

KHOVANOV HOMOTOPY TYPE, BURNSIDE CATEGORY, AND PRODUCTS

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ABSTRACT. In this paper, we give a new construction of a Khovanov homotopy type. We show that this construction gives a space stably homotopy equivalent to the Khovanov homotopy types constructed in [LS14a] and [HKK] and, as a corollary, that those two constructions give equivalent spaces. We show that the construction behaves well with respect to disjoint unions, connected sums and mirrors, verifying several conjectures from [LS14a]. Finally, combining these results with computations from [LS14c] and the refined s -invariant from [LS14b] we obtain new results about the slice genera of certain knots.

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1. INTRODUCTION

In [LS14a], we introduced a stable homotopy refinement of Khovanov homology. That is, to each link diagram K we associated a spectrum $\mathcal{X}_{Kh}(K)$, with the following properties:

- (1) The spectrum $\mathcal{X}_{Kh}(K)$ is a formal desuspension of a CW complex.
- (2) The spectrum $\mathcal{X}_{Kh}(K)$ comes with a wedge-sum decomposition $\mathcal{X}_{Kh}(K) = \bigvee_j \mathcal{X}_{Kh}^j(K)$.
- (3) For each j , the cellular cochain complex of (the CW complex corresponding to) $\mathcal{X}_{Kh}^j(K)$ is isomorphic to the Khovanov complex $\mathcal{C}_{Kh}^{*,j}(K)$ in quantum grading j , via an isomorphism taking the standard generators for $\mathcal{C}_{cell}^*(\mathcal{X}_{Kh}^j(K))$ to the standard generators for $\mathcal{C}_{Kh}^{*,j}(K)$.
- (4) For each j , the stable homotopy type of $\mathcal{X}_{Kh}^j(K)$ is an invariant of the isotopy class of the link represented by the diagram K .

There is also a reduced version $\tilde{\mathcal{X}}_{Kh}(K)$ of $\mathcal{X}_{Kh}(K)$, which satisfies properties (1)–(4) with \mathcal{C}_{Kh} replaced by the reduced Khovanov complex $\tilde{\mathcal{C}}_{Kh}$.

The goal of this paper is to study the behavior of $\mathcal{X}_{Kh}(K)$ under disjoint unions and connected sums of links. In particular, we will prove:

Theorem 1. [LS14a, Conjecture 10.3] *Let L_1 and L_2 be links, and $L_1 \amalg L_2$ their disjoint union. Then*

$$(1.1) \quad \mathcal{X}_{Kh}^j(L_1 \amalg L_2) \simeq \bigvee_{j_1+j_2=j} \mathcal{X}_{Kh}^{j_1}(L_1) \wedge \mathcal{X}_{Kh}^{j_2}(L_2).$$

Moreover, if we fix a basepoint p in L_1 , not at a crossing, and consider the corresponding basepoint for $L_1 \amalg L_2$, then

$$(1.2) \quad \tilde{\mathcal{X}}_{Kh}^j(L_1 \amalg L_2) \simeq \bigvee_{j_1+j_2=j} \tilde{\mathcal{X}}_{Kh}^{j_1}(L_1) \wedge \mathcal{X}_{Kh}^{j_2}(L_2).$$

Theorem 2. [LS14a, Conjecture 10.4] *Let L_1 and L_2 be based links and $L_1\#L_2$ the connected sum of L_1 and L_2 , where we take the connected sum near the basepoints. Then*

$$(1.3) \quad \tilde{\mathcal{X}}_{Kh}^j(L_1\#L_2) \simeq \bigvee_{j_1+j_2=j} \tilde{\mathcal{X}}_{Kh}^{j_1}(L_1) \wedge \tilde{\mathcal{X}}_{Kh}^{j_2}(L_2).$$

(We also compute the unreduced Khovanov spectrum of a connected sum [LS14a, Conjecture 10.6], as Theorem 8.)

These theorems, though unsurprising themselves, have some interesting corollaries:

Corollary 1.4. *For any n there exists a link L_n so that the operation*

$$\text{Sq}^n : Kh^{i,j}(L_n) \rightarrow Kh^{i+n,j}(L_n)$$

is non-zero, for some $i, j \in \mathbb{Z}$. Similarly, there exists a knot K_n so that the operation

$$\text{Sq}^n : \widetilde{Kh}^{i,j}(K_n) \rightarrow \widetilde{Kh}^{i+n,j}(K_n)$$

is non-zero, for some $i, j \in \mathbb{Z}$.

Corollary 1.5. *Let K be one of the knots 9_{42} , 10_{136} , $m(11_{19}^n)$, $m(11_{20}^n)$, 11_{70}^n , or 11_{96}^n . (Here m denotes the mirror.) Let L be a knot which is the closure of a positive braid. Letting g_4 denote the four-ball genus, we have*

$$g_4(K\#L) = g_4(K) + g_4(L).$$

Remark 1.6. There is some disagreement for nomenclature of knots regarding mirrors. We follow the convention from the Knot Atlas [BM], see Figure 1.1; alternatively, one can deduce our convention from the value of the signature $\sigma(K)$ in Table 11.1 (which, for us, is positive for positive knots). The value of $g_4(K)$ can be extracted from Figure 1.1 or Table 11.1; and $g_4(L)$ equals the genus of the Seifert surface obtained by applying Seifert's algorithm to the positive braid closure knot diagram for L [Ras10, Theorem 4].

Corollary 1.5 is not implied by computations of Rasmussen's s -invariant or the Heegaard Floer τ -invariant or the signature; see Table 11.1. At least for 9_{42} , the result is not implied by the Heegaard Floer concordance invariant Υ ; the Heegaard Floer d invariant of $+1$ surgery [Krc14]; or Hom-Rasmussen-Wu's ν^+ -invariant [Ras03, HW] [Hom14]. (These observations are not entirely independent.)

The construction of $\mathcal{X}_{Kh}(K)$ from [LS14a] uses the notion of flow categories, a notion introduced by Cohen-Jones-Segal in the context of Morse theory and Floer theory. To give a proof of Theorems 1 and 2 in this language seems tedious at best: it involves understanding the combinatorics of (broken) Morse flows on product manifolds, which turns out to be rather intricate.

Fortunately, there is another, more abstract construction of a Khovanov stable homotopy type, due to Hu-Kriz-Kriz [HKK], from which Theorems 1 and 2 follow easily. Roughly, they turn the Khovanov cube into a functor from the cube category to the Burnside category of finite sets and correspondences. They then apply the Elmendorf-Mandell infinite loop space machine to obtain a functor from the cube to symmetric spectra, and then (after adding some extra objects to the cube) take homotopy colimits to obtain a spectrum.

In this paper, we describe three additional constructions:

- (1) A framing-free reformulation of the construction from [LS14a] for a special family of flow categories, called cubical flow categories, in Section 3.
- (2) A version similar to the construction from [HKK] as a homotopy colimit, but using the thickened cube instead of the Elmendorf-Mandell machinery, in Section 4.
- (3) An intermediate object between the two in Section 5, using little k -cubes.

We then prove:

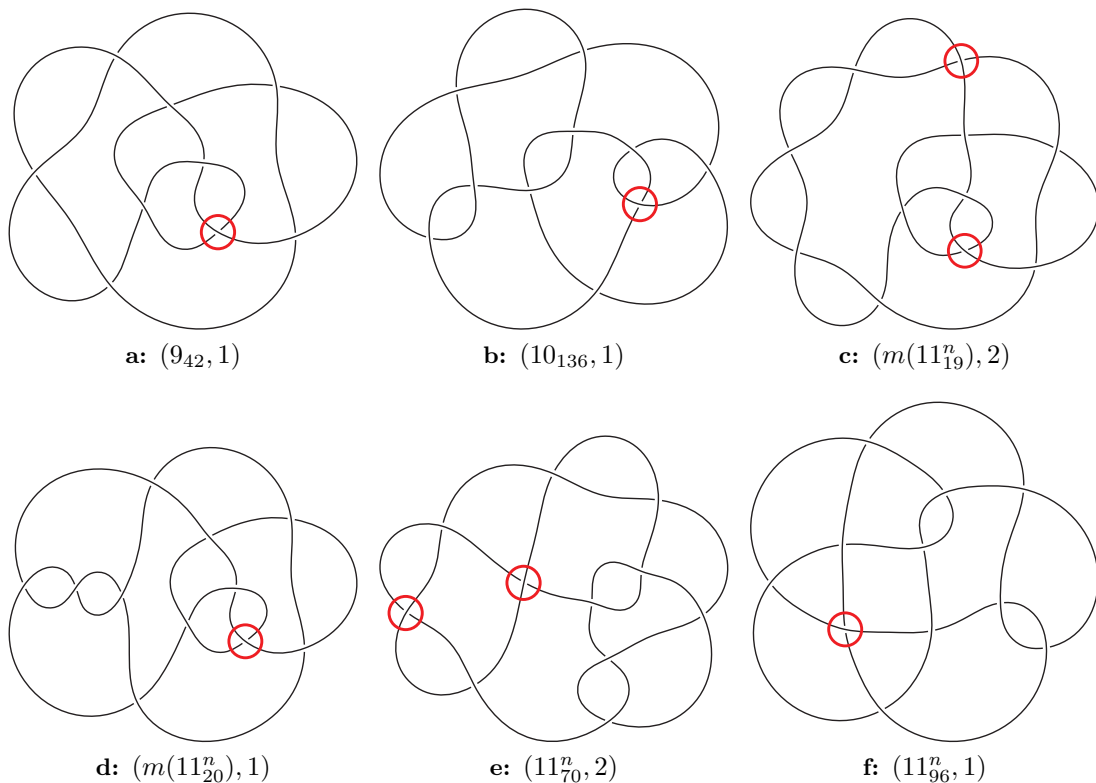


FIGURE 1.1. **The knots appearing in Corollary 1.5.** We have labeled the knot K by the pair $(K, g_4(K))$. The value of $g_4(K)$ is extracted from Knotinfo [CL]; the knot diagrams have been produced using Knotilus [FFFR]. Crossings giving a minimal unknotting for each knot are circled.

Theorem 3. *The Khovanov stable homotopy types constructed in Section 3, Section 4, and Section 5, the Khovanov stable homotopy type constructed in [HKK], and the Khovanov stable homotopy type constructed in [LS14a] are all stably homotopy equivalent.*

This theorem is proved in parts:

- Theorem 4 asserts that the cubical flow category realization (Section 3) agrees with the Cohen-Jones-Segal construction used in [LS14a].
- Theorem 6 asserts that the cubical flow category realization (Section 3) agrees with the little k -cubes realization (Section 5).
- Theorem 5 asserts that the little k -cubes realization (Section 5) agrees with the homotopy colimit description (Section 4).
- Theorem 7 asserts that the homotopy colimit description (Section 4) agrees with the construction in [HKK].

Theorems 1 and 2 follow easily. Corollary 1.4 follows immediately from these theorems and computations in [LS14c]. We also obtain, via a TQFT-style argument, that the Khovanov homotopy type of the mirror

knot $m(K)$ is the Spanier-Whitehead dual to the Khovanov homotopy type of K (Theorem 9). Finally, Corollary 1.5 follows from these results, the refined s invariant in [LS14b], the computations in [LS14c], and a brief further argument.

This paper is organized as follows. Section 2 has background on flow categories and related topics, on Khovanov homology, on some (2-)category theory and categories of interest, and on relevant facts about homotopy colimits. In Section 3 we introduce a special class of flow categories, cubical flow categories, which live over the cube, and give a reformulation of the Cohen-Jones-Segal realization for this class of flow categories (“cubical realization”). (The Khovanov flow category of [LS14a] is a cubical flow category; this is a crucial tool in its construction.) In Section 4 we show that cubical flow categories are equivalent to 2-functors from the cube to the Burnside category, and give a different, choice-free way to realize such a functor. In Section 5 we give a smaller but less canonical way to realize a 2-functor from the cube to the Burnside category, and prove the two ways to realize such a functor are equivalent. Section 6 shows that the realization from Section 5 agrees with the cubical realization from Section 3. Section 7 is a brief interlude to summarize these results and recall the Khovanov homotopy type. Section 8 shows that these realizations agree with the Hu-Kriz-Kriz-construction [HKK].

In Sections 9–11 we use these reformulated realizations to prove new properties of the Khovanov homotopy type. The realizations of the product (smash product) and disjoint union (wedge sum) of functors from the cube to the Burnside category have properties as one would expect; Section 9 uses these properties to study the Khovanov homotopy type of a disjoint union and connected sum, Theorems 1, 2, and 8, and verifies Corollary 1.4. Section 10 uses a TQFT-style argument suggested by the referee for [LS14c] to deduce a formula for the Khovanov homotopy type of a mirror, verifying another conjecture from [LS14a]. Finally, Section 11 gives an additivity property for the refined s invariant introduced in [LS14b] and obtains Corollary 1.5.

Remark 1.7. It may be interesting to compare the homotopy colimit definition of the Khovanov homotopy type with [ET14]; but see also [ELST].

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2. BACKGROUND

In this section, we review some background material which, while familiar in the homotopy theory community, may not be familiar to low-dimensional topologists.

2.1. Basic notation. The “cube” $\{0, 1\}^n$ will appear in a number of contexts in this paper, as will some auxiliary notions related to it:

- There is a partial order on $\{0, 1\}^n$ defined by $v \geq w$ if v is obtained from w by replacing some 0’s by 1’s. Define $v > w$ if $v \geq w$ and $v \neq w$, and \leq and $<$ in the corresponding ways. The maximum and minimum elements under this partial order are denoted $\vec{1}$ and $\vec{0}$, respectively.
- We denote the Manhattan (or ℓ^1) norm on $\{0, 1\}^n$ by $|v| = \sum_{i=1}^n v_i$.

- A *sign assignment* s on the cube is the following: For every $u > v$ with $|u| - |v| = 1$, we associate an element $s_{u,v} \in \mathbb{F}_2$ such that for any $u > w$ with $|u| - |w| = 2$, we have

$$\sum_{\substack{v \\ u > v > w}} (s_{u,v} + s_{v,w}) = 1.$$

A number of categories will appear in this paper:

- The category **Sets** of finite sets and set maps.
- The Burnside category \mathcal{B} of sets and correspondences (see Section 2.7).
- The cube category $\underline{2}^n = \{1 \rightarrow 0\}^n$ (see Section 2.2).
- The category \mathbf{Top}_\bullet of (well) based topological spaces.
- The subcategory \mathbf{CW}_\bullet of \mathbf{Top}_\bullet generated by based CW complexes and cellular maps.
- The category \mathcal{S} of spectra. For concreteness, we can take the category of symmetric spectra in topological spaces [MMSS01].
- The category $R\text{-Mod}$ of R -modules.
- The category **Permu** of permutative categories (see Section 8.4).
- The category $\mathbf{Cob}_{\text{emb}}^{1+1}$ of oriented 1-manifolds embedded in S^2 and oriented cobordisms embedded in $[0, 1] \times S^2$ (see Section 8.1).

Some other notation:

- Let $\mathbb{R}_+ = [0, \infty)$.
- Let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the set of non-negative integers.

2.2. The cube category. Let $\underline{2}^1$ denote the category with two objects, denoted 0 and 1, and a single non-identity morphism, from 1 to 0:

$$\underline{2}^1 = \{1 \rightarrow 0\}.$$

For $n \in \mathbb{Z}$, $n > 1$ let $\underline{2}^n = \underline{2}^1 \times \underline{2}^{n-1}$. That is, $\underline{2}^n$ is the small category with object set $\{0, 1\}^n$. Given objects $v, w \in \{0, 1\}^n$ the morphism set $\text{Hom}(v, w)$ is empty unless $v \geq w$, and if $v \geq w$ the set $\text{Hom}(v, w)$ has a single element, which we will denote $\varphi_{v,w}$.

The reader is warned that many authors refer to the opposite category $(\underline{2}^n)^{\text{op}}$ as the cube category.

We can (and will) view $\underline{2}^n$ as a 2-category with no non-identity 2-morphisms. (That is, for $f, g \in \text{Hom}(v, w)$, we define $\text{Hom}(f, g)$ to be empty unless $f = g$ and to have a single element if $f = g$.)

2.3. The Khovanov construction. The Khovanov homology, and several of its generalizations, are all constructed from the cube of resolutions of a link diagram. Let K be a link diagram in S^2 with n crossings, numbered c_1, \dots, c_n . Each of these crossings can be resolved locally in two different ways, called the 0-resolution and the 1-resolution; see for instance [Kho00, Figure 14]. Therefore, to each $v \in \{0, 1\}^n$ there is an associated complete resolution $\mathcal{P}(v)$ obtained by replacing the crossing c_i by its 0-resolution if $v_i = 0$ and its 1-resolution if $v_i = 1$. The complete resolution $\mathcal{P}(v)$ consists of a collection of disjoint circles in S^2 .

For any $u \geq v \in \{0, 1\}^n$, there is a cobordism embedded in $[0, 1] \times S^2$ that connects the resolutions $\mathcal{P}(u)$ and $\mathcal{P}(v)$: to obtain $\mathcal{P}(u)$, one attaches embedded 1-handles to $\mathcal{P}(v)$ using certain arcs, embedded near the crossings c_i for all i for which $u_i > v_i$, as the cores of these new 1-handles. This is illustrated in Figure 2.1; see also [Kho00, Figure 18]. (This is also the first step in Bar-Natan’s “picture world” approach to Khovanov homology [Bar05].) For the special case when $|u| - |v| = 1$, the cobordism either *merges* two circles into one, or *splits* a single circle into two.

To construct the Khovanov complex, we apply a $(1 + 1)$ -dimensional TQFT to this cube of cobordisms to obtain a commutative cube of abelian groups. Specifically, consider the rank-2 Frobenius algebras over

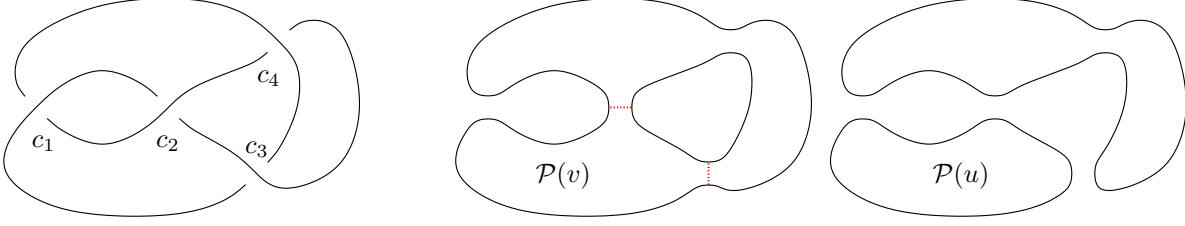


FIGURE 2.1. **Some resolutions of a knot diagram.** Left: A knot diagram for the figure-eight knot with crossings c_1, \dots, c_4 . Right: The complete resolutions at the vertices $v = (1, 0, 0, 0)$ and $u = (1, 1, 1, 0)$, along with the embedded 1-handle cores representing the embedded cobordism between $\mathcal{P}(u)$ and $\mathcal{P}(v)$.

$\mathbb{Z}[h, t]$ with basis $\{x_+, x_-\}$ and multiplication and comultiplication given by

$$\begin{aligned} x_+ \otimes x_+ &\xrightarrow{m} x_+ & x_+ \otimes x_- &\xrightarrow{m} x_- & x_- \otimes x_+ &\xrightarrow{m} x_- & x_- \otimes x_- &\xrightarrow{m} hx_- + tx_+ \\ x_+ &\xrightarrow{S} x_+ \otimes x_- + x_- \otimes x_+ - hx_+ \otimes x_+ & x_- &\xrightarrow{S} x_- \otimes x_- + tx_+ \otimes x_+, \end{aligned}$$

and the corresponding $(1+1)$ -dimensional TQFT. Applying this TQFT to the cube of resolutions of K gives a commutative cube $A: (\mathbb{Z}^n)^{\text{op}} \rightarrow \mathbb{Z}[h, t]\text{-Mod}$. Explicitly, given a vertex v , a *Khovanov generator over v* is a labeling of the circles in $\mathcal{P}(v)$ by elements of the set $\{x_+, x_-\}$. The module $A(v)$ is freely generated by the set of Khovanov generators $F(v)$ over v . For an edge of the cube $\varphi_{u,v}$ (where $u > v$ and $|u| - |v| = 1$), $A(\varphi_{u,v}): A(v) \rightarrow A(u)$ is the multiplication or comultiplication above, depending on whether the edge is a merge or split, respectively.

Definition 2.1. Fix a sign assignment s on the cube (in the sense of Section 2.1). The chain complex $\mathcal{C}(K)$ is the totalization of A with respect to s . That is, the chain group is defined to be $\mathcal{C}(K) = \bigoplus_{v \in \{0,1\}^n} A(v)$ and the differential $\delta: \mathcal{C}(K) \rightarrow \mathcal{C}(K)$ is defined by stipulating the component $\delta_{u,v}$ of δ that maps from $A(v)$ to $A(u)$ to be

$$\delta_{u,v} = \begin{cases} (-1)^{s_{u,v}} A(\varphi_{u,v}) & \text{if } u > v \text{ and } |u| - |v| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The *Khovanov chain complex* $(\mathcal{C}_{Kh}(K), \delta_{Kh})$ is the specialization $h = t = 0$. The homology of $\mathcal{C}_{Kh}(K)$ is the *Khovanov homology* $Kh(K)$.

(The Khovanov complex was introduced by Khovanov [Kho00]. The specializations $(h, t) = (0, 1)$ and $(h, t) = (1, 0)$ were studied by Lee [Lee05] and Bar-Natan [Bar05], respectively; the case of general (h, t) was studied by Khovanov [Kho06b], Naot [Nao06], and others.)

The homological grading of the summand $A(v) \subseteq \mathcal{C}(K)$ is $|v|$ minus the number of negative crossings in the link diagram. There is additionally an internal grading, called the *quantum grading*, that persists throughout. Modulo some global grading shift, the quantum grading of any Khovanov generator in $F(v)$ is

$$|v| + \#\{\text{circles in } \mathcal{P}(v) \text{ labeled } x_+\} - \#\{\text{circles in } \mathcal{P}(v) \text{ labeled } x_-\}.$$

The quantum gradings of the formal variables (h, t) are $(-2, -4)$.

In the presence of a basepoint on the link diagram, and after setting $t = 0$, there is also a reduced theory. In the reduced theory, for any $v \in \{0, 1\}^n$, we only consider the Khovanov generators in $F(v)$ that label the pointed circle in $\mathcal{P}(v)$ as x_- . The reduced Khovanov chain complex and the reduced Khovanov homology are denoted $\tilde{\mathcal{C}}_{Kh}$ and \tilde{Kh} , respectively. (Alternatively, we could have defined a reduced theory by

only considering the Khovanov generators that label the pointed circle as x_+ ; when $h = t = 0$, these two reduced theories agree on the nose.)

The (reduced) Khovanov homology is an invariant of the (pointed) link, and not just of the (pointed) link diagram. More generally, the chain homotopy type of the chain complex $\mathcal{C}(K)$ over $\mathbb{Z}[h, t]$ is a link invariant as well.

2.4. Manifolds with corners and $\langle n \rangle$ -manifolds. The original construction of the Khovanov stable homotopy type from [LS14a] relies on Cohen-Jones-Segal's notion of flow categories [CJS95], which in turn (implicitly) uses a particular notion of manifolds with corners, called $\langle n \rangle$ -manifolds [Jän68, Lau00]. For the reader's convenience, we review the relevant definitions here.

A k -dimensional manifold with corners is a topological space X together with an atlas $\{(U_\alpha, \phi_\alpha: U_\alpha \rightarrow (\mathbb{R}_+)^k)\}$ modeled on open subsets of $(\mathbb{R}_+)^k$, so that the transition functions are smooth. Given a point x in a chart (U, ϕ) let $c(x)$ be the number of coordinates in $\phi(x)$ which are 0; $c(x)$ is independent of the choice of chart. The *codimension- i boundary* of X is $\{x \in X \mid c(x) = i\}$. A k -dimensional manifold with corners X has a well-defined tangent space TX , which is an \mathbb{R}^k -plane bundle; a Riemannian metric on X means a Riemannian metric on TX .

A *facet* of X is the closure of a connected component of the codimension-1 boundary of X . (If X is a polytope then this agrees with the usual definition of facets.) A *multifacet* of X is a (possibly empty) union of disjoint facets of X . A manifold with corners X is a *multifaceted manifold* if every $x \in X$ belongs to exactly $c(x)$ facets of X . A $\langle n \rangle$ -manifold is a multifaceted manifold X along with an ordered n -tuple $(\partial_1 X, \dots, \partial_n X)$ of multifacets of X such that: $\bigcup_i \partial_i X = \partial X$; and for all distinct i, j , $\partial_i X \cap \partial_j X$ is a multifacet of both $\partial_i X$ and $\partial_j X$. (The number n need not be the dimension of X .) See [Lau00] for more details. ([Lau00] uses the terms 'connected face', 'face', and 'manifold with faces' for 'facet', 'multifacet', and 'multifaceted manifold', respectively. We have to change the terminology since 'face' means something different for polytopes in Section 2.6.) Given a $\langle n \rangle$ -manifold X and a vector $v \in \{0, 1\}^n$ let $X(v) = \bigcap_{v_i=0} \partial_i X$, with the convention that $X(\vec{1}) = X$.

As illustrative examples, an n -gon (polygon with n sides) is a multifaceted manifold if $n > 1$, while a 1-gon (disk with one corner on the boundary) is a manifold with corners but not a multifaceted manifold. Only the $2n$ gons can be made into $\langle 2 \rangle$ -manifolds, though $(2n + 1)$ -gons can be viewed as 2-dimensional $\langle 3 \rangle$ -manifolds. Of the Platonic solids only the tetrahedron, cube, and dodecahedron are manifolds with corners, and all three are multifaceted manifolds. The cube can be made into a $\langle 3 \rangle$ -manifold by defining $\partial_1 X$ to be the front and back facets, $\partial_2 X$ to be the top and bottom facets, and $\partial_3 X$ to be the left and right facets. The tetrahedron and dodecahedron can not be given the structure of $\langle 3 \rangle$ -manifolds, although both can be made into $\langle 4 \rangle$ -manifolds. An even more fundamental example is $(\mathbb{R}_+)^n$ itself, which is a $\langle n \rangle$ -manifold by setting $\partial_i(\mathbb{R}_+)^n = \{v \in \mathbb{R}_+^n \mid v_i = 0\}$. Similarly, $\mathbb{R}^N \times (\mathbb{R}_+^n)$ is an $(n + N)$ -dimensional $\langle n \rangle$ -manifold.

Given an $\langle n \rangle$ -manifold X and an $\langle m \rangle$ -manifold Y , the product $X \times Y$ inherits the structure of a $\langle n + m \rangle$ -manifold, by declaring

$$\partial_i(X \times Y) = \begin{cases} (\partial_i X) \times Y & 1 \leq i \leq n \\ X \times (\partial_{i-n} Y) & n + 1 \leq i \leq n + m. \end{cases}$$

We end this subsection with the definition of neat embeddings. Consider $\langle n \rangle$ -manifolds X and Y and fix a Riemannian metric on Y . A *neat embedding* of X into Y is a smooth map $f: X \rightarrow Y$ so that:

- $f^{-1}(Y(v)) = X(v)$ for all $v \in \{0, 1\}^n$.
- $f|_{X(v)}: X(v) \rightarrow Y(v)$ is an embedding for each $v \in \{0, 1\}^n$.
- For all $w < v \in \{0, 1\}^n$, $f(X(v))$ is perpendicular to $Y(w)$ with respect to the Riemannian metric on Y , and in particular is transverse to $Y(w)$.

2.5. Flow categories. Next we recall some notions about flow categories, from [CJS95] (see also [LS14a, Section 3]), partly to fix terminology for this paper.

Definition 2.2. A *flow category* \mathcal{C} is a topological category whose objects $\text{Ob}(\mathcal{C})$ form a discrete space equipped with a grading function $\text{gr}: \text{Ob}(\mathcal{C}) \rightarrow \mathbb{Z}$ and whose morphisms satisfy the following conditions:

- (FC-1) For any $x \in \text{Ob}(\mathcal{C})$, $\text{Hom}(x, x) = \{\text{Id}\}$.
- (FC-2) For distinct $x, y \in \text{Ob}(\mathcal{C})$ with $\text{gr}(x) - \text{gr}(y) = k$, $\text{Hom}(x, y)$ is a (possibly empty) compact $(k - 1)$ -dimensional $\langle k - 1 \rangle$ -manifold; and
- (FC-3) The composition maps combine to produce a diffeomorphism of $\langle k - 2 \rangle$ -manifolds

$$\coprod_{\substack{z \in \text{Ob}(\mathcal{C}) \setminus \{x, y\} \\ \text{gr}(z) - \text{gr}(y) = i}} \text{Hom}(z, y) \times \text{Hom}(x, z) \cong \partial_i \text{Hom}(x, y).$$

(In [CJS95], it is not required that the space of objects be discrete. In Morse theory, this corresponds to allowing Morse-Bott functions.)

The identity morphisms in a flow category are somewhat special, and it is often convenient to ignore them. So, for objects x, y in \mathcal{C} , the *moduli space from x to y* , $\mathcal{M}(x, y)$, is defined to be $\text{Hom}(x, y)$ if $x \neq y$, and empty if $x = y$. (In Morse theory, this corresponds to the moduli space of non-constant downwards gradient flows from x to y .)

For any flow category \mathcal{C} , let $\Sigma^k \mathcal{C}$ denote the flow category obtained by increasing the gradings of each object by k .

Definition 2.3. For each integer i , fix an integer $D_i \geq 0$, and let \mathbf{D} denote this sequence. A *neat embedding* of a flow category \mathcal{C} relative to \mathbf{D} is a collection $J_{x,y}$ of neat embeddings (with the standard Riemannian metric on the target space)

$$J_{x,y}: \mathcal{M}(x, y) \hookrightarrow \mathbb{R}^{D_{\text{gr}(y)}} \times \mathbb{R}_+ \times \mathbb{R}^{D_{\text{gr}(y)+1}} \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \times \mathbb{R}^{D_{\text{gr}(x)-1}}$$

of $(\text{gr}(x) - \text{gr}(y) - 1)$ -manifolds for all $x, y \in \text{Ob}(\mathcal{C})$, subject to the following:

- (1) For all integers i, j ,

$$\coprod_{\substack{x, y \\ \text{gr}(x)=i, \text{gr}(y)=j}} J_{x,y}: \coprod_{\substack{x, y \\ \text{gr}(x)=i, \text{gr}(y)=j}} \mathcal{M}(x, y) \rightarrow \mathbb{R}^{D_j} \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \times \mathbb{R}^{D_{i-1}}$$

is a neat embedding of $(i - j - 1)$ -manifolds.

- (2) For all $x, y, z \in \text{Ob}(\mathcal{C})$, and all points $(q, p) \in \mathcal{M}(y, z) \times \mathcal{M}(x, y)$

$$J_{x,z}(q \circ p) = (J_{y,z}(q), 0, J_{x,y}(p)).$$

A *coherent framing* for a neat embedding J is a collection of framings of the normal bundles $\nu_{J_{x,y}}$ of $J_{x,y}$ for all $x, y \in \text{Ob}(\mathcal{C})$, such that for all $x, y, z \in \text{Ob}(\mathcal{C})$, the product framing of $\nu_{J_{y,z}} \times \nu_{J_{x,y}}$ equals the pullback framing of $\circ^* \nu_{J_{x,z}}$, where \circ denotes composition.

Definition 2.4 ([LS14a, Definition 3.21]). A *framed flow category* is a flow category \mathcal{C} , along with a coherent framing for some neat embedding of \mathcal{C} (relative to some \mathbf{D}).

For a framed flow category, there is an *associated chain complex* $C^*(\mathcal{C})$, defined as follows. The n^{th} chain group C^n is the \mathbb{Z} -module freely generated by the objects of \mathcal{C} of grading n . The differential δ is of degree one. For $x, y \in \text{Ob}(\mathcal{C})$ with $\text{gr}(x) - \text{gr}(y) = 1$, the coefficient $\langle \delta y, x \rangle$ of x in $\delta(y)$ is the number of points in $\mathcal{M}(x, y)$, counted with sign. We say a framed flow category *refines* its associated chain complex.

Note that in order to define the associated chain complex, one only needs the framing of the 0-dimensional moduli spaces; and in order to check that one indeed gets a differential, one only needs to ensure that the framing extends to the 1-dimensional moduli spaces.

To a framed flow category, Cohen-Jones-Segal associate a based CW complex $|\mathcal{C}|$ whose cells (except the basepoint) correspond to the objects of the flow category [CJS95]. The following formulation of the Cohen-Jones-Segal construction is described in more detail in [LS14a, Definition 3.24].

Definition 2.5. Let \mathcal{C} be a framed flow category with a neat embedding j relative to some \mathbf{D} , and assume all objects of \mathcal{C} have grading in $[B, A]$ for some fixed $A, B \in \mathbb{Z}$. Using the framing of $\nu_{jx,y}$, extend $j_{x,y}$ to

$$\bar{j}_{x,y}: \mathcal{M}(x,y) \times [-\delta, \delta]^{D_{\text{gr}(y)} + \dots + D_{\text{gr}(x)} - 1} \rightarrow \mathbb{R}^{D_{\text{gr}(y)}} \times \mathbb{R}_+ \times \dots \times \mathbb{R}_+ \times \mathbb{R}^{D_{\text{gr}(x)} - 1}.$$

Choose δ small enough and T large enough so that the map $\prod_{x,y | \text{gr}(x)=i, \text{gr}(y)=j} \bar{j}_{x,y}$ is an embedding into $(-T, T)^{D_j} \times [0, T) \times \dots \times [0, T) \times (-T, T)^{D_i - 1}$ for all integers i, j . In the based CW complex $|\mathcal{C}|$, the cell associated to an object x of grading m is

$$\mathcal{C}(x) = \prod_{i=B}^{m-1} \left([0, T] \times [-T, T]^{D_i} \right) \times [-\delta, \delta]^{D_m + \dots + D_{A-1}}.$$

For any other object y with $\text{gr}(y) = n < m$, the embedding $\bar{j}_{x,y}$ identifies $\mathcal{C}(y) \times \mathcal{M}(x,y)$ with the following subset of $\partial\mathcal{C}(x)$,

$$\mathcal{C}_y(x) = \prod_{i=B}^{n-1} \left([0, T] \times [-T, T]^{D_i} \right) \times \{0\} \times \text{im}(\bar{j}_{x,y}) \times [-\delta, \delta]^{D_m + \dots + D_{A-1}}.$$

The attaching map for $\mathcal{C}(x)$ sends $\mathcal{C}_y(x) \cong \mathcal{C}(y) \times \mathcal{M}(x,y)$ via the projection map to $\mathcal{C}(y)$, and sends $\partial\mathcal{C}(x) \setminus \bigcup_y \mathcal{C}_y(x)$ to the basepoint.

[LS14a, Lemma 3.25] asserts that the above defines a CW complex, whose reduced cellular cochain complex is isomorphic (after shifting the gradings by $D_B + \dots + D_{A-1} - B$) to the chain complex associated to the framed flow category \mathcal{C} from Definition 2.4, and the isomorphism sends cells of $|\mathcal{C}|$ to the corresponding objects of \mathcal{C} . The Cohen-Jones-Segal realization of \mathcal{C} is the formal desuspension of $|\mathcal{C}|$ so that the gradings agree.

2.6. Permutohedra. We will be interested in a particular family of flow categories, in which the moduli spaces are unions of permutohedra. So, we recall some basic facts about permutohedra.

Before starting, let us fix some notations about polytopes, mostly following [Zie95]. Let $P \subset \mathbb{R}^n$ be a polytope or an \mathcal{H} -polyhedron (as described in [Zie95, Definition 0.1]). If there is an affine half-space of \mathbb{R}^n which contains P , then the intersection of its boundary with P is called a *face* of P ; and if $\dim(P) = d$, then we declare the entire polytope P to be its unique d -dimensional face. This gives a CW complex structure on P , with the cells being the faces. The faces of dimension 0, 1, $(d-1)$ are called vertices, edges, and facets, respectively. A d -dimensional polytope is called *simple* if every vertex is contained in exactly d facets; simple polytopes are multifaceted manifolds.

For $\sigma \in \mathcal{S}_n$ a permutation, let $v_\sigma = (\sigma^{-1}(1), \dots, \sigma^{-1}(n)) \in \mathbb{R}^n$. The $(n-1)$ -dimensional permutohedron Π_{n-1} is the convex hull in \mathbb{R}^n of the $n!$ points v_σ ; see [Zie95, Example 0.10]. Π_{n-1} lies in the affine subspace $\mathbb{A}^{n-1} := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_i x_i = n(n+1)/2\}$ of \mathbb{R}^n . As its name suggests, Π_{n-1} is $(n-1)$ -dimensional, and the v_σ are its vertices.

For each non-empty, proper subset S of $\{1, \dots, n\}$ of cardinality say k , let $H_S \subset \mathbb{A}^{n-1} \subset \mathbb{R}^n$ be the half-space $\{(x_1, \dots, x_n) \in \mathbb{A}^{n-1} \mid \sum_{i \in S} x_i \geq k(k+1)/2\}$. The permutohedron Π_{n-1} can also be defined as the intersection of the $2^n - 2$ half-spaces H_S . In fact, the facets of Π_{n-1} are exactly the $F_S := \Pi_{n-1} \cap \partial H_S$.

The facets F_S are identified with products of lower-dimensional permutohedra:

Lemma 2.6. *Let $a_1 < a_2 < \dots < a_k$ be the elements in S , and let $b_1 < b_2 < \dots < b_{n-k}$ be the elements in $\{1, 2, \dots, n\} \setminus S$. Then the map $f_S: \mathbb{R}^n \rightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$*

$$f_S(x_1, \dots, x_n) = ((x_{a_1}, \dots, x_{a_k}), (x_{b_1} - k, \dots, x_{b_{n-k}} - k))$$

identifies the facet $F_S \subset \mathbb{R}^n$ with $\Pi_{k-1} \times \Pi_{n-1-k} \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$.

Proof. It suffices to show that f_S takes the vertices of $F_S \subset \Pi_{n-1}$ to the vertices of $\Pi_{k-1} \times \Pi_{n-1-k}$. The vertices of F_S are the points (x_1, \dots, x_n) so that $\{x_{a_1}, \dots, x_{a_k}\} = \{1, \dots, k\}$ and $\{x_{b_1}, \dots, x_{b_{n-k}}\} = \{k+1, \dots, n\}$. It is immediate that f_S takes these vertices bijectively to the vertices of $\Pi_{k-1} \times \Pi_{n-1-k}$. \square

The permutohedron Π_{n-1} is simple, i.e., each vertex lies in exactly $n-1$ facets: v_σ lies in the facet $F_{\{\sigma(1), \dots, \sigma(k)\}}$ for each $1 \leq k < n$, and no others. Therefore, every d -dimensional face belongs to exactly $n-1-d$ facets; and the subsets corresponding to those facets are nested. Hence, d -dimensional faces correspond to sequences of $n-1-d$ nested proper non-empty subsets of $\{1, \dots, n\}$. Further:

Lemma 2.7. *The space Π_{n-1} can be treated as an $(n-1)$ -manifold by declaring*

$$\partial_i \Pi_{n-1} = \bigcup_{\substack{S \\ |S|=i}} F_S$$

for $1 \leq i \leq n-1$.

Proof. We must check:

- (1) Every point x belongs to $c(x)$ facets.
- (2) Each $\partial_i \Pi_{n-1}$ is a multifacet, that is, a disjoint union of facets.
- (3) $\bigcup_i \partial_i \Pi_{n-1} = \partial \Pi_{n-1}$.
- (4) For each $i \neq j$, $\partial_i \Pi_{n-1} \cap \partial_j \Pi_{n-1}$ is a multifacet of $\partial_i \Pi_{n-1}$ (and $\partial_j \Pi_{n-1}$).

Point (1) follows from the fact that Π_{n-1} is a simple polyhedron.

For point (2), we claim that if $|S| = |T| = i$ and $S \neq T$ then $F_S \cap F_T = \emptyset$; it follows that $\partial_i \Pi_{n-1}$ is the disjoint union of the facets F_S (with $|S| = i$). But if v_σ is a vertex in $F_S \cap F_T$, then

$$\begin{aligned} \sum_{j \in S} \sigma^{-1}(j) &= \sum_{j \in T} \sigma^{-1}(j) = i(i+1)/2 \\ \iff \{\sigma^{-1}(j) \mid j \in S\} &= \{\sigma^{-1}(j) \mid j \in T\} = \{1, \dots, i\} \\ \iff S = T &= \{\sigma(1), \dots, \sigma(i)\}. \end{aligned}$$

Point (3) is immediate from the definitions.

For point (4), suppose that $|S| = i$. After identifying F_S with $\Pi_{i-1} \times \Pi_{n-i-1}$ using Lemma 2.6, we get

$$F_S \cap \partial_j \Pi_{n-1} = \begin{cases} \Pi_{i-1} \times (\partial_{j-i} \Pi_{n-i-1}) & i < j \\ (\partial_j \Pi_{i-1}) \times \Pi_{n-i-1} & i > j. \end{cases}$$

Therefore, $F_S \cap \partial_j \Pi_{n-1}$ is a disjoint union of facets of $F_S \cong \Pi_{i-1} \times \Pi_{n-i-1}$. Since the F_S for $|S| = i$ are disjoint, $\partial_i \Pi_{n-1} \cap \partial_j \Pi_{n-1}$ is a disjoint union of facets of $\partial_i \Pi_{n-1}$ as well. \square

We will also use the following well-known cubical complex structure on Π_{n-1} : For any permutation $\sigma \in \mathcal{S}_n$, let C_σ be the convex hull of the barycenters of all the faces that contain the vertex v_σ .

Lemma 2.8. *Each C_σ is combinatorially equivalent to an $(n-1)$ -dimensional cube, and these cubes form a cubical complex subdivision of Π_{n-1} .*

Proof. We present the proof from [Ovc08, Section 3], for which Ovchinnikov cites Ziegler. Consider the following intersection of half-spaces

$$FC_\sigma = \bigcap_{S|v_\sigma \in F_S} H_S.$$

The space FC_σ is an $(n-1)$ -dimensional cone with cone point v_σ , with $n-1$ facets (corresponding to the facets of Π_{n-1} that contain v_σ), and therefore is a simplicial cone. (For comparison with [Ovc08, Section 3], FC_σ is the dual of the facet cone of the dual of v_σ (in the face fan of the dual of Π_{n-1}).)

Next, consider the vertex cone of v_σ (in the normal fan of Π_{n-1}), VC_σ . By definition, the cone point of VC_σ is the barycenter of Π_{n-1} , and the edges of VC_σ are obtained by dropping perpendiculars from the barycenter to the facets of Π_{n-1} that contain v_σ . The cone VC_σ is an $(n-1)$ -dimensional cone with $(n-1)$ edges, and therefore, VC_σ is a simplicial cone. The d -dimensional faces of VC_σ correspond to the $(n-1-d)$ -dimensional faces of Π_{n-1} that contain v_σ . Given corresponding faces f_{VC} of VC_σ and f of Π_{n-1} , f_{VC} is perpendicular to f and passes through the barycenter of f .

Therefore, C_σ is the intersection of the two simplicial cones FC_σ and VC_σ . Since the edges of VC_σ pass through the interiors of the facets of FC_σ , VC_σ and FC_σ intersect transversely, and therefore, $VC_\sigma \cap FC_\sigma$ is combinatorially equivalent to a cube.

The facets of C_σ are of two types: the ones contained in FC_σ , which are not identified with the facets of any other cube and lie in the boundary of Π_{n-1} ; and the ones contained in VC_σ , which are identified with facets of other cubes and lie in the interior of Π_{n-1} . Indeed the facets of the latter type correspond to the edges e of Π_{n-1} that contain v_σ : the facet corresponding to e is formed by taking the convex hull of the barycenters of all the faces of Π_{n-1} that contain e . With these identifications, it is clear that these cubes C_σ come together to form a cubical subdivision of the permutohedron. \square

2.7. The Burnside category. Given sets X and Y , a *correspondence* (sometimes also called a *span*) from X to Y is a set A and maps $s: A \rightarrow X$ and $t: A \rightarrow Y$ (for *source* and *target*). Given a correspondence (A, s_A, t_A) from X to Y and (B, s_B, t_B) from Y to Z the *composition* $(B, s_B, t_B) \circ (A, s_A, t_A)$ is the correspondence (C, s, t) from X to Z given by

$$C = B \times_Y A = \{(b, a) \in B \times A \mid t(a) = s(b)\} \quad s(b, a) = s_A(a) \quad t(b, a) = t_B(b).$$

Given correspondences (A, s_A, t_A) and (B, s_B, t_B) from X to Y , a *morphism of correspondences* from (A, s_A, t_A) to (B, s_B, t_B) is a bijection of sets $f: A \rightarrow B$ which commutes with the source and target maps, i.e., so that $s_A = s_B \circ f$ and $t_A = t_B \circ f$. Composition (of set maps) makes the set of correspondences from X to Y into a category. Further, composition of correspondences makes (Sets, Correspondences, Morphisms of correspondences) into a weak 2-category (bicategory in the language of [Bén67]). By the *Burnside category* we mean the sub-2-category of finite sets and finite correspondences. We denote the Burnside category by \mathcal{B} . (More typically, one defines the Burnside category of a group G in terms of G -sets and G -equivariant correspondences; for us, G is the trivial group.)

We will typically drop the maps s and t from the notation, referring simply to a correspondence A from X to Y .

As mentioned above, the Burnside category is a weak 2-category: the identity and associativity axioms only hold up to natural isomorphism. That is, given a set X , the *identity correspondence* of X is simply the set X itself (with the identity map as source and target maps). Given another correspondence A from W to X there is a natural isomorphism

$$\lambda: X \times_X A \xrightarrow{\cong} A$$

defined by $\lambda(x, a) = a$. Similarly, given a correspondence B from X to Y there is a natural isomorphism

$$\rho: B \times_X X \xrightarrow{\cong} B$$

defined by $\rho(b, x) = b$. Finally, given correspondences A from W to X , B from X to Y and C from Y to Z there is a natural isomorphism

$$\alpha: (C \times_Y B) \times_X A \rightarrow C \times_Y (B \times_X A)$$

defined by $\alpha((c, b), a) = (c, (b, a))$.

This distinction between weak and strict 2-categories may seem superficial here, but the distinction between weak and strict 2-functors, to which we turn next, will be crucial.

2.8. 2-functors. From Section 4 on, we will be interested in functors from the cube category $\underline{2}^n$ to the Burnside category \mathcal{B} . These will be lax functors, a notion that we recall here:

Definition 2.9. Given (weak) 2-categories \mathcal{C} and \mathcal{D} , a *lax 2-functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of the following data:

- For each object $x \in \text{Ob}(\mathcal{C})$ an object $F(x) \in \text{Ob}(\mathcal{D})$.
- For each pair of objects $x, y \in \text{Ob}(\mathcal{C})$ a functor $F_{x,y}: \text{Hom}(x, y) \rightarrow \text{Hom}(F(x), F(y))$.
- For each object $x \in \text{Ob}(\mathcal{C})$ a 2-morphism $F_{\text{Id}_x}: \text{Id}_{F(x)} \rightarrow F_{x,x}(\text{Id}_x)$.
- For any three objects $x, y, z \in \text{Ob}(\mathcal{C})$ a natural transformation

$$F_{x,y,z}(\cdot, \cdot): F_{y,z}(\cdot) \circ_1 F_{x,y}(\cdot) \rightarrow F_{x,z}(\cdot \circ_1 \cdot).$$

(Here, both $F_{y,z}(\cdot) \circ_1 F_{x,y}(\cdot)$ and $F_{x,z}(\cdot \circ_1 \cdot)$ are functors $\text{Hom}(y, z) \times \text{Hom}(x, y) \rightarrow \text{Hom}(F(x), F(z))$; \circ_1 denotes the 1-composition in \mathcal{C} or \mathcal{D} , not the composition of functors.)

These data must satisfy the following compatibility conditions:

(Fu-1) For each pair of objects $x, y \in \text{Ob}(\mathcal{C})$, the following diagrams commute:

$$\begin{array}{ccc} \text{Id}_{F(y)} \circ_1 F_{x,y}(\cdot) & \xrightarrow{\lambda} & F_{x,y}(\cdot) \\ F_{\text{Id}_y} \circ_1 \text{Id} \downarrow & & \uparrow F_{x,y}(\lambda) \\ F_{y,y}(\text{Id}_y) \circ_1 F_{x,y}(\cdot) & \xrightarrow{F_{x,y,y}} & F_{x,y}(\text{Id}_y \circ_1 \cdot) \end{array} \qquad \begin{array}{ccc} F_{x,y}(\cdot) \circ_1 \text{Id}_{F(x)} & \xrightarrow{\rho} & F_{x,y}(\cdot) \\ \text{Id} \circ_1 F_{\text{Id}_x} \downarrow & & \uparrow F_{x,y}(\rho) \\ F_{x,y}(\cdot) \circ_1 F_{x,x}(\text{Id}_x) & \xrightarrow{F_{x,x,y}} & F_{x,y}(\cdot \circ_1 \text{Id}_x). \end{array}$$

(Here, λ is the natural isomorphism from $\text{Id}_y \circ_1 \cdot$ to the identity functor of \mathcal{C} or \mathcal{D} , as appropriate, and ρ is the natural isomorphism from $\cdot \circ_1 \text{Id}_x$ to the identity functor of \mathcal{C} or \mathcal{D} .)

(Fu-2) For each quadruple of objects $x, y, z, w \in \text{Ob}(\mathcal{C})$, the following diagram commutes:

$$\begin{array}{ccc} (F_{y,z}(\cdot) \circ_1 F_{x,y}(\cdot)) \circ_1 F_{w,x}(\cdot) & \xrightarrow{\alpha} & F_{y,z}(\cdot) \circ_1 (F_{x,y}(\cdot) \circ_1 F_{w,x}(\cdot)) \\ F_{x,y,z}(\cdot, \cdot) \circ_1 \cdot \downarrow & & \downarrow \cdot \circ_1 F_{w,x,y}(\cdot, \cdot) \\ F_{x,z}(\cdot \circ_1 \cdot) \circ_1 F_{w,x}(\cdot) & & F_{y,z}(\cdot) \circ_1 F_{w,y}(\cdot \circ_1 \cdot) \\ F_{w,x,z}(\cdot \circ_1 \cdot, \cdot) \downarrow & & \downarrow F_{w,y,z}(\cdot, \cdot \circ_1 \cdot) \\ F_{w,z}((\cdot \circ_1 \cdot) \circ_1 \cdot) & \xrightarrow{F_{w,z}(\alpha)} & F_{w,z}(\cdot \circ_1 (\cdot \circ_1 \cdot)). \end{array}$$

(Here, α is the natural isomorphism from $((\cdot \circ_1 \cdot) \circ_1 \cdot)$ to $(\cdot \circ_1 (\cdot \circ_1 \cdot))$, the two different orders of triple-compositions, associated to \mathcal{C} or \mathcal{D} , as appropriate.)

(See for instance [Bén67, Definition 4.1 and Remark 4.2], where lax 2-functors are called homomorphisms.)

We will often drop the subscript from F : given an element $f \in \text{Hom}_{\mathcal{C}}(x, y)$ and a lax 2-functor $F: \mathcal{C} \rightarrow \mathcal{D}$ we will often write $F(f)$ for $F_{x,y}(f)$.

Definition 2.10. [Bén67, Remark 4.2] We call a lax 2-functor $F: \mathcal{C} \rightarrow \mathcal{D}$ *strictly unitary* if for all objects $x \in \text{Ob}(\mathcal{C})$, $F_{x,x}(\text{Id}_x) = \text{Id}_{F(x)}$ and F_{Id_x} is the identity 2-morphism.

As mentioned above, we will be mainly interested in 2-functors from $\underline{2}^n$ to \mathcal{B} , which moreover will be strictly unitary. In this case, Definition 2.9 simplifies substantially:

Lemma 2.11. *A strictly unitary 2-functor $F: \underline{2}^n \rightarrow \mathcal{B}$ is determined by the following data:*

- For each object $v \in \text{Ob}(\underline{2}^n) = \{0, 1\}^n$ the set $X_v = F(v)$.
- For each pair of objects $v, w \in \text{Ob}(\underline{2}^n)$ such that $v > w$, a correspondence $A_{v,w} = F(\varphi_{v,w})$ from X_v to X_w .
- For each triple of objects $u, v, w \in \text{Ob}(\underline{2}^n)$ such that $u > v > w$, a bijection $F_{u,v,w}: A_{v,w} \times_{X_v} A_{u,v} \rightarrow A_{u,w}$.

These data satisfy the compatibility condition:

(CF-1) *For any $u, v, w, x \in \text{Ob}(\underline{2}^n)$ with $u > v > w > x$, the following diagram commutes:*

$$\begin{array}{ccc} A_{w,x} \times_{X_w} A_{v,w} \times_{X_v} A_{u,v} & \xrightarrow{\text{Id} \times F_{u,v,w}} & A_{w,x} \times_{X_w} A_{u,w} \\ \downarrow F_{v,w,x} \times \text{Id} & & \downarrow F_{u,w,x} \\ A_{v,x} \times_{X_v} A_{u,v} & \xrightarrow{F_{u,v,x}} & A_{u,x}. \end{array}$$

(Here, we have suppressed some non-confusing parentheses.)

Moreover, any collection of data satisfying this compatibility condition determines a strictly unitary 2-functor $F: \underline{2}^n \rightarrow \mathcal{B}$.

Proof. Since F is strictly unitary, we have $F(\varphi_{v,v}) = \text{Id}_{X_v}$ (which is X_v , viewed as a correspondence from itself to itself). If $v \not\geq w$ then $\text{Hom}(v, w) = \emptyset$, so $\varphi_{v,w}$ is the unique functor from the empty category. Thus, the $\varphi_{v,w}$ are entirely specified by the correspondences $A_{v,w} = F_{v,w}(\varphi_{v,w})$ with $v > w$. Next, the source $\text{Hom}(v, w) \times \text{Hom}(u, v)$ of $F_{u,v,w}$ is nonempty if and only if $u \geq v \geq w$, in which case $\text{Hom}(v, w) \times \text{Hom}(u, v)$ consists of the single element $(\varphi_{v,w}, \varphi_{u,v})$. Since $\underline{2}^n$ is a strict 2-category and F is strictly unitary, Condition (Fu-1) is equivalent to the statement that $F_{v,v,w}: A_{v,w} \times_{X_v} X_v \rightarrow A_{v,w}$ is the canonical isomorphism ρ , and $F_{v,w,w}: X_w \times_{X_w} A_{v,w} \rightarrow A_{v,w}$ is the canonical isomorphism λ . So, F is determined by the specified data. Condition (Fu-2) is equivalent to Condition (CF-1); in Condition (CF-1) we have abused notation to identify the two sides of the top row of Condition (Fu-2), and the bottom arrow in Condition (Fu-2) is an equality because $\underline{2}^n$ is a strict 2-category and F is strictly unitary. The result follows. \square

Lemma 2.12. *Up to natural isomorphism, a strictly unitary 2-functor $F: \underline{2}^n \rightarrow \mathcal{B}$ is determined by the sets $F(v)$, the correspondences $F(\varphi_{v,w})$ with $v > w$ and $|v| - |w| = 1$, and the maps $F_{u,v',w}^{-1} \circ F_{u,v,w}: F(\varphi_{v,w}) \circ F(\varphi_{u,v}) \rightarrow F(\varphi_{v',w}) \circ F(\varphi_{u,v'})$ with $u > v, v' > w$ and $|u| - |w| = 2$.*

Proof. This follows from the observations that:

- Any morphism in $\underline{2}^n$ is a composition of maps associated to edges, so, $F(\varphi_{v,w})$ is determined for all $v \geq w$.
- Any two (directed) edge paths from v to w in $\underline{2}^n$ are related by sequence of swaps across 2-dimensional faces, so $F_{u,v,w}$ is determined for all $u \geq v \geq w$.

Further details are left to the reader. \square

2.9. Homotopy colimits and homotopy coherent diagrams. One step in the new construction of the Khovanov homotopy type will be taking an iterated mapping cone, which can be described as a homotopy colimit. So, we briefly review the notion of homotopy colimits here.

Given a diagram $F: \mathcal{D} \rightarrow \mathbf{Top}_\bullet$ of based topological spaces then the *homotopy colimit* of F , $\text{hocolim } F = \text{hocolim}_{\mathcal{D}} F$, is another based topological space. Similarly, if $F: \mathcal{D} \rightarrow \mathcal{S}$ is a diagram of spectra then we can again form the homotopy colimit of F , $\text{hocolim } F$, which is a spectrum. We will give a construction in a slightly more general setting presently, but first we note some key properties of the homotopy colimit, all of which hold for both diagrams of spaces and diagrams of spectra:

- (ho-1) Suppose that $F, G: \mathcal{C} \rightarrow \mathcal{S}$ are diagrams and $\eta: F \rightarrow G$ is a natural transformation. Then η induces a map $\text{hocolim } \eta: \text{hocolim } F \rightarrow \text{hocolim } G$. If $\eta(c)$ is a stable homotopy equivalence for each $c \in \text{Ob}(\mathcal{C})$ then $\text{hocolim } \eta$ is a stable homotopy equivalence as well.
- (ho-2) Suppose that $F, G: \mathcal{C} \rightarrow \mathcal{S}$ are diagrams and $F \vee G: \mathcal{C} \rightarrow \mathcal{S}$ is the diagram obtained by taking their wedge sum (i.e., $(F \vee G)(v) = F(v) \vee G(v)$). Then the natural map $\text{hocolim } F \vee G \rightarrow \text{hocolim } F \vee \text{hocolim } G$ is an equivalence.
- (ho-3) Suppose that $F: \mathcal{C} \rightarrow \mathcal{S}$ and $G: \mathcal{D} \rightarrow \mathcal{S}$. Then there is an induced functor $F \wedge G: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{S}$, with $(F \wedge G)(v \times w) = F(v) \wedge G(w)$. Then there is a natural weak equivalence $(\text{hocolim } F) \wedge (\text{hocolim } G) \rightarrow \text{hocolim}(F \wedge G)$.
- (ho-4) Let $G: \mathcal{C} \rightarrow \mathcal{D}$ be a map of diagrams (i.e., a functor between small categories). Given $d \in \text{Ob}(\mathcal{D})$, the *undercategory* of d has objects $\{(c, f) \mid c \in \mathcal{C}, f: d \rightarrow G(c)\}$, and $\text{Hom}((c, f), (c', f')) = \{g: c \rightarrow c' \mid f' = G(g) \circ f\}$. Let $d \downarrow G$ denote the undercategory of d . The functor G is called *homotopy cofinal* if for each $d \in (\mathcal{D})$, $d \downarrow G$ has contractible nerve.

Now, let $F: \mathcal{D} \rightarrow \mathbf{Top}_\bullet$ or \mathcal{S} be a diagram. Then there is an induced functor $F \circ G: \mathcal{C} \rightarrow \mathbf{Top}_\bullet$ or \mathcal{S} . Suppose that G is homotopy cofinal. Then

$$\text{hocolim } F \circ G \simeq \text{hocolim } F.$$

(In the case of homotopy limits, this is [BK72, Cofinality Theorem XI,9.2]. We note that the homotopy colimit can be characterized by knowing that the mapping space $\text{Map}(\text{hocolim } F, Z)$ is equivalent to the homotopy limit of the diagram of spaces $\text{Map}(F, Z)$.)

Turning to the generalization, we will sometimes find it convenient (e.g., in Section 5) to talk about homotopy colimits of diagrams which are only homotopy-commutative, but where the homotopies are part of the data, and are coherent up to higher homotopies (also part of the data). In this setting, we will make use of a particular construction, which we spell out now. We start with an appropriate notion of diagrams:

Definition 2.13. [Vog73, Definition 2.3] A *homotopy coherent diagram in \mathbf{Top}_\bullet* consists of:

- A small category \mathcal{C} .
- For each $x \in \text{Ob}(\mathcal{C})$ a space $F(x) \in \mathbf{Top}_\bullet$.
- For each $n \geq 1$ and each sequence

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n$$

of composable morphisms in \mathcal{C} a continuous map

$$F(f_n, \dots, f_1): [0, 1]^{n-1} \times F(x_0) \rightarrow F(x_n)$$

with $F(f_n, \dots, f_1)([0, 1]^{n-1} \times \{*\}) = *$, the basepoint in $F(x_n)$.

Letting (t_1, \dots, t_{n-1}) denote points in $[0, 1]^{n-1}$, these maps F are required to satisfy the conditions:

$$(2.14) \quad F(f_n, \dots, f_1)(t_1, \dots, t_{n-1}) = \begin{cases} F(f_n, \dots, f_2)(t_2, \dots, t_{n-1}) & f_1 = \text{Id} \\ F(f_n, \dots, f_{i+1}, f_{i-1}, \dots, f_1)(t_1, \dots, t_{i-1} \cdot t_i, \dots, t_{n-1}) & f_i = \text{Id}, 1 < i < n \\ F(f_{n-1}, \dots, f_1)(t_1, \dots, t_{n-2}) & f_n = \text{Id} \\ [F(f_n, \dots, f_{i+1})(t_{i+1}, \dots, t_{n-1})] \circ [F(f_i, \dots, f_1)(t_1, \dots, t_{i-1})] & t_i = 0 \\ F(f_n, \dots, f_{i+1} \circ f_i, \dots, f_1)(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{n-1}) & t_i = 1. \end{cases}$$

(A homotopy coherent diagram is what Vogt [Vog73] calls an $h\mathcal{C}$ -diagram. We are restricting to the case that his topological category \mathcal{C} is discrete.)

We will abuse notation and denote a homotopy coherent diagram as above by $F: \mathcal{C} \rightarrow \text{Top}_\bullet$. It will be clear from context when we mean a commutative diagram or a homotopy coherent diagram.

Example 2.15. Any commutative diagram $F: \mathcal{C} \rightarrow \text{Top}_\bullet$ can be viewed as a homotopy coherent diagram by defining

$$F(f_n, \dots, f_1)(t_1, \dots, t_{n-1}) = F(f_n \circ \dots \circ f_1).$$

Remark 2.16. Associated to an ordinary category \mathcal{C} , there is a simplicial category $\mathfrak{C}[\mathcal{C}]$, introduced in [Lei75] and further developed by many authors (e.g. [Cor82, DK80, Lur09]) such that homotopy coherent diagrams $\mathcal{C} \rightarrow \text{Top}_\bullet$ are precisely the same (up to the replacement of the continuous function $t_{i-1} \cdot t_i$ by an equivalent piecewise linear one) as simplicial functors from $\mathfrak{C}[\mathcal{C}]$ to spaces.

Definition 2.17. [Vog73, Paragraph (5.10)] Given a homotopy coherent diagram $F: \mathcal{C} \rightarrow \text{Top}_\bullet$, the *homotopy colimit* of F is defined by

$$(2.18) \quad \text{hocolim } F = \{*\} \amalg \coprod_{n \geq 0} \coprod_{x_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} x_n} [0, 1]^n \times F(x_0) / \sim,$$

where the second coproduct is over n -tuples of composable morphisms in \mathcal{C} and the case $n = 0$ corresponds to the objects $x_0 \in \text{Ob}(\mathcal{C})$. Letting (t_1, \dots, t_n) denote points in $[0, 1]^n$ and p a point in $F(x_0)$, the equivalence relation \sim is given by

$$(f_n, \dots, f_1; t_1, \dots, t_n; p) \sim \begin{cases} (f_n, \dots, f_2; t_2, \dots, t_n; p) & f_1 = \text{Id} \\ (f_n, \dots, f_{i+1}, f_{i-1}, \dots, f_1; t_1, \dots, t_{i-1}t_i, \dots, t_n; p) & f_i = \text{Id}, i > 1 \\ (f_n, \dots, f_{i+1}; t_{i+1}, \dots, t_n; F(f_i, \dots, f_1)(t_1, \dots, t_{i-1}, p)) & t_i = 0 \\ (f_n, \dots, f_{i+1} \circ f_i, \dots, f_1; t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n; p) & t_i = 1, i < n \\ (f_{n-1}, \dots, f_1; t_1, \dots, t_{n-1}; p) & t_n = 1 \\ * & p = *, \end{cases}$$

where $*$ denotes the basepoint.

Observation 2.19. The first three cases in the compatibility condition (2.14) for a homotopy coherent diagram imply that F is determined by its values on sequences of non-identity morphisms (and on objects). If one restricts to only non-identity morphisms, however, the compatibility condition for F becomes more complicated. In the special case that \mathcal{C} has no isomorphisms except for identity maps, however, the compatibility condition for sequences (f_n, \dots, f_1) of non-identity morphisms is simply the last two cases of Formula (2.14).

Similarly, the first two relations in the definition of \sim mean that we can write

$$\text{hocolim } F = \{*\} \amalg \coprod_{n \geq 0} \coprod_{\substack{x_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} x_n \\ \forall i \in \{1, \dots, n\}, f_i \neq \text{Id}}} [0, 1]^n \times F(x_0) / \sim',$$

for some equivalence relation \sim' , the difference being that we consider only non-identity morphisms when $n > 0$. The equivalence relation \sim' is more complicated than \sim . In the special case that \mathcal{C} has no isomorphisms except for identity maps, \sim' is simply given by the last four cases of the definition of \sim .

In the homotopy coherent diagrams and homotopy colimits considered in this paper, the categories \mathcal{C} will have no non-identity isomorphisms, and so we will work with these smaller formulations.

There is a notion of a morphism between homotopy coherent diagrams (h -morphisms [Vog73, Definition 2.7]) $F, G: \mathcal{C} \rightarrow \text{Top}_\bullet$, which relaxes the notion of a morphism (natural transformation) between diagrams. In particular, a morphism $F \rightarrow G$ of homotopy coherent diagrams includes the data of maps $F(v) \rightarrow G(v)$ for each $v \in \text{Ob}(\mathcal{C})$; we call these maps the *underlying maps* of the morphism. There is also the notion of a (simplicial) homotopy of morphisms [Vog73, Definition 2.7], and hence the notion of a homotopy equivalence of homotopy coherent diagrams. A special case is that any morphism of homotopy coherent diagrams whose underlying maps are homotopy equivalences is a homotopy equivalence of diagrams [Vog73, Proposition 4.6]. Further:

Proposition 2.20. [Vog73, Theorem 5.12] *If $F, G: \mathcal{C} \rightarrow \text{Top}_\bullet$ are homotopy equivalent diagrams then $\text{hocolim } F \simeq \text{hocolim } G$.*

There is also a rectification result, that any homotopy coherent diagram can be made coherent:

Proposition 2.21. [Vog73, Theorem 5.6] *Given any homotopy coherent diagram $F: \mathcal{C} \rightarrow \text{Top}_\bullet$ there is an honest diagram $G: \mathcal{C} \rightarrow \text{Top}_\bullet$ which is homotopy equivalent to F .*

Finally:

Proposition 2.22. [Vog73, Section 8] *If $F: \mathcal{C} \rightarrow \text{Top}_\bullet$ is an honest diagram then the homotopy colimits of F , viewed as an honest diagram and as a homotopy coherent diagram, are homotopy equivalent.*

Corollary 2.23. *Properties (ho-1)–(ho-4), or their obvious analogues, hold for homotopy colimits of homotopy coherent diagrams.*

2.10. Box maps. The method of realizing a functor $2^n \rightarrow \mathcal{B}$ in Section 5, and the proof that it agrees with the other realization methods, will rely on a particular class of maps between spheres which we describe here.

Fix once and for all an identification $S^k = [0, 1]^k / \partial$. (We will usually view S^k as a pointed space, and assume that this identification identifies the basepoint and $\partial[0, 1]^k$.) Let B be a box in \mathbb{R}^k with edges parallel to the coordinate axes, i.e., $B = [a_1, b_1] \times \dots \times [a_k, b_k]$ for some a_1, \dots, b_k . Then there is a standard homeomorphism $B \rightarrow [0, 1]^k$, given by $(x_1, \dots, x_k) \mapsto (\frac{x_1 - a_1}{b_1 - a_1}, \dots, \frac{x_k - a_k}{b_k - a_k})$, and so an induced identification $S^k \cong B / \partial B$.

Next, suppose we are given disjoint sub-boxes $B_1, \dots, B_\ell \subset B$. Then there is an induced map

$$(2.24) \quad S^k = B / \partial B \rightarrow B / (B \setminus (\mathring{B}_1 \cup \dots \cup \mathring{B}_\ell)) = \bigvee_{a=1}^{\ell} B_a / \partial B_a = \bigvee_{a=1}^{\ell} S^k \rightarrow S^k,$$

where the last map is the identity on each summand (so the composition has degree k). This construction is continuous with respect to the location of the sub-boxes. That is, if we let $E(B, \ell)$ denote the space of ℓ -tuples

of disjoint boxes in B (which is a subspace of $(\mathbb{R}^{2k})^\ell$) then there is a continuous map $E(B, \ell) \rightarrow \text{Map}(S^k, S^k)$. (In other words, this is a composition of the coaction of the little k -cubes operad with the fold map.)

More generally, suppose that we have index sets A and Y ; $|A|$ disjoint sub-boxes $B_a \subset B$, $a \in A$, labeled by elements of A ; and a map of sets $A \rightarrow Y$. Then there is an induced map

$$(2.25) \quad S^k = B/\partial B \rightarrow B/(B \setminus (\bigcup_{a \in A} \overset{\circ}{B}_a)) = \bigvee_{a \in A} B_a/\partial B_a = \bigvee_{a \in A} S^k \rightarrow \bigvee_{y \in Y} S^k.$$

(Formula (2.24) is the special case that Y has a single element.) Generalizing further, given a correspondence A from X to Y with source and target maps $s: A \rightarrow X$ and $t: A \rightarrow Y$; boxes B_x , $x \in X$; and disjoint sub-boxes $B_a \subset B_{s(a)}$, $a \in A$ there is an induced map

$$(2.26) \quad \bigvee_{x \in X} S^k \rightarrow \bigvee_{y \in Y} S^k.$$

gotten by applying Equation (2.25) on each summand (using the index sets $s^{-1}(x)$ and Y). We will say that any map as in Formula (2.26) *refines* the correspondence A . Let $E(\{B_x\}, s)$ be the space of collections of disjoint labeled sub-boxes $\{B_a \subset B_{s(a)} \mid a \in A\}$, so Equation (2.26) (together with the map of index sets t) gives a map $E(\{B_x\}, s) \rightarrow \text{Map}(\bigvee_{x \in X} S^k, \bigvee_{y \in Y} S^k)$. Given $e \in E(\{B_x\}, s)$ let $\Phi(e, A): \bigvee_{x \in X} S^k \rightarrow \bigvee_{y \in Y} S^k$ be the corresponding map.

In a slightly different direction, suppose that the boxes B_a are not necessarily all disjoint, but for each $y \in Y$, $\{B_a \mid t(a) = y\}$ are disjoint. Then for each $x \in X$ and $y \in Y$ we obtain a map $S_x^k \rightarrow S_y^k$ by Equation (2.24). From the universal properties of products and coproducts, these maps assemble to give a map

$$(2.27) \quad \bigvee_{x \in X} S^k \rightarrow \prod_{y \in Y} S^k.$$

Let $F(\{B_x\}, s, t)$ denote the space of such labeled sub-boxes, so Equation (2.27) gives a map $F(\{B_x\}, s, t) \rightarrow \text{Map}(\bigvee_{x \in X} S^k, \prod_{y \in Y} S^k)$.

By a *box map* we mean a map of one of the forms (2.26) or (2.27). A *disjoint box map* is a map of the form (2.26). A composition of box maps is a box map: given composable box maps F and G , the preimages under F of the boxes for G are the boxes for $G \circ F$. (Note that we have not defined, and will not need, box maps from products.)

Example 2.28. The diagonal map $S^k \rightarrow \prod_{i=1}^m S^k$ is a box map: take $X = \{x\}$ to have a single element, A and Y to have m elements each, t to be a bijection, and each box B_a to be all of B_x .

The value (to us) of these constructions comes from the following:

Lemma 2.29. *If B is k -dimensional then the space $E(B, \ell)$ is $(k-2)$ -connected. More generally, $E(\{B_x\}, s)$ and $F(\{B_x\}, s, t)$ are $(k-2)$ -connected.*

Proof. The space $E(B, \ell)$ is homotopy equivalent to the ordered configuration space of ℓ points in the interior of B , i.e., $B^\ell \setminus \Delta$, where Δ is the fat diagonal. Since Δ is a finite union of smooth submanifolds of codimension k , the result for $E(B, \ell)$ follows. Next, $E(\{B_x\}, s) \cong \prod_{x \in X} E([0, 1]^k, |s^{-1}(x)|)$ is a product of $(k-2)$ -connected spaces, and hence is $(k-2)$ -connected. Finally, $F(\{B_x\}, s, t) \cong \prod_{(x,y) \in X \times Y} E([0, 1]^k, |s^{-1}(x) \cap t^{-1}(y)|)$, and so again is $(k-2)$ -connected. \square

Informally, Lemma 2.29 says that the space of box maps is highly connected. (For this statement to be correct, we must think of a box map as not just the map but also the labeling of the boxes.) Indeed, the space of box maps is also highly connected in relative terms. Specifically, for any set map $s: A \rightarrow X$, define

the space $E^\circ(\{B_x\}, s)$ to be the subspace of $E(\{B_x\}, s)$ where we require the box B_a to lie in the interior of the box $B_{s(a)}$ for all $a \in A$. Let $A_0 \subset A$ be a subset and let $s_0: A_0 \rightarrow X$ denotes the restriction of s . There is a map $E^\circ(\{B_x\}, s) \rightarrow E^\circ(\{B_x\}, s_0)$ gotten by forgetting the boxes labeled by $A \setminus A_0$.

Lemma 2.30. *If the boxes $\{B_x\}$ are k -dimensional, then for any $i \leq k - 1$ and any commutative diagram*

$$\begin{array}{ccc} S^{i-1} & \longrightarrow & E^\circ(\{B_x\}, s) \\ \downarrow & & \downarrow \\ D^i & \longrightarrow & E^\circ(\{B_x\}, s_0), \end{array}$$

there exists a lift $D^i \rightarrow E^\circ(\{B_x\}, s)$ making the diagram commute.

Proof. By induction, we may assume that $A \setminus A_0$ consists of a single element, say a_1 , and let $x_1 = s(a_1)$ and $\ell = |s_0^{-1}(x_1)|$. The projection $\pi: E^\circ(\{B_x\}, s) \rightarrow E^\circ(\{B_x\}, s_0)$ is seen to be a fiber bundle by the following argument.

Let $\pi_\bullet: E_\bullet \rightarrow E^\circ(\{B_x\}, s_0)$ be the fiber bundle where the fiber over a point is the complement of the ℓ boxes in the interior of B_{x_1} . (It is easy to see that π_\bullet is a fiber bundle, by triangulating the complement of the ℓ boxes for instance.) Now for any $z \in E^\circ(\{B_x\}, s_0)$, construct a coordinate chart on $\pi^{-1}(z)$ by the following variables: the center C of the box B_{a_1} viewed as a point in $\pi_\bullet^{-1}(z)$; the ‘aspect ratio’ R of the box B_{a_1} , presented as a $(k - 1)$ -tuple of ratios of the k side-lengths of B_{a_1} ; and a proportion $P \in (0, 1)$ of the size of the box B_{a_1} relative to the size of the largest box with the same center and same aspect ratio that lies in B_{x_1} in the complement of the interiors of other ℓ boxes. This identifies $\pi^{-1}(z)$ with $\pi_\bullet^{-1}(z) \times (0, \infty)^{k-1} \times (0, 1)$, and the identification holds for small open sets around z . But π_\bullet is a fiber bundle, and therefore, so is π .

The fiber over each point is homeomorphic to the space of boxes in the complement of ℓ disjoint boxes in the interior of B_{x_1} . The fiber, being homotopy equivalent to the complement of ℓ points in \mathbb{R}^k , is $(k - 2)$ -connected, so the statement follows. \square

3. CUBICAL FLOW CATEGORIES

The Khovanov flow category from [LS14a] is defined as a kind of cover of another flow category, the cube flow category (Definition 3.1). After reviewing the cube flow category and some of its basic properties in Section 3.1, in Section 3.2 we abstract the notion of a cover of the cube flow category, into a cubical flow category (Definition 3.7). In Section 3.3 we give a slightly different notion of neat embeddings for cubical flow categories. Using this notion, in Section 3.4 we give a realization procedure for a cubical flow category, the cubical realization. In Section 3.5 we show that the cubical realization and the Cohen-Jones-Segal realization (reviewed in Section 2.5) give homotopy equivalent spaces.

3.1. Cube flow category. We start by recalling the cube flow category from [LS14a, Definition 4.1]. There we gave a definition in terms of Morse flows on $[0, 1]^n$. Here, we give a more directly combinatorial definition in terms of permutohedra (see also Remark 3.3):

Definition 3.1. The objects of the cube flow category $\mathcal{C}_C(n)$ are the same as the objects of the cube category $\underline{2}^n$, i.e., tuples $u = (u_1, \dots, u_n) \in \{0, 1\}^n$. The grading on the objects is defined by $\text{gr}(u) = |u| = \sum_i u_i$.

The space $\mathcal{M}(u, v)$ is defined to be empty unless $u > v$. If $u > v$ and $\text{gr}(u) - \text{gr}(v) = k > 0$ then we define $\mathcal{M}(u, v) = \Pi_{k-1}$, the $(k - 1)$ -dimensional permutohedron. Note that, by Lemma 2.7, $\mathcal{M}(u, v)$ is a $(k - 1)$ -manifold.

The composition map $\mathcal{M}(v, w) \times \mathcal{M}(u, v) \rightarrow \mathcal{M}(u, w)$ is defined as follows. Assume that $u > v > w$, $\text{gr}(u) - \text{gr}(v) = k$ and $\text{gr}(v) - \text{gr}(w) = l$. Suppose that $u_{a_1} = \dots = u_{a_{k+l}} = 1$ and $w_{a_1} = \dots = w_{a_{k+l}} = 0$

(i.e., the a_i are the coordinates in which u and w differ), where $a_1 < a_2 < \dots < a_{k+l}$. Let S be the subset of $\{1, \dots, k+l\}$ such that $v_{a_s} = 1$ for $s \in S$. (The set S has cardinality l .) By Lemma 2.6, there is a corresponding facet $F_S \subset \Pi_{k+l-1}$, and F_S is identified with $\Pi_{l-1} \times \Pi_{k-1} = \mathcal{M}(v, w) \times \mathcal{M}(u, v)$. The composition map is the corresponding inclusion map $\mathcal{M}(v, w) \times \mathcal{M}(u, v) = F_S \hookrightarrow \mathcal{M}(u, w)$.

Lemma 3.2. *Definition 3.1 defines a flow category.*

Proof. Conditions (FC-1) and (FC-2) of Definition 2.2 are immediate from the definitions and Lemma 2.7. For Condition (FC-3), it is enough to recall from Lemma 2.7 that $\partial_l \Pi_{k+l-1} = \bigcup_{|S|=l} F_S$.

Finally, we need to check that this defines a category, or in other words, that composition is associative. Towards this end, for any $u > v$ with $\text{gr}(u) - \text{gr}(v) = k > 0$, it is convenient to treat $\mathcal{M}(u, v) = \Pi_{k-1}$ as a subset of $\prod_{i|u_i > v_i} \mathbb{R}$ instead of \mathbb{R}^k , where the two ambient spaces are identified by linearly ordering the coordinates $a_1 < a_2 < \dots < a_k$ in which u and v differ. With this viewpoint, for $u > v > w$, with $\text{gr}(u) - \text{gr}(v) = k$ and $\text{gr}(v) - \text{gr}(w) = l$, the composition map $\Pi_{l-1} \times \Pi_{k-1} \rightarrow \Pi_{k+l-1}$ is induced from the map

$$\prod_{i|v_i > w_i} \mathbb{R} \times \prod_{i|u_i > v_i} \mathbb{R} \xrightarrow{+(\vec{0}, \vec{l})} \prod_{i|v_i > w_i} \mathbb{R} \times \prod_{i|u_i > v_i} \mathbb{R} \xrightarrow{\cong} \prod_{i|u_i > w_i} \mathbb{R},$$

where the first map adds l to each of the coordinates of $\prod_{i|u_i > v_i} \mathbb{R}$, and the second map is coordinate-wise identification. (See also Lemma 2.6.) Now, given $u > v > w > x$, with $\text{gr}(u) - \text{gr}(v) = k$, $\text{gr}(v) - \text{gr}(w) = l$ and $\text{gr}(w) - \text{gr}(x) = m$, the corresponding double compositions are:

$$\begin{array}{ccc} \prod_{i|w_i > x_i} \mathbb{R} \times \prod_{i|v_i > w_i} \mathbb{R} \times \prod_{i|u_i > v_i} \mathbb{R} & \xrightarrow{+(\vec{0}, \vec{0}, \vec{l})} & \prod_{i|w_i > x_i} \mathbb{R} \times \prod_{i|u_i > v_i} \mathbb{R} \\ \downarrow +(\vec{0}, \vec{m}, \vec{0}) & & \downarrow +(\vec{0}, \vec{m}) \\ \prod_{i|v_i > x_i} \mathbb{R} \times \prod_{i|u_i > v_i} \mathbb{R} & \xrightarrow{+(\vec{0}, \vec{l} + \vec{m})} & \prod_{i|u_i > x_i} \mathbb{R}, \end{array}$$

where we have suppressed the reshuffling of factors from the notation. So, both compositions are given by $+(\vec{0}, \vec{m}, \vec{l} + \vec{m})$ (and the same reshuffling of factors). \square

Remark 3.3. We have not shown that the definition of the cube flow category from Definition 3.1 agrees with the Morse-theoretic definition from [LS14a, Definition 4.1], as doing so would seem to require a nontrivial digression about the smooth structures on moduli spaces of Morse flows. For the purposes of this paper (and future work), note that the combinatorial definition used here works just as well for the construction of the Khovanov homotopy type in [LS14a], and all the results stated in [LS14a, LS14c, LS14b] remains true with this combinatorial definition. To be more precise, the only statements that involve the moduli spaces on the cube flow category coming from the Morse flows are [LS14a, Lemmas 4.2–4.3], which are immediate for the combinatorial definition. Therefore, when we talk about the Khovanov homotopy type in this paper (and future work), we mean the homotopy type defined using the cube flow category from Definition 3.1.

Before moving on to the definition of cubical flow categories in Section 3.2, we digress a little to study the cubical complex structure from Lemma 2.8 on the permutohedron $\mathcal{M}(u, v)$.

Definition 3.4. For $u > v$ in $\{0, 1\}^n$, define the space

$$M_{u,v} = \left(\prod_{u=u^0 > \dots > u^m = v} [0, 1]^{m-1} \right) / \sim$$

where, for each chain $u = u^0 > \dots > u^m = v$, each $1 \leq i \leq m-1$, and each point (t_1, \dots, t_{m-2}) in the cube $[0, 1]^{m-2}$ corresponding to the chain $u^0 > \dots > u^{i-1} > u^{i+1} > \dots > u^m$ the equivalence relation \sim identifies

$$(t_1, \dots, t_{m-2})_{u=u^0 > \dots > u^{i-1} > u^{i+1} > \dots > u^m = v} \sim (t_1, \dots, t_{i-1}, 1, t_i, \dots, t_{m-2})_{u=u^0 > \dots > u^m = v}.$$

For $u > v > w$ in $\{0, 1\}^n$, define a map $M_{v,w} \times M_{u,v} \rightarrow M_{u,w}$ by:

$$\begin{aligned} ((t_1, \dots, t_{\ell-1})_{v=v^0 > \dots > v^\ell = w}, (s_1, \dots, s_{m-1})_{u=u^0 > \dots > u^m = v}) \\ \mapsto (s_1, \dots, s_{m-1}, 0, t_1, \dots, t_{\ell-1})_{u=u^0 > \dots > u^m = v^0 > \dots > v^\ell = w}. \end{aligned}$$

Lemma 3.5. *There are homeomorphisms $h_{u,v}: M_{u,v} \xrightarrow{\cong} \mathcal{M}(u,v)$ from the spaces of Definition 3.4 to the permutohedra $\mathcal{M}(u,v)$ so that:*

- (1) *The $h_{u,v}$ identify the cubes in the definition of $M_{u,v}$ with some of the cubes in the cubical complex structure on $\mathcal{M}(u,v)$ from Lemma 2.8.*
- (2) *The $h_{u,v}$ identify the points in the cubes for $M_{u,v}$ where some coordinate is 0 with the points in $\partial\mathcal{M}(u,v)$.*
- (3) *For any $u > v > w$, the following diagram commutes:*

$$\begin{array}{ccc} M_{v,w} \times M_{u,v} & \xrightarrow[h_{v,w} \times h_{u,v}]{\cong} & \mathcal{M}(v,w) \times \mathcal{M}(u,v) \\ \downarrow & & \downarrow \\ M_{u,w} & \xrightarrow[h_{u,w}]{\cong} & \mathcal{M}(u,w). \end{array}$$

Here, the left vertical arrow is the map from Definition 3.4, and the right vertical arrow is the composition in $\mathcal{C}_C(n)$.

Proof. The chain $c = \{u = u^0 > \dots > u^m = v\}$ in $\{0, 1\}^n$ corresponds to the $(|u| - |v| - m)$ -dimensional face $F_c = \mathcal{M}(u^{m-1}, u^m) \times \dots \times \mathcal{M}(u^0, u^1)$ of the permutohedron $\mathcal{M}(u,v)$. If we take the convex hull of the barycenters of all the faces of the permutohedron that contain F_c , we get an $(m-1)$ -dimensional cube C_c which appears in the cubical complex structure from Lemma 2.8. (If F_c is a vertex of the permutohedron, or equivalently if c is a maximal chain, then the cube C_c is one of the C_σ from Lemma 2.8.) We will identify C_c with the cube $[0, 1]^{m-1}$ corresponding to c in $M_{u,v}$.

Let t_1, \dots, t_{m-1} be the coordinates of $[0, 1]^{m-1}$. As a first step, we identify the vertices of C_c and $[0, 1]^{m-1}$. A vertex of C_c corresponds to a barycenter of some face containing F_c , which in turn corresponds to some subchain c' of c ; the corresponding vertex in $[0, 1]^{m-1}$ has $t_i = 0$ if $u^i \in c'$, and has $t_i = 1$ otherwise. The identification on the vertices leads to our desired identification as follows. Construct the simplicial complex subdivision of C_c (respectively, $[0, 1]^{m-1}$) by joining every face to the barycenter of $\mathcal{M}(u,v)$ (respectively, the vertex $\vec{1} \in [0, 1]^{m-1}$), and extend the identification linearly over each simplex.

It is fairly straightforward to check that such identifications induce a cubical complex homeomorphism between the cubical complex $M_{u,v}$ and the cubical complex structure on $\mathcal{M}(u,v)$, and these homeomorphisms satisfy the conditions of the lemma. Further details are left to the reader. \square

Remark 3.6. To connect this with Remark 2.16, the above essentially proves that the cube flow category $\mathcal{C}_C(n)$ is the category $\mathfrak{C}[\underline{2}^n]$, and so functors out of $\mathcal{C}_C(n)$ are equivalent to coherent diagrams on $\underline{2}^n$.

3.2. Definition of a cubical flow category.

Definition 3.7. A *cubical flow category* is a flow category \mathcal{C} equipped with a grading-preserving functor $\mathfrak{f}: \Sigma^k \mathcal{C} \rightarrow \mathcal{C}_C(n)$ for some $k \in \mathbb{Z}, n \in \mathbb{N}$ so that for each $x, y \in \text{Ob}(\mathcal{C})$, $\mathfrak{f}: \mathcal{M}(x, y) \rightarrow \mathcal{M}(\mathfrak{f}(x), \mathfrak{f}(y))$ is a (trivial) covering map.

Thus, if $\mathfrak{f}: \mathcal{C} \rightarrow \mathcal{C}_C(n)$ is a cubical flow category and $x, y \in \text{Ob}(\mathcal{C})$ then $\text{Hom}(x, y)$ is empty unless $\mathfrak{f}(x) \geq \mathfrak{f}(y)$. If $x = y$, then $\text{Hom}(x, y) = \{\text{Id}\}$, and if $\mathfrak{f}(x) = \mathfrak{f}(y)$ but $x \neq y$ then $\text{Hom}(x, y)$ is empty. If $\mathfrak{f}(x) > \mathfrak{f}(y)$, then the moduli space $\mathcal{M}_{\mathcal{C}}(x, y) = \text{Hom}(x, y)$ is a (possibly empty) disjoint union of permutohedra.

Convention 3.8. Sometimes we suppress the grading information if it is inessential to the discussion, and drop the grading shift Σ^k from the notation.

A framing of the cube flow category $\mathcal{C}_C(n)$ (in the sense of Definition 2.4) induces a sign assignment s on the cube (see Section 2.1) as follows: For $u > v$ with $|u| - |v| = 1$, $s_{u,v} = 0$ if the point $\mathcal{M}(u, v)$ is framed positively, and $s_{u,v} = 1$ otherwise. Furthermore, every sign assignment on the cube is induced from an essentially unique framing of $\mathcal{C}_C(n)$; see [LS14a, Section 4.2]. The chain complex associated to $\mathcal{C}_C(n)$, framed according to some sign assignment s , is defined as follows. The chain group is generated by the vertices of the cube $\{0, 1\}^n$, and differential is given by

$$\delta(v) = \sum_{\substack{u > v \\ |u| - |v| = 1}} (-1)^{s_{u,v}} u.$$

It is an acyclic chain complex, and is often referred to as the *cube chain complex*.

Furthermore, if $(\mathcal{C}, \mathfrak{f}: \mathcal{C} \rightarrow \mathcal{C}_C(n))$ is a cubical flow category, then any sign assignment s on the cube induces a framing of the 0-dimensional moduli spaces in \mathcal{C} as well: All the points in $\mathcal{M}_{\mathcal{C}}(x, y)$ are framed positively if $s_{\mathfrak{f}(x), \mathfrak{f}(y)} = 0$; otherwise, all the points in $\mathcal{M}_{\mathcal{C}}(x, y)$ are framed negatively. The pullback of the (essentially unique) framing on $\mathcal{C}_C(n)$ inducing s produces an essentially canonical extension of this framing to the entire cubical flow category \mathcal{C} . The chain complex associated to \mathcal{C} for this framing has the following differential: For $x, y \in \text{Ob}(\mathcal{C})$ with $\text{gr}(x) - \text{gr}(y) = 1$, the coefficient of x in δy is

$$\langle \delta(y), x \rangle = \begin{cases} (-1)^{s_{\mathfrak{f}(x), \mathfrak{f}(y)}} & \mathfrak{f}(x) > \mathfrak{f}(y) \\ 0 & \text{otherwise.} \end{cases}$$

Note that this chain complex only depends on the sign assignment s (and not the entire framing of $\mathcal{C}_C(n)$).

Even though the definition of cubical flow categories seems fairly restrictive, there are many examples:

Example 3.9. Given any (finite) simplicial complex S_{\bullet} with vertices v_1, \dots, v_n , there is a corresponding cubical flow category $(\mathcal{C}, \mathfrak{f}: \Sigma \mathcal{C} \rightarrow \mathcal{C}_C(n))$, defined as follows. The objects of \mathcal{C} correspond to the simplices of S_{\bullet} , which in turn can be viewed as non-empty subsets of $\{v_1, \dots, v_n\}$. Given an object in \mathcal{C} , corresponding to a subset $T \subseteq \{v_1, \dots, v_n\}$, define $\mathfrak{f}(T)$ to be the vector in $\{0, 1\}^n$ whose i^{th} coordinate is 1 if and only if $v_i \in T$. Note that the map \mathfrak{f} is injective on objects. Let \mathcal{C} be the full subcategory of $\mathcal{C}_C(n)$ generated by the objects in the image of \mathfrak{f} . The chain complex associated to \mathcal{C} is isomorphic to the simplicial chain complex for S_{\bullet} .

(One can think of the category \mathcal{C} as coming from choosing a Morse function on each simplex in S_{\bullet} with a unique interior maximum and no other interior critical points, and so that these Morse functions are compatible under restriction. The moduli spaces in \mathcal{C} are then the corresponding Morse moduli spaces.)

Example 3.10. The Khovanov flow category [LS14a, Definition 5.3] $\mathcal{C}_K(K)$ associated to a link diagram K (with n crossings c_1, \dots, c_n) is, by construction, a cubical flow category. For any $v \in \{0, 1\}^n$, the subset of $\text{Ob}(\mathcal{C}_K(K))$ that maps to v are precisely the Khovanov generators over v (as defined in Section 2.3):

$$\mathfrak{f}^{-1}(v) = F(v).$$

For any $u > v \in \{0, 1\}^n$ with $|u| - |v| = 1$, and any $x \in \mathfrak{f}^{-1}(u) = F(u)$ and $y \in \mathfrak{f}^{-1}(v) = F(v)$, the moduli space has the following description:

$$\mathcal{M}_{\mathcal{C}_K(K)}(x, y) = \begin{cases} \{\text{pt}\} & \text{if } x \text{ appears in } \delta_{Kh}(y), \\ \emptyset & \text{otherwise.} \end{cases}$$

Therefore, the chain complex associated to $\mathcal{C}_K(K)$ is isomorphic to the Khovanov chain complex $\mathcal{C}_{Kh}(K)$ (from Definition 2.1).

3.3. Cubical neat embeddings. Consider the cube flow category $\mathcal{C}_C(n)$, and fix a tuple $\mathbf{d} = (d_0, \dots, d_{n-1}) \in \mathbb{N}^n$ and a real number $R > 0$. For any $u > v$ in $\text{Ob}(\mathcal{C}_C(n)) = \{0, 1\}^n$, let

$$E_{u,v} = \left[\prod_{i=|v|}^{|u|-1} (-R, R)^{d_i} \right] \times \mathcal{M}_{\mathcal{C}_C(n)}(u, v).$$

Equip $E_{u,v}$ with the Riemannian metric induced from the standard metric on the Euclidean space after viewing the permutohedron $\mathcal{M}_{\mathcal{C}_C(n)}(u, v)$ as a polyhedron in $\mathbb{R}^{|u|-|v|}$. For any $u > v > w$ in $\text{Ob}(\mathcal{C}_C(n))$, there is a map $E_{v,w} \times E_{u,v} \rightarrow E_{u,w}$ given by:

$$\begin{aligned} E_{v,w} \times E_{u,v} &= \prod_{i=|w|}^{|v|-1} (-R, R)^{d_i} \times \mathcal{M}_{\mathcal{C}_C(n)}(v, w) \times \prod_{i=|v|}^{|u|-1} (-R, R)^{d_i} \times \mathcal{M}_{\mathcal{C}_C(n)}(u, v) \\ &\cong \prod_{i=|w|}^{|u|-1} (-R, R)^{d_i} \times \mathcal{M}_{\mathcal{C}_C(n)}(v, w) \times \mathcal{M}_{\mathcal{C}_C(n)}(u, v) \\ &\xrightarrow{\text{Id} \times \circ} \prod_{i=|w|}^{|u|-1} (-R, R)^{d_i} \times \mathcal{M}_{\mathcal{C}_C(n)}(u, w). \end{aligned}$$

Definition 3.11. Fix a cubical flow category $(\mathcal{C}, \mathfrak{f}: \Sigma^k \mathcal{C} \rightarrow \mathcal{C}_C(n))$. A *cubical neat embedding* ι of $(\mathcal{C}, \mathfrak{f})$ (or, more succinctly, of \mathcal{C}) relative to a tuple $\mathbf{d} = (d_0, \dots, d_{n-1}) \in \mathbb{N}^n$ consists of neat embeddings

$$\iota_{x,y}: \mathcal{M}_{\mathcal{C}}(x, y) \hookrightarrow E_{\mathfrak{f}(x), \mathfrak{f}(y)},$$

such that:

- (1) For each $x, y \in \text{Ob}(\mathcal{C})$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{C}}(x, y) & \xrightarrow{\iota_{x,y}} & E_{\mathfrak{f}(x), \mathfrak{f}(y)} \\ & \searrow \mathfrak{f} & \downarrow \text{projection} \\ & & \mathcal{M}_{\mathcal{C}_C(n)}(\mathfrak{f}(x), \mathfrak{f}(y)). \end{array}$$

- (2) For each $u, v \in \text{Ob}(\mathcal{C}_C(n))$, the induced map

$$\prod_{\substack{x,y \\ \mathfrak{f}(x)=u, \mathfrak{f}(y)=v}} \iota_{x,y}: \prod_{\substack{x,y \\ \mathfrak{f}(x)=u, \mathfrak{f}(y)=v}} \mathcal{M}_{\mathcal{C}}(x, y) \rightarrow E_{u,v}$$

is a neat embedding.

(3) For each $x, y, z \in \text{Ob}(\mathcal{C})$, the following commutes:

$$\begin{array}{ccc} \mathcal{M}_{\mathcal{C}}(y, z) \times \mathcal{M}_{\mathcal{C}}(x, y) & \longrightarrow & \mathcal{M}_{\mathcal{C}}(x, z) \\ \downarrow & & \downarrow \\ E_{\mathfrak{f}(y), \mathfrak{f}(z)} \times E_{\mathfrak{f}(x), \mathfrak{f}(y)} & \longrightarrow & E_{\mathfrak{f}(x), \mathfrak{f}(z)}. \end{array}$$

In order to construct the cubical realization, we need to extend these embeddings $\iota_{x,y}$ to maps

$$\bar{\iota}_{x,y} : \left[\prod_{i=|\mathfrak{f}(y)|}^{|\mathfrak{f}(x)|-1} [-\epsilon, \epsilon]^{d_i} \right] \times \mathcal{M}_{\mathcal{C}}(x, y) \rightarrow E_{\mathfrak{f}(x), \mathfrak{f}(y)},$$

for some $\epsilon > 0$, so that that the analogue of the diagram from Condition (3) still commutes, and the extension of the map from Condition (2) is still an embedding. One way to choose such a family of extensions would be to coherently frame the normal bundles of the embeddings $\iota_{x,y}$ (in a similar sense to Definition 2.3) and then use the construction from Definition 2.5. Instead, we will use the following explicit extension.

For any $x, y \in \text{Ob}(\mathcal{C})$, let u and v denote $\mathfrak{f}(x)$ and $\mathfrak{f}(y)$, respectively, and let $\pi_{u,v}^R$ and $\pi_{u,v}^M$ denote the projections of $\left[\prod_{i=|v|}^{|u|-1} (-R, R)^{d_i} \right] \times \mathcal{M}_{\mathcal{C}(n)}(u, v)$ onto the two factors. Given sufficiently small $\epsilon > 0$, extend the embedding $\iota_{x,y}$ to a map

$$(3.12) \quad \bar{\iota}_{x,y} : \left[\prod_{i=|v|}^{|u|-1} [-\epsilon, \epsilon]^{d_i} \right] \times \mathcal{M}_{\mathcal{C}}(x, y) \rightarrow E_{u,v} = \left[\prod_{i=|v|}^{|u|-1} (-R, R)^{d_i} \right] \times \mathcal{M}_{\mathcal{C}(n)}(u, v)$$

$$(a, \gamma) \xrightarrow{\bar{\iota}_{x,y}} (a + \pi_{u,v}^R \iota_{x,y}(\gamma), \pi_{u,v}^M \iota_{x,y}(\gamma)).$$

The definition of $\bar{\iota}$ ensures that the analogue of the diagram from Condition (3) still commutes. For ϵ sufficiently small, the extension of the map from Condition (2) is still an embedding; we make this a requirement on ϵ .

Convention 3.13. When talking about extensions $\bar{\iota}_{x,y}$ of cubical neat embeddings, we will always assume that ϵ is chosen to be small in the sense that the induced map

$$\prod_{\substack{x,y \\ \mathfrak{f}(x)=u, \mathfrak{f}(y)=v}} \bar{\iota}_{x,y} : \prod_{\substack{x,y \\ \mathfrak{f}(x)=u, \mathfrak{f}(y)=v}} \left[\prod_{i=|v|}^{|u|-1} [-\epsilon, \epsilon]^{d_i} \right] \times \mathcal{M}_{\mathcal{C}}(x, y) \rightarrow E_{u,v}$$

is an embedding.

Remark 3.14. In [LS14a], we used a framing of the normal bundles, rather than the kind of explicit extension above, to trivialize tubular neighborhoods. When identifying the cubical realization with the Cohen-Jones-Segal realization (see Section 3.5), we will need an isotopy between these two trivializations.

Let $V_0 = \prod_{i=|v|}^{|u|-1} \mathbb{R}^{d_i} \times \{0\} \subset (TE_{u,v})|_{\iota_{x,y}(\mathcal{M}_{\mathcal{C}}(x,y))}$ and let V_1 be the normal bundle to $\iota_{x,y} \mathcal{M}_{\mathcal{C}}(x, y)$. Since $\pi_{u,v}^M \circ \iota_{x,y}$ is a covering map, the projection $d\pi_{u,v}^R : V_1 \rightarrow V_0$ is an isomorphism. For $t \in [0, 1]$ let $\pi_t = t \text{Id} + (1-t)d\pi_{u,v}^R$ and let $V_t = \pi_t(V_1)$. The V_t are a 1-parameter family of subbundles connecting V_0 to V_1 , and each V_t is a complement to $T(\iota_{x,y} \mathcal{M}_{\mathcal{C}}(x, y))$.

The bundle V_0 is a trivial bundle, and in particular framed; pushing forward this framing by π_t gives a framing of each V_t . Exponentiating these framings gives a 1-parameter family of extensions $\bar{\iota}_{x,y}^t$ of $\iota_{x,y}$, each

satisfying the analogue of Condition (3). The framing $\bar{\iota}_{x,y}^0$ is our explicit extension $\bar{\iota}_{x,y}$ from Equation (3.12), and $\bar{\iota}_{x,y}^1$ is an extension coming from coherently framing the normal bundles of $\iota_{x,y}$. Since each V_t is a complement to $T(\iota_{x,y}\mathcal{M}_{\mathcal{C}}(x,y))$, each $\bar{\iota}_{x,y}^t$ satisfies the analog of Condition (2) for some ϵ_t ; compactness allows us to choose a uniform ϵ for which each $\bar{\iota}_{x,y}^t$ satisfies the analog of Condition (2). This produces the required isotopy between the extension from Equation (3.12) and an extension coming from some framing of the normal bundle.

3.4. Cubical realization.

Definition 3.15. Fix a cubical neat embedding ι of a cubical flow category $(\mathcal{C}, \mathbf{f}; \Sigma^k \mathcal{C} \rightarrow \mathcal{C}_C(n))$, relative to a tuple $\mathbf{d} = (d_0, \dots, d_{n-1}) \in \mathbb{N}^d$, and fix $\epsilon > 0$ satisfying Convention 3.13. We build a CW complex $\|\mathcal{C}\| = \|\mathcal{C}\|_{\mathbf{f}, \iota}$ from this data as follows:

- The CW complex has one cell for each $x \in \text{Ob}(\mathcal{C})$. Letting u denote $\mathbf{f}(x)$, this cell is given by

$$\mathcal{C}(x) = \prod_{i=0}^{|u|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \widetilde{\mathcal{M}}_{\mathcal{C}_C(n)}(u, \vec{0}),$$

where $\widetilde{\mathcal{M}}_{\mathcal{C}_C(n)}(u, \vec{0})$ is defined to be $[0, 1] \times \mathcal{M}_{\mathcal{C}_C(n)}(u, \vec{0})$ if $u \neq \vec{0}$, or the point $\{0\}$ if $u = \vec{0}$.

- For any $x, y \in \text{Ob}(\mathcal{C})$ with $\mathbf{f}(x) = u > \mathbf{f}(y) = v$, the cubical neat embedding ι furnishes an embedding

$$\begin{aligned} & \mathcal{C}(y) \times \mathcal{M}_{\mathcal{C}}(x, y) \\ &= \prod_{i=0}^{|v|-1} [-R, R]^{d_i} \times \prod_{i=|v|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \widetilde{\mathcal{M}}_{\mathcal{C}_C(n)}(v, \vec{0}) \times \mathcal{M}_{\mathcal{C}}(x, y) \\ &\cong \prod_{i=0}^{|v|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \widetilde{\mathcal{M}}_{\mathcal{C}_C(n)}(v, \vec{0}) \times \left(\prod_{i=|v|}^{|u|-1} [-\epsilon, \epsilon]^{d_i} \times \mathcal{M}_{\mathcal{C}}(x, y) \right) \\ &\xrightarrow{\text{Id} \times \bar{\iota}_{x,y}} \prod_{i=0}^{|v|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \widetilde{\mathcal{M}}_{\mathcal{C}_C(n)}(v, \vec{0}) \times \left(\prod_{i=|v|}^{|u|-1} [-R, R]^{d_i} \times \mathcal{M}_{\mathcal{C}_C(n)}(u, v) \right) \\ &\cong \prod_{i=0}^{|u|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \widetilde{\mathcal{M}}_{\mathcal{C}_C(n)}(v, \vec{0}) \times \mathcal{M}_{\mathcal{C}_C(n)}(u, v) \\ &\hookrightarrow \prod_{i=0}^{|u|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \partial(\widetilde{\mathcal{M}}_{\mathcal{C}_C(n)}(u, \vec{0})) \\ &\subset \partial \mathcal{C}(x). \end{aligned}$$

Here, the second inclusion comes from the composition map if $v \neq \vec{0}$, or the inclusion $\{0\} \hookrightarrow [0, 1]$ if $v = \vec{0}$. Let $\mathcal{C}_y(x) \subset \partial \mathcal{C}(x)$ denote the image of this embedding.

- The attaching map for $\mathcal{C}(x)$ sends $\mathcal{C}_y(x) \cong \mathcal{C}(y) \times \mathcal{M}_{\mathcal{C}}(x, y)$ by the projection map to $\mathcal{C}(y)$ and sends the complement of $\cup_y \mathcal{C}_y(x)$ in $\partial \mathcal{C}(x)$ to the basepoint.

The *cubical realization* of $(\mathcal{C}, \mathbf{f})$ is defined to be the formal desuspension $\mathcal{X}(\mathcal{C}) = \Sigma^{-(k+d_0+\dots+d_{n-1})} \|\mathcal{C}\|$. (The desuspension ensures that the gradings of the objects in \mathcal{C} agree with the dimensions of the corresponding cells in $\mathcal{X}(\mathcal{C})$.)

Lemma 3.16. *The attaching maps in the cubical realization are well-defined.*

Proof. As in the proof of [LS14a, Lemma 3.25], we must show that for any $x, y, z \in \text{Ob}(\mathcal{C})$ with $\text{gr}(x) > \text{gr}(y) > \text{gr}(z)$, the dashed arrow in the following diagram exists such that the diagram commutes.

$$\begin{array}{ccccc}
 \mathcal{C}_z(x) \cap \mathcal{C}_y(x) & \hookrightarrow & \mathcal{C}_y(x) & \longrightarrow & \mathcal{C}(y) \\
 \downarrow & & & & \downarrow \\
 \mathcal{C}_z(x) & & & & \partial\mathcal{C}(y) \\
 \downarrow & \dashrightarrow & & & \downarrow \\
 \mathcal{C}(z) & \longleftarrow & & & \mathcal{C}_z(y)
 \end{array}$$

Let u, v, w denote $\mathfrak{f}(x), \mathfrak{f}(y), \mathfrak{f}(z)$, respectively. We may assume $u > v > w$; otherwise, it is not hard to verify that $\mathcal{C}_z(x)$ and $\mathcal{C}_y(x)$ are disjoint. In a similar vein to Definition 3.15, let $\mathcal{C}_{y,z}(x) \subset \partial\mathcal{C}(x)$ be the image of the following embedding:

$$\begin{aligned}
 & \mathcal{C}(z) \times \mathcal{M}_{\mathcal{C}}(y, z) \times \mathcal{M}_{\mathcal{C}}(x, y) \\
 &= \prod_{i=0}^{|w|-1} [-R, R]^{d_i} \times \prod_{i=|w|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \widetilde{\mathcal{M}}_{\mathcal{C}(n)}(w, \vec{0}) \times \mathcal{M}_{\mathcal{C}}(y, z) \times \mathcal{M}_{\mathcal{C}}(x, y) \\
 &\cong \prod_{i=0}^{|w|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \widetilde{\mathcal{M}}_{\mathcal{C}(n)}(w, \vec{0}) \times \left(\prod_{i=|w|}^{|v|-1} [-\epsilon, \epsilon]^{d_i} \times \mathcal{M}_{\mathcal{C}}(y, z) \right) \\
 &\quad \times \left(\prod_{i=|v|}^{|u|-1} [-\epsilon, \epsilon]^{d_i} \times \mathcal{M}_{\mathcal{C}}(x, y) \right) \\
 &\xrightarrow{\text{Id} \times \bar{\iota}_{y,z} \times \bar{\iota}_{x,y}} \prod_{i=0}^{|v|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \widetilde{\mathcal{M}}_{\mathcal{C}(n)}(w, \vec{0}) \\
 &\quad \times \left(\prod_{i=|w|}^{|v|-1} [-R, R]^{d_i} \times \mathcal{M}_{\mathcal{C}(n)}(v, w) \right) \times \left(\prod_{i=|v|}^{|u|-1} [-R, R]^{d_i} \times \mathcal{M}_{\mathcal{C}(n)}(u, v) \right) \\
 &\cong \prod_{i=0}^{|u|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \widetilde{\mathcal{M}}_{\mathcal{C}(n)}(w, \vec{0}) \times \mathcal{M}_{\mathcal{C}(n)}(v, w) \times \mathcal{M}_{\mathcal{C}(n)}(u, v) \\
 &\hookrightarrow \prod_{i=0}^{|u|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \partial(\widetilde{\mathcal{M}}_{\mathcal{C}(n)}(u, \vec{0})) \\
 &\subset \partial\mathcal{C}(x).
 \end{aligned}$$

(As in Definition 3.15, the second inclusion usually comes from the composition map; the only special case is if $w = \vec{0}$, when it comes partly from the inclusion $\{0\} \hookrightarrow [0, 1]$.)

We claim that $\mathcal{C}_{y,z}(x) = \mathcal{C}_z(x) \cap \mathcal{C}_y(x)$. The direction $\mathcal{C}_{y,z}(x) \subseteq \mathcal{C}_z(x) \cap \mathcal{C}_y(x)$ is immediate since the inclusion

$$\mathcal{C}(z) \times \mathcal{M}_{\mathcal{C}}(y, z) \times \mathcal{M}_{\mathcal{C}}(x, y) \xrightarrow{\cong} \mathcal{C}_{y,z}(x) \subset \partial\mathcal{C}(x)$$

factors in both of the following ways (using Condition (3) of Definition 3.11):

$$\begin{aligned} \mathcal{C}(z) \times \mathcal{M}_{\mathcal{C}}(y, z) \times \mathcal{M}_{\mathcal{C}}(x, y) &\xrightarrow{\text{Id} \times \circ} \mathcal{C}(z) \times \mathcal{M}_{\mathcal{C}}(x, z) \xrightarrow{\cong} \mathcal{C}_z(x) \subset \partial\mathcal{C}(x) \\ \mathcal{C}(z) \times \mathcal{M}_{\mathcal{C}}(y, z) \times \mathcal{M}_{\mathcal{C}}(x, y) &\xrightarrow{\cong} \mathcal{C}_z(y) \times \mathcal{M}_{\mathcal{C}}(x, y) \hookrightarrow \mathcal{C}(y) \times \mathcal{M}_{\mathcal{C}}(x, y) \xrightarrow{\cong} \mathcal{C}_y(x) \subset \partial\mathcal{C}(x). \end{aligned}$$

The other direction requires more work. We will abuse notation slightly and identify points with their images under the composition map in $\mathcal{C}(n)$. View any point $p \in \mathcal{C}_z(x) \cap \mathcal{C}_y(x)$ as a point (p_1, p_2, p_3, p_4) in the following subspace of $\partial\mathcal{C}(x)$:

$$\left(\prod_{i=0}^{|w|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \right) \times \prod_{i=|w|}^{|v|-1} [-R, R]^{d_i} \times \prod_{i=|v|}^{|u|-1} [-R, R]^{d_i} \times \partial(\widetilde{\mathcal{M}}_{\mathcal{C}(n)}(u, \vec{0})).$$

For p to lie in $\mathcal{C}_z(x)$, p_4 must lie in the subspace

$$\widetilde{\mathcal{M}}_{\mathcal{C}(n)}(w, \vec{0}) \times \mathcal{M}_{\mathcal{C}(n)}(u, w);$$

similarly, for p to lie in $\mathcal{C}_y(x)$, p_4 must lie in the subspace

$$\widetilde{\mathcal{M}}_{\mathcal{C}(n)}(v, \vec{0}) \times \mathcal{M}_{\mathcal{C}(n)}(u, v).$$

Therefore, p_4 must lie in the subspace $\widetilde{\mathcal{M}}_{\mathcal{C}(n)}(w, \vec{0}) \times \mathcal{M}_{\mathcal{C}(n)}(v, w) \times \mathcal{M}_{\mathcal{C}(n)}(u, v)$. Write p_4 also in component form as $(p_{4,1}, p_{4,2}, p_{4,3})$. Since p lies in $\mathcal{C}_y(x)$, we know $(p_3, p_{4,3}) \in \text{im}(\bar{\iota}_{x,y})$, and since p lies in $\mathcal{C}_z(x)$, we know $(p_2, p_3, p_{4,2}, p_{4,3}) \in \text{im}(\bar{\iota}_{x,z})$. Moreover, since ι is a cubical neat embedding (Definition 3.11),

$$\text{im}(\bar{\iota}_{x,z}) \cap \left(\prod_{i=|w|}^{|v|-1} [-R, R]^{d_i} \times \mathcal{M}_{\mathcal{C}(n)}(v, w) \times \prod_{i=|v|}^{|u|-1} [-R, R]^{d_i} \times \mathcal{M}_{\mathcal{C}(n)}(u, v) \right) = \text{im} \left(\prod_{y' | \mathfrak{f}(y')=v} \bar{\iota}_{y',z} \times \bar{\iota}_{x,y'} \right),$$

and therefore, there exists y' with $\mathfrak{f}(y') = v$ such that $(p_3, p_{4,3}) \in \text{im}(\bar{\iota}_{x,y'})$ and $(p_2, p_{4,2}) \in \text{im}(\bar{\iota}_{y',z})$. Condition (2) from Definition 3.11 (but with $\bar{\iota}$ instead of ι) ensures that $y' = y$, and then it is straightforward to see that p lies in $\mathcal{C}_{y,z}(x)$. This completes the proof that $\mathcal{C}_{y,z}(x) = \mathcal{C}_z(x) \cap \mathcal{C}_y(x)$.

Define the dashed arrow from $\mathcal{C}_z(x) \cap \mathcal{C}_y(x) = \mathcal{C}_{y,z}(x) \cong \mathcal{C}(z) \times \mathcal{M}_{\mathcal{C}}(y, z) \times \mathcal{M}_{\mathcal{C}}(x, y)$ to $\mathcal{C}_z(y) \cong \mathcal{C}(z) \times \mathcal{M}_{\mathcal{C}}(y, z)$ to be the projection map. From the definition of $\mathcal{C}_{y,z}(x)$, it is easy to verify that the resulting diagram commutes. \square

Proposition 3.17. *Up to stable homotopy equivalence, the cubical realization is independent of the choice of tuple \mathbf{d} . More precisely, let ι be a cubical neat embedding relative to $\mathbf{d} = (d_0, \dots, d_{n-1})$ and let $\|\mathcal{C}\|$ be the cubical realization corresponding to ι . Fix a tuple $\mathbf{d}' = (d'_0, \dots, d'_{n-1})$ with $d'_i \geq d_i$ for all i . There is an induced cubical neat embedding of \mathcal{C} relative to \mathbf{d}' , gotten by identifying the space $E_{u,v}$ for ι with the subspace*

$$\prod_{i=|v|}^{|u|-1} (-R, R)^{d_i} \times \{0\}^{d'_i - d_i} \times \mathcal{M}_{\mathcal{C}(n)}(u, v)$$

of the space $E'_{u,v}$ for ι' . Let $\|\mathcal{C}'\|$ be the cubical realization corresponding to ι' and let $N = |\mathbf{d}'| - |\mathbf{d}| = \sum_{i=0}^{n-1} d'_i - \sum_{i=0}^{n-1} d_i$. Then there is a homotopy equivalence

$$\Sigma^N \|\mathcal{C}\| \simeq \|\mathcal{C}'\|,$$

taking cells to the corresponding cells.

Proof. The proof is the same as Case (3) in the proof of [LS14a, Lemma 3.27]. \square

One can also show, by following the proofs of [LS14a, Lemmas 3.25–3.27], that the stable homotopy type of $\|\mathcal{C}\|_\iota$ is independent of the cubical neat embedding ι and the parameters R and ϵ . Since this result also follows from Theorem 4, we do not give further details here.

We conclude this section by returning to our simplicial complex example:

Proposition 3.18. *Let S_\bullet be a simplicial complex with n vertices and let $(\mathcal{C}, \mathfrak{f}: \Sigma\mathcal{C} \rightarrow \mathcal{C}_C(n))$ be the corresponding cubical flow category, as in Example 3.9. Let $|S_\bullet|_+$ denote the disjoint union of the geometric realization of S_\bullet and a basepoint. Then there is a stable homotopy equivalence*

$$\mathcal{X}(\mathcal{C}) \simeq |S_\bullet|_+.$$

Moreover, the stable homotopy equivalence may be chosen so that it induces an isomorphism between the simplicial cochain complex of S_\bullet and the reduced cellular cochain complex of $\mathcal{X}(\mathcal{C})$ (with the CW complex structure from Definition 3.15).

We leave the proof as a (somewhat involved) exercise to the reader.

3.5. Cubical realization agrees with the Cohen-Jones-Segal realization. The main aim of this subsection is to prove the following.

Theorem 4. *Let $(\mathcal{C}, \mathfrak{f}: \mathcal{C} \rightarrow \mathcal{C}_C(n))$ be a cubical flow category. For any cubical neat embedding ι relative to any tuple $\mathbf{d} = (d_0, \dots, d_{n-1})$, and parameters R, ϵ , the cubical realization $\mathcal{X}(\mathcal{C})$ (from Definition 3.15) is stably homotopy equivalent to the Cohen-Jones-Segal realization (from Section 2.5) of \mathcal{C} with framing induced from some framing of $\mathcal{C}_C(n)$. Furthermore, the stable homotopy equivalence sends cells to corresponding cells by degree ± 1 maps.*

Proof. Using Proposition 3.17, we may assume that the d_i 's are sufficiently large so that there is a neat embedding j of the cube flow category $\mathcal{C}_C(n)$ relative to $\mathbf{D} = (\dots, 0, \dots, 0, d_0, \dots, d_{n-1}, 0, \dots, 0 \dots)$ in the sense of Definition 2.3. Fix some framing of the cube flow category $\mathcal{C}_C(n)$, and construct the extension \bar{j} as in Definition 2.5. Let δ and T be the corresponding parameters. After scaling if necessary, we may assume $\delta = R$, and after increasing T if necessary, we may assume $T \geq 1$.

Note that $\bar{j} \circ \iota$ is a neat embedding (in the sense of Definition 2.3) of our flow category \mathcal{C} relative to \mathbf{D} ; it has an extension $\bar{j} \circ \bar{\iota}$ (again, as in Definition 2.5) with respect to the parameters ϵ and T , and $\bar{j} \circ \bar{\iota}$ is (isotopic to) the extension coming from the framing of \mathcal{C} induced from the framing of $\mathcal{C}_C(n)$; see also Remark 3.14.

Let $\|\mathcal{C}\|$ be the CW complex constructed from the cubical neat embedding ι and its extension $\bar{\iota}$ with respect to the parameters ϵ and R ; and let $|\mathcal{C}|$ be the CW complex constructed from the neat embedding $\bar{j} \circ \iota$ and its extension $\bar{j} \circ \bar{\iota}$ with respect to the parameters ϵ and T . Cells of both $\|\mathcal{C}\|$ and $|\mathcal{C}|$ correspond to objects of \mathcal{C} , and hence to each other. We will produce a quotient map from $|\mathcal{C}|$ to $\|\mathcal{C}\|$ which will send each cell via a degree ± 1 map to the corresponding cell. It follows that the quotient map is a stable homotopy equivalence.

For any $x \in \text{Ob}(\mathcal{C})$ with $\mathfrak{f}(x) = u \in \text{Ob}(\mathcal{C}_C(n))$, the cell associated to x in $\|\mathcal{C}\|$ is

$$\mathcal{C}(x)' = \begin{cases} \prod_{i=0}^{|u|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times [0, 1] \times \mathcal{M}_{\mathcal{C}_C(n)}(u, \vec{0}) & \text{if } u \neq \vec{0}, \\ \prod_{i=0}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \{0\} & \text{if } u = \vec{0}. \end{cases}$$

while the cell associated to x in $|\mathcal{C}|$ is

$$\mathcal{C}(x) = \prod_{i=0}^{|u|-1} [-T, T]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \prod_{i=0}^{|u|-1} [0, T].$$

Let $[\mathcal{C}(x)]$ (respectively $[\mathcal{C}(x)']$) denote the image of $\mathcal{C}(x)$ (respectively $\mathcal{C}(x)'$) in $|\mathcal{C}|$ (respectively $\|\mathcal{C}\|$). We will define a map $\mathcal{C}(x) \rightarrow [\mathcal{C}(x)']$ and check that this induces a well-defined map $[\mathcal{C}(x)] \rightarrow [\mathcal{C}(x)']$.

If $u = \vec{0}$, identify $\mathcal{C}(x)' \cong \prod_{i=0}^{n-1} [-\epsilon, \epsilon]^{d_i} \cong \mathcal{C}(x)$. If $u \neq \vec{0}$, the embedding

$$\begin{aligned} \bar{j}_{u, \vec{0}}: \mathcal{M}_{\mathcal{C}(n)}(u, \vec{0}) \times \prod_{i=0}^{|u|-1} [-R, R]^{d_i} &\hookrightarrow (-T, T)^{d_0} \times [0, T] \times \cdots \times [0, T] \times (-T, T)^{d_{|u|-1}} \\ &\cong \prod_{i=1}^{|u|-1} [0, T] \times \prod_{i=0}^{|u|-1} (-T, T)^{d_i} \end{aligned}$$

induces the following codimension-zero embedding of $\mathcal{C}(x)'$ into $\mathcal{C}(x)$:

$$\begin{aligned} \mathcal{C}(x)' &= \prod_{i=0}^{|u|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times [0, 1] \times \mathcal{M}_{\mathcal{C}(n)}(u, \vec{0}) \\ &\cong \mathcal{M}_{\mathcal{C}(n)}(u, \vec{0}) \times \prod_{i=0}^{|u|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times [0, 1] \\ &\xrightarrow{\bar{j}_{u, \vec{0}} \times \text{Id}} \prod_{i=1}^{|u|-1} [0, T] \times \prod_{i=0}^{|u|-1} [-T, T]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times [0, 1] \\ &\cong \prod_{i=0}^{|u|-1} [-T, T]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times [0, 1] \times \prod_{i=1}^{|u|-1} [0, T] \\ &\hookrightarrow \prod_{i=0}^{|u|-1} [-T, T]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \prod_{i=0}^{|u|-1} [0, T] \\ &= \mathcal{C}(x). \end{aligned}$$

(Here, the \cong arrows correspond to the obvious reshuffling of the factors, and the second inclusion is induced from the inclusion $[0, 1] \hookrightarrow [0, T]$.) In either case, map $\mathcal{C}(x)$ to $[\mathcal{C}(x)']$ by identifying the image of this embedding with $\mathcal{C}(x)'$, and quotienting everything else to the basepoint. To see that this gives a well-defined, continuous map from the CW complex $|\mathcal{C}|$ to the CW complex $\|\mathcal{C}\|$, we need to check that for any other $y \in \text{Ob}(\mathcal{C})$ with $\mathfrak{f}(y) = v < u$, the following commutes:

$$\begin{array}{ccc} \mathcal{C}(y)' \times \mathcal{M}_{\mathcal{C}}(x, y) & \hookrightarrow & \mathcal{C}(y) \times \mathcal{M}_{\mathcal{C}}(x, y) \\ \downarrow & & \downarrow \\ \mathcal{C}(x)' & \hookrightarrow & \mathcal{C}(x). \end{array}$$

(The horizontal arrows are induced from the inclusions defined above. The right vertical arrow is the inclusion defined in Definition 2.5 for $\bar{j} \circ \bar{i}$, while the left vertical arrow is the inclusion defined in Definition 3.15 for \bar{i} .)

When $v = \vec{0}$, after removing the constant factor of $\prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i}$ and doing some consistent reshuffling, the diagram is

$$\begin{array}{ccc} \{0\} \times \prod_{i=0}^{|u|-1} [-\epsilon, \epsilon]^{d_i} \times \mathcal{M}_{\mathcal{C}}(x, y) & \xrightarrow{\cong} & \prod_{i=0}^{|u|-1} [-\epsilon, \epsilon]^{d_i} \times \mathcal{M}_{\mathcal{C}}(x, y) \\ \downarrow (\kappa_1, \bar{\iota}_{x, y}) & & \downarrow (\kappa_3^0, \text{Id}) \circ \bar{\jmath}_{u, \vec{0}} \circ \bar{\iota}_{x, y} \\ [0, 1] \times \prod_{i=0}^{|u|-1} [-R, R]^{d_i} \times \mathcal{M}_{\mathcal{C}(n)}(u, \vec{0}) & \xrightarrow{(\kappa_2, \bar{\jmath}_{u, \vec{0}})} & \prod_{i=0}^{|u|-1} [0, T] \times \prod_{i=0}^{|u|-1} [-T, T]^{d_i}, \end{array}$$

where κ_1 is the inclusions $\{0\} \hookrightarrow [0, 1]$, κ_2 is the inclusion $[0, 1] \hookrightarrow [0, T]$, and κ_3^ℓ (for $0 \leq \ell < |u|$) is the inclusion

$$\prod_{i=\ell+1}^{|u|-1} [0, T] \cong \{0\} \times \prod_{i=\ell+1}^{|u|-1} [0, T] \xrightarrow{(\kappa_2 \circ \kappa_1, \text{Id})} \prod_{i=\ell}^{|u|-1} [0, T].$$

Therefore, the diagram commutes.

When $v \neq \vec{0}$, once again after removing the constant factor of $\prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i}$, and some consistent reshuffling, the diagram factors as

$$\begin{array}{ccc} \left[\begin{array}{l} [0, 1] \times \mathcal{M}_{\mathcal{C}(n)}(v, \vec{0}) \times \prod_{i=0}^{|v|-1} [-R, R]^{d_i} \\ \times \prod_{i=|v|}^{|u|-1} [-\epsilon, \epsilon]^{d_i} \times \mathcal{M}_{\mathcal{C}}(x, y) \end{array} \right] & \xrightarrow{(\kappa_2, \bar{\jmath}_{v, \vec{0}}, \text{Id})} & \left[\begin{array}{l} \prod_{i=0}^{|v|-1} [0, T] \times \prod_{i=0}^{|v|-1} [-T, T]^{d_i} \\ \times \prod_{i=|v|}^{|u|-1} [-\epsilon, \epsilon]^{d_i} \times \mathcal{M}_{\mathcal{C}}(x, y) \end{array} \right] \\ \downarrow (\text{Id}, \bar{\iota}_{x, y}) & & \downarrow (\text{Id}, \bar{\jmath}_{u, \vec{0}} \circ \bar{\iota}_{x, y}) \\ \left[\begin{array}{l} [0, 1] \times \mathcal{M}_{\mathcal{C}(n)}(v, \vec{0}) \times \prod_{i=0}^{|u|-1} [-R, R]^{d_i} \\ \times \mathcal{M}_{\mathcal{C}(n)}(u, v) \end{array} \right] & \xrightarrow{(\kappa_2, \bar{\jmath}_{v, \vec{0}}, \bar{\jmath}_{u, v})} & \left[\begin{array}{l} \prod_{i=0}^{|v|-1} [0, T] \times \prod_{i=0}^{|u|-1} [-T, T]^{d_i} \times \prod_{i=|v|+1}^{|u|-1} [0, T] \end{array} \right] \\ \downarrow (\text{Id}, \circ) \circ \rho & & \downarrow \rho \circ (\text{Id}, \kappa_3^{|v|}) \\ \left[\begin{array}{l} [0, 1] \times \prod_{i=0}^{|u|-1} [-R, R]^{d_i} \times \mathcal{M}_{\mathcal{C}(n)}(u, \vec{0}) \end{array} \right] & \xrightarrow{(\kappa_2, \bar{\jmath}_{u, \vec{0}})} & \left[\begin{array}{l} \prod_{i=0}^{|u|-1} [0, T] \times \prod_{i=0}^{|u|-1} [-T, T]^{d_i} \end{array} \right], \end{array}$$

where ρ throughout denotes some (further) consistent reshuffling of factors, and κ_2, κ_3^ℓ from before. The top square commutes automatically; the bottom square commutes since Condition (2) of Definition 2.3 holds for $\bar{\jmath}$ as well (because the extension was defined via coherent framings of the normal bundles of j).

Therefore, we get a well-defined map from $|\mathcal{C}|$ to $\|\mathcal{C}\|$ which sends the cell $[\mathcal{C}(x)]$ to the corresponding cell $[\mathcal{C}(x)']$ by a degree ± 1 map. It is easy to check that the grading shifts match up, and therefore, we get the required stable homotopy equivalence. \square

4. FUNCTORS FROM THE CUBE TO THE BURNSIDE CATEGORY AND THEIR REALIZATIONS

In this section we give a reformulation of cubical flow categories, as functors from the cube category to the Burnside category. We then give a choice-free way of realizing such a functor, in terms of a thickening construction. (Another, smaller but choice-dependent way to realize such a functor is given in Section 5.) The proof that this realization agrees with the cubical realization is deferred until Section 6.

4.1. Cubical flow categories are functors from the cube to the Burnside category.

Construction 4.1. Fix a cubical flow category $\mathfrak{f}: \mathcal{C} \rightarrow \mathcal{C}_C(n)$. We will construct a strictly unitary 2-functor $F: \underline{2}^n \rightarrow \mathcal{B}$, as follows. By Lemma 2.11, it suffices to define the sets X_v ($v \in \{0, 1\}^n$), correspondences $A_{v,w}$ ($v > w \in \{0, 1\}^n$) and $F_{u,v,w}$ ($u > v > w \in \{0, 1\}^n$). We do so as follows:

- Given $v \in \{0, 1\}^n$ define $F(v) = \mathfrak{f}^{-1}(v)$.
- Given $v > w$, define $A_{v,w}$ to be the set of path components of

$$\coprod_{\substack{x \in \mathfrak{f}^{-1}(v) \\ y \in \mathfrak{f}^{-1}(w)}} \text{Hom}(x, y).$$

We will write $\pi_0(X)$ for the set of path components of X . Then the source (respectively target) map $s: A_{v,w} \rightarrow X_v$ (respectively $t: A_{v,w} \rightarrow X_w$) is defined by $s(\pi_0(\text{Hom}(x, y))) = x$ (respectively $t(\pi_0(\text{Hom}(x, y))) = y$).

- Given $u > v > w$ the composition map in \mathcal{C} induces a map

$$\circ: \left(\coprod_{\substack{y \in \mathfrak{f}^{-1}(v) \\ z \in \mathfrak{f}^{-1}(w)}} \text{Hom}(y, z) \right) \times_{\mathfrak{f}^{-1}(v)} \left(\coprod_{\substack{x \in \mathfrak{f}^{-1}(u) \\ y \in \mathfrak{f}^{-1}(v)}} \text{Hom}(x, y) \right) \rightarrow \coprod_{\substack{x \in \mathfrak{f}^{-1}(u) \\ z \in \mathfrak{f}^{-1}(w)}} \text{Hom}(x, z).$$

Taking path components gives a map $A_{v,w} \times_{X_v} A_{u,v} \rightarrow A_{u,w}$, which we define to be $F_{u,v,w}$.

Lemma 4.2. *Construction 4.1 defines a strictly unitary 2-functor.*

Proof. By Lemma 2.11, we only need to check the compatibility Condition (CF-1), which is immediate from associativity of composition in \mathcal{C} . \square

Construction 4.3. Fix a strictly unitary functor $F: \underline{2}^n \rightarrow \mathcal{B}$. We will construct a cubical flow category $\mathfrak{f}: \mathcal{C} \rightarrow \mathcal{C}_C(n)$, as follows.

- $\text{Ob}(\mathcal{C}) = \coprod_{v \in \{0, 1\}^n} F(v)$. The functor \mathfrak{f} sends an object $x \in F(v)$ to v .
- For any object x , $\text{Hom}(x, x)$ consists of the identity morphism.
- Given objects x and y , with $v = \mathfrak{f}(x) > \mathfrak{f}(y) = w$, consider the set

$$B_{x,y} = s^{-1}(x) \cap t^{-1}(y) \subset A_{v,w} = F(\varphi_{v,w}).$$

Define $\text{Hom}(x, y) = B_{x,y} \times \mathcal{M}_{\mathcal{C}_C(n)}(v, w)$. The map $\mathfrak{f}: \text{Hom}(x, y) \rightarrow \text{Hom}(\mathfrak{f}(x), \mathfrak{f}(y))$ is projection to the permutohedron $\mathcal{M}_{\mathcal{C}_C(n)}(v, w)$.

- Given objects x, y, z with $\mathfrak{f}(x) > \mathfrak{f}(y) > \mathfrak{f}(z)$ define the composition map $\text{Hom}(y, z) \times \text{Hom}(x, y) \rightarrow \text{Hom}(x, z)$ as follows. Let $u = \mathfrak{f}(x)$, $v = \mathfrak{f}(y)$, $w = \mathfrak{f}(z)$. The 2-functor includes a map $F_{u,v,w}: A_{v,w} \times_{F(v)} A_{u,v} \rightarrow A_{u,w}$. The composition map in $\mathcal{C}_C(n)$ gives a map $\circ: \mathcal{M}_{\mathcal{C}_C(n)}(v, w) \times \mathcal{M}_{\mathcal{C}_C(n)}(u, v) \rightarrow \mathcal{M}_{\mathcal{C}_C(n)}(u, w)$. Define the composition map in \mathcal{C} to be

$$F_{u,v,w} \times \circ: (B_{y,z} \times \mathcal{M}_{\mathcal{C}_C(n)}(v, w)) \times (B_{x,y} \times \mathcal{M}_{\mathcal{C}_C(n)}(u, v)) \rightarrow (B_{x,z} \times \mathcal{M}_{\mathcal{C}_C(n)}(u, w)).$$

(That is, we apply $F_{u,v,w}$ to the B factors and \circ to the \mathcal{M} factors.)

Lemma 4.4. *Construction 4.3 defines a cubical flow category.*

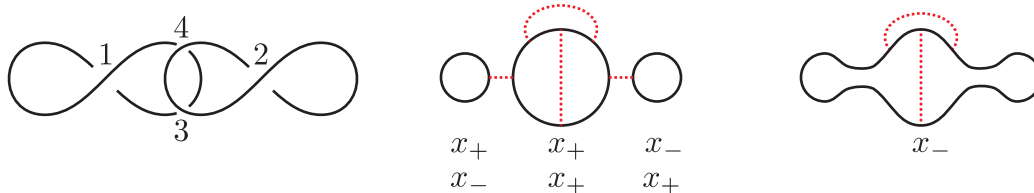


FIGURE 4.1. **An example showing that the Lee complex does not come from a functor $\underline{2}^n \rightarrow \mathcal{B}$ that extends F_{Kh} .** Left: a particular diagram for the two-component unlink, and an ordering of its crossings. Center: the corresponding resolution configuration (the $\bar{0}$ -resolution with dashed lines recording the crossings) and two labelings of this resolution. Both labelings are in the image of $F(\varphi_{1100,0000})$ of the labeling x_- of the circle in the $(1, 1, 0, 0)$ -resolution (right). Further, the two labelings give incompatible restrictions on the map $F_{(1111,1110,1100)}^{-1} \circ F_{(1111,1101,1100)}$ associated to the subcube $(1, 1, *, *)$.

Proof. The proof is similar to the proof of Lemma 3.2, and is left to the reader. \square

It is straightforward to verify that Constructions 4.1 and 4.3 are inverse to each other, in a sense which does not seem worth spelling out precisely.

Example 4.5. Construction 4.1 when applied to the Khovanov flow category from Example 3.10 yields a functor $F_{Kh} = F_{Kh}^K: \underline{2}^n \rightarrow \mathcal{B}$. For any $v \in \{0, 1\}^n$, the set $F_{Kh}(v)$ consists precisely of the Khovanov generators over v , denoted $F(v)$ in Section 2.3. For $u > v \in \{0, 1\}^n$ with $|u| - |v| = 1$, and for $x \in F_{Kh}(u), y \in F_{Kh}(v)$, the set

$$B_{x,y} = s^{-1}(x) \cap t^{-1}(y) \subseteq A_{u,v} = F_{Kh}(\varphi_{u,v})$$

consists of one element if x appears in $\delta_{Kh}(y)$ (see Definition 2.1), and is empty otherwise. The maps $F_{u,v,w}$ when $|u| - |w| = 2$ are defined using the *ladybug matching* [LS14a, Section 5.4]; see also Section 8.1.

Remark 4.6. In Section 2.3, we discussed a generalization of the Khovanov theory that works over the ring $\mathbb{Z}[h, t]$: a functor from $(\underline{2}^n)^{\text{op}}$ to the category of (graded) $\mathbb{Z}[h, t]$ -modules. Setting $h = t = 0$ recovers Khovanov's original functor to $\mathbb{Z}\text{-Mod}$ [Kho00], while setting $(h, t) = (0, 1)$ gives the theory studied by Lee [Lee05], and setting $(h, t) = (1, 0)$ gives a theory introduced by Bar-Natan [Bar05].

There is a natural functor $(\mathcal{B})^{\text{op}} \rightarrow \mathbb{Z}\text{-Mod}$ given as follows: to a set X , associate the Abelian group $\mathbb{Z}\langle X \rangle$ freely generated by elements of X ; to a correspondence (A, s, t) from X to Y , associate the following map $\mathbb{Z}\langle Y \rangle \rightarrow \mathbb{Z}\langle X \rangle$:

$$(4.7) \quad y \mapsto \sum_{x \in X} \# |\{a \in A \mid s(a) = x, t(a) = y\}| x.$$

Example 4.5 lifts Khovanov's functor $(\underline{2}^n)^{\text{op}} \rightarrow \mathbb{Z}\text{-Mod}$ to the functor $F_{Kh}: \underline{2}^n \rightarrow \mathcal{B}$. It is natural to ask whether any other specializations of h and t comes from a strictly unitary 2-functor $\underline{2}^n \rightarrow \mathcal{B}$. Any candidate must have $h = 0$ and $t \in \mathbb{N}$, since the coefficients in Equation (4.7) need to be positive and integral. The special case $(h, t) = (0, 1)$ (i.e., Lee's theory) does not come from any functor $\underline{2}^n \rightarrow \mathcal{B}$ for arbitrary link diagrams extending the functor F_{Kh} , as can be seen by considering the diagram in Figure 4.1. The question of whether there is such an extension for $h = 0, t > 1$ is, as far as we know, open.

4.2. The thickened diagram. Fix a small category \mathcal{D} , which we regard as a strict 2-category whose only 2-morphisms are identity maps. Fix also a strictly unitary lax 2-functor $F: \mathcal{D} \rightarrow \mathcal{B}$, i.e., a \mathcal{D} -diagram in \mathcal{B} . In this section, we will associate to (\mathcal{D}, F) a new (1-)category $\widehat{\mathcal{D}}$ and, for each $k \geq 1$, an (honest) functor $\widehat{F}_k: \widehat{\mathcal{D}} \rightarrow \mathbf{Top}_\bullet$. There will also be natural transformations $\Sigma \circ \widehat{F}_k \rightarrow \widehat{F}_{k+1}$ (where Σ denotes suspension), so that we get an induced diagram $\widehat{F}: \widehat{\mathcal{D}} \rightarrow \mathcal{S}$. To realize a functor from the cube category to the Burnside category we will apply this construction and then take an iterated mapping cone; see Section 4.3.

We start by defining $\widehat{\mathcal{D}}$:

- The objects of $\widehat{\mathcal{D}}$ are composable pairs of morphisms $u \xrightarrow{f} v \xrightarrow{g} w$ in \mathcal{D} .
- The morphisms in $\widehat{\mathcal{D}}$ are commutative diagrams: given composable pairs $u \xrightarrow{f} v \xrightarrow{g} w$ and $u' \xrightarrow{f'} v' \xrightarrow{g'} w'$,

$$\mathrm{Hom}((f, g), (f', g')) = \left\{ (\alpha: u \rightarrow u', \beta: v' \rightarrow v, \gamma: w \rightarrow w') \mid \begin{array}{ccccc} u & \xrightarrow{f} & v & \xrightarrow{g} & w \\ \downarrow \alpha & & \uparrow \beta & & \downarrow \gamma \\ u' & \xrightarrow{f'} & v' & \xrightarrow{g'} & w' \end{array} \text{ commutes} \right\}.$$

(Note the direction of the middle vertical arrow.)

- Composition of morphisms is given by stacking diagrams vertically: $(\alpha', \beta', \gamma') \circ (\alpha, \beta, \gamma) = (\alpha' \circ \alpha, \beta \circ \beta', \gamma' \circ \gamma)$.

Example 4.8. For $\mathcal{D} = \mathbb{2}^1$, $\widehat{\mathcal{D}}$ has four objects: $1 \rightarrow 1 \rightarrow 1$, $1 \rightarrow 1 \rightarrow 0$, $1 \rightarrow 0 \rightarrow 0$ and $0 \rightarrow 0 \rightarrow 0$. There are unique morphisms

$$(1 \rightarrow 1 \rightarrow 1) \longrightarrow (1 \rightarrow 1 \rightarrow 0) \longleftarrow (1 \rightarrow 0 \rightarrow 0) \longrightarrow (0 \rightarrow 0 \rightarrow 0).$$

(Again, note the direction of the middle arrow.)

Example 4.9. Given small categories \mathcal{C} and \mathcal{D} , there is an obvious isomorphism $\widehat{\mathcal{C} \times \mathcal{D}} \cong \widehat{\mathcal{C}} \times \widehat{\mathcal{D}}$.

Next we define \widehat{F}_k . On objects, we define

$$\widehat{F}_k(u \xrightarrow{f} v \xrightarrow{g} w) = \bigvee_{a \in F(f)} \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} S^k.$$

Unpacking this a bit, recall that $F(f)$ (respectively $F(g)$) is a correspondence from $F(u)$ to $F(v)$ (respectively $F(v)$ to $F(w)$). For each element a in the correspondence $F(f)$ we take the product over those b in $F(g)$ so that the source of b is the same as the target of a .

To define \widehat{F}_k on morphisms fix a commutative diagram

$$(4.10) \quad \begin{array}{ccccc} u & \xrightarrow{f} & v & \xrightarrow{g} & w \\ \downarrow \alpha & & \uparrow \beta & & \downarrow \gamma \\ u' & \xrightarrow{f'} & v' & \xrightarrow{g'} & w'. \end{array}$$

We must construct a map

$$\bigvee_{a \in F(f)} \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} S^k \rightarrow \bigvee_{a' \in F(f')} \prod_{\substack{b' \in F(g') \\ s(b')=t(a')}} S^k.$$

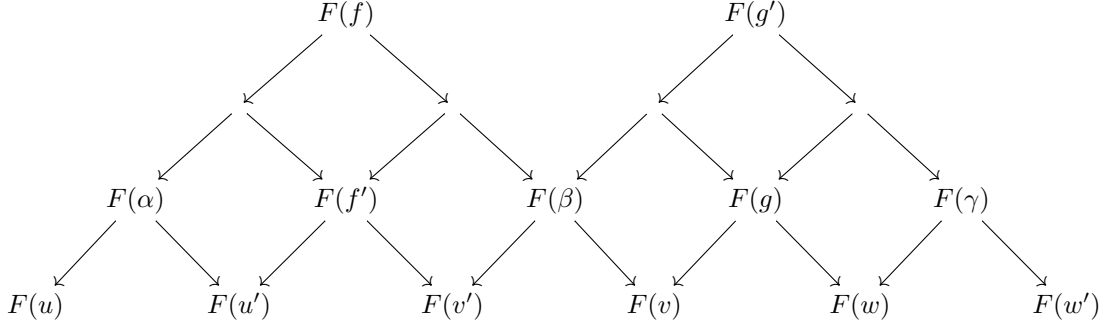


FIGURE 4.2. **Constructing the functor from the thickened diagram.** In the first row of arrows, leftwards arrows are source maps, and rightwards arrows are target maps. All squares are fiber products. To define the map (4.11), fix $a \in F(f)$ and $b \in F(g)$. The element a specifies $a' \in F(f')$ and $y \in F(\beta)$. We then consider the elements $b' \in F(g')$ which map down to y and b . The diagonal map Δ_b corresponds to this set. Any such b' has source equal to the target of a' .

It suffices to construct this map one a at a time, so fix $a \in F(f)$. The maps $F_{u,u',v'}$ and $F_{u,v',v}$ induce a bijection

$$F(f) \cong F(\beta) \times_{F(v')} F(f') \times_{F(u')} F(\alpha);$$

let $(y, a', x) \in F(\beta) \times_{F(v')} F(f') \times_{F(u')} F(\alpha)$ be the triple corresponding to a . The map \widehat{F}_k will send $\prod_{\substack{b \in F(g) \\ s(b)=t(a)}} S^k$, the summand corresponding to a , to $\prod_{\substack{b' \in F(g') \\ s(b')=t(a')}} S^k$, the summand corresponding to a' .

Next, the maps $F_{v',v,w}$ and $F_{v',w,w'}$ induce a bijection

$$F(g') \cong F(\gamma) \times_{F(w)} F(g) \times_{F(v)} F(\beta).$$

Consider the map

$$(4.11) \quad \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} S^k \xrightarrow{\prod_b \Delta_b} \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} \prod_{\substack{b'=(z,\bar{b},\bar{y}) \in F(g') \\ \bar{b}=b \\ \bar{y}=y}} S^k \cong \prod_{\substack{b' \in F(g') \\ b'=(z,b,y) \\ s(b)=t(a)}} S^k,$$

where Δ_b is the diagonal map $S^k \rightarrow \prod_{b'} S^k$. Notice that $\{b' \in F(g') \mid b' = (z, b, y), s(b) = t(a)\}$ is a subset of $\{b' \in F(g') \mid s(b') = t(a')\}$ since $s(b') = s(y) = t(a')$. We can extend the map (4.11) to a map

$$\prod_{\substack{b \in F(g) \\ s(b)=t(a)}} S^k \longrightarrow \prod_{\substack{b' \in F(g') \\ s(b')=t(a')}} S^k$$

by mapping to the basepoint in the remaining factors. This is the desired map.

(It can be helpful to think of this argument diagrammatically; see Figure 4.2.)

Lemma 4.12. *The construction above makes \widehat{F}_k into a functor whose values have natural actions of the symmetric group \mathcal{S}_k .*

We omit the proof, which is straightforward, albeit a bit elaborate.

Example 4.13. Consider the functor $F: \underline{2}^1 \rightarrow \mathcal{B}$ given by $F(1) = \{x, y\}$, $F(0) = \{z, w\}$, and $F(\varphi_{1,0}) = \{a, b, c, d\}$ with $s(a) = x$, $s(b) = s(c) = s(d) = y$, $t(a) = t(b) = t(c) = z$, $t(d) = w$. Graphically, F is given by:

$$1 : \begin{array}{ccc} & x & y \\ & \downarrow a & \downarrow d \\ & z & w \end{array} \begin{array}{c} \nearrow b,c \\ \searrow \end{array}$$

Recall the thickening $\widehat{\underline{2}}^1$ from Example 4.8. The induced diagram $\widehat{F}_k: \widehat{\underline{2}}^1 \rightarrow \mathbf{Top}_\bullet$ is given by

$$S_{x,x}^k \vee S_{y,y}^k \xrightarrow{\text{Id} \vee \Delta} (S_{x,a}^k) \vee (S_{y,b}^k \times S_{y,c}^k \times S_{y,d}^k) \longleftarrow S_{a,z}^k \vee S_{b,z}^k \vee S_{c,z}^k \vee S_{d,w}^k \longrightarrow S_{z,z}^k \vee S_{w,w}^k.$$

(Here, for instance, the sphere $S_{x,a}^k$ corresponds to the pair $(x, a) \in F(\varphi_{1,1}) \times F(\varphi_{1,0})$ over the object $1 \rightarrow 1 \rightarrow 0$.) We claim that the second map is the inclusion of the k -skeleton. To see this, note that it corresponds to the diagram

$$\begin{array}{ccccc} 1 & \xrightarrow{\varphi_{1,0}} & 0 & \xrightarrow{\varphi_{0,0}} & 0 \\ \downarrow \varphi_{1,1} & & \uparrow \varphi_{1,0} & & \downarrow \varphi_{0,0} \\ 1 & \xrightarrow{\varphi_{1,1}} & 1 & \xrightarrow{\varphi_{1,0}} & 0. \end{array}$$

The map decomposes along wedge sums. For b , for instance, we get the map $S_{b,z}^k \rightarrow (S_{y,b}^k \times S_{y,c}^k \times S_{y,d}^k)$ as follows:

$$\prod_{\{z\}} S^k = \left[\prod_{\substack{p \in F(\varphi_{0,0}) \\ s(p)=t(b)}} S^k \right] \xrightarrow{\Pi \Delta} \left[\prod_{\substack{q' \in F(\varphi_{1,0}) \\ q'=(r,p,b) \in F(\varphi_{0,0}) \times F(\varphi_{0,0}) \times F(\varphi_{1,0}) \\ s(p)=t(b)}} S^k \right] = \prod_{\{b\}} S^k \hookrightarrow \prod_{\{b,c,d\}} S^k.$$

Similarly, the third map sends $S_{a,z}^k$, $S_{b,z}^k$ and $S_{c,z}^k$ by the identity map to $S_{z,z}^k$ and $S_{d,w}^k$ by the identity map to $S_{w,w}^k$.

Finally, we discuss the natural transformations $S^n \wedge \widehat{F}_k \rightarrow \widehat{F}_{n+k}$. We need $\mathcal{S}_n \times \mathcal{S}_k$ -equivariant maps

$$(4.14) \quad S^n \wedge \left(\bigvee_{a \in F(f)} \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} S^k \right) \cong \bigvee_{a \in F(f)} S^n \wedge \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} S^k \rightarrow \bigvee_{a \in F(f)} \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} S^{n+k},$$

(for each $u \xrightarrow{f} v \xrightarrow{g} w$) intertwining the maps $S^n \wedge \widehat{F}_k$ and \widehat{F}_{k+1} . There is a canonical map

$$\sigma^n: S^n \wedge \prod_i X_i \rightarrow \prod_i S^n \wedge X_i:$$

if we view $S^n \wedge X$ as $[0, 1]^n \times X / (\partial[0, 1]^n \times X \cup [0, 1]^n \times \{*\})$ (where $*$ is the basepoint) then the map σ^n is given by $\sigma^n(y, x_1, \dots, x_n) = ((y, x_1), \dots, (y, x_n))$. If all of the X_i are $(k-1)$ -connected then the map σ^n induces an isomorphism on π_i for $0 \leq i \leq 2k-2$. The map of Equation (4.14) is given by applying σ^n to each summand. It is routine to verify that these maps, $\mathcal{S}_n \times \mathcal{S}_k$ -equivariantly, intertwine $S^n \wedge \widehat{F}_k$ and \widehat{F}_{n+k} , i.e., define a natural transformation. Thus, we obtain a diagram of spectra $\widehat{F}: \widehat{\mathcal{F}} \rightarrow \mathcal{S}$.

Remark 4.15. The above amounts to a verification that we can more concisely express \widehat{F} within the category of spectra itself by the formula

$$\widehat{F}(u \xrightarrow{f} v \xrightarrow{g} w) = \bigvee_{a \in F(f)} \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} \mathbb{S}.$$

The wedge and product play the roles of coproduct and product within this category.

4.3. The realization.

Construction 4.16. Let $\widehat{\underline{2}}_+$ be the category obtained from $\widehat{\underline{2}}^n$ by adding a new object $*$ and a unique morphism $(u \rightarrow v \rightarrow w) \rightarrow *$ from each vertex $(u \rightarrow v \rightarrow w)$ of $\widehat{\underline{2}}^n$ with $w \neq \vec{0}$.

Given a functor $F: \underline{2}^n \rightarrow \mathcal{B}$, define $\widehat{F}^+: \widehat{\underline{2}}_+ \rightarrow \mathcal{S}$ by $\widehat{F}^+|_{\widehat{\underline{2}}^n} = \widehat{F}$ and $\widehat{F}^+(*) = \{\text{pt}\}$, a single point.

Let $|F|$ be the homotopy colimit of \widehat{F}^+ . We call $|F|$ the *realization* of F .

Example 4.17. Continuing with Example 4.13, we have

$$|F| = \text{hocolim} \left(\begin{array}{c} \mathbb{S}_{x,x} \vee \mathbb{S}_{y,y} \longrightarrow (\mathbb{S}_{x,a}) \vee (\mathbb{S}_{y,b} \times \mathbb{S}_{y,c} \times \mathbb{S}_{y,d}) \longleftarrow \mathbb{S}_{a,z} \vee \mathbb{S}_{b,z} \vee \mathbb{S}_{c,z} \vee \mathbb{S}_{d,w} \longrightarrow \mathbb{S}_{z,z} \vee \mathbb{S}_{w,w} \\ \downarrow \\ \{\text{pt}\} \end{array} \right),$$

where \mathbb{S} denotes the sphere spectrum.

Instead of $\widehat{\underline{2}}_+$, it will be convenient, sometimes, to work with the larger enlargement $\widehat{\underline{2}}_{\dagger}^n = \left(\widehat{\underline{2}}_+^n\right)$ of $\widehat{\underline{2}}^n$. Given a functor $F: \underline{2}^n \rightarrow \mathcal{B}$ extend \widehat{F} to a functor $\widehat{F}^\dagger: \widehat{\underline{2}}_{\dagger}^n \rightarrow \mathcal{S}$ by setting $\widehat{F}^\dagger|_{\widehat{\underline{2}}^n} = \widehat{F}$ and $\widehat{F}^\dagger(d) = \{\text{pt}\}$ if $d \notin \text{Ob}(\widehat{\underline{2}}^n)$.

Lemma 4.18. *For any functor $F: \underline{2}^n \rightarrow \mathcal{B}$ there is a stable homotopy equivalence $\text{hocolim } \widehat{F}^+ \simeq \text{hocolim } \widehat{F}^\dagger$.*

Proof. Consider the functor $G: \widehat{\underline{2}}_{\dagger}^n \rightarrow \widehat{\underline{2}}_+^n$ which is the identity on $\widehat{\underline{2}}^n$ and sends all objects not in $\widehat{\underline{2}}^n$ to $*$. We claim that G is homotopy cofinal. To see this, we divide the computation of the undercategories into three cases:

- The undercategory $* \downarrow G$ of $*$ is the full subcategory of $\widehat{\underline{2}}_{\dagger}^n$ spanned by the objects not in $\widehat{\underline{2}}^n$. The nerve of this category is homeomorphic to $\{\vec{x} = (x_1, \dots, x_n) \in [0, 1]^n \mid \exists i \text{ such that } x_i = 0\}$, which is contractible.
- For any object $d = (u \rightarrow v \rightarrow \vec{0})$ of $\widehat{\underline{2}}^n$, the undercategory $d \downarrow G$ of d is the full subcategory of $\widehat{\underline{2}}^n$ of objects d' for which there is a map $d \rightarrow d'$. The object d is an initial object for this subcategory, so the nerve is contractible.
- Fix an object $d = (u \rightarrow v \rightarrow w)$ of $\widehat{\underline{2}}^n$ with $w \neq \vec{0}$. The undercategory $d \downarrow G$ of d is the full subcategory of $\widehat{\underline{2}}_{\dagger}^n$ spanned by the objects d' not in $\widehat{\underline{2}}^n$ and the objects d' in $\widehat{\underline{2}}^n$ for which there is a map $d \rightarrow d'$. Let \mathcal{D} be the full subcategory of $\widehat{\underline{2}}_{\dagger}^n$ spanned by the objects d' not in $\widehat{\underline{2}}^n$ and let \mathcal{E} be the full subcategory of $\widehat{\underline{2}}_{\dagger}^n$ consisting of objects d' for which there is a morphism $d \rightarrow d'$. The categories \mathcal{D} and \mathcal{E} are each downwards closed in $d \downarrow G$, i.e., there are no morphisms out of \mathcal{D} or \mathcal{E} . Thus, the nerve of $d \downarrow G$ is the union of the nerves of \mathcal{D} and \mathcal{E} , glued along the nerve of $\mathcal{D} \cap \mathcal{E}$. We already saw in part (a) that the nerve of \mathcal{D} is contractible. The category \mathcal{E} has an initial object, d , and hence the nerve of \mathcal{E} is contractible. Finally, the realization of the category $\mathcal{D} \cap \mathcal{E}$ is similar

to the realization of \mathcal{D} (i.e., a union of coordinate hyperplanes), and so contractible. It follows that the realization of $d \downarrow G$ is contractible.

Thus, the functor G is homotopy cofinal. It is immediate from the definitions that $\widehat{F}^\dagger = \widehat{F}^+ \circ G$, so the result follows from property (ho-4) of homotopy colimits (Section 2.9). \square

4.4. An invariance property of the realization.

Lemma 4.19. *If $F, G: \underline{2}^n \rightarrow \mathcal{B}$ are naturally isomorphic 2-functors then the realizations of F and G are stably homotopy equivalent.*

Proof. A natural isomorphism T from F to G specifies:

- A bijection $T_v: F(v) \rightarrow G(v)$ for each $v \in \{0, 1\}^n$, and
- A bijection $T_{v,w}: F(\varphi_{v,w}) \rightarrow G(\varphi_{v,w})$ for each $v > w \in \{0, 1\}^n$

satisfying the conditions that

$$\begin{array}{ccccc}
 F(\varphi_{v,w}) & \xrightarrow{T_{v,w}} & G(\varphi_{v,w}) & & \\
 \swarrow s & & \swarrow s & & \\
 F(v) & & G(v) & & \\
 \searrow t & & \searrow t & & \\
 & F(w) & & G(w) & \\
 \swarrow T_v & & \swarrow T_w & & \\
 & & & &
 \end{array}$$

and

$$\begin{array}{ccc}
 F(\varphi_{v,w}) \times_{F(v)} F(\varphi_{u,v}) & \xrightarrow{F_{u,v,w}} & F(\varphi_{u,w}) \\
 \downarrow T_{v,w} \times T_{u,v} & & \downarrow T_{u,w} \\
 G(\varphi_{v,w}) \times_{G(v)} G(\varphi_{u,v}) & \xrightarrow{G_{u,v,w}} & G(\varphi_{u,w})
 \end{array}$$

commute. (See [Gra74, Section I,2.4] and note that an isomorphism in \mathcal{B} between two sets induces a bijection between them.) Given a natural transformation T , there is a corresponding map of diagrams defined as follows. Given $u > v > w$ we want a map

$$\bigvee_{a \in F(\varphi_{u,v})} \prod_{\substack{b \in F(\varphi_{v,w}) \\ s(b)=t(a)}} \mathbb{S} \rightarrow \bigvee_{a \in G(\varphi_{u,v})} \prod_{\substack{b \in G(\varphi_{v,w}) \\ s(b)=t(a)}} \mathbb{S}.$$

This map sends the wedge summand corresponding to $a \in F(\varphi_{u,v})$ to the wedge summand corresponding to $T_{u,v}(a)$, and sends the factor corresponding to $b \in F(\varphi_{v,w})$ to the factor corresponding to $T_{v,w}(b)$. It is straightforward to verify that this gives a map of diagrams, which is by definition an isomorphism. It follows that the map induces a stable homotopy equivalence of homotopy colimits. (In fact, if we work with diagrams of S^k 's, the map would be a homeomorphism.) \square

4.5. Products and realization. In the language of flow categories, the product is a rather complicated object. In the language of functors from the cube to the Burnside category, however, the product is quite simple:

Definition 4.20. Given functors $F: \underline{2}^m \rightarrow \mathcal{B}$ and $G: \underline{2}^n \rightarrow \mathcal{B}$, we define the *product* of F and G , $(F \times G): \underline{2}^{m+n} \rightarrow \mathcal{B}$, as follows:

- For $(u_1, u_2) \in \{0, 1\}^m \times \{0, 1\}^n = \{0, 1\}^{m+n}$ define

$$(F \times G)(u_1, u_2) = F(u_1) \times G(u_2).$$

- For $(u_1, u_2) > (v_1, v_2) \in \{0, 1\}^{m+n}$ define $(F \times G)(\varphi_{(u_1, u_2), (v_1, v_2)})$ to be the correspondence

$$\begin{array}{ccc} & F(\varphi_{u_1, v_1}) \times G(\varphi_{u_2, v_2}) & \\ s_F \times s_G \swarrow & & \searrow t_F \times t_G \\ F(u_1) \times G(u_2) & & F(v_1) \times G(v_2). \end{array}$$

- For $(u_1, u_2) > (v_1, v_2) > (w_1, w_2) \in \{0, 1\}^{m+n}$ define $(F \times G)_{(u_1, u_2), (v_1, v_2), (w_1, w_2)}$ by

$$(F \times G)_{(u_1, u_2), (v_1, v_2), (w_1, w_2)}(x, y) = (F_{u_1, v_1, w_1}(x), G_{v_1, v_2, w_2}(y)).$$

Lemma 4.21. *Definition 4.20 specifies a strictly unitary 2-functor.*

Proof. This is immediate from the definitions. \square

Note that smash products distribute across wedge sums. Moreover, while $X \wedge (Y \times Z)$ is not homotopy equivalent to $(X \wedge Y) \times (X \wedge Z)$, there is a natural map $X \wedge (Y \times Z) \rightarrow (X \wedge Y) \times (X \wedge Z)$ defined by $(x, y, z) \mapsto ((x, y), (x, z))$. This generalizes to a map

$$p: \left(\prod_{a \in A} X_a \right) \wedge \left(\prod_{b \in B} Y_b \right) \rightarrow \prod_{(a, b) \in A \times B} X_a \wedge Y_b.$$

The map p is natural in both factors.

Given functors $F: \mathcal{C} \rightarrow \mathbf{Top}_\bullet$ and $G: \mathcal{D} \rightarrow \mathbf{Top}_\bullet$, we can take the smash product of F and G to obtain a functor $F \wedge G: \mathcal{C} \times \mathcal{D} \rightarrow \mathbf{Top}_\bullet$. At a vertex (c, d) of $\mathcal{C} \times \mathcal{D}$, $(F \wedge G)(c, d) = F(c) \wedge G(d)$.

Lemma 4.22. *The thickening construction from Section 4.2 respects products, in the following sense. Fix functors $F: \mathcal{C} \rightarrow \mathcal{B}$ and $G: \mathcal{D} \rightarrow \mathcal{B}$. Let $F \times G: \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{B}$ be as in Definition 4.20. Then there is an isomorphism $q: \widehat{\mathcal{C}} \times \widehat{\mathcal{D}} \cong \widehat{\mathcal{C} \times \mathcal{D}}$ defined on objects by*

$$q([u_C \xrightarrow{f_C} v_C \xrightarrow{g_C} w_C] \times [u_D \xrightarrow{f_D} v_D \xrightarrow{g_D} w_D]) = [u_C \times u_D \xrightarrow{f_C \times f_D} v_C \times v_D \xrightarrow{g_C \times g_D} w_C \times w_D].$$

Moreover, the map p induces a natural transformation from $\widehat{F}_k \wedge \widehat{G}_l$ to $(\widehat{F \times G})_{k+l} \circ q$ so that on vertices the natural transformation is a weak homotopy equivalence up to dimension $k + l + \min(k, l) - 1$. Finally, these natural transformations respect the spectrum structure, and so induce maps of diagrams $\widehat{F} \wedge \widehat{G} \rightarrow (\widehat{F \times G}) \circ q$ so that the map at each vertex is a stable homotopy equivalence.

Proof. This is straightforward from the definitions. To illustrate, we describe the natural transformation at the level of vertices. At a vertex $[u_C \xrightarrow{f_C} v_C \xrightarrow{g_C} w_C] \times [u_D \xrightarrow{f_D} v_D \xrightarrow{g_D} w_D]$ we have

$$\widehat{F}(u_C \xrightarrow{f_C} v_C \xrightarrow{g_C} w_C) \wedge \widehat{G}(u_D \xrightarrow{f_D} v_D \xrightarrow{g_D} w_D) = \left(\bigvee_{a_C \in F(f_C)} \prod_{\substack{b_C \in F(g_C) \\ s(b_C) = t(a_C)}} S^k \right) \wedge \left(\bigvee_{a_D \in G(f_D)} \prod_{\substack{b_D \in G(g_D) \\ s(b_D) = t(a_D)}} S^l \right)$$

while

$$(\widehat{F \times G})_{k+l}(u_C \times u_D \xrightarrow{f_C \times f_D} v_C \times v_D \xrightarrow{g_C \times g_D} w_C \times w_D) = \bigvee_{\substack{a_C \in F(f_C) \\ a_D \in G(f_D)}} \prod_{\substack{b_C \in F(g_C) \\ b_D \in G(g_D) \\ s(b_C) = t(a_C) \\ s(b_D) = t(a_D)}} S^{k+l}.$$

The map p sends the first to the second in an obvious way, and is an equivalence up to dimension $k + l + \min(k, l) - 1$. Moreover, this map is $\mathcal{S}_k \times \mathcal{S}_l$ -equivariant and respects the structure maps of the spectrum. \square

Proposition 4.23. *Given functors $F: \underline{2}^n \rightarrow \mathcal{B}$ and $G: \underline{2}^m \rightarrow \mathcal{B}$, we have $|F \times G| \simeq |F| \wedge |G|$.*

Proof. By Lemma 4.18,

$$|F| \simeq \text{hocolim } \widehat{F}^\dagger \quad |G| \simeq \text{hocolim } \widehat{G}^\dagger \quad |F \times G| \simeq \text{hocolim } (\widehat{F \times G})^\dagger.$$

(Throughout this argument, \simeq means stable homotopy equivalence.) From the definitions, it follows that the natural transformation from $\widehat{F} \wedge \widehat{G}$ to $\widehat{(F \times G)}$ of Lemma 4.22 extends to a natural transformation from $\widehat{F}^\dagger \wedge \widehat{G}^\dagger$ to $(\widehat{F \times G})^\dagger$, which is an stable homotopy equivalence on objects. By point (ho-1) in Section 2.9,

$$\text{hocolim } (\widehat{F \times G})^\dagger \simeq \text{hocolim } (\widehat{F}^\dagger \wedge \widehat{G}^\dagger).$$

By point (ho-3) in Section 2.9,

$$\text{hocolim } (\widehat{F}^\dagger \wedge \widehat{G}^\dagger) \simeq (\text{hocolim } \widehat{F}^\dagger) \wedge (\text{hocolim } \widehat{G}^\dagger).$$

The result follows. \square

An even simpler operation is disjoint union:

Definition 4.24. Given functors $F, G: \underline{2}^n \rightarrow \mathcal{B}$ we define the *disjoint union* of F and G , $(F \amalg G): \underline{2}^n \rightarrow \mathcal{B}$, as follows:

- For $v \in \{0, 1\}^n$, $(F \amalg G)(v) = F(v) \amalg G(v)$.
- For $v > w \in \{0, 1\}^n$, $(F \amalg G)_{v,w}(\varphi_{v,w})$ is defined to be the correspondence

$$\begin{array}{ccc} & F(\varphi_{v,w}) \amalg G(\varphi_{v,w}) & \\ \swarrow s_F \amalg s_G & & \searrow t_F \amalg t_G \\ F(v) \amalg G(v) & & F(w) \amalg G(w). \end{array}$$

- For $u > v > w \in \{0, 1\}^n$, $(F \amalg G)_{u,v,w}$ is defined by the commutative diagram

$$\begin{array}{ccc} [F(\varphi_{v,w}) \amalg G(\varphi_{v,w})] \times_{F(v) \amalg G(v)} [F(\varphi_{u,v}) \amalg G(\varphi_{u,v})] & \xrightarrow{(F \amalg G)_{u,v,w}} & F(\varphi_{u,w}) \amalg G(\varphi_{u,w}), \\ \cong \downarrow & \nearrow F_{u,v,w} \amalg G_{u,v,w} & \\ [F(\varphi_{v,w}) \times_{F(v)} F(\varphi_{u,v})] \amalg [G(\varphi_{v,w}) \times_{G(v)} G(\varphi_{u,v})] & & \end{array}$$

where the vertical arrow is the obvious bijection.

Lemma 4.25. *Definition 4.24 specifies a strictly unitary 2-functor.*

Proof. This is immediate from the definitions. \square

Given diagrams $F, G: \mathcal{C} \rightarrow \mathbf{Top}_\bullet$ there is an induced diagram $F \vee G: \mathcal{C} \rightarrow \mathbf{Top}_\bullet$ with $(F \vee G)(v) = F(v) \vee G(v)$ and $(F \vee G)(f) = F(f) \vee G(f)$.

Lemma 4.26. *The thickening construction respects disjoint unions in the sense that given $F, G: \mathcal{C} \rightarrow \mathcal{B}$, $(\widehat{F \amalg G})_k \cong \widehat{F}_k \vee \widehat{G}_k$.*

Proof. Again, this is immediate from the definitions. \square

Proposition 4.27. *Given functors $F, G: \underline{2}^n \rightarrow \mathcal{B}$, $|F \amalg G| \simeq |F| \vee |G|$.*

Proof. Lemma 4.26 extends immediately to the statement that $\widehat{F \amalg G}^+ \cong \widehat{F}^+ \vee \widehat{G}^+$, and homotopy colimit commutes with wedge sum (point (ho-2) in Section 2.9). \square

5. BUILDING A SMALLER CUBE FROM LITTLE BOX MAPS

In this section we show that the realization can be understood in terms of a smaller diagram:

Theorem 5. *Let $\underline{2}_+^n$ be the result of adding a single object, which we will denote $*$, to $\underline{2}^n$ and declaring that*

$$\mathrm{Hom}(v, *) = \begin{cases} \text{one element } \varphi_{v,*} & v \neq \bar{0} \\ \emptyset & v = \bar{0} \end{cases} \quad \mathrm{Hom}(*, v) = \begin{cases} \{\mathrm{Id}\} & v = * \\ \emptyset & v \neq * \end{cases}$$

Fix a strictly unitary functor $F: \underline{2}^n \rightarrow \mathcal{B}$. Then there is a homotopy coherent diagram $\widetilde{F}^+: \underline{2}_+^n \rightarrow \mathcal{S}$ so that:

- (1) *For each $v \in \mathrm{Ob}(\underline{2}^n)$, $\widetilde{F}^+(v) = \bigvee_{a \in F(v)} \mathcal{S}$.*
- (2) *$\widetilde{F}^+(*)$ is a single point.*
- (3) *$\mathrm{hocolim} \widetilde{F}^+ \simeq |F|$.*

The diagram \widetilde{F}^+ is a special case of a more general construction, which we give in Section 5.1. We define \widetilde{F}^+ in Section 5.2, and prove that this homotopy colimit is the same as $|F|$ in Section 5.3.

5.1. Refining diagrams via box maps.

Definition 5.1. Fix a small category \mathcal{D} and a strictly unitary lax 2-functor $F: \mathcal{D} \rightarrow \mathcal{B}$ (i.e., a \mathcal{D} -diagram in \mathcal{B}). A k -dimensional spacial refinement of F is a homotopy coherent diagram $\widetilde{F}_k: \mathcal{D} \rightarrow \mathbf{Top}_\bullet$ so that

- For any $u \in \mathrm{Ob}(\mathcal{D})$, $\widetilde{F}_k(u) = \bigvee_{x \in F(u)} \mathcal{S}^k$;
- For any $u, v \in \mathrm{Ob}(\mathcal{D})$ and $f: u \rightarrow v$, $\widetilde{F}_k(f)$ is a (disjoint) box map which refines the correspondence $F(f)$ from $F(u)$ to $F(v)$ (see Section 2.10); and, more generally,
- For any sequence of morphisms

$$u_0 \xrightarrow{f_1} u_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} u_n$$

in \mathcal{D} and any $\vec{t} \in [0, 1]^{n-1}$, the map

$$\widetilde{F}_k(f_n, \dots, f_1)(\vec{t}): \bigvee_{x \in F(u_0)} \mathcal{S}^k \rightarrow \bigvee_{x \in F(u_n)} \mathcal{S}^k$$

is a box map refining the correspondence

$$F(f_n \circ \cdots \circ f_1) \cong F(f_n) \times_{F(u_{n-1})} \cdots \times_{F(u_1)} F(f_1)$$

from $F(u_0)$ to $F(u_n)$.

Proposition 5.2. *Let \mathcal{D} be a small category in which every sequence of composable non-identity morphisms has length at most n . Fix a \mathcal{D} -diagram F in \mathcal{B} .*

- (1) *If $k \geq n$ then there is a k -dimensional spacial refinement of F .*
- (2) *If $k \geq n + 1$ then any two k -dimensional spacial refinements of F are homotopic (as homotopy coherent diagrams).*
- (3) *If \widetilde{F}_k is a k -dimensional spacial refinement of F then the result of suspending each $\widetilde{F}_k(u)$ and $\widetilde{F}_k(f_n, \dots, f_1)(\vec{t})$ gives a $(k + 1)$ -dimensional spacial refinement of F .*

Proof. We start with point (1). Given $u \in \text{Ob}(\mathcal{D})$, define $\tilde{F}_k(u) = \bigvee_{x \in F(u)} S^k$; write the S^k summand corresponding to x as B_x/∂ , where B_x is a box in \mathbb{R}^k (e.g., $B_x = [0, 1]^k$). Next, by Observation 2.19 it suffices to consider only non-identity morphisms. For each non-identity morphism $f: u \rightarrow v$ in \mathcal{D} choose a box map which refines the correspondence $F(f)$. Let $e_f \in E(\{B_x \mid x \in F(u)\}, s_{F(f)})$ be the collection of little boxes corresponding to $F(f)$.

We have now defined \tilde{F}_k on vertices and arrows. The diagram does not commute, so it remains to define the coherence homotopies associated to sequences of composable morphisms. We will build these inductively. As a warm up, we will spell out the first case carefully before proceeding to the general case.

Fix a composable pair of morphisms $u \xrightarrow{f} v \xrightarrow{g} w$ in \mathcal{D} . There are two points in $E(\{B_x \mid x \in F(u)\}, s_{F(g \circ f)})$ associated to (g, f) . One is the point $e_{g \circ f}$. The other is defined as follows. The point e_g corresponds to a collection of boxes \mathcal{B}_g in $\{B_y \mid y \in F(v)\}$, labeled by elements of $F(g)$. The inverse image $\Phi(e_f, F(f))^{-1}(\mathcal{B}_g)$ of these boxes is a collection of boxes in $\{B_x \mid x \in F(u)\}$. The boxes $\Phi(e_f, F(f))^{-1}(\mathcal{B}_g)$ inherit a labeling by elements of $F(g) \times_{F(v)} F(f) \cong_{F(u), v, w} F(g \circ f)$. This labeling makes $\Phi(e_f, F(f))^{-1}(\mathcal{B}_g)$ into a second point in $E(\{B_x \mid x \in F(u)\}, s_{F(g \circ f)})$, which by abuse of notation we will call $e_g \circ e_f$.

By Lemma 2.29, since $k \geq 2$ (or else we would not have a composable pair (g, f)), the space $E(\{B_x \mid x \in F(u)\}, s_{F(g \circ f)})$ is connected, so we can find a path from $e_{g \circ f}$ to $e_g \circ e_f$. Fix such a path, and call it $e_{g, f}: [0, 1] \rightarrow E(\{B_x \mid x \in F(u)\}, s_{F(g \circ f)})$. Then $e_{g, f}$ defines a homotopy $\Phi(e_{g, f}, F(g \circ f))$ from $\Phi(e_g, F(g)) \circ \Phi(e_f, F(f))$ to $\Phi(e_{g \circ f}, F(g \circ f))$.

More generally, suppose that for any sequence $v_0 \xrightarrow{f_1} \dots \xrightarrow{f_\ell} v_\ell$ of non-identity morphisms we have chosen a map $e_{f_\ell, \dots, f_1}: [0, 1]^{\ell-1} \rightarrow E(\{B_x \mid x \in F(v_0)\}, s_{F(f_\ell \circ \dots \circ f_1)})$, and these maps are compatible in the following sense. Let $(t_1, \dots, t_{\ell-1})$ be the coordinates on $[0, 1]^{\ell-1}$. Then for any $1 \leq i \leq \ell-1$ we require that:

$$(5.3) \quad \begin{aligned} e_{f_\ell, \dots, f_1}(t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_{\ell-1}) &= e_{f_\ell, \dots, f_i}(t_{i+1}, \dots, t_{\ell-1}) \circ e_{f_{i-1}, \dots, f_1}(t_1, \dots, t_{i-1}) \\ e_{f_\ell, \dots, f_1}(t_1, \dots, t_{i-1}, 1, t_{i+1}, \dots, t_{\ell-1}) &= e_{f_\ell, \dots, f_i \circ f_{i-1}, \dots, f_1}(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{\ell-1}). \end{aligned}$$

Then given $v_0 \xrightarrow{f_1} \dots \xrightarrow{f_{\ell+1}} v_{\ell+1}$ there is a map $S^{\ell-1} = \partial([0, 1]^\ell) \rightarrow E(\{B_x \mid x \in F(v_0)\}, s_{F(f_{\ell+1} \circ \dots \circ f_1)})$ defined by

$$(5.4) \quad \begin{aligned} (t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_\ell) &\mapsto e_{f_{\ell+1}, \dots, f_{i+1}}(t_{i+1}, \dots, t_\ell) \circ e_{f_i, \dots, f_1}(t_1, \dots, t_{i-1}) \\ (t_1, \dots, t_{i-1}, 1, t_{i+1}, \dots, t_\ell) &\mapsto e_{f_{\ell+1}, \dots, f_{i+1} \circ f_i, \dots, f_1}(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_\ell). \end{aligned}$$

The inductive hypothesis implies that this map is continuous. Since $k \geq \ell + 1$, by Lemma 2.29, the space $E(\{B_x \mid x \in F(v_0)\}, s_{F(f_{\ell+1} \circ \dots \circ f_1)})$ is $(\ell-1)$ -connected, so the map (5.4) extends to a map $[0, 1]^\ell \rightarrow E(\{B_x \mid x \in F(v_0)\}, s_{F(f_{\ell+1} \circ \dots \circ f_1)})$. Define $e_{f_{\ell+1}, \dots, f_1}$ to be any such extension.

Now, Equation (2.26) gives a map

$$\Phi(e_{v_0, \dots, v_{\ell+1}}, F(f_{\ell+1} \circ \dots \circ f_1)): [0, 1]^\ell \times \bigvee_{x \in F(v_0)} S^k \rightarrow \bigvee_{x \in F(v_{\ell+1})} S^k.$$

It follows from the compatibility conditions (5.3) that these maps define a homotopy coherent diagram.

Next, for point (2), fix spacial refinements \tilde{F}_k and \tilde{F}'_k of F . Consider the category $\mathbb{2}^1 \times \mathcal{D}$. It suffices to define a homotopy coherent diagram $G: \mathbb{2}^1 \times \mathcal{D} \rightarrow \text{Top}_\bullet$ so that $G|_{\{0\} \times \mathcal{D}} = \tilde{F}_k$, $G|_{\{1\} \times \mathcal{D}} = \tilde{F}'_k$, and for any $u \in \text{Ob}(\mathcal{D})$, $G(\varphi_{1,0} \times \text{Id}_u)$ is a homotopy equivalence [Vog73, Proposition 4.6]. To define G , note that $G|_{\{0\} \times \mathcal{D}}$ and $G|_{\{1\} \times \mathcal{D}}$ are already specified. Let $G(\varphi_{1,0} \times \text{Id}_u)$ be the identity map. More generally, define (somewhat arbitrarily) $G(\varphi_{1,0} \times g) = \tilde{F}_k(g)$. It follows from the fact that both \tilde{F}_k and \tilde{F}'_k refine F that the resulting diagram G is homotopy commutative. Extend G to a homotopy coherent diagram inductively, as in the proof of point (1).

Finally, point (3) is immediate from the definitions. \square

5.2. A coherent cube of box maps.

Definition 5.5. Given a strictly unitary 2-functor $F: \underline{2}^n \rightarrow \mathcal{B}$ let $\widetilde{F}_k: \underline{2}^n \rightarrow \mathbf{Top}_\bullet$ be a spacial refinement of F . Let $\widetilde{F}_k^+: \underline{2}_+^n \rightarrow \mathbf{Top}_\bullet$ be the diagram obtained from \widetilde{F}_k by defining $\widetilde{F}_k^+(\ast)$ to be a single point. Let \widehat{F}^+ be the diagram obtained from \widetilde{F}_k^+ by replacing each vertex $\widetilde{F}_k^+(u)$ with its suspension spectrum.

Corollary 5.6. *Up to stable homotopy equivalence, the spectrum $\mathrm{hocolim} \widehat{F}^+$ depends only on the functor F . In fact, for any $k > n$, the homotopy type of $\mathrm{hocolim} \widetilde{F}_k^+$ is independent of the choices in its construction.*

Proof. This is immediate from Proposition 5.2, together with the fact that the homotopy colimits of homotopic homotopy coherent diagrams are homotopy equivalent ([Vog73, Theorem 5.12], quoted as Proposition 2.20). \square

As in Section 4.3, we can also work with a larger enlargement $\underline{2}_+^n = (\ast \leftarrow 1 \rightarrow 0)^{\times n}$ of $\underline{2}^n$. Extend \widetilde{F}_k to a functor $\widetilde{F}_k^\dagger: \underline{2}_+^n \rightarrow \mathbf{Top}_\bullet$ by setting $\widetilde{F}_k^\dagger|_{\underline{2}^n} = \widetilde{F}_k$ and $\widetilde{F}_k^\dagger(v) = \{\mathrm{pt}\}$ if v is an object which is not in $\underline{2}^n$, i.e., if some coordinate of v is \ast .

Lemma 5.7. *For any functor $F: \underline{2}^n \rightarrow \mathcal{B}$ and any spacial refinement \widetilde{F}_k of F there is a stable homotopy equivalence $\mathrm{hocolim} \widetilde{F}_k^+ \simeq \mathrm{hocolim} \widetilde{F}_k^\dagger$.*

Proof. The proof is similar to but easier than the proof of Lemma 4.18, and is left to the reader. \square

5.3. The realizations of the small cube and big cube agree. Before proving Theorem 5, we introduce an auxiliary category, the *arrow category of $\underline{2}^n$* , and study its relationship with $\underline{2}^n$ and $\widehat{\underline{2}}^n$. Given a small category \mathcal{C} , the arrow category of \mathcal{C} , which we denote $\mathrm{Arr}(\mathcal{C})$ has $\mathrm{Ob}(\mathrm{Arr}(\mathcal{C})) = \bigcup_{u,v \in \mathrm{Ob}(\mathcal{C})} \mathrm{Hom}(u,v)$ the set of morphisms in \mathcal{C} . Given objects $f: u \rightarrow v$ and $g: w \rightarrow x$ in the arrow category, $\mathrm{Hom}(f,g)$ consists of pairs $(\alpha: u \rightarrow w, \beta: v \rightarrow x)$ so that

$$(5.8) \quad \begin{array}{ccc} u & \xrightarrow{f} & v \\ \alpha \downarrow & & \downarrow \beta \\ w & \xrightarrow{g} & x \end{array}$$

commutes. Maps compose in the obvious way: $(\gamma, \delta) \circ (\alpha, \beta) = (\gamma \circ \alpha, \delta \circ \beta)$. In the special case of the cube category,

$$\mathrm{Arr}(\underline{2}^1) = (\varphi_{1,1} \longrightarrow \varphi_{1,0} \longrightarrow \varphi_{0,0}) \quad \mathrm{Arr}(\underline{2}^n) = (\mathrm{Arr}(\underline{2}^1))^n.$$

We will also need a version with extra objects added, analogous to $\underline{2}_+^n$. Let

$$\mathrm{Arr}(\underline{2}^1)_\dagger = (\ast \longleftarrow \varphi_{1,1} \longrightarrow \varphi_{1,0} \longrightarrow \varphi_{0,0}) \quad \mathrm{Arr}(\underline{2}^n)_\dagger = (\mathrm{Arr}(\underline{2}^1)_\dagger)^n.$$

For any small category \mathcal{C} there are functors $A: \mathcal{C} \rightarrow \mathrm{Arr}(\mathcal{C})$ and $B: \widehat{\mathcal{C}} \rightarrow \mathrm{Arr}(\mathcal{C})$. The functor A is defined by

$$A(u) = \mathrm{Id}_u \quad A(f: u \rightarrow v) = \begin{array}{ccc} u & \xrightarrow{\mathrm{Id}_u} & u \\ \downarrow f & & \downarrow f \\ v & \xrightarrow{\mathrm{Id}_v} & v. \end{array}$$

The functor B is defined by

$$B(u \xrightarrow{f} v \xrightarrow{g} w) = g \circ f$$

$$B \left(\begin{array}{ccccc} u & \xrightarrow{f} & v & \xrightarrow{g} & w \\ \downarrow \alpha & & \uparrow \beta & & \downarrow \gamma \\ u' & \xrightarrow{f'} & v' & \xrightarrow{g'} & w' \end{array} \right) = \begin{array}{ccc} u & \xrightarrow{g \circ f} & w \\ \alpha \downarrow & & \downarrow \gamma \\ u' & \xrightarrow{g' \circ f'} & w'. \end{array}$$

Specializing to the case that $\mathcal{C} = \underline{2}^n$, these functors have obvious extensions

$$A_{\dagger} : \underline{2}_{\dagger}^n \rightarrow \text{Arr}(\underline{2}^n)_{\dagger} \qquad B_{\dagger} : \widehat{\underline{2}}_{\dagger}^n \rightarrow \text{Arr}(\underline{2}^n)_{\dagger}.$$

Everything here is a product of the 1-dimensional case, which is given by:

$$\begin{array}{ccccccc} * & \longleftarrow & 1 & \longrightarrow & 0 & & \\ | & & | & & | & & \\ \downarrow & & \downarrow & & \downarrow & & \\ * & \longleftarrow & \varphi_{1,1} & \longrightarrow & \varphi_{1,0} & \longrightarrow & \varphi_{0,0} \end{array} \qquad \begin{array}{ccccccc} * & \longleftarrow & 111 & \longrightarrow & 110 & \longleftarrow & 100 & \longrightarrow & 000 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ * & \longleftarrow & \varphi_{1,1} & \longrightarrow & \varphi_{1,0} & \longrightarrow & \varphi_{0,0} \end{array}$$

The dashed arrows denote A_{\dagger} , and the dotted arrows denote B_{\dagger} .

To relate various diagrams, we will need to know A_{\dagger} and B_{\dagger} are homotopy cofinal:

Lemma 5.9. *The functor A_{\dagger} is homotopy cofinal.*

Proof. Recall from point (ho-4) in Section 2.9 that homotopy cofinality means that each undercategory $d \downarrow A_{\dagger}$ has contractible nerve. Since taking undercategories commutes with taking products, it suffices to verify the one-dimensional case. This verification is straightforward, and is left to the reader. \square

Lemma 5.10. *The functor B_{\dagger} is homotopy cofinal.*

Proof. As in the proof of Lemma 5.9, it suffices to verify the 1-dimensional case, which is straightforward. \square

Next we see that for any small category \mathcal{C} , any functor $F: \mathcal{C} \rightarrow \mathcal{B}$ lifts to a functor $\vec{F}: \text{Arr}(\mathcal{C}) \rightarrow \mathcal{B}$ so that $\vec{F} \circ A = F$. For $f \in \text{Ob}(\text{Arr}(\mathcal{C}))$, $F(f)$ is a correspondence, and in particular a set; define $\vec{F}(f) = F(f)$. This defines \vec{F} on $\text{Ob}(\text{Arr}(\mathcal{C}))$. Given a diagram as in Formula (5.8), define $\vec{F}(\alpha, \beta) = F(g \circ \alpha)$ (which is exactly the same as $F(\beta \circ f)$, but merely in bijection with $F(g) \circ F(\alpha)$ and $F(\beta) \circ F(f)$). The source and target maps are given by

$$\begin{array}{ccc} F(\beta) \times_{F(v)} F(f) & \xleftarrow{F_{u,v,x}^{-1}} & \vec{F}(\alpha, \beta) & \xrightarrow{F_{u,w,x}^{-1}} & F(g) \times_{F(w)} F(\alpha) \\ \downarrow & & & & \downarrow \\ \vec{F}(f) = F(f) & & & & F(g) = \vec{F}(g). \end{array}$$

For any pair of composable morphisms

$$\begin{array}{ccc}
 u & \xrightarrow{f} & v \\
 \alpha \downarrow & & \downarrow \beta \\
 w & \xrightarrow{g} & x \\
 \gamma \downarrow & & \downarrow \delta \\
 y & \xrightarrow{h} & z,
 \end{array}$$

$\vec{F}_{f,g,h}$ should specify an isomorphism from $\vec{F}(\gamma, \delta) \times_{F(g)} \vec{F}(\alpha, \beta) = F(\delta \circ g) \times_{F(g)} F(g \circ \alpha)$ to $F(\delta \circ g \circ \alpha) = F(\gamma \circ \alpha, \delta \circ \beta)$. Define this isomorphism to be the composition of the isomorphisms

$$\begin{aligned}
 F(\delta \circ g) \times_{F(g)} F(g \circ \alpha) &\cong F(\delta) \times_{F(x)} F(g) \times_{F(g)} F(g) \times_{F(w)} F(\alpha) \\
 &\cong F(\delta) \times_{F(x)} F(g) \times_{F(w)} F(\alpha) \\
 &\cong F(\delta \circ g \circ \alpha).
 \end{aligned}$$

Lemma 5.11. *These maps make \vec{F} into a strictly unitary, lax 2-functor, and $\vec{F} \circ A = F$.*

We leave the proof as an exercise to the reader.

Lemma 5.12. *In $\text{Arr}(\underline{2}^n)_\dagger$, any sequence of composable, non-identity morphisms has length at most $2n$.*

Proof. This is immediate from the definitions. \square

Corollary 5.13. *If $k \geq 2n$ then any strictly unitary functor $\vec{F}: \text{Arr}(\underline{2}^n) \rightarrow \mathcal{B}$ admits a k -dimensional spacial refinement $\vec{\vec{F}} = \vec{F}: \text{Arr}(\underline{2}^n) \rightarrow \text{Top}_\bullet$.*

Proof. This is immediate from Lemma 5.12 and Proposition 5.2. \square

Lemma 5.14. *Fix $k > 2n$. Given $F: \underline{2}^n \rightarrow \mathcal{B}$, let $\vec{\vec{F}}$ be a k -dimensional spacial refinement of \vec{F} and consider the homotopy coherent diagram $\vec{\vec{F}} \circ B: \widehat{\underline{2}}^n \rightarrow \text{Top}_\bullet$. There is a morphism of homotopy coherent diagrams $G_k: \vec{\vec{F}} \circ B \rightarrow \widehat{F}_k$ so that on each object the underlying map induces an isomorphism on H_i for $i \leq 2k - 1$.*

Proof. Recall that a morphism from $\vec{\vec{F}} \circ B$ to \widehat{F}_k is a diagram over $\underline{2}^1 \times \widehat{\underline{2}}^n$ whose restriction to $\{1\} \times \widehat{\underline{2}}^n$ is $\vec{\vec{F}} \circ B$ and whose restriction to $\{0\} \times \widehat{\underline{2}}^n$ is \widehat{F}_k . We will build such a diagram inductively, using box maps from wedges to products.

On $\{0\} \times \widehat{\underline{2}}^n$ and $\{1\} \times \widehat{\underline{2}}^n$, of course, G is already specified. Notice that for each object $(u \xrightarrow{\varphi_{u,v}} v \xrightarrow{\varphi_{v,w}} w) \in \text{Ob}(\widehat{\underline{2}}^n)$, the space $(\vec{\vec{F}} \circ B)(u \rightarrow v \rightarrow w) = \bigvee_{a \in F(\varphi_{v,w} \circ \varphi_{u,v})} S^k$ is the k -skeleton of $\widehat{F}_k(u \rightarrow v \rightarrow w) = \bigvee_{a \in F(\varphi_{u,v})} \prod_{b \in F(\varphi_{v,w}), s(b)=t(a)} S^k$. For each arrow of the form $\varphi_{1,0} \times \text{Id}_{u \rightarrow v \rightarrow w}$, define $G(\varphi_{1,0} \times \text{Id}_{u \rightarrow v \rightarrow w})$ to be the inclusion of the k -skeleton. More generally, given a morphism (α, β, γ) as in Formula (4.10), define $G(\varphi_{1,0} \times (\alpha, \beta, \gamma))$ to be the composition $G(\varphi_{1,0} \times \text{Id}_{u' \rightarrow v' \rightarrow w'}) \circ (\vec{\vec{F}} \circ B)(\alpha, \beta, \gamma)$. (Factoring in the other order would work just as well, though it would give a different map.)

The result is a homotopy commutative diagram G , so that the restriction to $\{0\} \times \widehat{\underline{2}}^n$ is commutative and the restriction to $\{1\} \times \widehat{\underline{2}}^n$ is homotopy coherent. Moreover, each of the maps (or composition of maps) from the 1-side to the 0-side is a (possibly non-disjoint) box map. These box maps satisfy the combinatorial compatibility condition required to define homotopies between them. So, since $k \geq 2n + 1$ and any sequence of composable arrows has length at most $2n + 1$, we can extend this diagram to a homotopy coherent one. \square

Corollary 5.15. *There is a stable homotopy equivalence $\text{hocolim}(\tilde{F} \circ B)^\dagger \simeq \text{hocolim} \hat{F}^\dagger$.*

Proof. The morphism of diagrams G_k from Lemma 5.14 extends uniquely to a morphism of thickened diagrams $G_k^\dagger: (\tilde{F} \circ B)^\dagger \rightarrow \hat{F}_k^\dagger$. Further, the diagram \tilde{F} and the morphisms G_k^\dagger can be chosen so that G_{k+1}^\dagger is the suspension of G_k^\dagger . It follows that there is an induced map of diagrams of spectra G^\dagger , and the underlying maps of G^\dagger are equivalences. \square

Proof of Theorem 5. Let $\vec{F}: \text{Arr}(\mathbb{2}^n) \rightarrow \mathcal{B}$ be the functor from Lemma 5.11. By Corollary 5.13 there is a spacial refinement \tilde{F} of \vec{F} . The composition $\tilde{F} = \tilde{F} \circ A$ is a spacial refinement of F . We will show that the corresponding diagram $\tilde{F}^+: \mathbb{2}_+^n \rightarrow \mathcal{S}$ satisfies the conditions of the theorem. Indeed, all of the conditions except that the homotopy colimit is $|F|$ are immediate. We compute the homotopy colimit.

By Lemma 5.7,

$$\text{hocolim}_{\mathbb{2}_+^n} \tilde{F}^+ \simeq \text{hocolim}_{\mathbb{2}_+^n} \tilde{F}^\dagger.$$

By Lemma 5.9, Lemma 5.10, and Property (ho-4) of homotopy colimits,

$$\text{hocolim}_{\mathbb{2}_+^n} \tilde{F}^\dagger \simeq \text{hocolim}_{\text{Arr}(\mathbb{2}^n)_+} \tilde{F}^\dagger \simeq \text{hocolim}_{\mathbb{2}_+^n} \tilde{F}^\dagger \circ B^\dagger.$$

By Corollary 5.15, there is a homotopy equivalence

$$\text{hocolim}_{\mathbb{2}_+^n} \tilde{F}^\dagger \circ B^\dagger \simeq \text{hocolim}_{\mathbb{2}_+^n} \hat{F}^\dagger.$$

By Lemma 4.18,

$$\text{hocolim}_{\mathbb{2}_+^n} \hat{F}^\dagger \simeq \text{hocolim}_{\mathbb{2}_+^n} \hat{F}^+ = |F|,$$

proving the result. \square

6. A CW COMPLEX STRUCTURE ON THE REALIZATION OF THE SMALL CUBE

In this section, we prove that the realization in terms of little cubes (Section 5) is stably homotopy equivalent to the cubical realization. We start by studying the cell structure on the little cubes realization (Section 5):

Proposition 6.1. *Let $F: \mathbb{2}^n \rightarrow \mathcal{B}$ be a strictly unitary 2-functor and $\tilde{F}_k: \mathbb{2}^n \rightarrow \text{Top}_\bullet$ a spacial refinement of F (Definition 5.1). Then the homotopy colimit of \tilde{F}_k^+ carries a CW complex structure whose cells except the basepoint correspond to the elements of the set $\coprod_{u \in \{0,1\}^n} F(u)$.*

Proof. Per Observation 2.19, when taking the homotopy colimit we may (and will) consider only chains of non-identity arrows.

For $u \in \text{Ob}(\mathbb{2}^n)$ and $x \in F(u)$, let B_x be the box that is associated to x during the construction of \tilde{F}_k^+ ; that is, $B_x/\partial B_x$ is the S^k -summand corresponding to x in $\tilde{F}_k^+(u) = \bigvee_{x \in F(u)} S^k$. Following Definition 2.17 (and with \sim denoting the same equivalence relation), we can write the homotopy colimit as

$$\begin{aligned} \text{hocolim} \tilde{F}_k^+ &= \left(\{*\} \amalg \coprod_{u \in \{0,1\}^n} \left(\coprod_{m \geq 0} \coprod_{\substack{u=u^0 \xrightarrow{f_1} \dots \xrightarrow{f_m} u^m \\ u^i \in \{0,1\}^n \cup \{*\} \\ f_i \neq \text{Id}}} [0,1]^m \right) \times \left(\bigvee_{x \in F(u)} B_x/\partial B_x \right) \right) / \sim \\ &= \left(\left[\{*\} \amalg \coprod_{u \in \{0,1\}^n} \left(\coprod_{m \geq 0} \coprod_{\substack{u=u^0 \xrightarrow{f_1} \dots \xrightarrow{f_m} u^m \\ u^i \in \{0,1\}^n \cup \{*\} \\ f_i \neq \text{Id}}} [0,1]^m \right) / \sim_1 \right] \times \left(\coprod_{x \in F(u)} B_x \right) \right) / \sim_2. \end{aligned}$$

where we have broken up the identification \sim into a two-step identification \sim_1 and \sim_2 , defined as follows:

$$\begin{aligned} & (f_m, \dots, f_1; t_1, \dots, t_m) \\ & \sim_1 \begin{cases} (f_m, \dots, f_{i+1} \circ f_i, \dots, f_1; t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_m) & t_i = 1, i < m \\ (f_{m-1}, \dots, f_1; t_1, \dots, t_{m-1}) & t_m = 1 \end{cases} \\ & (f_m, \dots, f_1; t_1, \dots, t_m; y) \\ & \sim_2 \begin{cases} (f_m, \dots, f_{i+1}; t_{i+1}, \dots, t_m; \tilde{F}_k^+(f_i, \dots, f_1)(t_1, \dots, t_{i-1})(y)) & t_i = 0 \\ * & y \in \partial B_x. \end{cases} \end{aligned}$$

Now fix $u \in \{0, 1\}^n$, and let us study the cubical complex

$$(6.2) \quad M_u := \left(\coprod_{m \geq 0} \coprod_{\substack{u = u^0 \xrightarrow{f_1} \dots \xrightarrow{f_m} u^m \\ u^i \in \{0, 1\}^n \cup \{*\} \\ f_i \neq \text{Id}}} [0, 1]^m \right) / \sim_1.$$

If $u = \vec{0}$, then M_u is a single point, which we write as $\{0\}$ for reasons that will soon be apparent. When $u \neq \vec{0}$, we divide the chains $u = u^0 \xrightarrow{f_1} \dots \xrightarrow{f_m} u^m$ into two types: the ones ending at $\vec{0}$ or $*$, and the ones ending at neither. In the first case, when $u^m \in \{\vec{0}, *\}$, the facet $[0, 1]^{m-1} \times \{1\}$ is identified with the cube $[0, 1]^{m-1}$ coming from the sub-chain $u = u^0 \xrightarrow{f_1} \dots \xrightarrow{f_{m-1}} u^{m-1}$. Therefore, we can write

$$\begin{aligned} M_u &= \left(\left(\coprod_{m \geq 1} \coprod_{\substack{u = u^0 \xrightarrow{f_1} \dots \xrightarrow{f_m} \vec{0} \\ f_i \neq \text{Id}}} [0, 1]^{m-1} \times [0, 1] \right) \amalg \left(\coprod_{m \geq 1} \coprod_{\substack{u = u^0 \xrightarrow{f_1} \dots \xrightarrow{f_m} * \\ f_i \neq \text{Id}}} [0, 1]^{m-1} \times [0, 1] \right) \right. \\ & \quad \left. \amalg \left(\coprod_{m \geq 1} \coprod_{\substack{u = u^0 > \dots > u^{m-1} \\ u^i \in \{0, 1\}^n \setminus \{\vec{0}\}}} [0, 1]^{m-1} \right) \right) / \sim_1 \\ &\cong \left(\left(\coprod_{m \geq 1} \coprod_{\substack{u = u^0 \xrightarrow{f_1} \dots \xrightarrow{f_m} \vec{0} \\ f_i \neq \text{Id}}} [0, 1]^{m-1} \times [0, 1] \right) \amalg \left(\coprod_{m \geq 1} \coprod_{\substack{u = u^0 \xrightarrow{f_1} \dots \xrightarrow{f_m} * \\ f_i \neq \text{Id}}} [0, 1]^{m-1} \times [1, 2] \right) \right. \\ & \quad \left. \amalg \left(\coprod_{m \geq 1} \coprod_{\substack{u = u^0 > \dots > u^{m-1} \\ u^i \in \{0, 1\}^n \setminus \{\vec{0}\}}} [0, 1]^{m-1} \right) \right) / \sim_1 \\ &= \left(\coprod_{m \geq 1} \coprod_{\substack{u = u^0 > \dots > u^{m-1} \\ u^i \in \{0, 1\}^n \setminus \{\vec{0}\}}} [0, 1]^{m-1} \right) / \sim_1 \times [0, 2] \end{aligned}$$

where the second identification is via the linear map $[0, 1] \rightarrow [1, 2]$ that sends 0 to 2 and 1 to 1. This quotient space is just $M_{u, \vec{0}} \times [0, 2]$, where $M_{u, \vec{0}} \cong \mathcal{M}_{\mathcal{C}(n)}(u, \vec{0})$ is the cubical complex from Definition 3.4.

Therefore, we can write

$$\text{hocolim } \tilde{F}_k^+ = \left(\{*\} \amalg \left[\coprod_{u \in \{0, 1\}^n \setminus \{\vec{0}\}} \coprod_{x \in F(u)} \mathcal{M}_{\mathcal{C}(n)}(u, \vec{0}) \times [0, 2] \times B_x \right] \amalg \left[\coprod_{x \in F(\vec{0})} \{0\} \times B_x \right] \right) / \sim_2.$$

Observe that this gives the required CW complex on the homotopy colimit, where the cell corresponding to x is

$$\mathcal{C}(x) = \begin{cases} \mathcal{M}_{\mathcal{C}(n)}(u, \vec{0}) \times [0, 2] \times B_x & \text{if } u \neq \vec{0} \\ \{0\} \times B_x & \text{if } u = \vec{0}. \end{cases}$$

The identification \sim_2 glues the boundary $\partial\mathcal{C}(x)$ to lower-dimensional cells. Specifically, everything over ∂B_x is identified with the basepoint $*$. If $u \neq \vec{0}$, everything over $\{2\} \subset \partial([0, 2])$ is identified with $*$ as well, and everything over $\{0\} \subset \partial([0, 2])$ is identified with cells corresponding to $u = \vec{0}$. Finally, points on $\partial\mathcal{M}_{\mathcal{C}(n)}(u, \vec{0})$ correspond to points on some cube $[0, 1]^l \subset \mathcal{M}_{\mathcal{C}(n)}(u, \vec{0})$ where some coordinate is 0 (Lemma 3.5 (2)), and such points are identified by \sim_2 to points of $\mathcal{M}_{\mathcal{C}(n)}(u', \vec{0})$ for some $u > u' > \vec{0}$. \square

Indeed, it is not hard to show that if \mathcal{C} is the cubical flow category corresponding to F (from Construction 4.3) then its associated chain complex (from Definition 2.4) is isomorphic to the reduced cellular cochain complex of the above CW complex, via an isomorphism sending the objects of \mathcal{C} to the corresponding cells in the CW complex. We will not prove this now, since it follows from Theorem 6.

Theorem 6. *Let $(\mathcal{C}, \mathfrak{f}: \mathcal{C} \rightarrow \mathcal{C}_C(n))$ be a cubical flow category, and let $F: \underline{2}^n \rightarrow \mathcal{B}$ be the corresponding functor (Construction 4.1). Then the cubical realization of \mathcal{C} (Definition 3.15) is stably homotopy equivalent to the realization of F as the homotopy colimit of the homotopy coherent diagram $\tilde{F}^+: \underline{2}_+^n \rightarrow \mathcal{S}$ from Theorem 5; and the homotopy equivalence sends the cells in the CW complex structure on the homotopy colimit from Proposition 6.1 to the corresponding cells in the cubical realization of \mathcal{C} via maps of degree ± 1 .*

Proof. Fix a cubical neat embedding ι of \mathcal{C} relative to $\mathbf{d} = (d_0, \dots, d_{n-1})$ and let $k = \sum_i d_i$. Let ϵ and R be the parameters from Section 3.3, and let

$$\bar{\iota}_{x,y}: \prod_{i=|f(y)|}^{|f(x)|-1} [-\epsilon, \epsilon]^{d_i} \times \mathcal{M}_{\mathcal{C}}(x, y) \rightarrow \left[\prod_{i=|f(y)|}^{|f(x)|-1} [-R, R]^{d_i} \right] \times \mathcal{M}_{\mathcal{C}(n)}(f(x), f(y))$$

be the extension of ι from Formula (3.12). Recall that given $u \in \text{Ob}(\underline{2}^n)$ and $x \in F(u)$ the cubical realization $\|\mathcal{C}\|$ has a corresponding cell

$$\mathcal{C}(x) = \begin{cases} \prod_{i=0}^{|u|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \times [0, 1] \times \mathcal{M}_{\mathcal{C}(n)}(u, \vec{0}) & \text{if } u \neq \vec{0}, \\ \prod_{i=0}^{n-1} [-\epsilon, \epsilon]^{d_i} \times \{0\} & \text{if } u = \vec{0}. \end{cases}$$

The strategy of the proof is to use the cubical neat embedding to build a particular spacial refinement \tilde{F}_k of F and construct a map from $\text{hocolim } \tilde{F}_k^+$ to the cubical realization of F that sends cells to cells by degree ± 1 maps, and hence is a stable homotopy equivalence.

The diagram \tilde{F}_k is defined as follows. The box associated to $x \in F(u)$ is

$$B_x = \prod_{i=0}^{|u|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i}.$$

Next, consider a sequence of composable non-identity morphisms $u = u^0 \xrightarrow{f_1} \dots \xrightarrow{f_m} u^m = v$ in $\underline{2}^n$. We will define

$$\tilde{F}_k^+(f_m, \dots, f_1) = \Phi(e_{t_1, \dots, t_{m-1}}, F(\varphi_{u,v})): [0, 1]^{m-1} \times \tilde{F}_k^+(u) \rightarrow \tilde{F}_k^+(v)$$

for an appropriate $[0, 1]^{m-1}$ -parameter family of boxes $e_{t_1, \dots, t_{m-1}}: [0, 1]^{m-1} \rightarrow E(\{B_x \mid x \in F(u)\}, s_{F(\varphi_{u,v})})$. In other words, if for $\gamma \in F(\varphi_{u,v})$ we write $B_\gamma = \prod_{i=0}^{|\gamma|-1} [-R, R]^{d_i} \times \prod_{i=|\gamma|}^{n-1} [-\epsilon, \epsilon]^{d_i}$ then $e_{t_1, \dots, t_{m-1}}$ is a

$[0, 1]^{m-1}$ -parameter family of embeddings (as disjoint sub-boxes)

$$\coprod_{\gamma|s(\gamma)=x} B_\gamma \hookrightarrow B_x, \quad \forall x \in F(u).$$

To define $e_{t_1, \dots, t_{m-1}}$, fix $\gamma \in F(\varphi_{u,v})$ with $s(\gamma) = x$, and let $y = t(\gamma)$. Let v_γ be the section of the covering map $\mathcal{M}_{\mathcal{C}}(x, y) \rightarrow \mathcal{M}_{\mathcal{C}(n)}(u, v)$ whose image is the path component corresponding to γ . Consider the map

$$\begin{aligned} \mathcal{M}_{\mathcal{C}(n)}(u, v) \times B_\gamma &= \mathcal{M}_{\mathcal{C}(n)}(u, v) \times \prod_{i=0}^{|v|-1} [-R, R]^{d_i} \times \prod_{i=|v|}^{n-1} [-\epsilon, \epsilon]^{d_i} \\ &\xrightarrow{\langle v_\gamma, \text{Id} \rangle} \mathcal{M}_{\mathcal{C}}(x, y) \times \prod_{i=0}^{|v|-1} [-R, R]^{d_i} \times \prod_{i=|v|}^{n-1} [-\epsilon, \epsilon]^{d_i} \\ &\cong \prod_{i=0}^{|v|-1} [-R, R]^{d_i} \times \left(\prod_{i=|v|}^{|u|-1} [-\epsilon, \epsilon]^{d_i} \times \mathcal{M}_{\mathcal{C}}(x, y) \right) \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \\ &\xrightarrow{\langle \text{Id}, \bar{t}_{x,y}, \text{Id} \rangle} \prod_{i=0}^{|v|-1} [-R, R]^{d_i} \times \left(\prod_{i=|v|}^{|u|-1} [-R, R]^{d_i} \times \mathcal{M}_{\mathcal{C}(n)}(u, v) \right) \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \\ &\xrightarrow{\langle \text{Id}, \pi^R, \text{Id} \rangle} \prod_{i=0}^{|v|-1} [-R, R]^{d_i} \times \prod_{i=|v|}^{|u|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} \\ &\cong \prod_{i=0}^{|u|-1} [-R, R]^{d_i} \times \prod_{i=|u|}^{n-1} [-\epsilon, \epsilon]^{d_i} = B_x \end{aligned}$$

and the induced map

$$(6.3) \quad \mathcal{M}_{\mathcal{C}(n)}(u, v) \times \coprod_{\substack{\gamma \in F(\varphi_{u,v}) \\ s(\gamma)=x}} B_\gamma \rightarrow B_x.$$

It follows from the definition of cubical neat embeddings and the formula for $\bar{t}_{x,y}$ that for any point $\text{pt} \in \mathcal{M}_{\mathcal{C}(n)}(u, v)$, the restriction $\{\text{pt}\} \times \prod_{\gamma \in F(\varphi_{u,v})|s(\gamma)=x} B_\gamma \rightarrow B_x$ is an inclusion of disjoint sub-boxes. Therefore, we may view the map from Equation (6.3) as a $\mathcal{M}_{\mathcal{C}(n)}(u, v)$ -parameter family of sub-boxes $\prod_{\gamma|s(\gamma)=x} B_\gamma \subset B_x$.

The chain $u = u^0 > \dots > u^m = v$ corresponds to some cube $[0, 1]^{m-1}$ in the cubical complex $M_{u,v}$ from Definition 3.4, which via Lemma 3.5 is identified with some cube $[0, 1]^{m-1} \subset \mathcal{M}_{\mathcal{C}(n)}(u, v)$. Restrict the map from Formula (6.3) to $[0, 1]^{m-1} \times \prod_{\gamma|s(\gamma)=x} B_\gamma$ to obtain the required $[0, 1]^{m-1}$ -parameter family of sub-boxes $\prod_{\gamma|s(\gamma)=x} B_\gamma \subset B_x$.

We check that \widetilde{F}_k is indeed a homotopy coherent diagram. For any sequence of composable non-identity morphisms $u = u^0 \xrightarrow{f_1} \dots \xrightarrow{f_m} u^m = v$ in $\underline{2}^n$, we need to show that

$$\begin{aligned} \widetilde{F}_k(f_m, \dots, f_1)(t_1, \dots, t_{i-1}, 1, t_{i+1}, \dots, t_{m-1}) \\ &= \widetilde{F}_k(f_m, \dots, f_{i+1} \circ f_i, \dots, f_1)(t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_{m-1}) \\ \widetilde{F}_k(f_m, \dots, f_1)(t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_{m-1}) \\ &= \widetilde{F}_k(f_m, \dots, f_{i+1})(t_{i+1}, \dots, t_{m-1}) \circ \widetilde{F}_k(f_i, \dots, f_1)(t_1, \dots, t_{i-1}). \end{aligned}$$

The first equation is immediate from Definition 3.4 since the facet of $[0, 1]^{m-1}$ which has $t_i = 1$ is identified with the cube $[0, 1]^{m-2}$ coming from the sequence of composable morphisms $u = u^0 \xrightarrow{f_1} \dots \xrightarrow{f_{i-1}} u^{i-1} \xrightarrow{f_{i+1} \circ f_i} u^{i+1} \xrightarrow{f_{i+2}} \dots \xrightarrow{f_m} u^m = v$. The second equation follows from Lemma 3.5 (3) since the facet of $[0, 1]^{m-1}$ that has $t_i = 0$ lies in the facet $\mathcal{M}_{\mathcal{C}(n)}(u^i, v) \times \mathcal{M}_{\mathcal{C}(n)}(u, u^i)$ of $\mathcal{M}_{\mathcal{C}(n)}(u, v)$ and is identified with the product $[0, 1]^{m-i-1} \times [0, 1]^{i-1}$, coming from the sequences $u^i \xrightarrow{f_{i+1}} \dots \xrightarrow{f_m} u^m = v$ and $u^0 \xrightarrow{f_1} \dots \xrightarrow{f_i} u^i$, respectively. Since the maps \widetilde{F}_k^+ were defined via cubical neat embeddings that satisfied Definition 3.11 (3), the second equation holds.

We will now construct a cellular map from the CW complex structure on hocolim \widetilde{F}_k^+ (from Proposition 6.1) to the CW complex for the cubical realization, $\|\mathcal{C}\|$, sending cells to the corresponding cells with degree ± 1 . (Recall that the non-basepoint cells in either CW complex correspond to objects $\text{Ob}(\mathcal{C})$.) This will complete the proof.

For any $u \in \text{Ob}(\underline{2}^n)$ and any $x \in F(u)$, the cell associated to x in hocolim \widetilde{F}_k^+ is

$$\mathcal{C}(x)' = \begin{cases} \mathcal{M}_{\mathcal{C}(n)}(u, \vec{0}) \times [0, 2] \times B_x & \text{if } u \neq \vec{0} \\ \{0\} \times B_x & \text{if } u = \vec{0}, \end{cases}$$

while the cell associated to x in $\|\mathcal{C}\|$ is

$$\mathcal{C}(x) = \begin{cases} \mathcal{M}_{\mathcal{C}(n)}(u, \vec{0}) \times [0, 1] \times B_x & \text{if } u \neq \vec{0} \\ \{0\} \times B_x & \text{if } u = \vec{0}. \end{cases}$$

Map $\mathcal{C}(x)'$ to $\mathcal{C}(x)$ by the quotient map $[0, 2] \rightarrow [0, 2]/[1, 2] \cong [0, 1]$, and the identity map on all other factors. This map certainly has degree ± 1 on each cell. To check that it produces a well-defined map on CW complexes, we must check that it commutes with the attaching maps. Everything over ∂B_x was quotiented to the basepoint on either side. If $u \neq \vec{0}$, everything over $\{2\} \subset [0, 2]$ was quotiented to the basepoint for $\mathcal{C}(x)'$, while everything over $\{1\} \subset [0, 1]$ was quotiented to the basepoint for $\mathcal{C}(x)$. Therefore, we only need to consider $u \neq \vec{0}$ and concentrate on the attaching maps on the portion of the boundary of $\mathcal{C}(x)$ (respectively, $\mathcal{C}(x)'$) that lives over $\partial \mathcal{M}_{\mathcal{C}(n)}(u, \vec{0})$ or $\{0\} \subset \partial([0, 1])$ (respectively, $\{0\} \subset \partial([0, 2])$).

Consider the subcomplex

$$\widetilde{M}_u := \left(\coprod_{\substack{u=u^0 > \dots > u^m \\ u^i \in \{0,1\}^n}} [0, 1]^m \right) / \sim_1 \cong \mathcal{M}_{\mathcal{C}(n)}(u, \vec{0}) \times [0, 1]$$

of the cubical complex $M_u \cong \mathcal{M}_{\mathcal{C}(n)}(u, \vec{0}) \times [0, 2]$ from Equation (6.2) in the proof of Proposition 6.1. We are interested in the part of $\partial\mathcal{C}(x)'$ that lives over the following subset N_u of $\partial\widetilde{M}_u$:

$$\begin{aligned} N_u &:= \left(\partial\mathcal{M}_{\mathcal{C}(n)}(u, \vec{0}) \times [0, 1] \right) \cup \left(\mathcal{M}_{\mathcal{C}(n)}(u, \vec{0}) \times \{0\} \right) \\ &= \left(\prod_{u=u^0 > \dots > u^m} \prod_{i=1}^m [0, 1]^{i-1} \times \{0\} \times [0, 1]^{m-i} \right) / \sim_1 . \end{aligned}$$

Let $p \in \partial\mathcal{C}(x)'$ be some point living over N_u , and write $p = (p_1, p_2)$, where $p_1 \in N_u$ and $p_2 \in B_x$. Assume p_1 lies in the cube $[0, 1]^m$ corresponding to some chain $u = u^0 > \dots > u^m$. Let $p_{1,1}, \dots, p_{1,m}$ be the coordinates of p_1 as a point in the cube, and assume $p_{1,\ell} = 0$. Let ϕ denote the restriction of $\widetilde{F}_k(\varphi_{u^{\ell-1}, u^\ell}, \dots, \varphi_{u^0, u^1})$ to $[0, 1]^{\ell-1} \times (B_x / \partial B_x)$. Under the CW complex attaching map (denoted \sim_2 in the proof of Proposition 6.1), p is attached to the point

$$\left((p_{1,\ell+1}, \dots, p_{1,m}), \phi((p_{1,1}, \dots, p_{1,\ell-1}), p_2) \right) \in [0, 1]^{m-\ell} \times \bigvee_{y \in F(u^\ell)} (B_y / \partial B_y),$$

where $[0, 1]^{m-\ell}$ is the cube in \widetilde{M}_{u^ℓ} corresponding to the chain $u^\ell > \dots > u^m$. The map ϕ is constructed as a $[0, 1]^{m-\ell}$ -parameter family of box maps. Therefore, the attaching map glues p to the basepoint $*$ unless p_2 lies in the interior of one of the sub-boxes at the point $(p_{1,1}, \dots, p_{1,\ell-1})$ in the family; and if p_2 lies in the interior of some box B_{γ_0} then p is glued to the point

$$q := ((p_{1,\ell+1}, \dots, p_{1,m}), q_2) \in [0, 1]^{m-\ell} \times B_{y_0} \subset \widetilde{M}_{u^\ell} \times B_{y_0} \subset \mathcal{C}(y_0)'$$

where $y_0 = t(\gamma_0)$ and $q_2 := \phi((p_{1,1}, \dots, p_{1,\ell-1}), p_2) \in B_{y_0}$.

We need to check that p , now viewed as a point in $\partial\mathcal{C}(x)$, is also glued to q , now viewed as a point in $\mathcal{C}(y_0)$, in the CW complex $\|\mathcal{C}\|$. In the construction of $\|\mathcal{C}\|$ (Definition 3.15), we extended the cubical neat embedding

$$\iota_{x, y_0} : \mathcal{M}_{\mathcal{C}}(x, y_0) \hookrightarrow \mathcal{M}_{\mathcal{C}(n)}(u^\ell, \vec{0}) \times \prod_{i=|u^\ell|}^{|u|-1} (-R, R)^{d_i}$$

to an embedding

$$\bar{\iota}_{x, y_0} : \mathcal{M}_{\mathcal{C}}(x, y_0) \times \prod_{i=|u^\ell|}^{|u|-1} [-\epsilon, \epsilon]^{d_i} \hookrightarrow \mathcal{M}_{\mathcal{C}(n)}(u^\ell, \vec{0}) \times \prod_{i=|u^\ell|}^{|u|-1} [-R, R]^{d_i},$$

and used it to define embeddings $\iota : \mathcal{M}_{\mathcal{C}}(x, y_0) \times B_{y_0} \hookrightarrow \mathcal{M}_{\mathcal{C}(n)}(u, u^\ell) \times B_x$ and $j : \mathcal{M}_{\mathcal{C}}(x, y_0) \times \mathcal{C}(y_0) \hookrightarrow \partial\mathcal{C}(x)$. (Note that $\widetilde{\mathcal{M}}_{\mathcal{C}(n)}(u^\ell, \vec{0}) = \widetilde{M}_{u^\ell}$.)

For any path component γ of $\mathcal{M}_{\mathcal{C}}(x, y_0)$, let ι_γ and j_γ denote the restrictions $\iota|_{\gamma \times B_{y_0}}$ and $j|_{\gamma \times \mathcal{C}(y_0)}$; and as before, let v_γ denote the section of $\mathcal{M}_{\mathcal{C}}(x, y_0) \rightarrow \mathcal{M}_{\mathcal{C}(n)}(u, u^\ell)$ whose image is γ . Since ι_{x, y_0} satisfies Definition 3.11 (1), and its extension $\bar{\iota}_{x, y_0}$ was defined via Equation (3.12), there exists a map $\mu_\gamma : B_{y_0} \rightarrow \mathcal{M}_{\mathcal{C}(n)}(u, u^\ell) \times B_x$, so that $\iota_\gamma(v_\gamma(a), b) = (a, \mu_\gamma(a, b))$, for all $(a, b) \in \mathcal{M}_{\mathcal{C}(n)}(u, u^\ell) \times B_{y_0}$.

Let $q_1 = (p_{1,\ell+1}, \dots, p_{1,m}) \in [0, 1]^{m-\ell}$ and $q' = (p_{1,1}, \dots, p_{1,\ell-1}) \in [0, 1]^{\ell-1}$; treat q' as a point in $\mathcal{M}_{\mathcal{C}(n)}(u, u^\ell)$, after viewing the cube $[0, 1]^{\ell-1}$ as the cube corresponding to the chain $u = u^0 > \dots > u^\ell$ in the cubical complex structure on $\mathcal{M}_{\mathcal{C}(n)}(u, u^\ell)$ from Lemma 3.5. Let κ be the inclusion map $\mathcal{M}_{\mathcal{C}(n)}(u, u^\ell) \times \widetilde{M}_{u^\ell} \times B_x \hookrightarrow \mathcal{C}(x)$. Since $\widetilde{F}_k(\varphi_{u^{\ell-1}, u^\ell}, \dots, \varphi_{u^0, u^1})$ (used to define ϕ) is defined via Equation (6.3) using the

cubical neat embeddings ι_{x,y_0} (which are used to define the maps ι_γ , and consequently, μ_γ), p_2 is in the interior of the box B_{γ_0} , and $\phi(q', p_2) = q_2$, it follows that $\mu_{\gamma_0}(q', q_2) = p_2$. Therefore,

$$\begin{aligned} J_{\gamma_0}(\nu_\gamma(q'), q) &= J_{\gamma_0}(\nu_\gamma(q'), q_1, q_2) = \kappa(q', q_1, \mu_{\gamma_0}(q', q_2)) \\ &= \kappa(q', q_1, p_2) = \kappa((p_{1,1}, \dots, p_{1,\ell-1}), (p_{1,\ell+1}, \dots, p_{1,m}), p_2) \\ &= ((p_{1,1}, \dots, p_{1,\ell-1}, 0, p_{1,\ell+1}, \dots, p_{1,m}), p_2) = (p_1, p_2) = p. \end{aligned}$$

The last equation is justified by the fact that the cubical complex structures on $\mathcal{M}_{\mathcal{E}_C(n)}(u, u^\ell)$ and \widetilde{M}_{u^ℓ} respect the product structure on facets of $\widetilde{M}_u = \mathcal{M}_{\mathcal{E}_C(n)}(u, \vec{0}) \times [0, 1]$ (Lemma 3.5 (3)). Therefore, p is glued to q in the CW complex $\|\mathcal{E}\|$ as well. \square

7. THE KHOVANOV HOMOTOPY TYPE

We pause briefly to review where we stand. We have introduced a special kind of flow categories, cubical flow categories (Section 3), and shown that the data of a cubical flow category is equivalent to a strictly unitary 2-functor from the cube $\underline{2}^n$ to the Burnside 2-category (Section 4). Given a cubical flow category (or functor from the cube to the Burnside category) we have four ways of realizing the functor as a spectrum:

- The original Cohen-Jones-Segal realization (Section 2.5).
- The cubical realization, a modification of the Cohen-Jones-Segal construction taking into account the map to the cube (Section 3.4).
- Thickening the diagram, producing a canonical diagram in spectra, and taking the homotopy mapping cone (Section 4).
- Using the ‘‘little box’’ construction to produce a homotopy coherent cube in spectra, and then taking the homotopy mapping cone (Section 5.2).

Moreover, Theorems 4, 5, and 6 together imply that, up to stable homotopy equivalence, these realizations all agree.

For the rest of the paper, we turn to a particular cubical flow category: the Khovanov flow category constructed in [LS14a] (see also Section 2.3 and Examples 3.10, 4.5). Given a link diagram K , let $F_{Kh}(K): \underline{2}^n \rightarrow \mathcal{B}$ be the Khovanov functor constructed in Example 4.5. (A more direct description of $F_{Kh}(K)$ was given by Hu-Kriz-Kriz [HKK]; see Section 8.1.) Let $\mathcal{X}_{Kh}(K)$ be the result of applying Construction 4.16 to $F_{Kh}(K)$ (though, up to stable homotopy equivalence, this is the same as applying any of the other three realization constructions). Similar constructions can be carried out for the reduced Khovanov flow category, from [LS14a, Section 8]; let $F_{\widetilde{Kh}}(K): \underline{2}^n \rightarrow \mathcal{B}$ be the corresponding functor and $\widetilde{\mathcal{X}}_{Kh}(K)$ the corresponding space.

8. RELATIONSHIP WITH HU-KRIZ-KRIZ

The goal of this section is to prove:

Theorem 7. *Fix a link diagram K . Let $M(K)$ be the homotopy type associated to K in [HKK, Theorem 5.4]. Then $\mathcal{X}_{Kh}(K)$ is stably homotopy equivalent to $M(K)$.*

The Hu-Kriz-Kriz construction has four steps:

- (1) Construct a 2-functor $\underline{2}^n \rightarrow \mathcal{B}$. (Note that the category $\underline{2}^n$ is denoted I^n in [HKK], and the category \mathcal{B} is denoted \mathcal{S}_2 .)
- (2) Use the Elmendorf-Mandell machine [EM06] to turn the 2-functor $\underline{2}^n \rightarrow \mathcal{B}$ into an A_∞ -functor $B_2(\underline{2}^n)' \rightarrow \mathcal{S}$, where $B_2(\underline{2}^n)'$ is an auxiliary category which we will review below.
- (3) Use the rectification result [EM06, Theorem 1.4] to lift this composition to a strict functor $\underline{2}^n \rightarrow \mathcal{S}$.

- (4) Expand $\underline{2}^n$ to a category \mathcal{I} , analogous to the expansion of $\underline{2}^n$ to $\underline{2}_\dagger^n$, extend the functor $\underline{2}^n \rightarrow \mathcal{S}$ to a functor $\mathcal{I} \rightarrow \mathcal{S}$, and take the homotopy colimit.

We will prove that the two constructions agree, step-by-step.

8.1. The functors from the cube to the Burnside category agree. The $(1+1)$ -dimensional embedded cobordism category $\mathbf{Cob}_{\text{emb}}^{1+1}$ is defined as follows. The objects of $\mathbf{Cob}_{\text{emb}}^{1+1}$ are oriented 1-manifolds C embedded in S^2 along with a 2-coloring of the components of $S^2 \setminus C$, by the colors “white” and “black”, so that if $B(S^2 \setminus C)$ denotes the closure of the black region, then C is the oriented boundary of $B(S^2 \setminus C)$. The morphisms from C_1 to C_0 are oriented cobordisms Σ embedded in $[0, 1] \times S^2$ satisfying $\Sigma \cap (\{i\} \times S^2) = \{i\} \times C_i$ for $i \in \{0, 1\}$, along with a 2-coloring of $([0, 1] \times S^2) \setminus \Sigma$, so that if $B(([0, 1] \times S^2) \setminus \Sigma)$ denotes the closure of the black region, then Σ is oriented as the boundary of $B(([0, 1] \times S^2) \setminus \Sigma)$. The 2-morphisms are isotopy classes of isotopies of cobordisms relative boundary.

The Hu-Kriz-Kriz functor $\underline{2}^n \rightarrow \mathcal{B}$ is constructed in two steps [HKK, Section 5]. First [HKK, Section 4.3], given a link diagram L in S^2 with n crossings c_1, \dots, c_n , along with a checkerboard coloring of the link diagram, there is a lax 2-functor from $\underline{2}^n$ to $\mathbf{Cob}_{\text{emb}}^{1+1}$. This functor was partially described in Section 2.3: to $v \in \{0, 1\}^n$, associate the complete resolution $\mathcal{P}(v)$ which is a collection of disjoint circles in S^2 . The checkerboard coloring for L induces a 2-coloring of the complement of these circles; orient the circles as the boundary of the black region. To $u > v \in \{0, 1\}^n$, associate the embedded cobordism $\Sigma \subset [0, 1] \times S^2$, which is a product cobordism outside a neighborhood of the crossings where u and v differ, and has a saddle for each such crossing. We declare the cobordism to be running from $\mathcal{P}(u) = \Sigma \cap (\{1\} \times S^2)$ to $\mathcal{P}(v) = \Sigma \cap (\{0\} \times S^2)$. Up to isotopy, Σ is independent of the order of the saddles, and in fact the isotopies changing the order of saddles are themselves well-defined up to isotopy. Thus, this construction gives a lax 2-functor $\underline{2}^n \rightarrow \mathbf{Cob}_{\text{emb}}^{1+1}$.

Second [HKK, Section 3.4], there is a lax 2-functor $\mathcal{L}: \mathbf{Cob}_{\text{emb}}^{1+1} \rightarrow \mathcal{B}$ defined as follows. On objects, \mathcal{L} sends a 1-manifold $C \subset S^2$ to the set of all possible labelings of the components of C by elements of $\{x_+, x_-\}$: $\mathcal{L}(C) = \prod_{C_i \in \pi_0(C)} \{x_+, x_-\}$. The value of \mathcal{L} on morphisms is more complicated. For any embedded cobordism Σ and any connected component Σ_0 of Σ , consider the 2-coloring of $([0, 1] \times S^2) \setminus \Sigma_0$ that agrees with the given 2-coloring of $([0, 1] \times S^2) \setminus \Sigma$ near Σ_0 , and let $B(([0, 1] \times S^2) \setminus \Sigma_0)$ denote the closure of the black region. Observe that

$$\begin{aligned} H_1(([0, 1] \times S^2) \setminus \Sigma_0) / H_1((\{0, 1\} \times S^2) \setminus \partial \Sigma_0) &\cong \mathbb{Z}^{2g(\Sigma_0)} \\ H_1(B(([0, 1] \times S^2) \setminus \Sigma_0)) / H_1(B((\{0, 1\} \times S^2) \setminus \partial \Sigma_0)) &\cong \mathbb{Z}^{g(\Sigma_0)}. \end{aligned}$$

A *valid labeling* of a cobordism $\Sigma \subset [0, 1] \times S^2$ consists of:

- A labeling of each boundary component of Σ by x_+ or x_- , and
- A labeling of each genus 1 component Σ_0 of Σ by α or $-\alpha$, where $\{\pm\alpha\}$ are the generators of $H_1(B(([0, 1] \times S^2) \setminus \Sigma_0)) / H_1(B((\{0, 1\} \times S^2) \setminus \partial \Sigma_0)) \cong \mathbb{Z}$.

so that:

- Each connected component of Σ has genus 0 or 1.
- For each genus 0 connected component of Σ , the number of boundary components in $\{0\} \times S^2$ labeled x_- plus the number of boundary components in $\{1\} \times S^2$ labeled x_+ is 1.
- For each genus 1 connected component of Σ , all boundary components in $\{0\} \times S^2$ are labeled x_+ and all boundary components in $\{1\} \times S^2$ are labeled x_- .

(See [HKK, Formula (12)].) Define $\mathcal{L}(\Sigma)$ to be the set of valid labelings of Σ . The source and target maps of $\mathcal{L}(\Sigma)$ send a labeling of Σ to the induced labeling of the boundary components.

The composition 2-isomorphism $\mathcal{L}(\Sigma_0) \circ \mathcal{L}(\Sigma_1) \xrightarrow{\cong} \mathcal{L}(\Sigma_0 \circ \Sigma_1)$ is obvious except for the situation when one gets a genus 1 cobordism $\Sigma \subset [0, 1] \times S^2$ by stacking two genus 0 cobordisms $\Sigma_0 \subset [0, \frac{1}{2}] \times S^2$ and

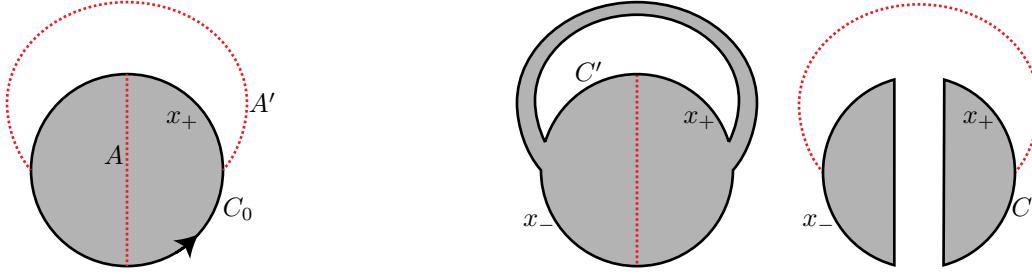


FIGURE 8.1. **The ladybug matching.** Left: The ladybug configuration. Right: Two intermediate configurations, obtained by attaching embedded 1-handles along the two arcs. Black regions are indicated by dark shading.

$\Sigma_1 \subset [\frac{1}{2}, 1] \times S^2$; for simplicity further assume that Σ is connected. Given valid labelings on Σ_0 and Σ_1 that agree on $(\partial\Sigma_0) \cap (\partial\Sigma_2) \subset \{\frac{1}{2}\} \times S^2$, we want to construct a valid labeling on Σ , which essentially amounts to labeling Σ by α or $-\alpha$. It follows from the labeling conditions that there is a unique component C of $(\partial\Sigma_0) \cap (\partial\Sigma_2)$ that is non-separating in Σ and is labeled x_+ . Orient C as the boundary of the black region, and let C_b and C_w be the push-offs of C into the black and the white regions, respectively. One of C_b and C_w is a generator of $H_1((\{0, 1\} \times S^2) \setminus \Sigma)/H_1(\{0, 1\} \times S^2 \setminus \partial\Sigma) \cong \mathbb{Z}^2$ and the other one is zero. If C_b is the generator, label Σ by $[C]$. If C_w is the generator, let D be a curve on Σ , oriented so that the algebraic intersection number $D \cdot C = 1$; and label Σ by $[D]$.

It is not hard to see that choosing the other checkerboard coloring on the link diagram yields a naturally isomorphic functor $\underline{2}^n \rightarrow \mathcal{B}$. On the other hand, one could have required $D \cdot C$ to be -1 instead of 1 ; this would have produced a different functor. This global choice is essentially the choice of ladybug matching from [LS14a, Section 5.4].

Lemma 8.1. *The 2-functor $F_{HKK} : \underline{2}^n \rightarrow \mathcal{B}$ constructed in [HKK] is naturally isomorphic to the 2-functor F_{Kh} constructed in Example 4.5 by applying Construction 4.1 to the Khovanov flow category from [LS14a].*

Proof. The two functors F_{HKK} and F_{Kh} are identical on the objects. By Lemma 2.12, we only need to show that they agree on the edges and that the composition 2-isomorphisms for the two functors agree on the 2-dimensional faces of the cube.

For $u > v \in \{0, 1\}^n$ with $|u| - |v| = 1$, the corresponding embedded cobordism is a merge or split. Let F denote either F_{HKK} or F_{Kh} . It is straightforward from the definitions that for any $x \in F(u)$ and $y \in F(v)$, $s^{-1}(x) \cap t^{-1}(y) \subset F(\varphi_{u,v})$ is empty if x does not appear in the Khovanov differential of y , and consists of one point otherwise, so in particular there is a unique bijection $F_{HKK}(\varphi_{u,v}) \cong F_{Kh}(\varphi_{u,v})$.

Now consider $u > w \in \{0, 1\}^n$ with $|u| - |w| = 2$, and let v, v' be the two intermediate vertices. The composition 2-isomorphisms produce a bijection between $F(\varphi_{v,w}) \circ F(\varphi_{u,v})$ and $F(\varphi_{v',w}) \circ F(\varphi_{u,v'})$. We want to show that these two bijections are the same.

Fix $x \in F(u)$ and $z \in F(w)$, and consider $s^{-1}(x) \cap t^{-1}(z) \subset F(\varphi_{u,w})$. This set can have 0, 1, or 2 elements. The only nontrivial case to check is when the set has 2 elements, which occurs precisely when x and z are related by a ladybug configuration (Figure 8.1).

Therefore, assume x and z are related by a ladybug configuration, and without loss of generality assume that the corresponding embedded genus 1 cobordism Σ is connected. The composition 2-isomorphisms for F are unchanged under isotopy in S^2 : this is [LS14a, Lemma 5.8] for F_{Kh} , and is immediate from the definition for F_{HKK} . Therefore, we may further assume that the ladybug configuration is as shown in Figure 8.1, in the following sense. The circle C_0 is the complete resolution at w , and it is labeled x_+ by z ; and the circle C_1 , the

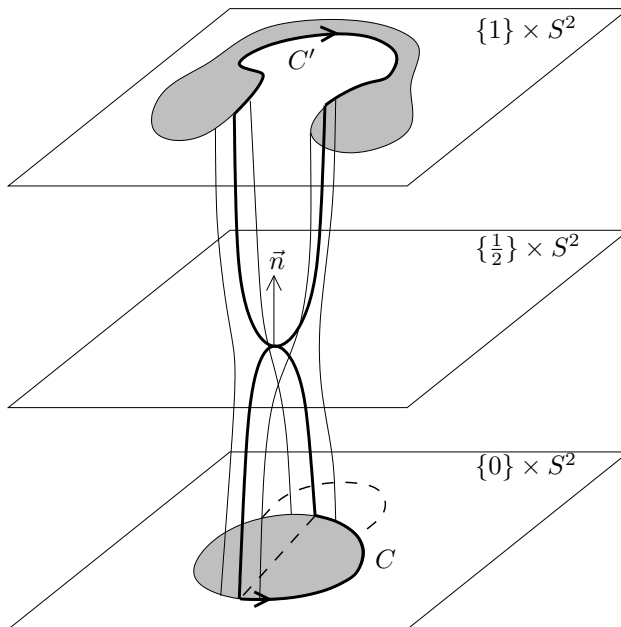


FIGURE 8.2. **The embedded cobordism Σ .** The cobordism connects C_0 at the bottom to C_1 at the top. The portion of the cobordism near the saddle point p , the oriented curves C and C' , and the normal vector \vec{n} are shown.

complete resolution at u , is obtained by attaching embedded 1-handles along the arcs A and A' , and C_1 is labeled x_- by x . The embedded cobordism $\Sigma \subset [0, 1] \times S^2$ connects C_1 to C_0 , that is, $\Sigma \cap (\{i\} \times S^2) = \{i\} \times C_i$. Further, the black region contains the arc A . Consider the labelings of the circles at the complete resolutions at v and v' shown in Figure 8.1; the corresponding generators $y \in F(v)$ and $y' \in F(v')$ are matched by the ladybug matching for F_{Kh} [LS14a, Figure 5.1b].

To show that y and y' are also matched by F_{HKK} , let C (respectively, C') be the circle that is labeled x_+ by y (respectively, y'); therefore, C (respectively, C') is a homology generator of the white (respectively, black) component of the complement of Σ . Isotope the cobordism Σ inside $[0, 1] \times S^2$ relative boundary so that both the saddles occur at $\{\frac{1}{2}\} \times S^2$. Isotope the curves C and C' on Σ so that they intersect transversally at one point, C lies in $[0, \frac{1}{2}] \times S^2$, and C' lies in $[\frac{1}{2}, 1] \times S^2$. Therefore, C and C' intersect at the saddle point p corresponding to the arc A . The portion of the cobordism Σ near p is shown in Figure 8.2. Let \vec{n} be the normal vector to Σ at p pointing away from the black region, and let \vec{t} and \vec{t}' be the tangent vectors to C and C' at p . It is clear from Figure 8.2 that $[\vec{n}, \vec{t}', \vec{t}]$ constitute a positive basis; consequently in Σ , oriented as the boundary of the black region, the intersection number $C' \cdot C = 1$. Therefore, for both y and y' , the surface Σ is labeled by $[C']$, and thus y and y' are matched by F_{HKK} . \square

8.2. Iterated mapping cones. The easiest part of the identification is to see that the Hu-Kriz-Kriz notion of iterated mapping cone agrees with ours. Specifically, to take the iterated mapping cone of their functor $K(F_{HKK}): \underline{2}^n \rightarrow \mathcal{S}$ (see Sections 8.1 and 8.4) they enlarge $\underline{2}^n$ slightly to a category \mathcal{I} , and extend F_{HKK} to a functor $\tilde{F}_{HKK}: \mathcal{I} \rightarrow \mathcal{S}$ by declaring that the new vertices in \mathcal{I} map to $\{*\}$, a one-point space [HKK, Section 5.2]. We observe that their enlargement is the same as $\underline{2}_+^n$:

Lemma 8.2. *There is an isomorphism $\mathcal{I} \cong \underline{2}_+^n$ which commutes with the inclusions of $\underline{2}^n$:*

$$\begin{array}{ccc} \mathcal{I} & \xrightarrow{\cong} & \underline{2}_+^n \\ & \searrow & \nearrow \\ & \underline{2}^n & \end{array}$$

Proof. The category \mathcal{I} has, as objects, pairs $(J \subseteq \{1, \dots, n\}, \phi: J \rightarrow \{0, 1\})$, and there is a morphism $\phi \rightarrow \psi$ (which is unique) if and only if ϕ is a restriction of ψ . The cube $\underline{2}^n$ sits in \mathcal{I} as the subcategory $\{(J, \phi) \mid 0 \notin \text{im}(\phi)\}$. Recall that $\underline{2}_+^n = (\underline{2}_+^1)^n$, and $\text{Ob}(\underline{2}_+^1) = \{0, 1, *\}$. Given an object $o = (v_1, \dots, v_n) \in \underline{2}_+^n$, define $J = \{i \mid v_i \in \{0, *\}\}$ and $\phi(i) = \begin{cases} 0 & v_i = * \\ 1 & v_i = 0 \end{cases}$. With this dictionary, the rest of the verification is straightforward. \square

8.3. Another kind of homotopy coherent diagram. Hu-Kriz-Kriz use a slightly different notion from Vogt of homotopy coherent diagrams (Section 2.9). It is defined in two steps. First, given a small category \mathcal{C} , let \mathcal{C}' be the 2-category with the same objects as \mathcal{C} ,

$$\text{Hom}_{\mathcal{C}'}(x, y) = \coprod_{x=x_0, x_1, \dots, x_n=y} \text{Hom}_{\mathcal{C}}(x_{n-1}, x_n) \times \text{Hom}_{\mathcal{C}}(x_{n-2}, x_{n-1}) \times \cdots \times \text{Hom}_{\mathcal{C}}(x_0, x_1)$$

the set of finite sequences of composable morphisms starting at x and ending at y , and a unique 2-morphism from (f_n, \dots, f_1) to (g_m, \dots, g_1) whenever $f_n \circ \cdots \circ f_1 = g_m \circ \cdots \circ g_1$ (compare [HKK, Section 4.1]). Composition is given by concatenation of sequences. There is a projection $\mathcal{C}' \rightarrow \mathcal{C}$, where we view \mathcal{C} as a 2-category with only identity 2-morphisms, which sends (f_n, \dots, f_1) to $f_n \circ \cdots \circ f_1$.

Second, given a 2-category \mathcal{D} , we can form a topological category $B_2\mathcal{D}$ by replacing each Hom category in \mathcal{D} by its nerve.

Combining these notions, the analogue of a homotopy coherent diagram in [HKK] is a continuous functor $B_2(\mathcal{C}') \rightarrow \mathcal{S}$, or more generally an A_∞ functor $B_2(\mathcal{C}') \rightarrow \mathcal{S}$. Note that there is a projection $B_2(\mathcal{C}') \rightarrow \mathcal{C}$ induced by the projection $\mathcal{C}' \rightarrow \mathcal{C}$ (and the triviality that $B_2(\mathcal{C}) = \mathcal{C}$). So, given a functor $F: \mathcal{C} \rightarrow \mathcal{S}$ (say) there is an induced continuous functor $B_2(F'): B_2(\mathcal{C}') \rightarrow \mathcal{S}$.

Lemma 8.3. *For any small category \mathcal{C} , the projection map $B_2(\mathcal{C}') \rightarrow \mathcal{C}$ is a homotopy equivalence on each Hom space.*

Proof. The category $\text{Hom}_{\mathcal{C}'}(x, y)$ decomposes as a disjoint union of subcategories

$$\text{Hom}_{\mathcal{C}'}(x, y) = \coprod_{f \in \text{Hom}(x, y)} \{(f_n, \dots, f_1) \mid f_n \circ \cdots \circ f_1 = f\},$$

and for each of these subcategories, every object is initial. \square

The notion of homotopy colimits extends easily to continuous functors from topological categories: in Formula (2.18), say, one replaces the disjoint union over sequences of composable morphisms with the disjoint union of products $\coprod_{x_0, \dots, x_n} \text{Hom}(x_{n-1}, x_n) \times \cdots \times \text{Hom}(x_0, x_1) \times [0, 1]^n \times F(x_0)$, and quotient by the same equivalence relation from Definition 2.17. The properties of homotopy colimits stated in Section 2.9 extend without change to continuous diagrams from topological categories.

Lemma 8.4. *Let $F: \mathcal{C} \rightarrow \mathcal{S}$ be a diagram from a small category \mathcal{C} . Then there is a homotopy equivalence $\text{hocolim}_{\mathcal{C}} F \simeq \text{hocolim}_{B_2(\mathcal{C}')} B_2(F')$.*

Proof. Writing Π for the projection $B_2(\mathcal{C}') \rightarrow \mathcal{C}$, we can factor $B_2(F') = F \circ \Pi$. By Lemma 8.3, the projection Π is a quasi-equivalence, and in particular homotopy cofinal, so the result follows from Property (ho-4) of homotopy colimits. \square

8.4. The Elmendorff-Mandell machine. The Elmendorff-Mandell machine [EM06] is a functor K from permutative categories to spectra, though constructions of this type go back to Segal [Seg74]. Rather than explain how the machine works, we list the properties we will need:

- (EM-1) Given a permutative category \mathcal{C} and an object x in \mathcal{C} there is an induced map $K(x): \mathbb{S} \rightarrow K(\mathcal{C})$. Further, this is natural in the sense that given a functor of permutative categories $F: \mathcal{C} \rightarrow \mathcal{D}$ the following diagram commutes:

$$\begin{array}{ccc} \mathbb{S} & & \\ K(x) \downarrow & \searrow^{K(F(x))} & \\ K(\mathcal{C}) & \xrightarrow{K(F)} & K(\mathcal{D}). \end{array}$$

- (EM-2) Given permutative categories \mathcal{C} and \mathcal{D} , $K(\mathcal{C} \times \mathcal{D}) = K(\mathcal{C}) \times K(\mathcal{D})$. (Note that the Cartesian product is both the categorical product and coproduct in the category of permutative categories.)
- (EM-3) If $F: \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence of permutative categories then $K(F): K(\mathcal{C}) \rightarrow K(\mathcal{D})$ is a stable homotopy equivalence.
- (EM-4) Note that the category **Sets** of finite sets, with disjoint union, is equivalent to a permutative category [Isb69]; to keep the exposition clear we will continue to use the name **Sets** for this category. The map $\mathbb{S} \rightarrow K(\mathbf{Sets})$ induced by a 1-element set (and property (EM-1)) is a stable homotopy equivalence. (This is a version of the Barratt-Priddy-Quillen theorem.)

In fact, the category **Permu** of permutative categories is a 2-category. Given a 2-category \mathcal{C} in which all 2-morphisms are isomorphisms and a functor $F: \mathcal{C} \rightarrow \mathbf{Permu}$, there is an induced continuous functor $K(F): B_2(\mathcal{C}) \rightarrow \mathcal{S}$.

Given a set X , we can consider the category $\prod_{x \in X} \mathbf{Sets}$. Given a correspondence (C, s, t) from X to Y , there is an induced functor $\prod_{x \in X} \mathbf{Sets} \rightarrow \prod_{y \in Y} \mathbf{Sets}$ which sends

$$(A_x)_{x \in X} \mapsto \left(\bigcup_{x \in X} (s^{-1}(x) \cap t^{-1}(y)) \times A_x \right)_{y \in Y}.$$

(Note that the union operation in the above formula is actually a disjoint union.) An isomorphism between correspondences (C, s, t) and (C', s', t') can be viewed as simply a relabeling of the elements of C ; and this relabeling induces a natural isomorphism between the two functors. One can verify, after some work, that this defines a lax 2-functor $\mathcal{B} \rightarrow \mathbf{Permu}$. (One method to carry out this verification is to note that this category is naturally isomorphic to the category \mathbf{Sets}/X , and that under this identification the functor induced by C is naturally isomorphic to the functor $A \mapsto C \times_X A$.) Note that using properties (EM-2) and (EM-4), $K(\prod_{x \in X} \mathbf{Sets}) \simeq \prod_{x \in X} K(\mathbf{Sets}) \simeq \prod_{x \in X} \mathbb{S}$. With respect to this decomposition, however, the map $\prod_{x \in X} \mathbb{S} \rightarrow \prod_{y \in Y} \mathbb{S}$ induced by a correspondence $C: X \rightarrow Y$ is not obvious.

Now, given a category \mathcal{C} and a strictly unitary, lax 2-functor $F: \mathcal{C} \rightarrow \mathcal{B}$ there is an induced strict 2-functor $F': \mathcal{C}' \rightarrow \mathcal{B}$. Composing with the $\prod_{x \in (-)} \mathbf{Sets}$ construction gives a strict 2-functor $\mathcal{C}' \rightarrow \mathbf{Permu}$, which we will still denote F' , with $F'(u) = \prod_{x \in F(u)} \mathbf{Sets}$. Finally, applying the K -theory functor gives a functor $K(B_2(F')): B_2(\mathcal{C}') \rightarrow \mathcal{S}$. Hu-Kriz-Kriz apply this construction to the Khovanov functor $F_{HKK}: \underline{2}^n \rightarrow \mathcal{B}$.

To identify this construction and the thickening construction from Section 4 we use a sequence of intermediate diagrams, somewhat in the spirit of Section 5.3:

- (1) By Lemma 5.11, we can lift the functor $F: \underline{2}^n \rightarrow \mathcal{B}$ to a functor $\vec{F}: \text{Arr}(\underline{2}^n) \rightarrow \mathcal{B}$, so that $F = \vec{F} \circ A$. Composing with the composition map $B: \widehat{\underline{2}}^n \rightarrow \text{Arr}(\underline{2}^n)$ gives a functor $\vec{F} \circ B: \widehat{\underline{2}}^n \rightarrow \mathcal{B}$.
- (2) There is a homotopy equivalence $\text{hocolim } K(B_2(F'))^\dagger \simeq \text{hocolim } K(B_2((\vec{F} \circ B)'))^\dagger$; the argument is similar to the one in Section 5.3. By Lemma 5.9, the inclusion map $A_\dagger: \underline{2}^n_\dagger \rightarrow \text{Arr}(\underline{2}^n)_\dagger$ is homotopy cofinal, and by Lemma 5.11 we have $\vec{F} \circ A = F$. Hence, Property (ho-4) of homotopy colimits, Property (EM-3) of the K -theory functor, and Lemma 8.3 imply that $\text{hocolim } K(B_2(F'))^\dagger \simeq \text{hocolim } K(B_2(\vec{F}'))^\dagger$. By Lemma 5.10, the projection $B_\dagger: \widehat{\underline{2}}^n_\dagger \rightarrow \text{Arr}(\underline{2}^n)_\dagger$ is also homotopy cofinal, so Property (ho-4) of homotopy colimits, Property (EM-3) of the K -theory functor, and Lemma 8.3 imply that $\text{hocolim } K(B_2(\vec{F}'))^\dagger \simeq \text{hocolim } K(B_2((\vec{F} \circ B)'))^\dagger$.
- (3) Recall that \widehat{F} sends an object $u \xrightarrow{f} v \xrightarrow{g} w$ to $\bigvee_{a \in F(f)} \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} \mathbb{S}$. Further, the definition of \widehat{F} uses only the universal properties of product and coproduct. Thus, for *any* spectrum X we could define a functor $\widehat{F}_X: \widehat{\underline{2}}^n \rightarrow \mathcal{S}$ with $\widehat{F}_X(u \xrightarrow{f} v \xrightarrow{g} w) = \bigvee_{a \in F(f)} \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} X$. Moreover, this thickening procedure is clearly natural in X . Taking the special case $X = K(\text{Sets})$ and using Property (EM-4) gives a diagram $G: \widehat{\underline{2}}^n \rightarrow \mathcal{S}$ with

$$G(u \xrightarrow{f} v \xrightarrow{g} w) = \bigvee_{a \in F(f)} \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} K(\text{Sets}),$$

and a natural transformation of diagrams $\widehat{F} \rightarrow G$ which is an equivalence on objects. In particular, $\text{hocolim } \widehat{F}^\dagger \simeq \text{hocolim } G^\dagger$.

- (4) There is also a functor $G_p: \widehat{\underline{2}}^n \rightarrow \text{Permu}$ defined similarly to \widehat{F} , using the product and coproduct on Permu in place of the product and wedge sum of spaces, and using Sets in place of \mathbb{S} . That is,

$$G_p(u \xrightarrow{f} v \xrightarrow{g} w) = \prod_{a \in F(f)} \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} \text{Sets}.$$

Recall that finite products and coproducts of permutative categories are given by the Cartesian product. Thus, using Property (EM-2),

$$K(G_p)(u \xrightarrow{f} v \xrightarrow{g} w) = \prod_{a \in F(f)} \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} K(\text{Sets}).$$

Using the stable homotopy equivalence between wedge sum and product we get a natural transformation of diagrams from G to $K(G_p)$ which is an equivalence on objects. In particular, $\text{hocolim } G^\dagger \simeq \text{hocolim } K(G_p)^\dagger$.

- (5) By Lemma 8.4, there is a homotopy equivalence $\text{hocolim } K(G_p)^\dagger \simeq \text{hocolim } K(B_2(G'_p))^\dagger$.
- (6) To finish the identification, there are isomorphisms H from G'_p to $(\vec{F} \circ B)'$. On the objects, the natural transformation will induce the equivalence from

$$G'_p(u \xrightarrow{f} v \xrightarrow{g} w) = G_p(u \xrightarrow{f} v \xrightarrow{g} w) = \prod_{a \in F(f)} \prod_{\substack{b \in F(g) \\ s(b)=t(a)}} \text{Sets}$$

to

$$(\vec{F} \circ B)'(u \xrightarrow{f} v \xrightarrow{g} w) = (\vec{F} \circ B)(u \xrightarrow{f} v \xrightarrow{g} w) = \prod_{c \in F(g \circ f)} \text{Sets}$$

which comes from the natural bijection $F(g \circ f) \rightarrow F(g) \times_{F(v)} F(f)$ between their indexing sets, and the identification of coproducts and products in \mathbf{Permu} with Cartesian products. This does not strictly define a strict natural isomorphism $G'_p \rightarrow (\vec{F} \circ B)'$, but instead a pseudonatural equivalence: for a map $\varphi: (u \xrightarrow{f} v \xrightarrow{g} w) \rightarrow (u' \xrightarrow{f'} v' \xrightarrow{g'} w')$ in $\widehat{\underline{2}}^n$, there is a natural isomorphism of functors

$$(\vec{F} \circ B)'(\varphi) \circ H(u \xrightarrow{f} v \xrightarrow{g} w) \cong H(u' \xrightarrow{f'} v' \xrightarrow{g'} w') \circ G'_p(\varphi).$$

which respects composition in $\widehat{\underline{2}}^n$.

We define \mathcal{E} to be the category with objects 1 and 0 and a unique morphism between any pair of objects. Let $\mathcal{D} = \mathcal{E} \times \widehat{\underline{2}}^n_{\dagger}$, and let \mathcal{D}_0 and \mathcal{D}_1 be the full subcategories of \mathcal{D} spanned by the objects of $\{0\} \times \widehat{\underline{2}}^n_{\dagger}$ and $\{1\} \times \widehat{\underline{2}}^n_{\dagger}$, respectively. Note that the projection $\mathcal{D} \rightarrow \widehat{\underline{2}}^n_{\dagger}$ is an equivalence, and restricts to isomorphisms from both \mathcal{D}_0 and \mathcal{D}_1 to $\widehat{\underline{2}}^n_{\dagger}$.

The pseudonatural equivalence above defines a lax 2-functor, also denoted H , from \mathcal{D} to \mathbf{Permu} whose restriction to $\{1\} \times \widehat{\underline{2}}^n$ is G'_p and whose restriction to $\{0\} \times \widehat{\underline{2}}^n$ is $(\vec{F} \circ B)'$.

Therefore, by Property (EM-3), the maps $\mathcal{D}_0 \rightarrow \mathcal{D} \leftarrow \mathcal{D}_1$ give us a diagram

$$K(B_2(G'_p))^\dagger \rightarrow K(B_2(H)) \leftarrow K(B_2(\vec{F} \circ B)')^\dagger$$

where both arrows are homotopy equivalences of diagrams (in the sense of Vogt [Vog73]). Taking homotopy colimits, Proposition 2.20 gives

$$\mathrm{hocolim} K(B_2((\vec{F} \circ B)'))^\dagger \simeq \mathrm{hocolim} K(B_2(G'_p))^\dagger.$$

To summarize:

Proposition 8.5. *Given a strictly unitary 2-functor $F: \underline{2}^n \rightarrow \mathcal{B}$, there is a stable homotopy equivalence between the Hu-Kriz-Kriz realization $\mathrm{hocolim} K(B_2(F'))^\dagger$ and the realization $\mathrm{hocolim} \widehat{F}^\dagger$ from Section 4.3.*

Proof. By Lemma 8.2, the Hu-Kriz-Kriz realization is $\mathrm{hocolim}_{\widehat{\underline{2}}^n} K(B_2(F'))^\dagger$. By Step (2) (which uses Step (1)), $\mathrm{hocolim} K(B_2(F'))^\dagger \simeq \mathrm{hocolim} K(B_2((\vec{F} \circ B)'))^\dagger$. By Step (6), $\mathrm{hocolim} K(B_2((\vec{F} \circ B)'))^\dagger \simeq \mathrm{hocolim} K(B_2(G'_p))^\dagger$. By Step (5), $\mathrm{hocolim} K(B_2(G'_p))^\dagger \simeq \mathrm{hocolim} K(G_p)^\dagger$. By Step (4), $\mathrm{hocolim} K(G_p)^\dagger \simeq \mathrm{hocolim} G^\dagger$. Finally, by Step (3), $\mathrm{hocolim} G^\dagger \simeq \mathrm{hocolim} \widehat{F}^\dagger$. \square

8.5. Proof that the Khovanov homotopy types agree.

Proof of Theorem 7. Lemma 8.1 identifies the 2-functors $\underline{2}^n \rightarrow \mathcal{B}$ used in this paper and [HKK]. Proposition 8.5 identifies Hu-Kriz-Kriz's realization of this functor with ours. \square

9. KHOVANOV HOMOTOPY TYPE OF A DISJOINT UNION AND CONNECTED SUM

In this section we prove Theorems 1, 2, and 8. For the first two theorems, we merely need to show that the functor associated to a disjoint union (respectively connect sum) of links is the product of the functors of the individual links:

Proposition 9.1. *Let L_1 and L_2 be link diagrams, and let $L_1 \amalg L_2$ be their disjoint union. Order the crossings in $L_1 \amalg L_2$ so that all of the crossings in L_1 come before all of the crossings in L_2 . Then*

$$F_{Kh}^j(L_1 \amalg L_2) \cong \coprod_{j_1+j_2=j} F_{Kh}^{j_1}(L_1) \times F_{Kh}^{j_2}(L_2),$$

where \times denotes the product of functors (Definition 4.20), \amalg denotes the disjoint union of functors (Definition 4.24), and \cong denotes natural isomorphism of 2-functors. If we fix a basepoint on L_1 then

$$F_{\widetilde{Kh}}^j(L_1 \amalg L_2) \cong \coprod_{j_1+j_2=j} F_{\widetilde{Kh}}^{j_1}(L_1) \times F_{\widetilde{Kh}}^{j_2}(L_2).$$

If we fix basepoints on L_1 and L_2 and let $L_1 \# L_2$ denote the connected sum (at the basepoints) then

$$F_{\widetilde{Kh}}^j(L_1 \# L_2) \cong \coprod_{j_1+j_2=j} F_{\widetilde{Kh}}^{j_1}(L_1) \times F_{\widetilde{Kh}}^{j_2}(L_2).$$

Proof. We will prove the first statement; the proofs of the other two are similar. Let n_i be the number of crossings in L_i . To keep notation simple, write $F = F_{\widetilde{Kh}}^j(L_1 \amalg L_2)$, $X_v = F(v)$, $A_{v,w} = F(\varphi_{v,w})$, $G = \coprod_{j_1+j_2=j} F_{\widetilde{Kh}}^{j_1}(L_1) \times F_{\widetilde{Kh}}^{j_2}(L_2)$, $Y_v = G(v)$ and $B_{v,w} = G(\varphi_{v,w})$.

By Lemma 2.12, it suffices to construct bijections $\phi_v: X_v \xrightarrow{\cong} Y_v$ and $\psi_{v,w}: A_{v,w} \xrightarrow{\cong} B_{v,w}$ for all $v > w$ with $|v| - |w| = 1$ so that $\psi_{v,w}$ respects the source and target maps and for any $u > w$ with $|u| - |w| = 2$, the following diagram commutes:

$$(9.2) \quad \begin{array}{ccc} A_{v,w} \times_{X_v} A_{u,v} & \xrightarrow{\psi_{v,w} \times \psi_{u,v}} & B_{v,w} \times_{Y_v} B_{u,v} \\ \downarrow F_{u,v,w} & & \downarrow G_{u,v,w} \\ A_{u,w} & & B_{u,w} \\ \uparrow F_{u,v',w} & & \uparrow G_{u,v',w} \\ A_{v',w} \times_{X_{v'}} A_{u,v'} & \xrightarrow{\psi_{v',w} \times \psi_{u,v'}} & B_{v',w} \times_{Y_{v'}} B_{u,v'} \end{array}$$

Here, v and v' are the two vertices so that $u > v, v' > w$. Note that all arrows in this diagram are isomorphisms.

The map ϕ_v is the canonical identification between Khovanov generators for $L_1 \amalg L_2$ and pairs of a Khovanov generator for L_1 and a Khovanov generator for L_2 . There is a unique map $\psi_{v,w}: A_{v,w} \rightarrow B_{v,w}$ for $v > w$ with $|v| - |w| = 1$ which commutes with the source and target maps, because:

- (1) Given $x_v \in X_v$ and $x_w \in X_w$, $s^{-1}(x_v) \cap t^{-1}(x_w) \subset A_{v,w}$ is either empty (if x_v does not occur in the Khovanov differential of x_w) or consists of a single point (if x_v does occur in the Khovanov differential of x_w). Similar statements hold for Y_v and $B_{v,w}$. It follows that if $\psi_{v,w}$ exists then it is unique.
- (2) The canonical identification of Khovanov generators does, in fact, give a chain map. So, by the observations in the previous point, the map $\psi_{v,w}$ does exist.

Except in one case, the same argument shows that the diagram (9.2) commutes: typically, for each $x_u \in X_u$ and $x_w \in X_w$ (with $u > w$ and $|u| - |w| = 2$), $s^{-1}(x_u) \cap t^{-1}(x_w) \subset A_{u,w}$ is either empty or has a single element. The exceptional case is the case of a ladybug configuration, as in [LS14a, Section 5.4], see also Figure 8.1. In the ladybug case, either both crossings under consideration lie in L_1 or both crossings lie in L_2 , from which it follows easily that the diagram commutes. (This is immediate for the present case when we are considering the disjoint union $L_1 \amalg L_2$; the connect-sum case $L_1 \# L_2$ is also fairly obvious.) \square

Proof of Theorems 1 and 2. We will prove Formula (1.1); the proofs of Formulas (1.2) and (1.3) are essentially the same.

Fix a diagram for $L_1 \amalg L_2$ so that there are no crossings between L_1 and L_2 . Order the crossings in $L_1 \amalg L_2$ so that all of the crossings in L_1 come before all of the crossings in L_2 . By Proposition 9.1,

$$F_{Kh}^j(L_1 \amalg L_2) \cong \amalg_{j_1+j_2=j} F_{Kh}^{j_1}(L_1) \times F_{Kh}^{j_2}(L_2).$$

By Lemma 4.19, naturally isomorphic functors have stably homotopy equivalent realizations. By Propositions 4.23 and 4.27, the realization of $\amalg_{j_1+j_2=j} F_{Kh}^{j_1}(L_1) \times F_{Kh}^{j_2}(L_2)$ is $\bigvee_{j_1+j_2=j} \mathcal{X}_{Kh}^{j_1}(K_1) \wedge \mathcal{X}_{Kh}^{j_2}(K_2)$. \square

Proof of Corollary 1.4. For the first statement, consider the disjoint union L_n of n copies of the left-handed trefoil T . It follows from [LS14a, Proposition 9.2] that

$$\mathcal{X}_{Kh}(T) \simeq \Sigma^{-3}\mathbb{S} \vee \Sigma^{-2}\mathbb{S} \vee \mathbb{S} \vee \Sigma^{-4}\mathbb{RP}^2$$

(compare [LS14a, Example 9.4]). So, by Theorem 1,

$$\mathcal{X}_{Kh}(L_n) \simeq \Sigma^{-4n} \overbrace{(\mathbb{RP}^2 \wedge \cdots \wedge \mathbb{RP}^2)}^{n \text{ copies}} \vee Y$$

for some space Y . It follows from the Cartan formula that

$$\text{Sq}^n : H^n((\mathbb{RP}^2)^{\wedge n}; \mathbb{Z}/2\mathbb{Z}) \rightarrow H^{2n}((\mathbb{RP}^2)^{\wedge n}; \mathbb{Z}/2\mathbb{Z})$$

is non-trivial. This proves the first part of the result.

For the second part of the result, let K be the knot 15^2_{41127} and let K_n be the connect sum of n copies of K . According to the calculation in [Shu11, Figure 6], $\widetilde{Kh}^{-2,0}(K; \mathbb{Z}) \cong \mathbb{Z}$, $\widetilde{Kh}^{0,0}(K; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, and $\widetilde{Kh}^{i,0}(K; \mathbb{Z}) = 0$ for $i \neq -2, 0$. In particular, there is a class $\alpha \in \widetilde{Kh}^{-1,0}(K; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ so that $\text{Sq}^1(\alpha)$ is non-zero and $\text{Sq}^i(\alpha) = 0$ for $i > 1$. So, it follows from Theorem 2 and the Cartan formula that for the class $\beta = \alpha \wedge \cdots \wedge \alpha \in \widetilde{Kh}^{-n,0}(K_n)$, $\text{Sq}^n(\beta)$ is non-trivial. \square

Finally, we turn to the unreduced Khovanov homology of a connected sum. Consider the Khovanov homotopy type associated to the unknot, $\mathcal{X}_{Kh}(U) = \mathbb{S} \vee \mathbb{S}$, which is the suspension spectrum of $S^0 \vee S^0 = \{*, p_-, p_+\}$. The space $\mathbb{H}^1 := \mathcal{X}_{Kh}(U)$ has a product $\mu: \mathbb{H}^1 \wedge \mathbb{H}^1 \rightarrow \mathbb{H}^1$ induced by

$$p_- \wedge p_- \mapsto p_- \quad p_+ \wedge p_- \mapsto p_+ \quad p_- \wedge p_+ \mapsto p_+ \quad p_+ \wedge p_+ \mapsto *.$$

The induced map on reduced cohomology is the split map $Kh(U) \rightarrow Kh(U) \otimes Kh(U)$. (The generators of $Kh(U)$ are x_- corresponding to p_- and x_+ corresponding to p_+ .) The operation μ makes \mathbb{H}^1 into a ring spectrum. (The notation \mathbb{H}^1 is chosen to be reminiscent of the first of Khovanov's arc algebras H^n ; by a slight abuse of notation, we will sometimes view \mathbb{H}^1 as the 0-dimensional CW complex $\{*, p_-, p_+\}$.)

Next, given any n -crossing link diagram K and a basepoint $p \in K$, we make $\mathcal{X}_{Kh}(K)$ into a module spectrum over \mathbb{H}^1 . For concreteness, let us assume that $\mathcal{X}_{Kh}(K)$ is constructed via the box map realization from Section 5: that is, we start with the Khovanov functor $F_{Kh}: \underline{2}^n \rightarrow \mathcal{B}$; for some sufficiently large k (that will be suppressed from our notation), we construct a k -dimensional spacial refinement of F_{Kh} , which is a homotopy coherent diagram $\widetilde{F}_{Kh}: \underline{2}^n \rightarrow \mathbf{Top}_\bullet$; and then, after adding an extra object to get $\widetilde{F}_{Kh}^+: \underline{2}_+^n \rightarrow \mathbf{Top}_\bullet$, we define $\mathcal{X}_{Kh}(K)$ to be $\text{hocolim}(\widetilde{F}_{Kh}^+)$, modulo some desuspension (which we also suppress). Further assume that $\mathcal{X}_{Kh}(K)$ carries the CW complex structure from Proposition 6.1; the cells correspond to the Khovanov generators, which gives an identification between the reduced cellular cochain complex of $\mathcal{X}_{Kh}(K)$ and the Khovanov chain complex $\mathcal{C}_{Kh}(K)$.

We will construct the k -dimensional spacial refinement \widetilde{F}_{Kh} in a specific manner. Towards that end, let us recall the reduced Khovanov functors $F_{\pm Kh}: \underline{2}^n \rightarrow \mathcal{B}$. For $u \in \{0, 1\}^n$, define $F_{+Kh}(u)$ (respectively, $F_{-Kh}(u)$) to be the subset of $F_{Kh}(u)$ where the circle in the complete resolution $\mathcal{P}(u)$ containing the basepoint is labeled x_+ (respectively, x_-); for $u > v \in \{0, 1\}^n$, define the correspondence from $F_{+Kh}(u)$

to $F_{+Kh}(v)$ (respectively, from $F_{-Kh}(u)$ to $F_{-Kh}(v)$) to be the subset $s^{-1}(F_{+Kh}(u)) \cap t^{-1}(F_{+Kh}(v))$ (respectively, $s^{-1}(F_{-Kh}(u)) \cap t^{-1}(F_{-Kh}(v))$) of the correspondence from $F_{Kh}(u)$ to $F_{Kh}(v)$. It is straightforward from the definition of the Khovanov differential (Section 2.3) that this produces well-defined functors $F_{\pm Kh}: \underline{2}^n \rightarrow \mathcal{B}$. Furthermore, the map from $F_{+Kh}(u)$ to $F_{-Kh}(u)$ which relabels the pointed circle in $\mathcal{P}(u)$ from x_+ to x_- induces an isomorphism from F_{+Kh} to F_{-Kh} ; we often write $F_{\widetilde{Kh}}$ to denote either functor. Finally, for any $u \in \{0, 1\}^n$, $F_{Kh}(u) = F_{+Kh}(u) \amalg F_{-Kh}(u)$; and for any $u > v \in \{0, 1\}^n$, the correspondence $F_{Kh}(\varphi_{u,v})$ from $F_{Kh}(u)$ to $F_{Kh}(v)$ is the disjoint union of the correspondence $F_{+Kh}(\varphi_{u,v})$ from $F_{+Kh}(u)$ to $F_{+Kh}(v)$, the correspondence $F_{-Kh}(\varphi_{u,v})$ from $F_{-Kh}(u)$ to $F_{-Kh}(v)$, and some correspondence from $F_{-Kh}(u)$ to $F_{+Kh}(v)$.

We construct the spacial refinement \widetilde{F}_{Kh} of F_{Kh} in several steps. First construct a spacial refinement \widetilde{F}_{-Kh} of F_{-Kh} with the additional restriction that the box maps come from the subspaces $E^\circ(\{B_x\}, s)$ of $E(\{B_x\}, s)$, that is, the sub-boxes are contained in the interiors of the bigger boxes. Then use the natural isomorphism between F_{+Kh} and F_{-Kh} to get a spacial refinement \widetilde{F}_{+Kh} of F_{+Kh} . This ensures that the CW complexes $\text{hocolim}(\widetilde{F}_{+Kh}^+)$ and $\text{hocolim}(\widetilde{F}_{-Kh}^+)$ are canonically isomorphic. Finally, extend \widetilde{F}_{+Kh} and \widetilde{F}_{-Kh} to construct a spacial refinement \widetilde{F}_{Kh} of F_{Kh} , following the inductive argument in the proof of Proposition 5.2 (1). For the induction step, fix a length- ℓ sequence $v_0 \rightarrow \cdots \rightarrow v_\ell$ of non-identity morphisms in $\underline{2}^n$. There is a correspondence $F_{Kh}(\varphi_{v_0, v_\ell})$ and a subset $F_{+Kh}(\varphi_{v_0, v_\ell}) \amalg F_{-Kh}(\varphi_{v_0, v_\ell})$; let s be the source map of the correspondence $F_{Kh}(\varphi_{v_0, v_\ell})$ and s' the restriction of s to $F_{+Kh}(\varphi_{v_0, v_\ell}) \amalg F_{-Kh}(\varphi_{v_0, v_\ell})$. Induction and \widetilde{F}_{+Kh} and \widetilde{F}_{-Kh} give a diagram

$$\begin{array}{ccc} \partial([0, 1]^{\ell-1}) & \longrightarrow & E^\circ(\{B_x\}, s) \\ \downarrow & & \downarrow \\ [0, 1]^{\ell-1} & \longrightarrow & E^\circ(\{B_x\}, s'), \end{array}$$

where the right-hand vertical map forgets the boxes labeled by elements of $F_{Kh}(\varphi_{v_0, v_\ell}) \setminus (F_{+Kh}(\varphi_{v_0, v_\ell}) \amalg F_{-Kh}(\varphi_{v_0, v_\ell}))$. The inductive step is to construct a lift $[0, 1]^{\ell-1} \rightarrow E^\circ(\{B_x\}, s)$ making the diagram commute. Lemma 2.30 guarantees the existence of such a lift. Thus, induction implies that F_{Kh} has a spacial refinement extending \widetilde{F}_{+Kh} and \widetilde{F}_{-Kh} . We will call a spacial refinement \widetilde{F}_{Kh} so that the induced refinements \widetilde{F}_{+Kh} and \widetilde{F}_{-Kh} agree (as above) a *pointed spacial refinement*.

Given a pointed spacial refinement, define a map $\Psi: \mathcal{X}_{Kh}(K) \rightarrow \mathcal{X}_{Kh}(K)$ as follows. Notice that $\text{hocolim}(\widetilde{F}_{+Kh}^+)$ is a subcomplex of $\text{hocolim}(\widetilde{F}_{Kh}^+) = \mathcal{X}_{Kh}(K)$ and $\text{hocolim}(\widetilde{F}_{-Kh}^+)$ is the corresponding quotient complex. Define Ψ to be the composition

$$\mathcal{X}_{Kh}(K) \twoheadrightarrow \text{hocolim}(\widetilde{F}_{-Kh}^+) \xrightarrow{\cong} \text{hocolim}(\widetilde{F}_{+Kh}^+) \hookrightarrow \mathcal{X}_{Kh}(K),$$

where the first map is the quotient map, the second map is the canonical isomorphism, and the third map is the subcomplex inclusion. Note that Ψ is a cellular map. The induced map on $\mathcal{C}_{Kh}(K)$, the reduced cellular cochain complex of $\mathcal{X}_{Kh}(K)$, sends generators that label the pointed circle by x_- to zero and on the rest of the Khovanov generators relabels the pointed circle from x_+ to x_- .

Now we are ready to define the \mathbb{H}^1 -module structure on $\mathcal{X}_{Kh}(K)$. That is, we will define a map $\mathcal{X}_{Kh}(K) \wedge \{*, p_-, p_+\} = (\mathcal{X}_{Kh}(K) \times \{p_-\}) \vee (\mathcal{X}_{Kh}(K) \times \{p_+\})$ to $\mathcal{X}_{Kh}(K)$. On the first summand, the map $\mathcal{X}_{Kh}(K) \times \{p_-\} \rightarrow \mathcal{X}_{Kh}(K)$ is the projection to the first factor. On the second summand, the map is projection to the first factor composed with the map $\Psi: \mathcal{X}_{Kh}(K) \rightarrow \mathcal{X}_{Kh}(K)$ defined above. Since $\Psi \circ \Psi$ sends all of $\mathcal{X}_{Kh}(K)$ to the basepoint, $\mathcal{X}_{Kh}(K)$ becomes a strict module spectrum over \mathbb{H}^1 . Note that \mathbb{H}^1 is commutative, so we can view the action of \mathbb{H}^1 on $\mathcal{X}_{Kh}(K)$ as either a left or a right action. Further note that the induced map on the reduced cellular cochain complexes is the split map $\mathcal{C}_{Kh}(K) \rightarrow \mathcal{C}_{Kh}(K) \otimes Kh(U)$.

Proposition 9.3. *The quasi-isomorphism type of the \mathbb{H}^1 -module spectrum $\mathcal{X}_{Kh}(K)$ is an invariant of pointed links. That is, if (K, p) and (K', p') are pointed link diagrams representing isotopic pointed links, then there exist \mathbb{H}^1 -module spectra $\mathcal{X}_{Kh}(K) = X_0, X_1, \dots, X_{\ell-1}, X_\ell = \mathcal{X}_{Kh}(K')$, and for any adjacent pair X_i, X_{i+1} , either a map $X_i \rightarrow X_{i+1}$ or a map $X_{i+1} \rightarrow X_i$, which is both an \mathbb{H}^1 -module map and a stable homotopy equivalence.*

Proof. We first observe that the \mathbb{H}^1 -module structure is independent of the choice of box maps. The proof is essentially the same as Proposition 5.2 (2), but using Lemma 2.30 instead of Lemma 2.29.

Next we show that the quasi-isomorphism type of the \mathbb{H}^1 -module spectrum is invariant under Reidemeister moves. Following the standard argument from [Kho03, Section 3], we only need to consider Reidemeister moves that do not cross the marked point p . We follow the framework from [LS14a, Section 6]. Let K_0 and K_1 (with n_0 and n_1 crossings respectively) be pointed link diagrams related by any of the three Reidemeister moves of [LS14a, Figure 6.1], and assume $n_0 < n_1$. It is part of the standard arguments that $\mathcal{C}_{Kh}(K_0)$ can be identified with a subquotient complex of $\mathcal{C}_{Kh}(K_1)$, inducing a (two-step) zig-zag of isomorphisms connecting $Kh(K_0)$ and $Kh(K_1)$ (see also the proofs of [LS14a, Propositions 6.2–6.4]). Indeed, there exists a fixed vertex $w \in \{0, 1\}^{n_1 - n_0}$, so that for every $u \in \{0, 1\}^{n_0}$, $F_{Kh}(K_0)(u)$ is identified with a certain subset $S_u \subseteq F_{Kh}(K_1)((u, w))$, and for every $u > v \in \{0, 1\}^{n_0}$, the correspondence $F_{Kh}(K_0)(\varphi_{u,v})$ is identified with the subset $s^{-1}(S_u) \cap t^{-1}(S_v) \subseteq F_{Kh}(K_1)(\varphi_{(u,w),(v,w)})$. Furthermore, these identifications identify $F_{+Kh}(K_0)(u)$ with $S_u \cap F_{+Kh}(K_1)((u, w))$ (and consequently, $F_{-Kh}(K_0)(u)$ with $S_u \cap F_{-Kh}(K_1)((u, w))$).

Construct the \mathbb{H}^1 -module spectrum $\mathcal{X}_{Kh}(K_1)$ using some pointed spacial refinement for K_1 . Restricting to the subsets S_u (and the correspondences between them), we get a pointed spacial refinement for K_0 , which we use to construct the \mathbb{H}^1 -module spectrum $\mathcal{X}_{Kh}(K_0)$. With the CW complex structures from Proposition 6.1, $\mathcal{X}_{Kh}(K_0)$ can be identified with a subquotient complex of $\mathcal{X}_{Kh}(K_1)$, leading to a two-step zig-zag of maps connecting them. The maps are plainly \mathbb{H}^1 -equivariant, and since they induce isomorphisms on homology, they are stable homotopy equivalences. \square

For the rest of this section, fix a link diagram for $K_1 \amalg K_2$, which is a disjoint union of link diagrams for K_1 and K_2 , with n_1 and n_2 crossings respectively, and fix basepoints p_i on K_i so that the two basepoints are next to one another. The (derived) tensor product of the spectra $\mathcal{X}_{Kh}(K_1)$ and $\mathcal{X}_{Kh}(K_2)$ is the homotopy colimit of the diagram

$$(9.4) \quad \mathcal{X}_{Kh}(K_1) \wedge \mathcal{X}_{Kh}(K_2) \xleftarrow{\quad} \mathcal{X}_{Kh}(K_1) \wedge \mathbb{H}^1 \wedge \mathcal{X}_{Kh}(K_2) \xleftarrow{\quad} \mathcal{X}_{Kh}(K_1) \wedge \mathbb{H}^1 \wedge \mathbb{H}^1 \wedge \mathcal{X}_{Kh}(K_2) \xleftarrow{\quad} \cdots$$

where the maps are all possible ways of applying μ to a pair of consecutive factors. To be more precise, let Δ_{inj} be the category with one object $\mathbf{n} = \{0, \dots, n-1\}$ for each positive integer n and $\text{Hom}(\mathbf{m}, \mathbf{n})$ the set of order-preserving injections $\{0, \dots, m-1\} \rightarrow \{0, \dots, n-1\}$; for $n > 0$ and $0 \leq i \leq n$, let $f_{\mathbf{n}, i} \in \text{Hom}_{\Delta_{inj}}(\mathbf{n}, \mathbf{n} + \mathbf{1})$ be the morphism $\mathbf{n} \rightarrow \mathbf{n} + \mathbf{1}$ whose image is $\mathbf{n} + \mathbf{1} \setminus \{i\}$. (The category Δ_{inj} is the subcategory of the simplex category generated by the face maps, and the $f_{\mathbf{n}, i}$ are the face maps themselves.) Then the diagram (9.4) can be treated as a (strict) functor F_\otimes from Δ_{inj}^{op} to CW_\bullet , the category of pointed CW complexes. On objects, $F_\otimes(\mathbf{n}) = \mathcal{X}_{Kh}(K_1) \wedge (\bigwedge_{i=1}^{n-1} \mathbb{H}^1) \wedge \mathcal{X}_{Kh}(K_2)$. On morphisms, $F_\otimes(f_{\mathbf{n}, i}^{\text{op}})$ is the map $\mathcal{X}_{Kh}(K_1) \wedge (\bigwedge_{i=1}^n \mathbb{H}^1) \wedge \mathcal{X}_{Kh}(K_2) \rightarrow \mathcal{X}_{Kh}(K_1) \wedge (\bigwedge_{i=1}^{n-1} \mathbb{H}^1) \wedge \mathcal{X}_{Kh}(K_2)$ gotten by applying μ to the $(i+1)^{\text{th}}$ pair of consecutive factors. Let $\mathcal{X}_{Kh}(K_1) \otimes_{\mathbb{H}^1} \mathcal{X}_{Kh}(K_2) := \text{hocolim}(F_\otimes)$ denote the derived tensor product of $\mathcal{X}_{Kh}(K_1)$ and $\mathcal{X}_{Kh}(K_2)$.

Theorem 8. *There is a stable homotopy equivalence $\mathcal{X}_{Kh}(K_1 \# K_2) \simeq \mathcal{X}_{Kh}(K_1) \otimes_{\mathbb{H}^1} \mathcal{X}_{Kh}(K_2)$.*

The proof of Theorem 8 involves three components. First, we show how the functor $F_{Kh}(K_1 \# K_2)$ is determined by the functors $F_{Kh}(K_1)$ and $F_{Kh}(K_2)$; this is Lemmas 9.5 and 9.6, which take a little work but are purely combinatorial. Second, in Lemmas 9.7 and 9.9, we prove that Theorem 8 holds at the level of

cellular cochains, for an appropriate CW complex structure on $\mathcal{X}_{Kh}(K_1) \otimes_{\mathbb{H}^1} \mathcal{X}_{Kh}(K_2)$. This is essentially immediate from Segal's construction of homotopy colimits and the connected sum theorem for the Khovanov chain complex. Third, using the description of $F_{Kh}(K_1 \# K_2)$ in terms of $F_{Kh}(K_1)$ and $F_{Kh}(K_2)$ and carefully chosen spacial refinements, we produce a (strict) map from the diagram (9.4) to $\mathcal{X}_{Kh}(K_1 \# K_2)$, inducing the desired map of cellular cochains. This argument is Lemmas 9.10 and 9.11. From these three steps, Theorem 8 follows easily.

We begin by reconstructing the functor $F_{Kh}(K_1 \# K_2)$ from the functor $F_{Kh}(K_1 \amalg K_2): \underline{2}^{n_1+n_2} \rightarrow \mathcal{B}$. For $a, b \in \{*, +, -\}$, let $F_{abKh}(K_1 \amalg K_2): \underline{2}^{n_1+n_2} \rightarrow \mathcal{B}$ denote the functor where we only consider the Khovanov generators that label the circle containing p_1 by x_a if $a \in \{+, -\}$ and label the circle containing p_2 by x_b if $b \in \{+, -\}$, and we restrict the correspondences correspondingly. (If a or b is $*$, we make no restriction on the label of the corresponding circle.)

Lemma 9.5. *For $v \in \{0, 1\}^{n_1+n_2}$, the map from $F_{+-Kh}(K_1 \amalg K_2)(v)$ to $F_{-+Kh}(K_1 \amalg K_2)(v)$ that interchanges the labelings of the two pointed circles in $\mathcal{P}(v)$ induces an isomorphism from $F_{+-Kh}(K_1 \amalg K_2)$ to $F_{-+Kh}(K_1 \amalg K_2)$.*

Proof. The isomorphism from Proposition 9.1 identifies either functor to $F_{\widetilde{Kh}}(K_1) \times F_{\widetilde{Kh}}(K_2)$. The given map is the composition $F_{+-Kh}(K_1 \amalg K_2) \cong F_{\widetilde{Kh}}(K_1) \times F_{\widetilde{Kh}}(K_2) \cong F_{-+Kh}(K_1 \amalg K_2)$. \square

Let $F_{+/-Kh}(K_1 \amalg K_2)$ denote the functor $\underline{2}^{n_1+n_2} \rightarrow \mathcal{B}$ where we only consider the Khovanov generators that label at least one of the two pointed circles by x_- , and we restrict the correspondences correspondingly. (The notation $F_{+/-Kh}$ is the mnemonic ‘‘not ++’’.) That is, for all $u \in \{0, 1\}^{n_1+n_2}$, $F_{+/-Kh}(K_1 \amalg K_2) = F_{--Kh}(K_1 \amalg K_2)(u) \amalg F_{+-Kh}(K_1 \amalg K_2)(u) \amalg F_{-+Kh}(K_1 \amalg K_2)(u)$; and for all $u > v \in \{0, 1\}^{n_1+n_2}$, the correspondence $F_{+/-Kh}(K_1 \amalg K_2)(\varphi_{u,v})$ is the disjoint union of the correspondences $F_{--Kh}(K_1 \amalg K_2)(\varphi_{u,v})$, $F_{+-Kh}(K_1 \amalg K_2)(\varphi_{u,v})$, $F_{-+Kh}(K_1 \amalg K_2)(\varphi_{u,v})$, some correspondence P from $F_{--Kh}(K_1 \amalg K_2)(u)$ to $F_{+-Kh}(K_1 \amalg K_2)(v)$, and some correspondence Q from $F_{--Kh}(K_1 \amalg K_2)(u)$ to $F_{-+Kh}(K_1 \amalg K_2)(v)$. Let F be the functor obtained from $F_{+/-Kh}(K_1 \amalg K_2)$ by identifying $F_{+-Kh}(K_1 \amalg K_2)$ and $F_{-+Kh}(K_1 \amalg K_2)$ by the isomorphism from Lemma 9.5. That is, for all $u \in \{0, 1\}^{n_1+n_2}$,

$$F(u) = (F_{--Kh}(u) \amalg F_{+-Kh}(u) \amalg F_{-+Kh}(u)) / F_{+-Kh}(u) = F_{-+Kh}(u);$$

and for all $u > v \in \{0, 1\}^{n_1+n_2}$,

$$F(\varphi_{u,v}) = (F_{--Kh}(\varphi_{u,v}) \amalg F_{+-Kh}(\varphi_{u,v}) \amalg F_{-+Kh}(\varphi_{u,v}) \amalg P \amalg Q) / F_{+-Kh}(\varphi_{u,v}) = F_{-+Kh}(\varphi_{u,v}).$$

Lemma 9.6. *The functor F constructed above is isomorphic to $F_{Kh}(K_1 \# K_2)$ via the following map: For all $u \in \{0, 1\}^{n_1+n_2}$, the isomorphism sends $x \in F(u)$ to $y \in F_{Kh}(K_1 \# K_2)(u)$ where y labels the connect-sum circle by x_- if and only if x labels both the pointed circles by x_- , and x and y label all the circles that are disjoint from the connect-sum region identically.*

Proof. The proof is similar to the proof of Proposition 9.1. To keep the notations similar, let $X_v = F(v)$, $A_{u,v} = F(\varphi_{u,v})$, $G = F_{Kh}(K_1 \# K_2)$, $Y_v = G(v)$, and $B_{u,v} = G(\varphi_{u,v})$. The bijections $\phi_v: X_v \xrightarrow{\cong} Y_v$ are already provided to us. To construct bijections $\psi_{u,v}: A_{u,v} \xrightarrow{\cong} B_{u,v}$, for all $u > v$ with $|u| - |v| = 1$, we need to check the conditions (1) and (2) of the proof of Proposition 9.1.

For any u and any $z_u \in F_{+/-Kh}(K_1 \amalg K_2)(u)$, let $\pi(z_u)$ denote its image in X_u ; and for any $x_u \in X_u$, let $\iota^1(x_u)$ (respectively, $\iota^2(x_u)$) denote its preimage in $F_{*-Kh}(K_1 \amalg K_2)(u)$ (respectively, $F_{*-Kh}(K_1 \amalg K_2)(u)$). Then for any $u > v$, $z_u \in F_{+/-Kh}(K_1 \amalg K_2)(u)$, $x_v \in X_v$, one of the two subsets $s^{-1}(z_u) \cap t^{-1}(\iota^1(x_v)) \subseteq F_{*-Kh}(K_1 \amalg K_2)(\varphi_{u,v})$ and $s^{-1}(z_u) \cap t^{-1}(\iota^2(x_v)) \subseteq F_{*-Kh}(K_1 \amalg K_2)(\varphi_{u,v})$ is empty, and the other one is canonically identified with the subset $s^{-1}(\pi(z_u)) \cap t^{-1}(x_v) \subseteq A_{u,v}$. This follows from the fact that the correspondences in $F_{+/-Kh}(K_1 \amalg K_2)$ preserve two quantum gradings, the one coming from K_1 and the one

coming from K_2 ; however, the double quantum gradings of $\iota^1(x_v)$ and $\iota^2(x_v)$ are different, and therefore, at least one of $s^{-1}(z_u) \cap t^{-1}(\iota^1(x_v))$ and $s^{-1}(z_u) \cap t^{-1}(\iota^2(x_v))$ is empty.

From this observation, condition (1) is immediate. Condition (2) follows from additionally noting that the composition $F_{Kh}(K_1 \amalg K_2) \rightarrow X \xrightarrow{\phi} Y$ induces the cobordism map $\mathcal{C}_{Kh}(K_1 \# K_2) \rightarrow \mathcal{C}_{Kh}(K_1 \amalg K_2)$ associated to splitting at the connect-sum region, which is a chain map.

Finally, we need to check that diagram (9.2) is satisfied for all $u > w$ with $|u| - |w| = 2$. Using the above observation, this follows from the same arguments as in the proof of Proposition 9.1. \square

Next we observe that Theorem 8 holds at the level of cellular cochain complexes.

Lemma 9.7. *There exists a CW complex structure on $\mathcal{X}_{Kh}(K_1) \otimes_{\mathbb{H}^1} \mathcal{X}_{Kh}(K_2)$ so that the reduced cellular cochain complex is the following chain complex*

$$(9.8) \quad \mathcal{C}_{Kh}(K_1) \otimes \mathcal{C}_{Kh}(K_2) \rightarrow \mathcal{C}_{Kh}(K_1) \otimes Kh(U) \otimes \mathcal{C}_{Kh}(K_2) \rightarrow \mathcal{C}_{Kh}(K_1) \otimes Kh(U) \otimes Kh(U) \otimes \mathcal{C}_{Kh}(K_2) \rightarrow \cdots$$

with the differential given by

$$\begin{aligned} d(x_0 \otimes \cdots \otimes x_n) &= \sum_{i=0}^n (-1)^{\text{gr}(x_0) + \cdots + \text{gr}(x_{i-1})} x_0 \otimes \cdots \otimes x_{i-1} \otimes \delta(x_i) \otimes x_{i+1} \otimes \cdots \otimes x_n \\ &\quad + \sum_{i=0}^n (-1)^{i+n} x_0 \otimes \cdots \otimes x_{i-1} \otimes S(x_i) \otimes x_{i+1} \otimes \cdots \otimes x_n, \end{aligned}$$

where S denotes either the Khovanov Frobenius algebra comultiplication map $Kh(U) \rightarrow Kh(U) \otimes Kh(U)$ or the cobordism map $\mathcal{C}_{Kh}(K_1) \rightarrow \mathcal{C}_{Kh}(K_1) \otimes Kh(U)$ (respectively, $\mathcal{C}_{Kh}(K_2) \rightarrow Kh(U) \otimes \mathcal{C}_{Kh}(K_2)$) for splitting off a trivial unknot at p_1 (respectively, p_2), and δ denotes the Khovanov differential on the various chain complexes (which is zero on $\mathcal{C}_{Kh}(U) = Kh(U)$).

Proof. Consider the geometric realization of $F_{\otimes} : \Delta_{inj}^{\text{op}} \rightarrow \text{CW}_{\bullet}$, as constructed in [Seg74, Appendix A]:

$$\|F_{\otimes}\| = \left(\{*\} \amalg \coprod_n \Delta^{n-1} \times F_{\otimes}(\mathbf{n}) \right) / \sim$$

with $(f_*(\zeta), a) \sim (\zeta, F_{\otimes}(f^{\text{op}})(a))$ for all $\zeta \in \Delta^{m-1}$, $a \in F_{\otimes}(\mathbf{n})$, and $f \in \text{Hom}_{\Delta_{inj}}(\mathbf{m}, \mathbf{n})$, and $\Delta^{n-1} \times \{*\} \sim *$ for all n . (The map $f_* : \Delta^{m-1} \rightarrow \Delta^{n-1}$ is the face inclusion corresponding to f .) Equipped with the natural CW complex structure ([Seg74, Proposition A.1(i)]), its reduced cellular cochain complex is easily seen to be the one from Formula (9.8).

To identify $\|F_{\otimes}\|$ with $\text{hocolim}(F_{\otimes})$, use the construction of homotopy colimits of the strict functor F_{\otimes} via simplices, instead of cubes; Vogt showed the two definitions agree [Vog73, Corollary 8.5]. Using the simplicial model for the homotopy colimit, it is well known that $\text{hocolim}(F_{\otimes})$ is the barycentric subdivision of $\|F_{\otimes}\|$; see also [Seg74, Proposition A.3] where $|\text{simp}(\cdot)|$ plays the role of this space. \square

Lemma 9.9. *The cobordism map $S : \mathcal{C}_{Kh}(K_1 \# K_2) \rightarrow \mathcal{C}_{Kh}(K_1) \otimes \mathcal{C}_{Kh}(K_2)$ associated to splitting at the connected sum region induces a quasi-isomorphism*

$$\begin{array}{c} \mathcal{C}_{Kh}(K_1 \# K_2) \\ \downarrow S \\ \mathcal{C}_{Kh}(K_1) \otimes \mathcal{C}_{Kh}(K_2) \longrightarrow \mathcal{C}_{Kh}(K_1) \otimes Kh(U) \otimes \mathcal{C}_{Kh}(K_2) \longrightarrow \mathcal{C}_{Kh}(K_1) \otimes Kh(U) \otimes Kh(U) \otimes \mathcal{C}_{Kh}(K_2) \longrightarrow \cdots \end{array}$$

from $\mathcal{C}_{Kh}(K_1 \# K_2)$ to the chain complex from Formula (9.8).

Proof. The Khovanov complex $\mathcal{C}_{Kh}(K_1 \# K_2)$ is the cotensor product of $\mathcal{C}_{Kh}(K_1)$ and $\mathcal{C}_{Kh}(K_2)$ as comodules over $Kh(U)$ [LS14a, Lemma 10.5], while Formula (9.8) is the derived cotensor product of $\mathcal{C}_{Kh}(K_1)$ and $\mathcal{C}_{Kh}(K_2)$. Thus, the statement presumably follows from the fact that $\mathcal{C}_{Kh}(K_1)$ and $\mathcal{C}_{Kh}(K_2)$ are co-flat. Rather than going down this rabbit hole, dualize the complex (9.8) over \mathbb{Z} , which exchanges split map S and the merge map m , (derived) cotensor product and (derived) tensor product, and $\mathcal{C}_{Kh}(K)$ and $\mathcal{C}_{Kh}(m(K))$. (The last assertion is [Kho00, Proposition 32].) The result then follows from the fact that $\mathcal{C}_{Kh}(m(K_i))$ is free as a $Kh(U)$ -module and Khovanov's connected sum theorem [Kho03, Proposition 3.3]. \square

We turn to the third part of the argument, constructing compatible spacial refinements for $F_{Kh}(K_1 \# K_2)$ and $F_{Kh}(K_1 \amalg U \amalg \cdots \amalg U \amalg K_2)$.

Lemma 9.10. *Consider any spacial refinement $\tilde{F}_{++Kh}(K_1 \amalg K_2)$ of $F_{++Kh}(K_1 \amalg K_2)$ whose induced spacial refinements $\tilde{F}_{-+Kh}(K_1 \amalg K_2)$ and $\tilde{F}_{+-Kh}(K_1 \amalg K_2)$ of $F_{-+Kh}(K_1 \amalg K_2)$ and $F_{+-Kh}(K_1 \amalg K_2)$ agree. Then, identifying $\tilde{F}_{-+Kh}(K_1 \amalg K_2)$ and $\tilde{F}_{+-Kh}(K_1 \amalg K_2)$ produces a spacial refinement $\tilde{F}_{Kh}(K_1 \# K_2)$ of $F_{Kh}(K_1 \# K_2)$.*

Proof. It is immediate from the definitions that identifying $\tilde{F}_{-+Kh}(K_1 \amalg K_2)$ and $\tilde{F}_{+-Kh}(K_1 \amalg K_2)$ produces a spacial refinement of the functor F above. The isomorphism from Lemma 9.6 then produces the spacial refinement $\tilde{F}_{Kh}(K_1 \# K_2)$ of $F_{Kh}(K_1 \# K_2)$. \square

Let $\overline{\Delta}_{inj} = \Delta_{inj} \cup \mathbf{0}$ be the category obtained by adding an object $\mathbf{0} = \emptyset$ to Δ_{inj} and a unique morphism $\mathbf{0} \rightarrow \mathbf{n}$ for each n ; let $f_{\mathbf{0},0}$ denote the unique morphism from $\mathbf{0}$ to $\mathbf{1}$. We will now extend diagram (9.4) to construct a functor $\overline{\Delta}_{inj}^{\text{op}} \rightarrow \text{CW}_\bullet$.

Lemma 9.11. *There exists a functor $\overline{F}_\times : \overline{\Delta}_{inj}^{\text{op}} \rightarrow \text{CW}_\bullet$ satisfying the following:*

- (1) $\overline{F}_\times(\mathbf{0}) = \mathcal{X}_{Kh}(K_1 \# K_2)$ and $\overline{F}_\times(\mathbf{n}) = \mathcal{X}_{Kh}(K_1 \amalg K_2) \wedge (\bigwedge_{i=1}^{n-1} \mathbb{H}^1)$ for all $n > 0$.
- (2) Let F_\times denote the restriction $\overline{F}_\times|_{\Delta_{inj}^{\text{op}}}$. Then there is a natural transformation η from the functor F_\otimes of diagram (9.4) to F_\times , so that for all $n > 0$, $\eta_{\mathbf{n}} : F_\otimes(\mathbf{n}) \rightarrow F_\times(\mathbf{n})$ sends each cell in $\mathcal{X}_{Kh}(K_1) \wedge (\bigwedge_{i=1}^{n-1} \mathbb{H}^1) \wedge \mathcal{X}_{Kh}(K_2)$ to the corresponding cell in $\mathcal{X}_{Kh}(K_1 \amalg K_2) \wedge (\bigwedge_{i=1}^{n-1} \mathbb{H}^1)$ by a degree one map.
- (3) $\overline{F}_\times(f_{\mathbf{0},0}^{\text{op}})$ is a map $\mathcal{X}_{Kh}(K_1 \amalg K_2) \rightarrow \mathcal{X}_{Kh}(K_1 \# K_2)$ so that the induced map on reduced cellular cochains is the cobordism map

$$S : \mathcal{C}_{Kh}(K_1 \# K_2) \rightarrow \mathcal{C}_{Kh}(K_1 \amalg K_2)$$

induced by splitting at the connected sum region.

Proof. During the construction of \overline{F}_\times , we will use spacial refinements for $K_1 \amalg K_2$ that are pointed spacial refinements with respect to both p_1 and p_2 . We will call such spacial refinements *doubly pointed spacial refinements*. Doubly pointed spacial refinements are spacial refinements that agree on $F_{*+Kh}(K_1 \amalg K_2)$ and $F_{*-Kh}(K_1 \amalg K_2)$, and also on $F_{+*Kh}(K_1 \amalg K_2)$ and $F_{-*Kh}(K_1 \amalg K_2)$ (and, therefore, agree on $F_{++Kh}(K_1 \amalg K_2)$, $F_{+-Kh}(K_1 \amalg K_2)$, $F_{-+Kh}(K_1 \amalg K_2)$, and $F_{--Kh}(K_1 \amalg K_2)$). The CW complex $\mathcal{X}_{Kh}(K_1 \amalg K_2)$ constructed using any such doubly pointed spacial refinement can be viewed as a strict bimodule over \mathbb{H}^1 , with the two actions coming from the two basepoints p_1 and p_2 . Therefore, we can construct a strict functor $G : \overline{\Delta}_{inj}^{\text{op}} \rightarrow \text{CW}_\bullet$ by declaring $G(\mathbf{n}) = \mathcal{X}_{Kh}(K_1 \amalg K_2) \wedge (\bigwedge_{i=1}^{n-1} \mathbb{H}^1)$ and by defining the map $G(f_{\mathbf{n},i}^{\text{op}}) : \mathcal{X}_{Kh}(K_1 \amalg K_2) \wedge (\bigwedge_{i=1}^n \mathbb{H}^1) \rightarrow \mathcal{X}_{Kh}(K_1 \amalg K_2) \wedge (\bigwedge_{i=1}^{n-1} \mathbb{H}^1)$ to be the ring multiplication map applied to the i^{th} pair of consecutive \mathbb{H}^1 -factors if $0 < i < n$, and the bimodule map coming from p_1 (respectively, p_2) using the first (respectively, last) \mathbb{H}^1 factor if $i = 0$ (respectively, n).

Now we are in a position to construct F_\times . The construction proceeds in several stages.

- (F-1) We start with pointed spacial refinements $\tilde{F}_{Kh}(K_1)$ and $\tilde{F}_{Kh}(K_2)$ of $F_{Kh}(K_1)$ and $F_{Kh}(K_2)$ (using k_1 -dimensional and k_2 -dimensional boxes with $k_1 + k_2 = k$). Since $F_{Kh}(K_1 \amalg K_2) = F_{Kh}(K_1) \times F_{Kh}(K_2)$ (Proposition 9.1), $\tilde{F}_{Kh}(K_1 \amalg K_2) := \tilde{F}_{Kh}(K_1) \wedge \tilde{F}_{Kh}(K_2)$ is a doubly pointed spacial refinement for $K_1 \amalg K_2$.
- (F-2) Define $F_\times: \Delta_{inj}^{op} \rightarrow CW_\bullet$ to be the functor associated to this doubly pointed spacial refinement. This automatically satisfies the second part of Lemma 9.11 (1). To relate F_\otimes and F_\times , observe that for all $n > 0$, $F_\times(\mathbf{n}) = F_\times(\mathbf{1}) \wedge (\bigwedge_{i=1}^{n-1} \mathbb{H}^1)$ and $F_\otimes(\mathbf{n})$ is canonically isomorphic to $F_\otimes(\mathbf{1}) \wedge (\bigwedge_{i=1}^{n-1} \mathbb{H}^1)$ (preserving the order of the \mathbb{H}^1 -factors); therefore, it is enough to relate $F_\times(\mathbf{1}) = \text{hocolim}((\tilde{F}_{Kh}(K_1) \wedge \tilde{F}_{Kh}(K_2))^+)$ and $F_\otimes(\mathbf{1}) = \text{hocolim}((\tilde{F}_{Kh}(K_1))^+) \wedge \text{hocolim}((\tilde{F}_{Kh}(K_2))^+)$. However, the former is easily seen to be a quotient of the latter, with the quotient map sending each cell by a degree one map to the corresponding cell. This proves Lemma 9.11 (2).
- (F-3) The doubly pointed spacial refinement $\tilde{F}_{Kh}(K_1 \amalg K_2)$ induces a spacial refinement $\tilde{F}_{+/+Kh}(K_1 \amalg K_2)$ of $F_{+/+Kh}(K_1 \amalg K_2)$; and its induced spacial refinements $\tilde{F}_{-+Kh}(K_1 \amalg K_2)$ and $\tilde{F}_{+-Kh}(K_1 \amalg K_2)$ of $F_{-+Kh}(K_1 \amalg K_2)$ and $F_{+-Kh}(K_1 \amalg K_2)$ agree (with $F_{-+Kh}(K_1 \amalg K_2)$ and $F_{+-Kh}(K_1 \amalg K_2)$ identified by Lemma 9.5). Therefore, by Lemma 9.10, we get a (pointed) spacial refinement $\tilde{F}_{Kh}(K_1 \# K_2)$ for $K_1 \# K_2$ (with the basepoint chosen on either of the two strands near the connect sum region).
- (F-4) We use $\tilde{F}_{Kh}(K_1 \# K_2)$ to construct the CW complex $\bar{F}_\times(\mathbf{0}) = \mathcal{X}_{Kh}(K_1 \# K_2)$; this automatically satisfies the first part of Lemma 9.11 (1).
- (F-5) The space $\text{hocolim}((\tilde{F}_{+/+Kh}(K_1 \amalg K_2))^+)$ is a quotient complex of $\mathcal{X}_{Kh}(K_1 \amalg K_2) = \text{hocolim}((\tilde{F}_{Kh}(K_1 \amalg K_2))^+)$; it has two subcomplexes $\text{hocolim}((\tilde{F}_{-+Kh}(K_1 \amalg K_2))^+)$ and $\text{hocolim}((\tilde{F}_{+-Kh}(K_1 \amalg K_2))^+)$ that have a canonical isomorphism between them; and the coequalizer is canonically identified with $\mathcal{X}_{Kh}(K_1 \# K_2) = \text{hocolim}((\tilde{F}_{Kh}(K_1 \# K_2))^+)$. That is, we have a diagram

$$\begin{array}{ccc}
& \text{hocolim}((\tilde{F}_{-+Kh}(K_1 \amalg K_2))^+) \cong \text{hocolim}((\tilde{F}_{+-Kh}(K_1 \amalg K_2))^+) & \\
& \searrow \qquad \qquad \qquad \swarrow & \\
\mathcal{X}_{Kh}(K_1 \amalg K_2) = \text{hocolim}((\tilde{F}_{Kh}(K_1 \amalg K_2))^+) & \twoheadrightarrow & \text{hocolim}((\tilde{F}_{+/+Kh}(K_1 \amalg K_2))^+) \\
& & \downarrow \\
& & \text{hocolim}((\tilde{F}_{Kh}(K_1 \# K_2))^+) = \mathcal{X}_{Kh}(K_1 \# K_2).
\end{array}$$

Define the map $\bar{F}_\times(f_{\mathbf{0},0}^{op}): \bar{F}_\times(\mathbf{1}) = \mathcal{X}_{Kh}(K_1 \amalg K_2) \rightarrow \mathcal{X}_{Kh}(K_1 \# K_2) = \bar{F}_\times(\mathbf{0})$ to be the composition. This map is a cellular map sending the cells in $\mathcal{X}_{Kh}(K_1 \amalg K_2)$ to the corresponding cells in $\mathcal{X}_{Kh}(K_1 \# K_2)$ by degree one maps (with the correspondence described in Lemma 9.6). Therefore, this map satisfies Lemma 9.11 (3).

- (F-6) We have to define the map $\bar{F}_\times(\mathbf{2}) \rightarrow \bar{F}_\times(\mathbf{0})$ to be both $\bar{F}_\times(f_{\mathbf{0},0}^{op}) \circ \bar{F}_\times(f_{\mathbf{1},0}^{op})$ and $\bar{F}_\times(f_{\mathbf{0},0}^{op}) \circ \bar{F}_\times(f_{\mathbf{1},1}^{op})$; so we merely need to check that the latter two maps agree. The two maps $\bar{F}_\times(f_{\mathbf{1},0}^{op})$ and $\bar{F}_\times(f_{\mathbf{1},1}^{op})$

$$\mathcal{X}_{Kh}(K_1 \amalg K_2) \wedge \{*, p_-, p_+\} = (\mathcal{X}_{Kh}(K_1 \amalg K_2) \times \{p_-\}) \vee (\mathcal{X}_{Kh}(K_1 \amalg K_2) \times \{p_+\}) \rightarrow \mathcal{X}_{Kh}(K_1 \amalg K_2)$$

agree on the first summand: both denote the projection $\mathcal{X}_{Kh}(K_1 \amalg K_2) \times \{p_-\} \rightarrow \mathcal{X}_{Kh}(K_1 \amalg K_2)$. On the second summand, the two maps are compositions of the projection map $\mathcal{X}_{Kh}(K_1 \amalg K_2) \times \{p_+\} \rightarrow$

$\mathcal{X}_{Kh}(K_1 \amalg K_2)$ with the following two maps:

$$\begin{array}{ccc}
 & \text{hocolim } ((\tilde{F}_{Kh}(K_1 \amalg K_2))^+) & \\
 & \swarrow \quad \searrow & \\
 \text{hocolim } ((\tilde{F}_{-*Kh}(K_1 \amalg K_2))^+) & & \text{hocolim } ((\tilde{F}_{*-Kh}(K_1 \amalg K_2))^+) \\
 \downarrow \cong & & \downarrow \cong \\
 \text{hocolim } ((\tilde{F}_{+*Kh}(K_1 \amalg K_2))^+) & & \text{hocolim } ((\tilde{F}_{*+Kh}(K_1 \amalg K_2))^+) \\
 & \swarrow \quad \searrow & \\
 & \text{hocolim } ((\tilde{F}_{Kh}(K_1 \amalg K_2))^+) &
 \end{array}$$

Therefore, after composing with $\overline{F}_\times(f_{\underline{\mathbf{0}},0}^{\text{op}})$, we get the two maps

$$\begin{array}{ccc}
 & \text{hocolim } ((\tilde{F}_{Kh}(K_1 \amalg K_2))^+) & \\
 & \downarrow & \\
 & \text{hocolim } ((\tilde{F}_{-Kh}(K_1 \amalg K_2))^+) & \\
 \cong \swarrow & & \searrow \cong \\
 \text{hocolim } ((\tilde{F}_{+Kh}(K_1 \amalg K_2))^+) & & \text{hocolim } ((\tilde{F}_{-+Kh}(K_1 \amalg K_2))^+) \\
 & \swarrow \quad \searrow & \\
 & \text{hocolim } ((\tilde{F}_{+Kh}(K_1 \amalg K_2))^+) & \\
 & \downarrow & \\
 & \text{hocolim } ((\tilde{F}_{Kh}(K_1 \# K_2))^+) &
 \end{array}$$

(The first vertical map is the quotient map to $\text{hocolim } ((\tilde{F}_{-Kh}(K_1 \amalg K_2))^+)$, as the composition sends the rest of $\text{hocolim } ((\tilde{F}_{Kh}(K_1 \amalg K_2))^+)$ to the basepoint.) However, since $\text{hocolim } ((\tilde{F}_{Kh}(K_1 \# K_2))^+)$ is the coequalizer (see step (F-5)), these maps agree.

(F-7) The n different morphisms $\underline{\mathbf{n}} \rightarrow \underline{\mathbf{1}}$ in $\overline{\Delta}_{inj}^{\text{op}}$ induce the following two maps $\overline{F}_\times(\underline{\mathbf{n}}) \rightarrow \overline{F}_\times(\underline{\mathbf{1}})$: first apply the ring multiplication on the \mathbb{H}^1 -factors to get a map

$$\phi: \overline{F}_\times(\underline{\mathbf{n}}) = \mathcal{X}_{Kh}(K_1 \amalg K_2) \wedge \left(\bigwedge_{i=1}^{n-1} \mathbb{H}^1 \right) \rightarrow \mathcal{X}_{Kh}(K_1 \amalg K_2) \wedge \mathbb{H}^1 = \overline{F}_\times(\underline{\mathbf{2}}),$$

and then compose with the two maps $\overline{F}_\times(f_{\underline{\mathbf{1}},0}^{\text{op}})$ and $\overline{F}_\times(f_{\underline{\mathbf{1}},1}^{\text{op}})$. We have already defined the map $\overline{F}_\times(\underline{\mathbf{2}}) \rightarrow \overline{F}_\times(\underline{\mathbf{0}})$; define the map $\overline{F}_\times(\underline{\mathbf{n}}) \rightarrow \overline{F}_\times(\underline{\mathbf{0}})$ to be its composition with ϕ .

The functor \overline{F}_\times thus constructed satisfies the conditions of the lemma (see steps (F-2), (F-4), and (F-5)), thereby concluding the proof. \square

Remark 9.12. Steps (F-5) and (F-6) in the proof of Lemma 9.11 actually show that with the specific choices made in the proof, $\mathcal{X}_{Kh}(K_1 \# K_2)$ is the ordinary tensor product of $\mathcal{X}_{Kh}(K_1)$ and $\mathcal{X}_{Kh}(K_2)$ over $\{*, p_-, p_+\}$. However, these spaces are only defined up to stable homotopy equivalences, and the module structures are only defined up to quasi-isomorphisms (see Proposition 9.3); and the ordinary tensor product is not invariant under such equivalences, while the derived tensor product is.

Proof of Theorem 8. Let $F_\otimes: \Delta_{inj}^{\text{op}} \rightarrow \text{CW}_\bullet$ be the functor from diagram (9.4), and let $\overline{F}_\times: \overline{\Delta}_{inj}^{\text{op}} \rightarrow \text{CW}_\bullet$ be the functor constructed in Lemma 9.11. The derived tensor product $\mathcal{X}_{Kh}(K_1) \otimes_{\mathbb{H}^1} \mathcal{X}_{Kh}(K_2)$ is defined to be $\text{hocolim}(F_\otimes)$. The natural transformation $\eta: F_\otimes \rightarrow F_\times = \overline{F}_\times|_{\overline{\Delta}_{inj}^{\text{op}}}$ from Lemma 9.11 (2) is a stable homotopy equivalence on each object; therefore, by property (ho-1) in Section 2.9, $\text{hocolim}(F_\otimes) = \text{hocolim}(F_\times)$. Since $\mathbf{0}$ is a terminal object in $\overline{\Delta}_{inj}^{\text{op}}$, the functor \overline{F}_\times induces a map

$$\mathcal{X}_{Kh}(K_1) \otimes_{\mathbb{H}^1} \mathcal{X}_{Kh}(K_2) = \text{hocolim}(F_\times) \longrightarrow \overline{F}_\times(\mathbf{0}) = \mathcal{X}_{Kh}(K_1 \# K_2).$$

Using Lemma 9.7 and Lemma 9.11 (2), we can view $\text{hocolim}(F_\times)$ as a CW complex whose reduced cellular cochain complex is the complex from Formula (9.8). Lemma 9.11 (3) implies that the map $\text{hocolim}(F_\times) \rightarrow \overline{F}_\times(\mathbf{0})$ induces the quasi-isomorphism from Lemma 9.9 at the level of reduced cellular cochain complexes. Therefore, the map is a stable homotopy equivalence. \square

10. KHOVANOV HOMOTOPY TYPE OF A MIRROR

Given a spectrum X , let X^\vee denote the Spanier-Whitehead dual of X , which is an internal function object parametrizing maps $X \rightarrow \mathbb{S}$. If X is finite, the spectrum X^\vee is also a finite spectrum characterized by the existence of map of spectra

$$\mu: X \wedge X^\vee \rightarrow \mathbb{S}$$

such that the slant product map

$$(10.1) \quad \mu^*(\gamma)/\cdot: \widetilde{H}_*(X) \rightarrow \widetilde{H}^*(X^\vee)$$

is an isomorphism. (Here, $\gamma \in \widetilde{H}^0(\mathbb{S})$ is the fundamental class.) We will say that μ *witnesses S-duality between X and X^\vee* . See [Swi75, Proposition 14.37] and [Ada95, Section 9] for more details.

Theorem 9. [LS14a, Conjecture 10.1] *Let L be a link and let $m(L)$ denote the mirror of L . Then*

$$\mathcal{X}_{Kh}^j(m(L)) \simeq \mathcal{X}_{Kh}^{-j}(L)^\vee$$

Before proving the theorem we give a (presumably standard) reformulation of the Spanier-Whitehead duality criterion. We start with some (presumably familiar) homological algebra.

Lemma 10.2. *Let C_* and D_* be finitely-generated chain complexes of abelian groups and let $f: C_* \rightarrow D_*$ be a chain map. Suppose that for any field k , the induced map $f \otimes \text{Id}_k: C_* \otimes k \rightarrow D_* \otimes k$ is a quasi-isomorphism. Then f is a quasi-isomorphism.*

Proof. (Compare [Hat02, 3.A.7]) It suffices to prove that the mapping cone $\text{Cone}(f)$ of f is acyclic. Observe that $\text{Cone}(f \otimes \text{Id}_k) = \text{Cone}(f) \otimes k$, and by assumption $\text{Cone}(f \otimes \text{Id}_k)$ is acyclic for any field k . It follows from the universal coefficient theorem that if $E_* \otimes \mathbb{F}_p$ is acyclic for all primes p then E_* is acyclic. \square

Lemma 10.3. *Let C_* and D_* be finitely-generated chain complexes of abelian groups and $F: C_* \otimes D_* \rightarrow \mathbb{Z}$ a chain map. Write $D^* = \text{Hom}(D_*, \mathbb{Z})$ for the dual complex to D_* . Then the following conditions are equivalent:*

- (1) *The map $C_* \rightarrow D^*$ induced by F is a quasi-isomorphism.*
- (2) *For any field k , the map $H_*((C_* \otimes k)) \otimes H_*((D_* \otimes k)) \rightarrow k$ induced by F is a perfect pairing.*

Proof. (1) \implies (2) Tensoring with k , the map $f: C_* \otimes k \rightarrow D^* \otimes k = \text{Hom}(D_*, k)$ induced by F is a quasi-isomorphism. Suppose α is a non-trivial element of $H_*(C_* \otimes k)$. Then $f_*(\alpha) \neq 0 \in H_*(D^* \otimes k) = \text{Hom}(H_*(D_* \otimes k), k)$. Let $\beta \in H_*(D_* \otimes k)$ be an element on which $f_*(\alpha)$ evaluates non-trivially. Then $F_*(\alpha \otimes \beta) = f_*(\alpha)(\beta) \neq 0$.

Similarly, if $\beta \neq 0 \in H_*(D_* \otimes k)$ then there is an element $b \in H_*(D^* \otimes k)$ such that $b(\beta) \neq 0$. Since f is a quasi-isomorphism there is an $\alpha \in H_*(C_* \otimes k)$ so that $b = f(\alpha)$. Then $F_*(\alpha \otimes \beta) = f_*(\alpha)(\beta) = b(\beta) \neq 0$.

(2) \implies (1) We start by checking that for any field k , the map $f: C_* \otimes k \rightarrow D^* \otimes k$ is a quasi-isomorphism. To this end, suppose $\alpha \neq 0 \in H_*(C_* \otimes k)$. Then there exists $\beta \in H_*(D_* \otimes k)$ so that $F_*(\alpha \otimes \beta) \neq 0 \in k$. Then $f_*(\alpha)(\beta) = F_*(\alpha \otimes \beta) \neq 0$, so $f_*(\alpha) \neq 0$. Thus, f_* is injective. Next, given any element $b \in H_*(D^* \otimes k) = \text{Hom}(H_*(D_* \otimes k), k)$, since F_* is a perfect pairing there is an element $\alpha \in H_*(C_* \otimes k)$ so that $b(\cdot) = F_*(\alpha \otimes \cdot)$. Since $F_*(\alpha \otimes \cdot) = f_*(\alpha)(\cdot)$, we get $b = f_*(\alpha)$. So, f_* is surjective.

Thus, the map $f \otimes \text{Id}_k: C_* \otimes k \rightarrow D^* \otimes k$ is a quasi-isomorphism for any field k . So, by Lemma 10.2, the map $f: C_* \rightarrow D^*$ induced by F is a quasi-isomorphism, as desired. \square

Proposition 10.4. *Let X and Y be finite CW complexes and let $\mu: X \wedge Y \rightarrow S^n$ be a continuous map. Then μ witnesses S -duality between X and $\Sigma^{-n}Y$ if and only if for every field k and integer i , the induced map*

$$\tilde{H}_i(X; k) \otimes \tilde{H}_{n-i}(Y; k) \rightarrow \tilde{H}_n(S^n; k) = k$$

is a perfect pairing.

Proof. Let $C_*^{\text{cell}}(X)$ denote the reduced cellular chain complex of X and $C_{\text{cell}}^*(X)$ the reduced cellular cochain complex. The slant product map of Formula (10.1) is the map on homology induced by

$$\begin{aligned} C_*^{\text{cell}}(X) &\rightarrow C_{\text{cell}}^{n-*}(Y) = \text{Hom}(C_{n-*}^{\text{cell}}(Y), \mathbb{Z}) \\ x &\mapsto (y \mapsto \gamma(\mu_{\#}(x \wedge y))) \end{aligned}$$

In other words, this is the map induced by the pairing

$$\begin{aligned} C_*^{\text{cell}}(X) \otimes C_{n-*}^{\text{cell}}(Y) &\rightarrow C_n^{\text{cell}}(S^n) = \mathbb{Z} \\ x \otimes y &\mapsto \gamma(\mu_{\#}(x \wedge y)). \end{aligned}$$

So, the result is immediate from Lemma 10.3. \square

The other ingredients in the proof of Theorem 9 are some facts about functoriality of Khovanov homology and the Khovanov homotopy type.

Suppose that $F \subset [0, 1] \times \mathbb{R}^3$ is a link cobordism from L_0 to L_1 . Associated to F is a map $\Phi_F: Kh(L_0) \rightarrow Kh(L_1)$, well-defined up to multiplication by ± 1 ; see [Jac04, Kho06a, Bar05]. The following properties of Φ are immediate from the definition:

- (1) If $F = [0, 1] \times L$ is the identity cobordism from L to L then $\Phi_F = \pm \text{Id}: Kh(L) \rightarrow Kh(L)$.
- (2) If F_1 is a cobordism from L_0 to L_1 and F_2 is a cobordism from L_1 to L_2 , and $F_2 \circ F_1$ denotes the composition of F_2 and F_1 , then

$$\Phi_{F_2 \circ F_1} = \pm \Phi_{F_2} \circ \Phi_{F_1}.$$

- (3) If F is a cobordism from L_0 to L_1 and F' is a cobordism from L'_0 to L'_1 , and $F \amalg F'$ denotes the disjoint union of F and F' , which is a cobordism from $L_0 \amalg L'_0$ to $L_1 \amalg L'_1$, then for any field k ,

$$\begin{aligned} \Phi_{F \amalg F'} &= \pm \Phi_F \otimes \Phi_{F'}: Kh(L_0; k) \otimes Kh(L'_0; k) = Kh(L_0 \amalg L'_0; k) \\ &\rightarrow Kh(L_1 \amalg L'_1; k) = Kh(L_1; k) \otimes Kh(L'_1; k). \end{aligned}$$

Given a link L there is a canonical, genus 0 cobordism F from $L \amalg m(L)$ to the empty link.

Proposition 10.5. *For any field k , the map*

$$\Phi_F: Kh(L; k) \otimes Kh(m(L); k) = Kh(L \amalg m(L); k) \rightarrow Kh(\emptyset; k) = k$$

associated to the canonical cobordism from $L \amalg m(L)$ to the empty link is a perfect pairing.

Proof. This follows from Properties (1), (2) and (3) above via the usual snake-straightening argument in topological field theory (see, for instance, [Qui95, Lecture 7]). \square

As noted above, the cohomology groups of $\mathcal{X}_{Kh}(L)$ are the Khovanov homology of L . Since we are viewing Khovanov homology as covariant in the cobordism, it is more convenient to work with the homology groups of $\mathcal{X}_{Kh}(L)$. These can be understood as follows:

Lemma 10.6. *Let L be a link diagram. Then the cellular chain complex for $\mathcal{X}_{Kh}(L)$ agrees with the Khovanov complex for $m(L)$. In particular, $\tilde{H}_i(\mathcal{X}_{Kh}^j(L)) = Kh^{-i, -j}(m(L))$.*

Proof. This is immediate from the definitions. To wit, $\mathcal{X}_{Kh}(L)$ can be constructed as a CW complex whose reduced cellular cochain complex $C_{cell}^*(\mathcal{X}_{Kh}(L))$ is isomorphic to the Khovanov complex $\mathcal{C}_{Kh}(L)$ from Section 2.3. Therefore, the reduced cellular chain complex $C_{cell}^{*}(\mathcal{X}_{Kh}(L))$ is isomorphic to the dual complex $\text{Hom}(\mathcal{C}_{Kh}(L), \mathbb{Z})$. However, the dual complex is isomorphic to $\mathcal{C}_{Kh}(m(L))$, see [Kho00, Proposition 32]. In the language of Section 2.3, this isomorphism takes a Khovanov generator in $F_L(v)$ to a Khovanov generator in $F_{m(L)}(\vec{1} - v)$ by changing the labels on the circles of $\mathcal{P}_L(v) = \mathcal{P}_{m(L)}(\vec{1} - v)$ from x_+ to x_- and vice versa. The gradings do work out, and we get $\tilde{H}_i(\mathcal{X}_{Kh}^j(L)) = Kh^{-i, -j}(m(L))$. \square

Functoriality for the Khovanov spectrum has not yet been verified, but in [LS14b] we did associate maps to elementary cobordisms:

Proposition 10.7. *Let L_1 and L_2 be links in \mathbb{R}^3 and F a cobordism from L_1 to L_2 . Then there is a map of spectra*

$$\Psi_{m(F)}: \mathcal{X}_{Kh}(m(L_1)) \rightarrow \mathcal{X}_{Kh}(m(L_2))$$

so that the induced map $\Psi_{m(F),}: \tilde{H}_*(\mathcal{X}_{Kh}(m(L_1))) = Kh(L_1) \rightarrow Kh(L_2) = \tilde{H}_*(\mathcal{X}_{Kh}(m(L_2)))$ agrees with the cobordism map Φ_F up to sign.*

Proof. The corresponding statement in cohomology is immediate from [LS14b, Proposition 3.4, Lemma 3.6 and Lemma 3.7]; the homology statement comes from dualizing [LS14b, Proposition 3.4, Lemma 3.6 and Lemma 3.7] (cf. Lemma 10.6). \square

With these ingredients, we are now ready to verify the mirror theorem.

Proof of Theorem 9. The following argument was suggested to us by the referee for [LS14c].

By Proposition 10.4, it suffices to construct a map $\mathcal{X}_{Kh}(m(L)) \wedge \mathcal{X}_{Kh}(L) \rightarrow \mathbb{S}$ so that for any field k , the induced map

$$\tilde{H}_i(\mathcal{X}_{Kh}(m(L)); k) \otimes \tilde{H}_{-i}(\mathcal{X}_{Kh}(L); k) \rightarrow \tilde{H}_0(\mathbb{S}) = k$$

is a perfect pairing. By Proposition 10.5, applying Proposition 10.7 to the canonical cobordism from $L \amalg m(L)$ to the empty link gives such a cobordism. \square

Remark 10.8. Theorem 9 (perhaps) gives an obstruction to knots being amphicheiral: if K is amphicheiral then $\mathcal{X}_{Kh}^j(K)$ is Spanier-Whitehead dual to $\mathcal{X}_{Kh}^{-j}(K)$. Using KnotKit [See] (and the technique in [LS14c]), it is possible to verify that this obstruction does not give any additional restrictions on amphicheirality for knots up to 15 crossings, beyond those implied by Khovanov homology itself. Indeed, the only Khovanov homology-symmetric knots with 15 or fewer crossings for which Sq^2 is non-vanishing are 14_{8440}^n , 14_{9732}^n , 14_{21794}^n , 14_{22073}^n .

and 15_{139717}^n . In each of these cases, the non-Moore space summands of \mathcal{X}_{Kh} are copies of (various suspensions and desuspensions of) $\mathbb{R}P^5/\mathbb{R}P^2$ and $\mathbb{R}P^4/\mathbb{R}P^1$, and these summands (with appropriate grading shifts) are exchanged by $(i, j) \rightarrow (-i, -j)$. It remains open whether the homotopy type gives nontrivial restrictions for larger knots, though we expect that it does.

11. APPLICATIONS

In this section we give an application of the Künneth theorem for the Khovanov homotopy type to knot concordance. We begin with some background, continuing from Section 2.3. Recall that Rasmussen [Ras10] used the Lee deformation of the Khovanov complex [Lee05] (the specialization $(h, t) = (0, 1)$ of Definition 2.1) to define a concordance invariant $s_K^{\mathbb{Q}} \in 2\mathbb{Z}$. As the notation suggests, to define $s_K^{\mathbb{Q}}$, Rasmussen used Khovanov homology with coefficients in \mathbb{Q} , though any field of characteristic different from 2 would work as well. In [Bar05], Bar-Natan gave an analogue \mathcal{C}_{BN} of the Lee deformation that also works over \mathbb{F}_2 ; see also [Nao06, Tur06]. The Bar-Natan complex \mathcal{C}_{BN} is the specialization $(h, t) = (1, 0)$ of the universal complex \mathcal{C} of Definition 2.1. Let $s_K = s_K^{\mathbb{F}_2}$ denote the corresponding Rasmussen-type invariant; see also [LS14b, Section 2.2].

Since the formal variable h carried a quantum grading of (-2) , the differential in the Bar-Natan chain complex either preserves the quantum grading or increases it. Let $\mathcal{F}_q \mathcal{C}_{BN}$ denote the subcomplex of \mathcal{C}_{BN} supported in quantum grading q or higher. This defines a filtration on the Bar-Natan complex, and the associated graded object $\bigoplus_q \mathcal{F}_q \mathcal{C}_{BN} / \mathcal{F}_{q+2} \mathcal{C}_{BN}$ is the Khovanov chain complex \mathcal{C}_{Kh} .

In [LS14b], we constructed a refined s -invariant $s_+^{\text{Sq}^2}(K)$, defined as follows: The second Steenrod square Sq^2 for the Khovanov spectrum $\mathcal{X}_{Kh}(K)$ produces a map

$$\text{Sq}^2: Kh^{i,j}(K; \mathbb{F}_2) \rightarrow Kh^{i+2,j}(K; \mathbb{F}_2)$$

for each i, j . We consider configurations of the following form:

$$(11.1) \quad \begin{array}{ccccccc} \langle \tilde{a}, \tilde{b} \rangle & \longrightarrow & \langle \hat{a}, \hat{b} \rangle & \longleftarrow & \langle a, b \rangle & \longrightarrow & \langle \bar{a}, \bar{b} \rangle \\ \downarrow & & \downarrow & & \downarrow & & \parallel \\ Kh^{-2, s_K-1}(K; \mathbb{F}_2) & \xrightarrow{\text{Sq}^2} & Kh^{0, s_K-1}(K; \mathbb{F}_2) & \longleftarrow & H_0(\mathcal{F}_{s_K-1} \mathcal{C}_{BN}(K); \mathbb{F}_2) & \longrightarrow & H_0(\mathcal{C}_{BN}(K); \mathbb{F}_2). \end{array}$$

Then, we define [LS14b, Definition 1.2 and Lemma 4.2]

$$s_+^{\text{Sq}^2}(K) = \begin{cases} s_K & \text{if there does not exist such a configuration,} \\ s_K + 2 & \text{otherwise.} \end{cases}$$

In [LS14b, Theorem 1] we showed that $s_+^{\text{Sq}^2}(K)$ is a concordance invariant, and its absolute value is a lower bound for twice the four-ball genus:

$$2g_4(K) \geq |s_+^{\text{Sq}^2}(K)|.$$

Lemma 11.2. *Let K and L be two knots such that the following holds:*

- (1) $s_+^{\text{Sq}^2}(K) = s_K + 2$.
- (2) $Kh^{0, s_L+1}(L; \mathbb{F}_2) \xrightarrow{\text{Sq}^2} Kh^{2, s_L+1}(L; \mathbb{F}_2)$ is the zero map.
- (3) Either $Kh^{0, s_L+1}(L; \mathbb{F}_2) \xrightarrow{\text{Sq}^1} Kh^{1, s_L+1}(L; \mathbb{F}_2)$ is the zero map or $Kh^{-2, s_K-1}(K; \mathbb{F}_2) \xrightarrow{\text{Sq}^1} Kh^{-1, s_K-1}(K; \mathbb{F}_2)$ is the zero map.

Then $s_+^{\text{Sq}^2}(K \# L) = s_{K \# L} + 2 = s_K + s_L + 2$.

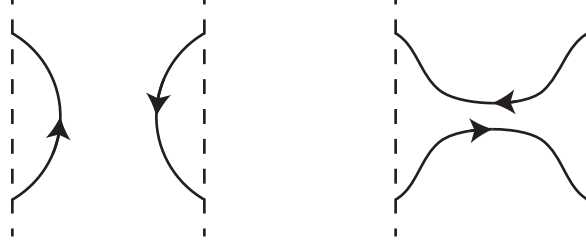


FIGURE 11.1. **Oriented connected sum.** Pieces of two knots and their connected sum, oriented coherently with the obvious saddle cobordism, are shown.

Proof. Consider the saddle cobordism from $K \amalg L$ to $K \# L$. Choose some orientations o_K , o_L , and $o_{K \# L}$, of the knots K , L , and $K \# L$, which are coherent with respect to the saddle cobordism (see Figure 11.1). Let $\{\pm o_K\}$, $\{\pm o_L\}$, and $\{\pm o_{K \# L}\}$ denote the sets of orientations of K , L , and $K \# L$. For any orientation o there is a corresponding generator $g(o)$ of $H_0(\mathcal{C}_{BN}; \mathbb{F}_2)$ [Lee05, Theorem 4.2]. We will use the following facts.

(1) $\mathcal{C}_{BN}(K \amalg L)$ is canonically identified with $\mathcal{C}_{BN}(K) \otimes \mathcal{C}_{BN}(L)$. The induced identification on the associated graded objects, $Kh(K \amalg L) \cong Kh(K) \otimes Kh(L)$, is the one induced from the equivalence from Theorem 1. (This is immediate from the definitions.)

(2) The sets $\{g(o_K), g(-o_K)\}$, $\{g(o_L), g(-o_L)\}$, $\{g(o_{K \# L}), g(-o_{K \# L})\}$, and $\{g(o_K) \otimes g(o_L), g(-o_K) \otimes g(o_L), g(o_K) \otimes g(-o_L), g(-o_K) \otimes g(-o_L)\}$ form bases of $H_0(\mathcal{C}_{BN}(K))$, $H_0(\mathcal{C}_{BN}(L))$, $H_0(\mathcal{C}_{BN}(K \# L))$, and $H_0(\mathcal{C}_{BN}(K \amalg L))$, respectively [Lee05, Section 4.4.3], [Tur06, Section 3.1].

(3) $g(o_L) + g(-o_L)$ has a cycle representative in $\mathcal{F}_{s_L+1}\mathcal{C}_{BN}(L)$ [LS14b, Proposition 2.6].

(4) The saddle cobordism map $\mathcal{C}_{BN}(K \amalg L) \rightarrow \mathcal{C}_{BN}(K \# L)$ preserves the homological grading and either increases the quantum grading or decreases it by exactly one. The induced map on Kh commutes with the map Sq^2 [LS14b, Theorem 4]. The induced map on $H_0(\mathcal{C}_{BN}; \mathbb{F}_2)$ is the following:

$$\begin{array}{ll} g(o_K) \otimes g(o_L) \mapsto g(o_{K \# L}) & g(-o_K) \otimes g(o_L) \mapsto 0 \\ g(o_K) \otimes g(-o_L) \mapsto 0 & g(-o_K) \otimes g(-o_L) \mapsto g(-o_{K \# L}). \end{array}$$

(This follows from the same argument as [Ras10, Proposition 4.1], since Turner's change of basis diagonalizes the Bar-Natan Frobenius algebra. Alternately, this claim is easy to check directly.)

Since $s_+^{Sq^2}(K) = s_K + 2$, there is a configuration as in Formula (11.1). Since $\{g(o_K), g(-o_K)\}$ also form a basis for $H_0(\mathcal{C}_{BN}(K); \mathbb{F}_2)$ (Fact (2)), after performing a change of basis if necessary, we may assume that $\bar{a} = g(o_K)$ and $\bar{b} = g(-o_K)$.

Using Fact (3), choose some configuration of the following form

$$\begin{array}{ccccc} \langle \bar{c} \rangle & \longleftarrow & \langle c \rangle & \longrightarrow & \langle g(o_L) + g(-o_L) \rangle \\ \downarrow & & \downarrow & & \downarrow \\ Kh^{0, s_L+1}(L; \mathbb{F}_2) & \longleftarrow & H_0(\mathcal{F}_{s_L+1}\mathcal{C}_{BN}(L); \mathbb{F}_2) & \longrightarrow & H_0(\mathcal{C}_{BN}(L); \mathbb{F}_2). \end{array}$$

Combining the two configurations and using the identification from Fact (1), we get the configuration

$$\begin{array}{ccccccc}
 \langle \tilde{a} \otimes \tilde{c}, \tilde{b} \otimes \tilde{c} \rangle & \longrightarrow & \langle \hat{a} \otimes \hat{c}, \hat{b} \otimes \hat{c} \rangle & \longleftarrow & \langle a \otimes c, b \otimes c \rangle & \longrightarrow & \langle g(o_K) \otimes (g(o_L) + g(-o_L)), \\
 & & & & & & g(-o_K) \otimes (g(o_L) + g(-o_L)) \rangle \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 Kh^{-2, s_K + s_L}(K \amalg L; \mathbb{F}_2) & \xrightarrow{\text{Sq}^2} & Kh^{0, s_K + s_L}(K \amalg L; \mathbb{F}_2) & \longleftarrow & H_0(\mathcal{F}_{s_K + s_L} \mathcal{C}_{BN}(K \amalg L); \mathbb{F}_2) & \longrightarrow & H_0(\mathcal{C}_{BN}(K \amalg L); \mathbb{F}_2).
 \end{array}$$

We should justify the leftmost horizontal arrow; that is, assuming $\text{Sq}^2(\tilde{a}) = \hat{a}$ and $\text{Sq}^2(\tilde{b}) = \hat{b}$, we need to show that $\text{Sq}^2(\tilde{a} \otimes \tilde{c}) = \hat{a} \otimes \hat{c}$ and $\text{Sq}^2(\tilde{b} \otimes \tilde{c}) = \hat{b} \otimes \hat{c}$. Since the identification $Kh(K \amalg L) \cong Kh(K) \otimes Kh(L)$ is induced from the identification $\mathcal{X}_{Kh}(K \amalg L) \cong \mathcal{X}_{Kh}(K) \wedge \mathcal{X}_{Kh}(L)$ of Theorem 1, cf. Fact (1),

$$\begin{aligned}
 \text{Sq}^2(\tilde{a} \otimes \tilde{c}) &= \text{Sq}^2(\tilde{a}) \otimes \tilde{c} + \text{Sq}^1(\tilde{a}) \otimes \text{Sq}^1(\tilde{c}) + \tilde{a} \otimes \text{Sq}^2(\tilde{c}) \\
 &= \hat{a} \otimes \hat{c}.
 \end{aligned}$$

The first equality is the Cartan formula; the second uses the lemma's hypotheses. Similarly, $\text{Sq}^2(\tilde{b} \otimes \tilde{c}) = \hat{b} \otimes \hat{c}$.

Now consider the image of this configuration under the saddle cobordism map. Using Fact (4), we get a configuration

$$\begin{array}{ccccccc}
 \langle \tilde{p}, \tilde{q} \rangle & \longrightarrow & \langle \hat{p}, \hat{q} \rangle & \longleftarrow & \langle p, q \rangle & \longrightarrow & \langle g(o_{K\#L}), g(-o_{K\#L}) \rangle \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 Kh^{-2, s_{K\#L} - 1}(K\#L; \mathbb{F}_2) & \xrightarrow{\text{Sq}^2} & Kh^{0, s_{K\#L} - 1}(K\#L; \mathbb{F}_2) & \longleftarrow & H_0(\mathcal{F}_{s_{K\#L} - 1} \mathcal{C}_{BN}(K\#L); \mathbb{F}_2) & \longrightarrow & H_0(\mathcal{C}_{BN}(K\#L); \mathbb{F}_2).
 \end{array}$$

By Fact (2), $\langle g(o_{K\#L}), g(-o_{K\#L}) \rangle = H_0(\mathcal{C}_{BN}(K\#L); \mathbb{F}_2)$; therefore, $s_+^{\text{Sq}^2}(K\#L) = s_{K\#L} + 2$. \square

We are almost ready to prove Corollary 1.5. First, we tabulate some invariants of the knots K that appear in its statement.

K	$\sigma(K)$	s_K	$\tau(K)$	$s_+^{\text{Sq}^2}(K)$	$g_4(K)$	$s_K^{\mathbb{Q}}$	$u(K)$
9_{42}	2	0	0	2	1	0	1
10_{136}	2	0	0	2	1	0	1
$m(11_{19}^n)$	4	2	1	4	2	2	2
$m(11_{20}^n)$	2	0	0	2	1	0	1
11_{70}^n	4	2	1	4	2	2	2
11_{96}^n	2	0	0	2	1	0	1

TABLE 11.1

The values of $s_K = s_K^{\mathbb{F}_2}$ and $s_+^{\text{Sq}^2}(K)$ are imported from [LS14c]; s_K can also be computed independently by Knotkit [See]. The values of the four-ball genus $g_4(K)$ and the unknotting number $u(K)$ are extracted from Knotinfo [CL]. The values of the signature $\sigma(K)$ and $s_K^{\mathbb{Q}}$ are extracted from the Knot Atlas [BM] (which follows the convention that positive knots have positive signature). The value of Ozsváth-Szabó's invariant $\tau(K)$ come from [BG12] (with the signs adjusted to agree with our conventions). We list $\tau(K)$, $s_K^{\mathbb{Q}}$, and $u(K)$ purely for the reader's interest.

Proof of Corollary 1.5. Certainly $g_4(K\#L) \leq g_4(K) + g_4(L)$, so we only need to show $g_4(K\#L) \geq g_4(K) + g_4(L)$.

First consider the knot $K \in \{9_{42}, 10_{136}, m(11_{19}^n), m(11_{20}^n), 11_{70}^n, 11_{96}^n\}$. In all cases $s_+^{\text{Sq}^2}(K) = s_K + 2 = 2g_4(K)$ (see Table 11.1).

Next consider the knot L . Draw L as the closure of a positive braid with n crossings and m strands. We will classify the generators of the Khovanov chain complex $\mathcal{C}_{Kh}(L)$ in quantum grading $n + 2 - m$ or less.

Let v be some vertex in the cube of resolutions for L . Assume there are c_v circles in the resolution at v . The smallest quantum grading over v is achieved by the Khovanov generator which labels all the c_v circles by x_- . The value of this smallest quantum grading is $n + |v| - c_v$.

Since all the crossings of L are positive, the resolution at the zero vertex $\vec{0}$ is the oriented resolution; therefore $c_{\vec{0}} = m$. If u is some vertex of weight one, i.e., $|u| = 1$, then the resolution at u is obtained from the oriented resolution by a merge; therefore $c_u = m - 1$. The resolution at any other vertex v is obtained from some weight one vertex by $|v| - 1$ merges or splits; therefore $c_v \leq m - 1 + |v| - 1 = m + |v| - 2$, with equality holding if and only if the resolution at v can be obtained from some weight one resolution by splits only.

It follows that the minimum quantum grading over any vertex $v \neq \vec{0}$ is at least $n - m + 2$; and over $\vec{0}$, the minimum quantum grading is $n - m$, which is attained only by the Khovanov generator that labels all the m circles by x_- . Observe that this is enough to compute the values of s_L , $g_4(L)$ and $g(L)$: $s_L \geq n - m + 1$, and Seifert's algorithm yields a surface of genus $(n - m + 1)/2$; therefore, the inequality

$$n - m + 1 \leq s_L \leq 2g_4(L) \leq 2g(L) \leq n - m + 1$$

leads to the equality

$$s_L = 2g_4(L) = 2g(L) = n - m + 1.$$

Next, we compute the Khovanov homology in these quantum gradings. We have $Kh^{*,q}(L; \mathbb{Z}) = 0$ for all $q < n - m$ and $Kh^{*,n-m}(L; \mathbb{Z})$ is \mathbb{Z} supported in homological grading zero. However, our interest lies in quantum grading $n - m + 2$; we show that $Kh^{*,n-m+2}(L; \mathbb{Z})$ is \mathbb{Z} supported in homological grading zero as well.

Number the m strands in the braid diagram from left to right. For $1 \leq i < m$, let n_i be the number of crossings in the diagram between the i^{th} and the $(i + 1)^{\text{st}}$ strands, i.e., the number of times σ_i occurs in the braid word. (Then $n = \sum_i n_i$, and since L is a knot, $n_i \geq 1$ for all i .) Number the n_i crossings from top to bottom. For $1 \leq i \leq m$, let $\mathbf{x}_i = (\vec{0}, x_i)$ be the Khovanov generator where x_i labels the i^{th} circle in the oriented resolution by x_+ , and the rest by x_- . For $1 \leq i < m$ and $\emptyset \neq J \subseteq \{1, \dots, n_i\}$, let $u_{i,J}$ be the weight $|J|$ vertex in the cube of resolutions where the 1-resolution is taken only at the crossings that appear in J between the i^{th} and $(i + 1)^{\text{st}}$ strands; and let $\mathbf{y}_{i,J}$ be the Khovanov generator living over $u_{i,J}$ where all the circles are labeled by x_- . From the discussion above it is clear that the Khovanov chain group $\mathcal{C}_{Kh}^{*,n+2-m}(L)$ is generated by these generators \mathbf{x}_i and $\mathbf{y}_{i,J}$. The differential is fairly straightforward:

$$\delta \mathbf{x}_i = \begin{cases} \sum_{j=1}^{n_1} \mathbf{y}_{1,\{j\}} & i = 1 \\ \sum_{j=1}^{n_i} \mathbf{y}_{i,\{j\}} + \sum_{j=1}^{n_{i-1}} \mathbf{y}_{i-1,\{j\}} & 1 < i < m \\ \sum_{j=1}^{n_{m-1}} \mathbf{y}_{m-1,\{j\}} & i = m \end{cases}$$

$$\delta \mathbf{y}_{i,J} = \sum_{\substack{J' \supset J \\ |J' \setminus J|=1}} \pm \mathbf{y}_{i,J'}.$$

Here we are using the standard sign assignment on the cube, and the signs are determined by the ordering of the n crossings.

Using the change of basis replacing \mathbf{x}_k by $\sum_{i=1}^k (-1)^i \mathbf{x}_i$ for all $1 \leq k \leq m$, it is easy to see that the chain complex $\mathcal{C}_{Kh}^{*,n+2-m}(L)$ is isomorphic to the following direct sum of cube complexes:

$$\mathcal{C}_{Kh}^{*,n+2-m}(L; \mathbb{Z}) \cong \mathbb{Z} \oplus \left(\bigoplus_{i=1}^{m-1} \left(\bigotimes_{j=1}^{n_i} (\mathbb{Z} \rightarrow \mathbb{Z}) \right) \right).$$

Therefore, $Kh^{*,n+2-m}(L; \mathbb{Z}) \cong \mathbb{Z}$, supported in homological grading zero (and is generated by the cycle $\sum_{i=1}^m (-1)^i \mathbf{x}_i$).

As an immediate consequence, we have that both the maps

$$\mathrm{Sq}^1: Kh^{0,s_L+1}(L; \mathbb{F}_2) \rightarrow Kh^{1,s_L+1}(L; \mathbb{F}_2) = 0$$

and

$$\mathrm{Sq}^2: Kh^{0,s_L+1}(L; \mathbb{F}_2) \rightarrow Kh^{2,s_L+1}(L; \mathbb{F}_2) = 0$$

vanish. Therefore, K and L satisfy the conditions of Lemma 11.2, so

$$2g_4(K \# L) \geq s_+^{\mathrm{Sq}^2}(K \# L) = s_K + s_L + 2 = 2g_4(K) + 2g_4(L). \quad \square$$

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