u(\tau,\tau_0)|_{L(X_{\tau_0},X_\tau)} \leq Ce^{\eta_+^u(\tau-\tau_0)}$, $|\Phi_-^s(\tau,\tau_0)|_{L(X_{\tau_0},X_\tau)} \leq Ce^{\eta_+^s(\tau-\tau_0)}$, *and we can choose any* $\eta_+^s > 0$.

We define $P_+^{cs}(\tau) := \Phi_+^s(\tau,\tau)$.

Proof.

Step1: Fourier Ansatz

The proof of this lemma is the central part of our analysis. Complexifying X, the subspaces

$$E^k = \{(ue^{ik\varphi}, ve^{ik\varphi}) \in X;\ \underline{u} = (u,v) \in (\mathbb{C}^N)^2\} \leq X_\tau$$

are invariant under (3.2). Of course, we are primarily interested in the real subspace, where we have a relation between the vectors in E^k and E^{-k}. In E^k the differential equation reads

$$u'' + \frac{1}{\tau}u' - \frac{k^2}{\tau^2}u = -D^{-1}(F_u(\lambda,0) + cik)u =: B_{\lambda,c}^k u.$$

If we expand $\underline{u}(\tau) = \sum_{k\in\mathbb{Z}} \underline{u}^k(\tau)e^{ik\varphi}$, then $|\underline{u}(\tau)|_{X_\tau}$ is equivalent to $(\sum_{k\in\mathbb{Z}} |\underline{u}^k|_{E_\tau^k}^2)^{1/2}$, where

$$|\underline{u}^k|_{E_\tau^k} = k^l(\frac{1}{\tau}|ku^k|_{\mathbb{C}^N} + |k^{1/2}u^k|_{\mathbb{C}^N} + |v|_{\mathbb{C}^N})$$

if $k \neq 0$ and $|\underline{u}^0|_{E^k_\tau} = |\underline{u}^0|_{(\mathbb{C}^N)^2}$.

By the above considerations we see that it is sufficient to construct the evolution operators on E^k_τ,and uniform exponential bounds on the norms in E^k_τ will carry over to X_τ.

Step2: Projections

According to the remarks in § 2, we decompose E^k into $E^k_{c,+} = P^c_+ E^k$ and $E^k_{h,+} = (1 - P^c_+)E^k$, where $P^c_+ = \operatorname{diag}(\tilde{P}^c_+(\lambda_0, c_0), \tilde{P}^c_+(\lambda_0, c_0))$ and $\tilde{P}^c_+(\lambda_0, c_0)$ projects on the negative part of the spectrum of B_{λ_0,c_0}.

Step3: Stable projections, estimates

We show that all solutions in $E^k_{c,+}$ are exponentially bounded in E^k_τ with an arbitrarily small exponent δ — keeping λ, c sufficiently close to λ_0, c_0.

As the range of $\tilde{P}^c_+(\lambda_0, c_0)$ is finite-dimensional, only finitely many modes k are involved in the computation. We therefore use the equivalent,standard, k- and τ-independent norm on $(\mathbb{C}^N)^2$. Decomposing $B^k_{\lambda,c}$ furthermore into Jordan blocks, it is sufficient to consider

$$u'' + \frac{1}{\tau}u' - \frac{k^2}{\tau^2}u + \Lambda(k, \lambda, c)u = 0$$

where $\Lambda(k_0, \lambda_0, c_0)$ is a Jordan block. The eigenvalue of $\Lambda(k_0, \lambda_0, c_0)$ belongs to $\overline{\mathbb{R}}_+$ as $u \in R(\tilde{P}^c_+(\lambda_0, c_0)$. If we add $\alpha' = -\alpha^2$, then $\tau = 1/\alpha$ and we see that at $\lambda = \lambda_0$, $c = c_0$, the origin $u = 0, u' = 0$ and $\alpha = 0$ (alias $\tau = \infty$) is an equilibrium with all eigenvalues of the linearization being situated on the imaginary axis. Exponential growth with rate $\eta^s_+ > 0$ arbitrarily small now follows from standard Gronwall estimates for bounded α, that is, choosing $\bar{r}$ bounded away from zero, and λ, c sufficiently close to λ_0, c_0. This proves the second inequality in (v).

Step4: Unstable projections, estimates

Now let $\underline{u} \in E^k_{h,+}$. Our aim is to decompose $E^k_{h,+}$ in subspaces of exponentially decaying and exponentially growing solutions. We set

$$\tilde{u}(\tau) = (\frac{k^2}{\tau^2} + B^k_{\lambda_0,c_0})^{1/2}u(\tau).$$

As here $B^k_{\lambda,c}$ does not have eigenvalues on $\overline{\mathbb{R}}_-$, we can use the standard square root cut along $\mathbb{R}_-$. Moreover the norm $|\underline{u}|_{E^k_\tau}$ is equivalent to $|\tilde{u}|_{\mathbb{C}^N} + |v|_{\mathbb{C}^N}$. Note that here we omitted the factor k^l, as it is independent of time and does not change the equations to be considered below. We write $\alpha = 1/\tau$ and $L(\alpha) = (k^2\alpha^2 + B^k_{\lambda_0,c_0})^{1/2}$. In the new variables, the differential equation on $E^k_{h,+}$ reads

$$\begin{aligned} \tilde{u}' &= L(\alpha)v + \partial_\alpha L(\alpha)\alpha' u \\ &= L(\alpha)v - \alpha^3 k^2 L^{-2}(\alpha)\tilde{u} \\ v' &= -\alpha v + L(\alpha)\tilde{u} \\ \alpha' &= -\alpha^2. \end{aligned} \tag{4.1}$$

Next, we set $|L^{-1}(\alpha)|\frac{d}{d\tau} = \frac{d}{ds}$ and obtain

$$\begin{aligned} \frac{d\tilde{u}}{ds} &= L|L^{-1}|v - \alpha^3 k^2 L^{-2}|L^{-1}|\tilde{u} \\ \frac{dv}{ds} &= -\alpha|L^{-1}|v + L|L^{-1}|\tilde{u} \\ \frac{d\alpha}{ds} &= -\alpha^2|L^{-1}| \end{aligned} \tag{4.2}$$

with $L = L(\alpha)$. The linearization at $\tilde{u} = v = 0$, $\alpha = 0$,

$$\begin{aligned} \frac{d\tilde{u}}{ds} &= L|L^{-1}|v \\ \frac{dv}{ds} &= L|L^{-1}|\tilde{u} \\ \frac{d\alpha}{ds} &= 0. \end{aligned} \tag{4.3}$$

admits a projection $P(\tilde{u}, v) = \frac{1}{2}(\tilde{u} + v, \tilde{u} + v)$, which is independent of k and α. Therefore, the flow $\tilde{\Phi}_0$ of equation (4.3) possesses uniform exponential dichotomies at $\tilde{u} = v = 0$. To see this, we first observe that for $s \le s_0$,

$$|\tilde{\Phi}_0(s, s_0)P|_{L(\mathbb{C}^{2N})} \le |e^{L|L^{-1}|(s-s_0)}|_{L(\mathbb{C}^{2N})}.$$

Now remember that by definition of the square root, the spectrum of L lies in the right half plane and is, for $|k| \to \infty, \alpha = 0$, asymptotic to $k^{1/2}e^{\pm i\pi/4}$. For finitely many k, we therefore obtain

$$|e^{-L|L^{-1}|t}|_{L(\mathbb{C}^{2N})} \le C_1 e^{-\eta_1 t}, \quad t > 0$$

with some constants $C_1, \eta_1 > 0$, independent of k, α. As $k \to \infty$, we consider first $\tilde{L} = k^{-1/2}L$. Of course $\tilde{L}|\tilde{L}^{-1}| = L|L^{-1}|$. For k large, the operator $\tilde{L}_0 = (k\alpha^2 + D^{-1}ci)^{1/2}$ is a small (uniformly in α, k) perturbation of $\tilde{L}$. As $D > 0$, the spectrum of $D^{-1}i$ lies on $i\mathbb{R}^+$. Therefore the spectrum of $\tilde{L}_0$ lies in the right half plane, uniformly bounded away from the imaginary axis, and we can diagonalize $\tilde{L}_0$ by a transformation which is independent of α and k to obtain

$$|e^{-\tilde{L}_0|\tilde{L}_0^{-1}|t}|_{L(\mathbb{C}^{2N})} \le C_2 e^{-\eta_2 t}, \quad t > 0$$

for some constants $C_2, \eta_2 > 0$, independent of α, k. By perturbation arguments, the same estimate holds true for $\tilde{L}$ and L and we conclude

$$|\tilde{\Phi}_0(s; s_0)P|_{L(\mathbb{C}^{2N})} \le Ce^{\eta(s-s_0)}, \quad s \le s_0$$

for some $C, \eta > 0$, independent of α and k. The calculation on $R(1-P)$ is the same and we obtain

$$|\tilde{\Phi}_0(s; s_0)(1-P)|_{L(\mathbb{C}^{2N})} \le |e^{-L|L^{-1}|(s-s_0)}|_{L(\mathbb{C}^{2N})} \le Ce^{-\eta(s-s_0)}, \quad s \ge s_0.$$

These two estimates together guarantee an exponential dichotomy for the equation (4.3). Equation (4.2) is a perturbation of (4.3). We show that the perturbation of the vector field is $O(\alpha)$, uniformly in k. By standard perturbation results on exponential dichotomies [2] this then proves that (4.2) possesses an exponential dichotomy with projection $\tilde{P}(k, \alpha)$, and constants $\tilde{C}, \tilde{\eta} > 0$, independent of k, α as long as α is bounded.

The error terms we have to deal with are $\alpha^3 k^2 L^{-2}|L^{-1}|$ and $\alpha|L^{-1}|$. Of course for finite k these terms are $O(\alpha)$. Consider now the first expression for large k:

$$\begin{aligned} \alpha^3 k^2 |L^{-3}| &= \alpha^3 k^2 |[\alpha^2 k^2 + B^k_{\lambda_0, c_0}]^{-3/2}| \\ &= \alpha^3 k^2 |[\alpha^2 k^2 + D^{-1}cik + O(1)]^{-1}| \cdot |D^{-1}cik + O(1)|^{-1/2} \\ &= |[1 + D^{-1}ci\frac{1}{\alpha^2 k}(1 + O(1/k))]^{-1}| \cdot O(\alpha k^{-1/2}). \end{aligned}$$

As $|[1+D^{-1}ci\frac{1}{\alpha^2 k}]^{-1}| \le C_3$ uniformly in α, k, the above expression is $\mathrm{O}(\alpha k^{-1/2})$, uniformly in k. Next we consider the second error term $\alpha|L^{-1}|$:

$$\begin{aligned}\alpha|L^{-1}| &= \alpha|[\alpha^2k^2 + D^{-1}cik + \mathrm{O}(1)]^{-1/2}| \\ &= \alpha k^{-1/2}|\tilde{L}_0^{-1}(1+\mathrm{O}(1/k))| \\ &= \mathrm{O}(\alpha k^{-1/2}).\end{aligned}$$

This proves uniform smallness of the perturbation. It remains to translate the exponential dichotomy rate $\tilde{\eta}$ into the correct time $\tau = \tau(s)$.

As $\frac{ds}{d\tau} = |L^{-1}(\alpha)|^{-1}$, it is sufficient to get α, k-uniform bounds $|L^{-1}(\alpha)|^{-1} \ge \eta_0 > 0$. This is precisely the type of estimate we developed above for $\alpha|L^{-1}|$. Indeed we showed that

$$|L^{-1}| = k^{-1/2}|\tilde{L}_0^{-1}|(1+\mathrm{O}(1/k))$$

and therefore η_0 can be chosen $\mathrm{O}(k^{1/2})$ as $k \to \infty$. This proves the lemma with $\eta_+^u = \eta_0\eta$ and η_+^s from step 3. □

4.3. Matching at $\bar{\tau} = 2\bar{r}$, the center space $E^c(\tau)$. We define $E^c(\tau) = \Phi_+^s(\tau,\tau)\Phi_-^u(\tau,\tau)X$, which, by the previous two lemmata, coincides with the definition of $E^c(\tau)$ as the initial values for Y_δ-bounded solutions, if we only choose δ small enough. In order to prove the claim on the dimension of $E^c(\tau)$, we need a transversality result from the theory of Bessel functions. Suppose first that $\underline{u}(\bar{\tau}) \in E^0$, the subspace of radially homogeneous functions. For $\tau \to -\infty$ the linear equation in E^0 is $u_{\tau\tau} = e^{2\tau}\Lambda u$ with some matrix Λ and clearly any solution is Y_δ-bounded as exponential rates of solutions coincide with the rates of the autonomous part $u_{\tau\tau} = 0$. So the negative orbit of $\underline{u}(\bar{\tau})$ is Y_δ-bounded. The positive orbit is bounded in Y_δ if and only if $\underline{u}(\bar{\tau}) \in P_+^c(\lambda_0, 0)E^0$; therefore $\dim E^c \cap E^0 = 2\dim \tilde{P}_+^c(\lambda_0,0)\mathbb{R}^N$. For the rest of this section we restrict to $(E^0)^\perp$, the non-homogeneous Fourier modes. Recall that $P_+^c = \mathrm{diag}\,(\tilde{P}_+^c(\lambda_0,c_0), \tilde{P}_+^c(\lambda_0,c_0))$ and $P_+^h = 1 - P_+^c$ projects on the hyperbolic part of (3.2) at $r = \infty$. We claim that

$$\Phi_+^s(\bar{\tau},\bar{\tau})\Phi_-^u(\bar{\tau},\bar{\tau})P_+^h\underline{u} = 0 \tag{4.4}$$

for $\underline{u} \in (E^0)^\perp$. We decompose into Fourier modes $e^{ik\varphi}$ and minimal Jordan blocks Λ, and we consider

$$u'' + \frac{1}{\tau}u' - \frac{k^2}{\tau^2}u + \Lambda(k,\lambda_0,c_0)u = 0.$$

If Λ is semi-simple, that is $\Lambda \in \mathbb{C}\setminus\overline{\mathbb{R}}_+$, then the solutions of this scalar ODE are the Bessel functions. Indeed, we can write this equation as

$$r^2u'' + ru' + (-k^2 + (r\sqrt{\Lambda})^2)u = 0$$

and therefore

$$u(r) = u_0 J_k(r\sqrt{\Lambda}) + u_1 Y_k(r\sqrt{\Lambda}).$$

As $J_k(r) = r^k(1+\mathrm{O}(r))$ and $Y_k(r) = r^{-k}(1+\mathrm{O}(r))$ for $r \to 0$, if $k \neq 0$, solutions bounded close to $r = 0$ satisfy $u_1 = 0$. At infinity the J_k behave like

$$J_k(r) = \sqrt{\frac{2}{\pi r}}[\cos(r - \frac{k\pi}{2} - \frac{\pi}{4}) + \mathrm{O}(1/r)].$$

Solutions $u(r) = u_0 J_k(r\sqrt{\Lambda})$ can only stay exponentially bounded by $e^{\delta r}$ as $r \to \infty$, for a small fixed δ, if $\sqrt{\Lambda}$ is real. But then Λ is real and positive, that is $u \in \tilde{P}^c_+ X$. This proves the required transversality result (4.4) for semi-simple eigenvalues.

If Λ is a Jordan block we can rescale the principal vectors — without changing the angle between stable and unstable subspaces — to make it a small perturbation of its semi-simple part. The transverse intersection persists for the non semi-simple Jordan block.

Now suppose $u \in \tilde{P}^c_+ X$. Then the above reasoning showed that for any such u there is exactly one Y_δ-bounded solution. This implies $\dim(E^c(\tau) \cap (E^0)^\perp) = \dim R(\tilde{P}^c_+(\lambda_0, c_0)(E^0)^\perp)$ and proves the claim (iii) in Theorem 1 on the dimension of the invariant manifold, once it is constructed as a graph over $\{E^c(\tau);\ \tau \in \mathbb{R}\}$.

5. Nonlinear equations, Proof of Theorem 1. With the estimates on the linearized equation at hand, it is fairly standard to construct invariant manifolds for the nonlinear equation. We consider equation (3.1).

PROPOSITION 4. *Under the conditions of Theorem 1, any Y^-_δ-bounded (or Y^+_δ-bounded) solution $\underline{u}(\tau, \tau_0)$ on $(-\infty, \tau_0]$ (or $[\tau_0, +\infty)$ respectively) satisfies*

$$\underline{u}(\tau,\tau_0) = \Phi^u_-(\tau,\tau_0)\underline{u}(\tau_0,\tau_0) + \int_{\tau_0}^{\tau} \Phi^u_-(\tau,\sigma)\mathcal{G}(\underline{u}(\sigma,\tau_0))d\sigma + \int_{-\infty}^{\tau} \Phi^s_-(\tau,\sigma)\mathcal{G}(\underline{u}(\sigma,\tau_0))d\sigma,$$

or

$$\underline{u}(\tau,\tau_0) = \Phi^s_+(\tau,\tau_0)\underline{u}(\tau_0,\tau_0) + \int_{\tau_0}^{\tau} \Phi^s_+(\tau,\sigma)\mathcal{G}(\underline{u}(\sigma,\tau_0))d\sigma + \int_{-\infty}^{\tau} \Phi^u_+(\tau,\sigma)\mathcal{G}(\underline{u}(\sigma,\tau_0))d\sigma,$$

respectively. On the other hand, the above integral equations possess for any $\underline{u}(\tau_0, \tau_0)$ a unique solution $\underline{u}(\tau, \tau_0)$ in $Y^\pm_\delta$ which depends C^K on $\underline{u}(\tau_0, \tau_0), \lambda, c, \tau$ and τ_0.

Proof. The integral operators are bounded operators on $Y^\pm_\delta$ and the Lipschitz constant of the nonlinearity $\mathcal{G}$ is small. Indeed

$$\mathrm{Lip}_{X_\tau}\mathcal{G} \le \mathrm{Lip}_{H^{l+1/2}\to H^l}[\tilde{F} - \tilde{F}_u]$$

which was supposed to be sufficiently small. Regularity of the unique fixed point can be proved as usually for center-manifolds; see [24] for example. ∎

We call the set

$$\{\Phi^u_-(\tau,\tau)\underline{u} + \int_{-\infty}^{\tau} \Phi^s_-(\tau,\sigma)\mathcal{G}(\underline{u}(\sigma,\tau))d\sigma =: \Psi_-(\Phi^u_-(\tau,\tau)\underline{u});\ \underline{u} \in X\}$$

the center-unstable manifold $\mathcal{M}^{cu}_-(\tau)$ at $-\infty$ and the set

$$\{\Phi^s_+(\tau,\tau)\underline{u} + \int_{\infty}^{\tau} \Phi^u_+(\tau,\sigma)\mathcal{G}(\underline{u}(\sigma,\tau))d\sigma =: \Psi_+(\Phi^s_+(\tau,\tau)\underline{u});\ \underline{u} \in X\}$$

the center-stable manifold $\mathcal{M}^{cs}_+(\tau)$ at $+\infty$ and we define

$$\mathcal{M}(\tau) = \mathcal{M}^{cu}_-(\tau) \cap \mathcal{M}^{cs}_+(\tau).$$

By definition, $\mathcal{M}(\tau) = \{$ initial values at time τ of Y_δ-bounded solutions $\}$. We have to show that $\mathcal{M}(\tau)$ is a smooth manifold, parameterized over $E^c(\tau)$.

Therefore, we have to solve $\Psi_+ - \Psi_- = 0$. The linearization is given by $\Phi^u_- - \Phi^s_+ = 0$. We already know that the kernel of this equation is exactly E^c, thus finite-dimensional. In order to apply the implicit function theorem we have to show that $\Phi^u_- - \Phi^s_+$ is surjective. We have to decompose a $\underline{u} \in E^k$ into two vectors belonging to the range of Φ^s_+ and Φ^u_- respectively, with estimates on the norms uniform with respect to k. The fact that we can decompose follows simply from the linear independence of the Bessel functions of the first and second kind J_k and Y_k (actually, we merely refer to purely imaginary arguments, the hyperbolic case, where the notation is I_k for the Bessel function bounded at $r = 0$, and K_k for the solution bounded at $r = \infty$). Estimates on the norms — for a fixed time τ — follow from uniform estimates on the Wronski-determinant

$$\det \begin{pmatrix} I_k(\tau) & K_k(\tau) \\ I_k'(\tau) & K_k'(\tau) \end{pmatrix}$$

which in turn are an immediate consequence of the Taylor expansions at $r = 0$ of the Bessel functions; see for example [25]. As in §4.3, Jordan blocks can be considered as a small perturbation. By Lyapunov-Schmidt reduction we can now solve $\Psi_+ - \Psi_- = 0$, parameterizing the set of solutions over the kernel of the linearization $E^c(\tau)$. This proves Theorem 1.

6. Local center-manifolds. If the nonlinearity $\tilde{F}$ does not have a small Lipschitz constant, which is usually the case in applications, we have to modify $\tilde{F}$.

We cut off $\tilde{F}$ outside a small neighborhood B_{ϵ_0} of zero with a smooth cut-off function in $H^{l+1/2}$, for example the norm, which is invariant under the action of $\mathcal{SO}(2)$. Therefore let $\chi \in C^\infty([0,\infty), \mathbb{R})$ with $\chi(t) = 1$ if $t \leq 1$ and $\chi(t) = 0$ if $t \geq 2$. Then define

$$\tilde{F}_{mod}(\lambda, u) = \chi(|u|^2_{H^{l+1/2}}/\epsilon_0)(\tilde{F}(\lambda, u) - \tilde{F}_u(\lambda, 0)u) + \tilde{F}_u(\lambda, 0)u.$$

The nonlinear part of $\tilde{F}_{mod}$ has an arbitrarily small Lipschitz constant if we make ϵ_0 sufficiently small, and thereby satisfies the conditions of Theorem 1. Any solution on the center-manifold to the modified nonlinearity $\tilde{F}_{mod}$, which has norm $\sup_\tau |\underline{u}(\tau)|_{X_\tau}$ small enough, will have $\sup_\tau |u(\tau)|_{H^{l+1/2}}$ small such that the modified nonlinearity coincides with the original nonlinearity on the solution $u(\tau)$, which is in consequence a solution to the original equation. Note that bounds on the norm in X_τ are by construction of $\mathcal{M}$ equivalent to bounds on the norms of the projection of the solution on $\{E^c(\tau);\ \tau \in \mathbb{R}\}$.

7. Regularity of solutions. The solutions $u(r, \varphi)$ we obtain are bounded in X_τ. By the smoothing property of the equation (which can be considered for any l, without changing $\mathcal{M}$), any solution is actually of class C^∞ with respect to $r > 0$ and φ, if F is — though $\mathcal{M}$ is not C^∞ in general! As $r \to \infty$, the angular derivatives $\partial^m_\varphi u(r, \varphi)$ are bounded for any m, which implies that the derivatives along curves $r \equiv const$ with respect to arclength $r d\varphi$ are of order $1/r^m$: patterns are slowly varying in the angular direction far away from the origin.

At $r = 0$ we have to be careful about smoothness of the solution. Suppose first that $E^c(\tau)$ does not contain solutions in the angular homogeneous subspace E^0. Then solutions in $E^c(\tau)$ are $\mathrm{O}(r) = \mathrm{O}(e^\tau)$ as $r \to 0$ and smooth in a neighborhood of the origin by interior elliptic regularity.

The homogeneous subspace can be — and has been — treated separately studying the ODE on Fix($\mathcal{SO}(2)$). Indeed there is a subspace of dimension N with solutions which actually stay bounded, whereas solutions outside this subspace have singularities of order $\log r$.

On the other hand, considering again τ-dynamics in $\mathcal{M}$, this subspace of homogeneous functions is fibered by strongly unstable fibers such that any solution in $\mathcal{M}$ converges with rate $\mathrm{O}(e^{\tau})$ to a solution in the homogeneous subspace and inherits its regularity.

8. Center-manifolds at infinity. We construct a finite-dimensional invariant manifold which contains all solutions which are bounded at $\tau = +\infty$ but do not decay too rapidly. Recall that $P_+^c = \mathrm{diag}\,(\tilde{P}_+^c, \tilde{P}_+^c)$ projects on the center part of (3.2) at $\alpha = 1/r = 1/\tau = 0$.

PROPOSITION 5. *Under the conditions of Theorem 1, consider equation (3.1) close to* $\underline{u} = 0$. *Fix* $\delta > 0$ *sufficiently small and* $K < \infty$.

Then there is an invariant C^K*-manifold* $\mathcal{M}_+^c$, *contained in* $\mathcal{M}_+^{cs}$ *and containing* $\mathcal{M}$, *given as a graph over* $\{R(P_+^c);\ \tau \in \mathbb{R}\}$, *smoothly depending on* λ, c.

Moreover there is a C^K*-flow on* $\mathcal{M}_+^c$ *such that any orbit is a solution of (3.1) and any solution* $\underline{u}(\tau)$ *of (3.1) with*

$$\sup_{\tau_0 \geq \tau > 2\bar{\tau}} e^{-\delta|\tau-\tau_0|} \frac{|\underline{u}(\tau)|_{X_\tau}}{|\underline{u}(\tau_0)|_{X_{\tau_0}}} < \infty$$

is contained in $\mathcal{M}_+^c$.

Proof. We start by constructing $\mathcal{M}_+^c$ for $0 < \alpha = 1/r \leq 1/2\bar{r}$ bounded. The manifold $\mathcal{M}_+^c$ is the union of center-unstable fibers of the zero-solution in the center-stable manifold $\mathcal{M}_+^{cs}$. These fibers can easily be shown to exist, using graph transformation (we have a smooth semi-flow on $\mathcal{M}_+^{cs}$) or a Lyapunov-Perron approach as in [16]. The dependence on time $\alpha = 1/\tau$ is smooth as fibers are mapped into each other by the flow.

We have to ensure that we can arrange to have $\mathcal{M}$ included in $\mathcal{M}_+^c$. This can be achieved by either starting the graph transformation with graphs that contain $\mathcal{M}$ (and 'feeding in' such graphs appropiately) or, referring to the Lyapunov-Perron approach of [16], including the manifold $\mathcal{M}$ in the fixed initial unstable fiber at $\tau = 2\bar{r}$ (see for example [16], at the end of §3).

We next have to continue this manifold for $\alpha > 1/2\bar{r}$, or, equivalently, for $t \to -\infty$. This will again be done using the methods from [16]. If we had an evolution type equation we would propagate the manifold $\mathcal{M}_+^c$ with the flow. Here we do not have a flow! By [16], the equation possesses an exponential dichotomy which permits to prove the existence of the center-unstable manifold $\mathcal{M}_-^{cu}$ (the union of unstable fibers over time τ), as pointed out in Lemma 2 and, furthermore, the existence of stable fibrations to any solution in $\mathcal{M}_-^{cu}$ for any fixed initial fiber at $\tau = 2\bar{r}$ (which is transversely intersecting $\mathcal{M}_-^{cu} \cap \{\tau = 2\bar{r}\}$). We are interested in the stable fibration induced by the manifold $\mathcal{M}_+^c$, which is of course not complementary to $\mathcal{M}_-^{cu}$. However the methods from [16] can be adapted in order to guarantee precisely the existence of such a manifold. In the following we indicate how to make the necessary changes.

We solve the integral equation for stable and unstable fibrations with the restriction that the fiber at the initial time $\tau = 2\bar{r}$ belongs to a fixed manifold transverse to $\mathcal{M}_-^{cu}$ which we can choose to contain $\mathcal{M}_+^c \cap \{\tau = 2\bar{r}\}$. On this smaller subspace the fixed point equation for stable and unstable fibers still defines a contraction mapping and the solution is the desired global continuation of $\mathcal{M}_+^c$. The smoothness of the union of the fibers as a manifold follows, because we can differentiate the fixed point equation with respect to the base solution in the center-unstable manifold $\mathcal{M}_{-}^{cu}$. The exponential properties of the new fixed point equation allow for a setting in the usual scale of exponentially weighted spaces [23], because the equation for the stable fiber at a fixed time τ only involves the finite time interval $[\tau, 1/2\bar{r}]$. We do not carry out the details which include only straightforward modifications of smoothness proofs for fibrations (note however that we do not care about the limit $\tau = -\infty$ — alias $r = 0$ — of the fibration which would lead to limitation in regularity of the fibration).

Of course the projected vector field is also smooth and thereby defines a smooth flow on the finite-dimensional manifold $\mathcal{M}_{+}^{c}$. □

The hypothesis $\tilde{F}(\lambda, 0) = 0$ was only needed in order to fix a reference solution in $\mathcal{M}_{+}^{cs}$, notably the zero solution. In general we could construct smooth fibrations along any solution in $\mathcal{M}_{+}^{cs}$.

The manifold $\mathcal{M}_{+}^{c}$ we constructed is very useful in order to describe bounded solutions near infinity, though most solutions on $\mathcal{M}_{+}^{c}$ are not bounded at the origin $r = 0$.

9. Hopf bifurcation and (λ, ω)-systems. We give the most simple non-trivial application of our main theorem. Suppose $D = id$, $\tilde{F}(\lambda, 0) = 0$, $\lambda \in \mathbb{R}$ and $N = 2$, that is $U \in \mathbb{R}^2$ which we identify with $\mathbb{C}$. Suppose that the homogeneous zero state undergoes a non-degenerate Hopf bifurcation in the space of homogeneous solutions:

$$\frac{d}{dU}\tilde{F}(\lambda, 0) = i\omega + \lambda, \quad \omega \neq 0.$$

We write U as a complex Fourier series $U(r, \varphi) = \sum_{k \in \mathbb{Z}} U^k(r) e^{ik\varphi}$. The spaces E^k are just the complex two-dimensional spans $\langle (e^{ik\varphi}, 0), (0, e^{ik\varphi}) \rangle$. The operator $B_{\lambda,c}$ acts on E^k as multiplication $B^k(\lambda, c) : U^k \to (cik - i\omega - \lambda)U^k$. Thereby $E^c(r) \leq E^{k_0}$ if $c_0 k_0 = \omega$. In other words, for any k(-armed spiral) there is a rotation speed $c = \omega/k$ such that rotating waves with this speed may bifurcate. Our analysis has shown that for other wave speeds, the homogeneous state is isolated as a rotating wave.

Let us comment on the symmetry. The flow on $\mathcal{M}$ projected on $E^c(r) \leq E^k$ is equivariant with respect to the action of $\mathcal{SO}(2)$:

$$(U, U') \to (Ue^{i\psi}, U'e^{i\psi}), \qquad \psi \in \mathbb{R}/2\pi\mathbb{Z} \simeq \mathcal{SO}(2).$$

This is exactly the same symmetry that authors usually *assumed* to be present in bifurcation equations, the so-called (λ, ω)-systems, modeling the creation of spiral waves; see [1]. We showed *rigorously* that the symmetry of (λ, ω)-systems, without any error term, is present in this type of bifurcations.

The actual solutions $U = U^k(r)e^{ik\varphi}$ of the linearized system in $E^c(\tau)$ are easily calculated: they solve $(U^k)'' + \frac{1}{r}(U^k)' = (k^2/r^2)U^k$ and are given as $U(r, \varphi) = U^c r^k e^{ik\varphi}, U^c \in \mathbb{C}$. Note that the invariant complement in E^k, spanned by $\tilde{U}(r, \varphi) = \tilde{U}^c r^{-k} e^{ik\varphi}, \tilde{U}^c \in \mathbb{C}$ converges as $E^c(r)$ to the same limit $\{(U, U');\ U' = 0\}$. This is the reason why we constructed $\mathcal{M}_{+}^{c}$ tangent to E^k in § 8. The equation on $\mathcal{M}_{+}^{c}$ is a non-autonomous, $\mathcal{SO}(2)$-equivariant ODE in $\mathbb{C}^2$ with linear part given by Bessel's differential equation. It can be smoothly extended to time $\tau = \infty (\alpha = 0)$ where the equation becomes autonomous. In order to determine existence and shape of rotating waves at $r = \infty$, we have to calculate expansions of the vector field on $\mathcal{M}_{+}^{c}$ and determine the ω-limit set of the two-dimensional slice $\mathcal{M}(\tau)$ in $\mathcal{M}_{+}^{c}$. We examine a simple model problem in the next sections.

10. An example. As an example we study the following reaction-diffusion system

$$\begin{aligned} u_t &= d_1\triangle u + \kappa u - v - au^3 \\ v_t &= d_2\triangle v + bu - \gamma v \end{aligned}$$

in the plane $x \in \mathbb{R}^2$. When $\kappa = \gamma$ and $b - \gamma\kappa > 0$, the pure reaction system undergoes a Hopf bifurcation in the origin $u = v = 0$. Rescaling u, v, t and x, we may assume the system to be in the particular form

$$\begin{aligned} u_t &= \triangle u + \alpha u - \beta v - au^3 \\ v_t &= \nu\triangle v + \beta u - \alpha v + \lambda v \end{aligned}$$

with $\beta^2 - \alpha^2 = 1$ and $\alpha, \beta > 0$. We assume in the following that λ is close to zero, that is, we are close to a Hopf bifurcation with eigenvalues $\pm i$ of the linearized reaction system. The rotating wave ansatz yields

$$\begin{array}{rcl} cu_\varphi &=& \triangle_{r,\varphi} u + \alpha u - \beta v - au^3 \\ cu_\varphi &=& \nu\triangle_{r,\varphi} v + \beta u - \alpha v + \lambda v \end{array} \tag{10.1}$$

where $\triangle_{r,\varphi} = \partial_{rr} + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\varphi\varphi}$. The linearization at $\lambda = 0, u = v = 0$ is

$$\begin{array}{rcl} cu_\varphi &=& \triangle_{r,\varphi} u + \alpha u - \beta v \\ cv_\varphi &=& \nu\triangle_{r,\varphi} v + \beta u - \alpha v. \end{array} \tag{10.2}$$

We now expand the solutions in Fourier series with respect to φ

$$(u, v) = \sum_{k\in\mathbb{Z}} (u^k, v^k) e^{ik\varphi}, \quad (u^{-k}, v^{-k}) = (\overline{u^k}, \overline{v^k}).$$

The linearization (10.2) then becomes an uncoupled system of ODEs for the Fourier coefficients

$$\begin{array}{rcl} \triangle_{r,k} u^k &=& (cik - \alpha)u^k + \beta v^k \\ \nu\triangle_{r,k} v &=& -\beta u^k + (cik + \alpha)v^k \end{array}$$

where $\triangle_{r,k} = \partial_{rr} + \frac{1}{r}\partial_r - \frac{k^2}{r^2}$. The right side has a kernel as a linear operator on $\mathbb{C}^2$ whenever $ck = 1$ and we therefore set $c = 1/k_0 + \mu$ with μ close to zero, having fixed $k_0 \in \mathbb{N}$ for the sequel.

Remember that together with the above equations we should write the equations for the complex conjugates, which are just the conjugate equations.

The eigenvector in the kernel is easily calculated as

$$w_0 = \beta u^k + \nu(i - \alpha)v^k, \quad \triangle_{r,k} w_0 = 0,$$

and

$$w_1 = -\beta u^k + (i + \alpha)v^k, \quad \triangle_{r,k} w_1 = (i - \alpha + \frac{i + \alpha}{\nu})w_1$$

is the complementary eigenvector to the eigenvalue $i - \alpha + (i + \alpha)/\nu$.

Proposition 5 implies the existence of a center-manifold $\mathcal{M}_+^c$ with a smooth vector field, tangent to the span of $w_0 e^{ik_0\varphi}$ and $\partial_r w_0 e^{ik_0\varphi}$ at any 'time' r. The vector field is obtained up to third order using the following strategy:

(i) write the linear equation for w_0, depending on parameters λ, μ; this gives the linear part of the vector field on $\mathcal{M}^c_+$.

(ii) calculate the quadratic (in w_0) expansion of $\mathcal{M}^c_+$ depending on time; this is zero, due to the absence of quadratic terms in the reaction.

(iii) evaluate the nonlinearity au^3 on $w_0 e^{ik_0\varphi}$.

(iv) project away non-critical Fourier modes.

(v) project on $< w_0 e^{ik_0\varphi} >$ along $< w_1 e^{ik_0\varphi} >$.

Carrying out the necessary calculations gives first, by projecting away the non-critical Fourier modes,

$$\triangle_{r,k} u^k = (i-\alpha)u^k + \beta v^k + i\mu u^k + a u^k |u^k|^2$$
$$\nu \triangle_{r,k} v = -\beta u^k + (i + \alpha v^k) + i\mu v^k - \lambda v^k$$

and therefore

$$\triangle_{r,k} w_0 = i\mu\beta u^k + \beta a u^k |u^k|^2 - (i-\alpha)\lambda v^k + (i-\alpha) i\mu v^k.$$

Transforming back

$$u^k = \frac{i+\alpha}{\beta(i+\alpha+\nu(i-\alpha))} w_0 + \mathrm{O}(w_1)$$
$$v^k = \frac{1}{i+\alpha+\nu(i-\alpha)} w_0 + \mathrm{O}(w_1)$$

gives, on $\mathcal{M}^c_+$, up to third order, the second order in time ordinary differential equation

$$\triangle_{r,k} w_0 = \frac{-2\mu - (i-\alpha)\lambda}{i+\alpha+\nu(i-\alpha)} w_0 + \frac{a}{\beta^2} \frac{1}{1+\nu\frac{i-\alpha}{i+\alpha}} \left| \frac{1}{1+\nu\frac{i-\alpha}{i+\alpha}} \right|^2 w_0 |w_0|^2.$$

The fifth order terms might of course destroy the second order structure of this equation, though keeping the structure of a local non-autonomous differential equation in $\mathbb{C}^2$.

We write new parameters $\lambda', a' \in \mathbb{C}$ such that the truncated equation takes the form

$$(w_0)_{rr} + \frac{1}{r}(w_0)_r - \frac{k^2}{r^2} w_0 = \lambda' w_0 + a' w_0 |w_0|^2. \tag{10.3}$$

Disregarded all our efforts in reducing and simplifying the problem, this equation is in general still hard to solve analytically. In the following section we study this problem, obtaining existence of bounded solutions $w(r)$ (and thereby solutions $(u(r,\varphi), v(r,\varphi))$ to (10.1)), when a' is almost real. This is actually the approach taken by [7, 11], who deal with a similar system.

By our explicit calculations, the imaginary part of a' will be small if the diffusion rate ν or the parameter α is close to zero.

The first condition has an interesting interpretation as the limit $d_2 \to 0$ is exactly the interesting limit in excitable media, though we admit that our equation is different from the typical models for excitable media (the null-clines of $\alpha u - au^3 - v$ are symmetric to the origin whereas this is not the usual assumption for excitable media, modeled for example by the Fitz-Hugh-Nagumo equation). We refer the reader to the interesting, though formal, work on spiral waves in excitable media reviewed in [22].

The second, alternative condition is merely an assumption on the location of equilibria in the pure reaction system, which are situated approximately at $u \sim \pm\sqrt{b/(a\alpha)}$ and zero.

The important point to notice at this stage is that the full equation on $\mathcal{M}^c_+$ is a small perturbation of the truncated equation close to the bifurcation point, that is, close to $\mathrm{Re}\,\lambda' = 0$. Indeed, scaling $|\,\mathrm{Re}\,\lambda'|r^2 = \tilde{r}^2$ and $w_0^2 = |\,\mathrm{Re}\,\lambda'|\tilde{w_0}^2$ makes the higher order terms $\mathrm{O}(|\,\mathrm{Re}\,\lambda'|)$. Structurally stable dynamics of the truncated equation persist for the full system on $\mathcal{M}^c_+$ for sufficiently small $|\,\mathrm{Re}\,\lambda'|$.

In this sense, we have established a rigorous proof of the validity of approximations of reaction-diffusion systems by λ-ω systems, at least when we restrict to the question of existence of rotating wave solutions. This was proved up to now only using formal multi-scale methods. The advantage of our approach is that it gives rigorous proofs and information on the domain of validity in parameter space of such kinds of approximations.

Furthermore, we should comment on the symmetry. The equation possesses, as announced, an $\mathcal{SO}(2)$-symmetry $w_0 \to w_0 e^{i\theta}$, $\theta \in S^1$. The additional reflection $(u, v) \to (-u, -v)$ in the original reaction-diffusion system does not yield any more symmetry in the bifurcation equation.

At $\mathrm{Im}\,a' = \mathrm{Im}\,\lambda' = 0$, there is the additional reflectional symmetry $w_0 \to \bar{w}_0$, fixing the real subspace. Note also that $\mathrm{Im}\,\lambda' = 0$ can be achieved by adjusting the wave speed c.

11. The bifurcation equations. During this section we omit the primes of λ and a. We begin with a study of possible asymptotic states of (10.4) at $r = \infty$. The limit equation

$$w'' = \lambda w + a w |w|^2, \quad {}' = \frac{d}{dr}$$

can be simplified by dividing out the symmetry with the new coordinates $z = z_R + i z_I = w'/w \in \mathbb{C}$ and $R = |w| \in \mathbb{R}_+$:

$$\begin{aligned} R' &= z_R R \\ z' &= -z^2 + \lambda + aR^2. \end{aligned}$$

Reversibility of the w-equation $(r \to -r)$ is translated into reversibility with respect to the reflection $z \to -z$ (and of course $r \to -r$). Any equilibrium of (11.1) corresponds to a periodic orbit of the w-equation which we call a rotating wave, as it is a relative equilibrium, for the dynamics in r, with respect to rotational symmetry $\mathcal{SO}(2)$. The asymptotic shape of a spiral wave behaving like such a rotating wave for large r is just a one-dimensional periodic wave-train, translation invariant in one space-direction. There are two types of equilibria. Type I has $R = 0$ and corresponds to the origin of the w-equation, and $z = \pm\sqrt{\lambda}$ are the blown up invariant manifolds of the equilibrium $w = w' = 0$. Type II has necessarily $z_R = 0$ and

$$R^2 = -\lambda_I / a_I, \quad z_I^2 = -\lambda_R + a_R \lambda_I / a_I.$$

A linear stability analysis gives that the type I equilibrium with $\mathrm{Re}\,\sqrt{\lambda} > 0$ is stable in $\{R = 0, z \in \bar{\mathbb{C}}\}$ and unstable in the direction of R. The equilibrium with $\mathrm{Re}\,\sqrt{\lambda} < 0$ is unstable in $\{R = 0, z \in \bar{\mathbb{C}}\}$ but stable in the direction of R. Along the type II equilibria the linearization is

$$L = \begin{pmatrix} 0 & R & 0 \\ 2a_R R & 0 & 2z_I \\ 2a_I R & -2z_I & 0 \end{pmatrix}, \qquad \det L = -4\lambda_I z_I, \quad \mathrm{trace}\, L = 0,$$

such that one equilibrium is 2d-unstable and the other is 2d-stable.

Bifurcations occur at $\mathrm{Re}\sqrt{\lambda} = 0$ where type I equilibria coalesce, the origin becomes a center, and when $a\bar{\lambda} \in \mathbb{R}$, where a reversible saddle-node bifurcation of the type II equilibria occurs.

For the non-autonomous system, we can interpret the manifold $\mathcal{M}$ as a shooting manifold, which is two-dimensional in (w, w')-space at any fixed time r, invariant under the symmetry and therefore yields a one-dimensional shooting curve in the reduced phase space (z, R). We focus here on asymptotically stationary behavior where the shooting curve intersects the stable manifold of an equilibrium of (11.1). These are possibly not the only asymptotic shapes at large distances from the center of rotation but they seem to be of sufficient physical relevance making reasonable such a restriction.

In the following, we distinguish two different cases which we refer to as the subcritical case, when $a_R > 0$, and the supercritical case, when $a_R < 0$. These terms are justified by the branching of equilibria of (11.1) at $\lambda_I = a_I = 0$. In our model problem of the preceding section, these two cases are distinguished by the sign of $a(1 - \nu(\alpha^2 - 1)/\beta^2)$.

We now study the real sub-system in the non-autonomous setting.

LEMMA 6. *(Supercritical)[5, 11] Suppose $a_R < 0$ and $\lambda_R > 0$. Then for any wave number $k_0 \in \mathbb{N}$, there exists a heteroclinic orbit $w(r) > 0$, with $\lim_{r\to 0} w(r) = 0$ and $\lim_{r\to\infty} w(r) = \sqrt{-\lambda_R/a_R}$. Moreover the heteroclinic orbit is transverse in the real subsystem: the center-manifold $\mathcal{M}$ intersects transversely the stable manifold of $\sqrt{-\lambda_R/a_R}$.*

Proof. The proof of this lemma can be found in [11], where the necessary modifications to the proof of a similar statement in [5] are indicated. □

LEMMA 7. *(Subcritical) Suppose $a_R > 0$ and $\lambda_R < 0$. Then for any $k_0 \in \mathbb{N}$, there exists a heteroclinic orbit $w(r)$, with $\lim_{r\to 0} w(r) = \lim_{r\to\infty} w(r) = 0$. Moreover the heteroclinic orbit is transverse in the real subsystem: the manifold $\mathcal{M}$ intersects transversely the stable manifold of the origin at $r = \infty$.*

Proof. The proof, together with a more detailed description of such solutions, can be found in [18]. □

We next examine the non-degenerate system with $a_I \neq 0$.

PROPOSITION 8. *(Supercritical, $a_I \neq 0$) Suppose $a_R < 0$ and $\lambda_R > 0$ and fix any wave number $k_0 \in \mathbb{N}$. Then for any a_I, λ_I sufficiently small and $\lambda_I/a_I - 1 \gg a_I^2$ there exists a heteroclinic orbit $w(r)$, with $\lim_{r\to 0} w(r) = 0$ and tending to a type II equilibrium as $r \to \infty$. The heteroclinic orbit is transverse. Moreover there exists a unique value $\lambda_I^0 = \mathrm{O}(a_I)$ such that the heteroclinic orbit tends to the other type II equilibrium as $r \to \infty$. This heteroclinic orbit is transversely unfolded by the parameter λ_I.*

Proof. We suppose $a_R = -1$ and $\lambda_R = 1$. We use singular perturbation methods in order to establish the existence of heteroclinic orbits for the perturbed system. At $a_I = \lambda_I = 0$, there is a curve of type II equilibria for the asymptotic equations at $r = \infty$, given by $z_I^2 = 1 - R^2$, which intersects transversely the real subspace at the equilibrium $z = 0, R = 1$. Therefore the center-stable manifold of this line of equilibria intersects transversely the shooting manifold $\mathcal{M}$ in (z, R, τ)-space. In the perturbed system, the line of equilibria persists as a normally hyperbolic slow manifold (see [3]). The heteroclinic as a transverse intersection persists as the intersection with a strong stable fiber of the slow manifold for a_I, λ_I small enough. On the slow manifold there are two equilibria $z_I^2 = -1 - \lambda_I/a_I$, which are close to the real subspace $\{z_I = 0\}$

if $-\lambda_I/a_I$ is close to, but bigger than one. By the above stability analysis, the equilibrium which is stable within the slow manifold has det $L > 0$ and thereby $\lambda_I z_I < 0$. We now have to examine the perturbation of the shooting manifold $\mathcal{M}$ by the complex perturbation terms involving λ_I and a_I. The derivative along the real heteroclinic at $\lambda_I = a_I = 0$ of the non-autonomous equation for z_I with respect to λ_I and a_I gives

$$z_I' = \lambda_I + a_I R^2 = a_I(\lambda_I/a_I + R^2).$$

Thereby the Melnikov integral along the heteroclinic gives a contribution $\mathrm{O}(a_I)$ which shows that the shooting manifold $\mathcal{M}$ intersects transversely a stable fiber of a point on the slow manifold with $z_I^0 = \mathrm{O}(a_I)$.

With these ingredients we can establish the existence of the desired connections. Firstly choosing λ_I as a parameter, the shooting manifold $\mathcal{M}$ crosses transversely the strong stable fibers of the slow manifold. The type II equilibria on the slow manifold are located at $\mathrm{O}(\sqrt{|\lambda_I/a_I + 1|})$. If $|z_I^0| < \sqrt{|\lambda_I/a_I - 1|}$, there is a heteroclinic trajectory connecting to the type II equilibrium which is stable within the slow manifold. If $(z_I^0)^2 = -\lambda_I/a_I - 1$, the heteroclinic trajectory connects to the type II equilibrium which is unstable on the slow manifold. This proves the proposition. □

PROPOSITION 9. *(Subcritical, $a_I \neq 0$) Suppose $a_R > 0$ and $\lambda_R < 0$ and fix any wave number $k_0 \in \mathbb{N}$. Then for any a_I sufficiently small, there exists a smooth function $\lambda_I = \lambda_I(a_I)$ such that there exists a heteroclinic orbit $w(r)$, with $\lim_{r\to 0} w(r) = \lim_{r\to\infty} w(r) = 0$. The heteroclinic orbit is transversely unfolded by the parameter λ_I.*

Proof. We suppose $a_R = 1$ and $\lambda_R = -1$. In the real subspace at $\lambda_I = a_I = 0$, the heteroclinic orbit joining the origin at $r = 0$ to the origin at $r = \infty$ is transverse by Lemma 7. Transverse to the real subspace, the origin is unstable at both, $r = 0$ and $r = \infty$: the heteroclinic is non-transverse in full-space. We now need the parameter λ_I (alias the speed of rotation) in order to obtain connections for specific values of the parameter $\lambda_I = \lambda_I(a_I)$. For this it is sufficient to show that the Melnikov integral with respect to the parameter λ_I along the heteroclinic does not vanish. The adjoint variational equation along the heteroclinic has a unique (up to scalar multiples) bounded solution which lies strictly in the half space $z_I > 0$, because $z_I = 0$ is invariant. The derivative of the vector field with respect to λ_I in the direction of this half space is just 1, which proves that the Melnikov integral is non-zero. In other words we can push through the stable and unstable manifolds by the help of λ_I with non-zero speed. This proofs the proposition. □

12. Conclusions. For a large class of reaction-diffusion systems we have shown the existence of spiral wave solutions. In contrast to the previous results on λ-ω systems, our reduction to a non-autonomous ODE is not based on the *assumption* that Fourier modes decouple. We merely show that, close to the threshold of instability of a homogeneous equilibrium, there is some kind of decoupling. The interaction between critical modes is in a smooth sense of higher order then the projection on the critical modes. Compared to similar reduction methods, technical complications arise here because the problem is non-autonomous, even in the principal part (from a regularity point of view).

As another advantage of our method we are able to determine explicitly coefficients in our bifurcation equations. These are in general still hard to analyse analytically — we considered a simple but interesting model problem in the last section — but can easily be studied numerically.

The reduction procedure can be applied to other problems, possibly involving higher-dimensional center-manifolds. A systematic treatise of such equations (as known for elliptic problems in infinite cylinders, exploiting reversibility, integrability and normal forms of the reduced bifurcation equations) would be interesting.

The rotating waves we discover are of various shape, depending on the nature of the bifurcation. In supercritical bifurcations, they are approximately archimedian spirals at large distances from the tip. Indeed, the derivative of the phase of u is given by z_I and approaches for large values of the radius r a constant but non-zero value. As a subtle difference we noticed that in the supercritical case there are two different types of asymptotic states, given by the two different types of equilibria $z_I = \pm\sqrt{-\lambda_I/a_I - 1}$ (see the preceding section). For the first type, z_I approaches its limit value exponentially at a uniform rate with respect to λ_I , whereas for the other type the exponential rate is close to zero. The sign of z_I has another important interpretation. If z_I is positive, then the arms turn in the sense of the rotation of the spiral; at a fixed ray, under time evolution of the reaction-diffusion system, the arms move towards the center of rotation. Similarly, $z_I < 0$ corresponds to an outwards movement of the arms. Therefore the waves, appearing for discrete wave speeds move outwards if $a_I < 0$ and inwards if $a_I > 0$.

The rotating waves bifurcating subcritically are isolated as rotating waves and appear for distinguished speeds of rotation. Their shape at large distances from the center of rotation is determined by the phase varying according to $\varphi \sim e^{-\mathrm{const}\cdot r}$ and their amplitude decaying to zero exponentially.

Though we do not carry out here a stability analysis, we comment on the difference between sub- and supercritical bifurcation. Linearizing the reaction-diffusion system along the subcritical waves in say $L^2(\mathbb{R}^2, \mathbb{R}^2)$ gives us a linearized operator for the period map whose continuous spectrum is strictly contained in the left half plane, bounded away from the imaginary axis. Zero is (at least) a triple eigenvalue due to the euclidean symmetry, generated by rotation and translations. An analysis of secondary bifurcations from this type of spiral waves, including meandering and drifting waves has been carried out in [19, 20] and [4].

The linearized period map along supercritical waves has zero in the essential spectrum and rigorous stability proofs seem to be hard. Hagan showed [7] that one-armed spiral waves might be stable whereas multi-armed waves ($k_0 \neq 1$) should be unstable.

REFERENCES

[1] D.S. COHEN, J.C. NEU, AND R.R. ROSALES, *Rotating spiral wave solutions of reaction-diffusion equations*, SIAM J. Appl. Math., 35 (1978), pp. 536–547.

[2] W.A. COPPEL, *Dichotomies in stability theory*, Lect. Notes Math., 629, Springer, Berlin, 1978.

[3] N. FENICHEL, *Geometric singular perturbation theory for ordinary differential equations*, J. Diff. Eq., 31 (1979), 53–98.

[4] B. FIEDLER, B. SANDSTEDE, A. SCHEEL, AND C. WULFF, *Bifurcation from relative equilibria with non-compact group actions: skew products, meanders and drifts*, Doc. Math. J. DMV, 1 (1996), pp. 479–505.

[5] J.M. GREENBERG, *Spiral waves for λ-ω systems,* SIAM J. Appl. Math., 39 (1980), pp. 301–309.

[6] J.M. GREENBERG, *Spiral waves for λ-ω systems, II,* Adv. Appl. Math., 2 (1989), pp. 450–455.

[7] P.S. HAGAN, *Spiral waves in reaction-diffusion equations,* SIAM J. Appl. Math. 42 (1982), pp. 762–786.

[8] D. HENRY, *Geometric theory of semilinear parabolic equations*, Lect. Notes Math., 804, Springer, New York, Heidelberg, 1981.
[9] K. KIRCHGÄSSNER, *Wave solutions of reversible systems and applications*, J. Diff. Eq., 45 (1982), pp. 113–127.
[10] N. KOPELL AND L.N. HOWARD, *Plane wave solutions to reaction-diffusion equations*, Studies in Appl. Math., 52 (1973), pp. 291–328.
[11] N. KOPELL AND L.N. HOWARD, *Target patterns and spiral solutions to reaction-diffusion equations with more than one space dimension*, Adv. Appl. Math., **2** (1981), pp. 417–449.
[12] I. MELBOURNE, *Steady-state bifurcation with euclidean symmetry,* Research Report UH/MD - 214.
[13] A. MIELKE, *A reduction principle for non-autonomous systems in infinite-dimensional spaces,* J. Diff. Eqns., 65 (1986), pp. 68–88.
[14] A. MIELKE, *Reduction of PDEs on domains with several unbounded directions: A first step towards modulation equations,* ZAMP, 43 (1992), pp. 449–470.
[15] K.J. PALMER, *Exponential dichotomies and transversal homoclinic points,* J. Diff. Eqns., **55** (1984), pp. 225–256.
[16] D. PETERHOF, B. SANDSTEDE, AND A. SCHEEL, *Exponential dichotomies for solitary-wave solutions of semilinear elliptic equations on infinite cylinders,* J. Diff. Eqns., to appear.
[17] A. SCHEEL, *Existence of fast travelling waves for some parabolic equations – a dynamical systems approach,* J. Dyn. Diff. Eqns., 8 (1996), pp. 469–548.
[18] A. SCHEEL, *Subcritical bifurcation to infinitely rotating waves,* J. Math. Anal. Appl., to appear.
[19] B. SANDSTEDE, A. SCHEEL, AND C. WULFF, *Center-manifold reduction for spiral waves*, C. R. Acad. Sci. Paris, Serie I, Math., 324 (1997), pp. 153–158.
[20] B. SANDSTEDE, A. SCHEEL, AND C. WULFF, *Dynamics of Spiral Waves on Unbounded Domains Using Center-Manifold Reductions*, J. Diff. Eq.,to appear.
[21] G. SCHNEIDER, *Hopf-bifurcation in spatially extended reaction-diffusion systems,* Preprint 1996.
[22] J.J. TYSON AND J.P. KEENER, *Singular perturbation theory of traveling waves in excitable media (a review),* Phys. D., 32 (1988), pp. 327–361.
[23] A. VANDERBAUWHEDE, *Center-manifolds, normal forms and elementary bifurcations,* Dynamics Reported, 2 (1989), pp. 89–169.
[24] A. VANDERBAUWHEDE AND G. IOOSS, *Center-manifold theory in infinite dimensions,* Dynamics Reported, 1 (1992), pp. 125–163.
[25] G.N. WATSON, *Theory of Bessel functions*, Cambridge Univ. Press 1922.