Stable power operations

Tyler Lawson, Saul Glasman

March 11, 2019

Abstract

For any E_∞ ring spectrum E, we show that there is an algebra $\mathcal{R}^E$ of stable power operations that acts naturally on the underlying spectrum of any E-algebra. Further, we show that there are maps of rings $E \to \mathcal{R}^E \to \mathrm{End}(E)$, where the latter determines a restriction from power operations to stable operations in the cohomology of spaces. In the case where E is the mod-p Eilenberg–Mac Lane spectrum, this realizes a natural quotient from Mandell's algebra of generalized Steenrod operations to the mod-p Steenrod algebra. More generally, this arises as part of a classification of endomorphisms of representable functors from an ∞-category $\mathcal{C}$ to spectra, with particular attention to the case where $\mathcal{C}$ is an $\mathcal{O}$-monoidal ∞-category.

1 Introduction

The Steenrod reduced power operations occupy the unusual position that they are often constructed in one way and used in another. We typically construct the Steenrod operations as *power operations*: the diagonal map $X \to X^p$ is equivariant for the action of the symmetric group Σ_p, and this produces cochain-level structure that results in cohomology-level operations. We typically think of the Steenrod operations as *stable operations*: they generate all possible stable natural transformations between cohomology groups, and as such are represented by maps between Eilenberg–Mac Lane spectra. In some sense, these are opposing perspectives. For example, if x is an element in $H^n(X;\mathbb{F}_2)$ then each of the identities $Sq^n(x) = x^2$ and $Sq^0(x) = x$ is obvious from one of these two points of view and not the other.

This divide becomes even more apparent if we replace ordinary mod-p cohomology with a generalized cohomology theory E^*, represented by a spectrum E. The stable operations on E-cohomology of degree n are in bijection with homotopy classes of maps $E \to \Sigma^n E$, and form a graded algebra which is always defined. Power operations are more subtle and need more from E. They require a refined multiplicative structure on E to define (e.g. see [BMMS86]) and have a more delicate composition structure which is only distributive on one side [Ber06, BW05, Rez09], but they appear in more contexts: they also appear in the E-homology of infinite loop spaces and the E-homology of commutative ring spectra. Restricting our attention to those power operations that preserve addition (called *additive* or *primitive*), we obtain a graded algebra. In the case where E is mod-p cohomology, this is called the algebra of (Araki–Kudo–)Dyer–Lashof operations or simply the Dyer–Lashof algebra.

In terms of stable homotopy theory, we may identify $E^n(X)$ with the homotopy group π_{-n} of the function spectrum $F(\Sigma^\infty_+ X, E) = E^X$. From this point of view, these groups have stable operations because $F(\Sigma^\infty_+ X, E)$ is a left module over the endomorphism algebra $F(E,E) = \mathrm{End}(E)$. On the other hand, if E is a commutative ring spectrum these groups have power operations because they are the homotopy groups of the commutative E-algebra E^X. If $\mathcal{S}$ is the category of spaces, these two roles played by E^X are encapsulated in the following diagram of categories that commutes up to natural isomorphism:

$$\begin{tikzcd}
\mathcal{S}^{op} \arrow[r, "F(\Sigma^\infty_+(-),E)"] \arrow[d, "E^{(-)}"'] & \mathrm{Mod}_{\mathrm{End}(E)} \arrow[d] \\
\mathrm{CAlg}_E \arrow[r] & \mathrm{Mod}_E
\end{tikzcd}$$

Here $\mathcal{S}$ is the category of spaces, Mod_R is the category of left R-modules, and CAlg_E is the category of commutative E-algebras.

In the case of ordinary mod-p cohomology, we have the Steenrod algebra $\mathcal{A}^*$ of stable operations on mod-p cohomology and the algebra $\mathcal{B}^*$ of generalized Steenrod operations [Man01, §5] acting on the homotopy of any commutative algebra over the Eilenberg-Mac Lane spectrum $H\mathbb{F}_p$. For the algebra $(H\mathbb{F}_p)^X$, these actions are related: there is a quotient map of algebras $\mathcal{B}^* \to \mathcal{A}^*$ that annihilates the two-sided ideal generated by a certain element $(1 - P^0)$.

Our goal is to systematically isolate what distinguishes the Steenrod operations and Araki–Kudo–Dyer–Lashof operations as special among power operations and gives them a relation to stable operations. For a given commutative E, we will construct a ring spectrum $\mathcal{R}^E$ of *stable power operations* that acts on the underlying spectrum of any commutative E-algebra, together with a map $\mathcal{R}^E \to \mathrm{End}(E)$ that determines how stable power operations act on the E-cohomology of spaces. Here are more precise statements of our main results.

Theorem 1.1. *For any commutative ring spectrum E, there is an associative ring spectrum $\mathcal{R}^E$ with a diagram of associative ring spectra*

$$E \to \mathcal{R}^E \to \mathrm{End}(E).$$

The ring $\mathcal{R}^E$ has a natural action on the underlying spectrum of a commutative E-algebra in a matter compatible with stable cohomology operations, in the sense that there *is a canonical lift in the following diagram:*

$$\begin{tikzcd}
\mathcal{S}^{op} \arrow[r, "F(\Sigma^\infty_+(-),E)"] \arrow[dd, "E^{(-)}"'] & \mathrm{Mod}_{\mathrm{End}(E)} \arrow[d] \\
 & \mathrm{Mod}_{\mathcal{R}^E} \arrow[d] \\
\mathrm{CAlg}_E \arrow[r] \arrow[ur, dotted] & \mathrm{Mod}_E
\end{tikzcd}$$

These three module categories, though initially appearing for quite different reasons, are connected in the following way. The ring $\mathrm{End}(E)$ is the ring of endomorphisms of the functor $E^{(-)}\colon \mathcal{S}^{op} \to \mathcal{S}p$; E is the ring of endomorphisms of the forgetful functor $\mathrm{Mod}_E \to \mathcal{S}p$; and $\mathcal{R}^E$ is the ring of endomorphisms of the forgetful functor $\mathrm{CAlg}_E \to \mathcal{S}p$. Moreover, these three functors are representable: the representing objects are cospectra.

- The functor $\mathcal{S}^{op} \to \mathcal{S}p$ is represented by the spectrum $E = \{E_n\}_{n\in\mathbb{Z}}$ itself. This is a spectrum object in $\mathcal{S}$ and hence a cospectrum object in $\mathcal{S}^{op}$.
- The forgetful functor $\mathrm{Mod}_E \to \mathcal{S}p$ is represented by the suspensions $\{\Sigma^{-n}E\}_{n\in\mathbb{Z}}$. These form a cospectrum object in Mod_E.
- The forgetful functor $\mathrm{CAlg}_E \to \mathcal{S}p$ is represented by the free E-algebras $\{\mathbb{P}_E(\Sigma^{-n}E)\}_{n\in\mathbb{Z}}$. These form a cospectrum object in CAlg_E.

Sufficient knowledge of these representing objects will enable us to calculate endomorphism rings. In the case of mod-p homology, we have the following.

Theorem 1.2. *For any prime p, the map $\mathcal{R}^{H\mathbb{F}_p} \to \mathrm{End}(H\mathbb{F}_p)$ of ring spectra becomes, on taking coefficient rings, a quotient map $(\mathfrak{B}^*)^\wedge \to \mathfrak{A}^*$ to the Steenrod algebra from the completion of the algebra of generalized Steenrod operations with respect to the excess filtration.*

As one benefit of having a spectrum level lift of the algebra $\mathfrak{B}^*$, we can employ several tools from the theory of secondary operations on the homotopy groups of ring and module spectra [BM09, Law18]. This means that we can immediately define secondary Dyer–Lashof operations, obtain relations such as Peterson–Stein relations, and establish compatibility of all of this structure with secondary Steenrod operations.

We note that the construction of $\mathcal{R}^E$ is outlined in course notes of Lurie [Lur07, Lecture 24], where it is used to show that power operations are compatible with geometric realization of simplicial objects.

1.1 Outline

We will begin in §2 and §3 by recalling a number of results from Lurie's work on functoriality of endomorphism objects, presentable categories, and the Yoneda embedding. The goal is the analogue of a result from basic category theory: if a category $\mathcal{D}$ is enriched in $\mathcal{V}$, then the enriched homomorphism objects $F(-,y)$ naturally take values in the category of left modules over the algebra $\mathrm{End}(y) = F(y,y)$ in $\mathcal{V}$. In §4 we will further discuss how spectrum objects in $\mathcal{C}$ represent functors $\mathcal{C}^{op} \to \mathcal{S}p$, and that a left A-module structure on Y is essentially equivalent to a lift of the represented functor to left A-modules. In §5 we will discuss how to calculate the universal example of the endomorphism algebra $\mathrm{End}(Y)$.

In §6 we will apply this to the case of stable presentable $\mathcal{O}$-monoidal ∞-categories and algebras in them. The spectra parametrizing stable operations turn out to be related to certain generalized symmetric power functors. We will briefly discuss the relation between stable operations and dual Goodwillie calculus.

In §7 we will give our main results regarding endomorphisms of commutative algebras and their relationship to stable cohomology operations. In §8, §9, and §10 we will give example calculations and show, in particular, the relationship with classical Dyer–Lashof operations. For this we will need some equivariant homotopy theory, which is carried out in the appendix §A.

1.2 Acknowledgements

The authors would like to thank Tobias Barthel, John Greenlees, Nick Kuhn, Akhil Mathew, Haynes Miller, Charles Rezk, and Dylan Wilson for discussions related to this work.

The second author was partially supported by NSF grant 1610408 and a grant from the Simons Foundation. He would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme HHH when work on this paper was undertaken. This work was supported by: EPSRC grant numbers EP/K032208/1 and EP/R014604/1.

2 Functoriality of endomorphisms

We will start with a brief rundown of results from [Lur17] on endomorphism objects.

- Operads and colored operads are encoded, in coherent category theory, by the notion of an ∞-operad [Lur17, §2].

- To the associative operad (whose algebras are associative algebras A), and to the left-module colored operad (whose algebras are pairs (A, M) of an algebra and a left A-module), there are associated ∞-operads $Assoc^{\otimes}$ and $\mathcal{LM}^{\otimes}$ [Lur17, 4.1.1.3, 4.2.1.7].

- A monoidal ∞-category $\mathcal{C}$ is expressed in terms of a coCartesian fibration $\mathcal{C}^{\otimes} \to Assoc^{\otimes}$, and an ∞-category $\mathcal{M}$ left-tensored over $\mathcal{C}$ is expressed as a coCartesian fibration $\mathcal{M}^{\otimes} \to \mathcal{LM}^{\otimes}$ [Lur17, 4.1.1.10, 4.2.1.19].

- A lax monoidal functor $\mathcal{C} \to \mathcal{D}$ is expressible as a map $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ over $Assoc^{\otimes}$, and similarly a lax functor $\mathcal{M} \to \mathcal{N}$ of categories left-tensored over $\mathcal{C}$ and $\mathcal{D}$ is expressible as a map $\mathcal{M}^{\otimes} \to \mathcal{N}^{\otimes}$ over $\mathcal{LM}^{\otimes}$. Strong monoidal functors and functors strongly compatible with left-tensoring are expressed in terms of coCartesian morphisms [Lur17, 2.1.3.6, 2.1.3.7].

- Right adjoints to strong monoidal functors and strongly compatible left-tensorings are lax [Lur17, 7.3.2.7].

- Associative algebra objects A in $\mathcal{C}$ are defined as certain sections of the map $\mathcal{C}^\otimes \to Assoc^\otimes$, and pairs of an associative algebra A in $\mathcal{C}$ and a left A-module M in $\mathcal{M}$ are defined as certain sections of the maps $\mathcal{M}^\otimes \to \mathcal{LM}^\otimes$ [Lur17, 2.1.3.1, 4.1.1.6, 4.2.1.13]. The category of algebras is denoted $\mathrm{Alg}(\mathcal{C})$ and the category of algebra-module pairs is denoted $\mathrm{LMod}(\mathcal{M})$, admitting forgetful functors to $\mathcal{C}$ and $\mathcal{C} \times \mathcal{M}$ respectively.

- In terms of these sections, a lax monoidal functor $\mathcal{C} \to \mathcal{D}$ therefore induces a functor $\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{D})$, and a lax tensored functor induces a compatible functor $\mathrm{LMod}(\mathcal{M}) \to \mathrm{LMod}(\mathcal{N})$.

- Further, a left-tensored category $\mathcal{M}$ is enriched over $\mathcal{C}$ if, for any objects M and M' in $\mathcal{M}$, there exists a function object $F_{\mathcal{M}}(M, M')$ in $\mathcal{C}$ such that maps $C \otimes M \to M'$ are equivalent to maps $C \to F_{\mathcal{M}}(M, M')$. Morphism objects are functorial in M and M' if they exist [Lur17, 4.2.1.28, 4.2.1.31].

- As a result, a lax monoidal functor $f : \mathcal{C} \to \mathcal{D}$ with a compatible lax functor $g : \mathcal{M} \to \mathcal{N}$ of left-tensored categories induces canonical maps $f(F_{\mathcal{M}}(M, M')) \to F_{\mathcal{N}}(gM, gM')$.

- If morphism objects exist, then for any fixed M the endomorphism object $\mathrm{End}(M) = F_{\mathcal{M}}(M, M)$ has a canonical algebra structure, and the category $\mathrm{LMod}(\mathcal{M}) \times_{\mathcal{M}} \{M\}$ of algebras acting on M is equivalent to the category of algebras $\mathrm{Alg}(\mathcal{C})_{/\mathrm{End}(M)}$ of algebras with a map $A \to \mathrm{End}(M)$ [Lur17, 4.7.1.40, 4.7.1.41].

- As a result, a lax monoidal functor $f : \mathcal{C} \to \mathcal{D}$ with a compatible lax functor $g : \mathcal{M} \to \mathcal{N}$ of left-tensored categories induces canonical maps of algebras $f(\mathrm{End}(M)) \to \mathrm{End}(gM)$, compatible with the $\mathrm{End}(M)$-module structure on M and the $\mathrm{End}(gM)$-module structure on $\mathrm{End}(gM)$.

3 Presentable categories and Yoneda embeddings

We will now discuss some results on presentable ∞-categories and their tensor product.

- There exist an ∞-category Pr^L (resp. Pr^R), whose objects are presentable ∞-categories and colimit-presenting (resp. accessible and limit-preserving) functors. The assignment that takes a colimit-preserving functor to a limit-preserving right adjoint gives an equivalence between these categories: $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D}) \cong \mathrm{Fun}^R(\mathcal{D}, \mathcal{C})$ [Lur09, 5.5.3.1, 5.5.3.4].

- There is a symmetric monoidal structure on Pr^L such that functors $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ are equivalent to functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ that preserve colimits in each variable separately, and there is an equivalence $\Upsilon \colon \mathcal{C} \otimes \mathcal{D} \cong \mathrm{Fun}^R(\mathcal{C}^{op}, \mathcal{D})$ [Lur17, 4.8.1.15, 4.8.1.17].

- A monoidal presentable category $\mathcal{C}$ is equivalent to a monoid in Pr^L, and a presentable category $\mathcal{M}$ left-tensored over $\mathcal{C}$ is equivalent to a left module over $\mathcal{C}$ in Pr^L. Under these conditions, $\mathcal{M}$ is enriched over $\mathcal{C}$ [Lur17, 4.2.1.33].

- For a monoidal presentable ∞-category $\mathcal{C}$, an associative algebra A in $\mathcal{C}$, and a presentable ∞-category $\mathcal{M}$ left-tensored over $\mathcal{C}$, there is an equivalence

$$\mathrm{LMod}_A(\mathcal{C}) \otimes_{\mathcal{C}} \mathcal{M} \xrightarrow{\sim} \mathrm{LMod}_A(\mathcal{M})$$

compatible with the underlying equivalence $\mathcal{C} \otimes_{\mathcal{C}} \mathcal{M} \to \mathcal{M}$ [Lur17, 4.8.4.6]. On objects, this sends a left A-module L in $\mathcal{C}$ and an object M in $\mathcal{M}$ to the left A-module $L \otimes M$ in $\mathcal{M}$.

- In particular, if $\mathcal{D}$ is presentable, we can take $\mathcal{M} = \mathcal{C} \otimes \mathcal{D}$ and get a natural commutative diagram

$$\begin{tikzcd}
\mathrm{LMod}_A(\mathcal{C} \otimes \mathcal{D}) \arrow[d] & \mathrm{LMod}_A(\mathcal{C}) \otimes \mathcal{D} \arrow[l] \arrow[r] \arrow[d] & \mathrm{Fun}^R(\mathcal{D}^{op}, \mathrm{LMod}_A(\mathcal{C})) \arrow[d] \\
\mathcal{C} \otimes \mathcal{D} & \mathcal{C} \otimes \mathcal{D} \arrow[l] \arrow[r] & \mathrm{Fun}^R(\mathcal{D}^{op}, \mathcal{C})
\end{tikzcd}$$

where the horizontal maps are equivalences. This allows us to construct a natural equivalence

$$\mathrm{LMod}_A(\mathcal{C} \otimes \mathcal{D}) \to \mathrm{Fun}^R(\mathcal{D}^{op}, \mathrm{LMod}_A(\mathcal{C}))$$

lifting the equivalence $\mathcal{C} \otimes \mathcal{D} \to \mathrm{Fun}^R(\mathcal{D}^{op}, \mathcal{C})$. We refer to this as the *Yoneda embedding for left A-modules.*

- For a small ∞-category S, there is a presheaf ∞-category $\mathcal{P}(S) = \mathrm{Fun}(S^{op}, \mathcal{S})$ which is presentable. For any set $\mathcal{K}$ of simplicial sets such that S has $\mathcal{K}$-indexed colimits, $\mathcal{P}(S)$ has a presentable subcategory $\mathcal{P}^{\mathcal{K}}(S) = \mathrm{Fun}^{\mathcal{K}}(S^{op}, \mathcal{S})$ of contravariant functors that take $\mathcal{K}$-indexed colimits to $\mathcal{K}$-indexed limits [Lur09, 5.1.0.1, 5.3.6.2, 5.5.1.1].

- There is a fully faithful Yoneda embedding $S \to \mathcal{P}(S)$ that preserves all limits that exist in S, and the Yoneda embedding factors through $\mathcal{P}^{\mathcal{K}}(S)$ [Lur09, 5.1.3.1, 5.1.3.2]. This sends an object X to a model for the functor $\mathrm{Map}_S(-, X)$.

- These presheaf categories have the universal property that, for any presentable ∞-category $\mathcal{C}$, we have

$$\mathrm{Fun}^L(\mathcal{P}^{\mathcal{K}}(S), \mathcal{C}) \cong \mathrm{Fun}^{\mathcal{K}}(S, \mathcal{C})$$

[Lur09, 5.1.5.6, 5.3.6.2]. This universal property gives rise to the identity

$$\mathcal{C} \otimes \mathcal{P}^{\mathcal{K}}(S) \cong \mathrm{Fun}^{\mathcal{K}}(S^{op}, \mathcal{C}).$$

- For a monoidal ∞-category $\mathcal{C}$, applying the Yoneda embedding for left A-modules to $\mathcal{P}^{\mathcal{K}}(S)$ produces a natural equivalence

$$\Upsilon_A\colon \ \mathrm{LMod}_A(\mathrm{Fun}^{\mathcal{K}}(S^{op},\mathcal{C})) \to \mathrm{Fun}^{\mathcal{K}}(S^{op},\mathrm{LMod}_A(\mathcal{C}))$$

  over $\mathrm{Fun}^{\mathcal{K}}(S^{op},\mathcal{C})$. In short, a left A-module structure on a functor is equivalent data to a lift of the functor to left A-modules.

4 The spectral Yoneda functor

We can specialize the constructions and results of §3 to the category of spectra. There are several important properties of spectra that make it simplify there.

- For a category S with pullbacks and a terminal object, there is a category

$$\mathcal{S}p(S) = \lim(\cdots \to S_* \xrightarrow{\Omega_{\mathcal{C}}} S_* \xrightarrow{\Omega_{\mathcal{C}}} S_* \xrightarrow{\Omega_{\mathcal{C}}} \cdots)$$

  of (Ω-)spectrum objects in $\mathcal{C}$. In particular,

$$\mathrm{Map}_{\mathcal{S}p(S)}(\{X_n\},\{Y_n\}) \simeq \lim_n \mathrm{Map}_{S_*}(X_n, Y_n)$$

  in this category.

- The category $\mathcal{S}p$ of spectra is presentable, and for any presentable category $\mathcal{C}$ there is an equivalence of the tensor product $\mathcal{S}p \otimes \mathcal{C}$ with the category

$$\mathcal{S}p(\mathcal{C}) = \lim(\cdots \to \mathcal{C}_* \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}_* \xrightarrow{\Omega_{\mathcal{C}}} \mathcal{C}_* \xrightarrow{\Omega_{\mathcal{C}}} \cdots)$$

  of spectrum objects in $\mathcal{C}$ [Lur17, 4.8.1.13]. In particular, we refer to the equivalence

$$\Upsilon\colon \ \mathcal{S}p(\mathcal{C}) \to \mathrm{Fun}^R(\mathcal{C}^{op},\mathcal{S}p)$$

  as the *spectral Yoneda embedding*.

- The category $\mathcal{S}p$ is idempotent under $\otimes$, and a presentable ∞-category $\mathcal{E}$ is a stable presentable ∞-category if and only if $\mathcal{E} \to \mathcal{S}p \otimes \mathcal{E}$ is an equivalence [Lur17, 4.8.2.18].

- If S is a small ∞-category that has pullbacks and a terminal object, the Yoneda embedding extends to a fully faithful embedding

$$\mathcal{S}p(S) \to \mathcal{S}p(\mathcal{P}(S)) \simeq \mathrm{Fun}(S^{op},\mathcal{S}p).$$

  If S has $\mathcal{K}$-indexed colimits, this naturally factors through the full subcategory $\mathrm{Fun}^{\mathcal{K}}(S^{op},\mathcal{S}p)$.

- Fix an associative algebra A in $\mathcal{S}p$. There is a full subcategory

$$\mathrm{LMod}_A(\mathcal{S}p(S)) \subset \mathrm{LMod}_A(\mathrm{Fun}(S^{op}, \mathcal{S}p)) \simeq \mathrm{Fun}(S^{op}, \mathrm{LMod}_A)$$

of left A-modules in the presheaf category whose underlying spectrum object lies in $\mathcal{S}p(S)$. If S has $\mathcal{K}$-filtered colimits, this naturally factors through the full subcategory $\mathrm{Fun}^{\mathcal{K}}(S^{op}, \mathrm{LMod}_A)$.

We can assemble this into the following statement.

Theorem 4.1. *Fix an algebra $A \in \mathcal{S}p$. For a small ∞-category S which has pullbacks, terminal objects, and $\mathcal{K}$-indexed colimits, there is a natural pullback diagram*

$$\begin{tikzcd} \mathrm{LMod}_A(\mathcal{S}p(S)) \arrow[r, "\Upsilon_A"] \arrow[d] & \mathrm{Fun}^{\mathcal{K}}(S^{op}, \mathrm{LMod}_A) \arrow[d] \\ \mathcal{S}p(S) \arrow[r, "\Upsilon"'] & \mathrm{Fun}^{\mathcal{K}}(S^{op}, \mathcal{S}p) \end{tikzcd}$$

where the horizontal maps are fully faithful embeddings. In particular, a left A-module structure on a spectrum object Y is equivalent to a lift of the functor $F(-,Y)$ from spectra to left A-modules.

Proposition 4.2. *The spectral Yoneda embedding is natural in the following senses.*

1. *Suppose S and T are categories with pullbacks and terminal objects and $g\colon S \to T$ is a functor with left adjoint f, and that S and T have $\mathcal{K}$-indexed colimits. Then there is an induced limit-preserving functor $g\colon \mathcal{S}p(S) \to \mathcal{S}p(T)$ with a commutative diagram*

$$\begin{tikzcd} \mathcal{S}p(S) \arrow[r, "\Upsilon"] \arrow[d, "g"'] & \mathrm{Fun}^{\mathcal{K}}(S^{op}, \mathcal{S}p) \arrow[d, "f^*"] \\ \mathcal{S}p(T) \arrow[r, "\Upsilon"'] & \mathrm{Fun}^{\mathcal{K}}(T^{op}, \mathcal{S}p). \end{tikzcd}$$

2. *Suppose A is an associative algebra in $\mathcal{S}p$. Then the above lifts to an essentially unique commutative diagram*

$$\begin{tikzcd} \mathrm{LMod}_A(\mathcal{S}p(S)) \arrow[r, "\Upsilon_A"] \arrow[d] & \mathrm{Fun}^{\mathcal{K}}(S^{op}, \mathrm{LMod}_A) \arrow[d, "f^*"] \\ \mathrm{LMod}_A(\mathcal{S}p(T)) \arrow[r, "\Upsilon_A"'] & \mathrm{Fun}^{\mathcal{K}}(T^{op}, \mathrm{LMod}_A). \end{tikzcd}$$

3. *Suppose $A \to B$ is a map of associative algebras in $\mathcal{S}p$. Then the forgetful maps from B-modules to A-modules are part of a commutative diagram*

$$\begin{tikzcd} \mathrm{LMod}_B(\mathcal{S}p(S)) \arrow[r, "\Upsilon_B"] \arrow[d] & \mathrm{Fun}^{\mathcal{K}}(S^{op}, \mathrm{LMod}_B) \arrow[d] \\ \mathrm{LMod}_A(\mathcal{S}p(S)) \arrow[r, "\Upsilon_A"'] & \mathrm{Fun}^{\mathcal{K}}(S^{op}, \mathrm{LMod}_A). \end{tikzcd}$$

Proof. Since g is a right adjoint, it preserves pullbacks and the terminal object and thus induces a functor $g\colon \mathcal{S}p(S) \to \mathcal{S}p(T)$. Under the spectral Yoneda embedding, for a spectrum $Y = \{Y_n\}$ in $\mathcal{S}p(S)$ the spectrum gY represents the functor $T^{op} \to \mathcal{S}p$ given by

$$X \mapsto \{\mathrm{Map}_T(X, gY_n)\} \simeq \{\mathrm{Map}_S(fX, Y_n)\}.$$

This makes the functors $\Upsilon(gY)$ and $(\Upsilon Y) \circ f$ naturally equivalent, as desired.

Suppose A is an associative algebra in $\mathcal{S}p$. For a left A-module Y, the object $gY \in \mathcal{S}p(T)$ represents a functor $\Upsilon(gY) \simeq \Upsilon(Y) \circ f$ with a chosen lift $\Upsilon_A(Y) \circ f$ from T^{op} to left A-modules, and is thus equivalent to a left A-module in $\mathcal{S}p(T)$. □

5 Stable endomorphisms

In this section we will briefly indicate how to determine function objects in categories of spectra. In particular, we would like to determine information about the homotopy groups of the endomorphism ring $\mathrm{End}_{\mathcal{S}p(\mathcal{C})}(Y)$, as well as discuss the relationship between these elements and stable cohomology operations.

Proposition 5.1. *Suppose $\mathcal{C}$ is a presentable ∞-category and $X, Y \in \mathcal{S}p(\mathcal{C})$. Then the function object $F_{\mathcal{S}p(\mathcal{C})}(X, Y) \in \mathcal{S}p$ is a spectrum whose k'th term is equivalent to a homotopy limit*

$$\mathrm{holim}_n \mathrm{Map}_{\mathcal{C}_*}(X_{n-k}, Y_n).$$

Proof. The function object $F(X, Y)$ has the universal property that maps $W \to F(X, Y)$ are equivalent to maps $W \otimes X \to Y$. Therefore, we can determine the structure of the spaces in the Ω-spectrum defining $F(X, Y)$:

$$\begin{aligned} F(X,Y)_k &\simeq \mathrm{Map}_{\mathcal{S}p}(S^{-k}, F(X,Y)) \\ &\simeq \mathrm{Map}_{\mathcal{S}p(\mathcal{C})}(S^{-k} \otimes X, Y) \\ &\simeq \mathrm{Map}_{\mathcal{S}p(\mathcal{C})}(S^{-k} \otimes (\operatorname*{hocolim}_n S^{k-n} \otimes \Sigma^\infty X_{n-k}), Y) \\ &\simeq \operatorname*{holim}_n \mathrm{Map}_{\mathcal{S}p(\mathcal{C})}(\Sigma^\infty X_{n-k}, S^n \otimes Y) \\ &\simeq \operatorname*{holim}_n \mathrm{Map}_{\mathcal{C}_*}(X_{n-k}, Y_n). \end{aligned}$$

□

Remark 5.2. Just as in the ordinary case, stable endomorphisms enjoy a compatibility with Mayer–Vietoris sequences. Suppose that Y is in $\mathcal{S}p(\mathcal{C})$ and that we have a homotopy pushout diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

of objects in $\mathcal{C}$. Then we obtain a homotopy pullback diagram

$$\begin{tikzcd} F(A,Y) & F(B,Y) \arrow[l] \\ F(C,Y) \arrow[u] & F(D,Y) \arrow[l] \arrow[u] \end{tikzcd}$$

of spectra, inducing a natural Mayer–Vietoris sequence:

$$\cdots \longrightarrow [\Sigma^\infty D, Y] \longrightarrow [\Sigma^\infty B, Y] \oplus [\Sigma^\infty C, Y] \longrightarrow [\Sigma^\infty A, Y] \longrightarrow \cdots$$

We know that the pullback diagram of spectra lifts to a diagram of left modules over $\mathrm{End}(Y)$; it is still a pullback diagram there. As a result, the natural action of the coefficient ring $\mathrm{End}(Y)_*$ on $[-,Y]$ is compatible with the connecting homomorphism in the Mayer–Vietoris sequence.

Definition 5.3. For any small category S and any functor $F\colon S^{op} \to \mathcal{S}p$, let $\mathrm{End}(F)$ be the enriched endomorphism object of F in $\mathrm{Fun}(S^{op}, \mathcal{S}p)$.

Since F is naturally a left $\mathrm{End}(F)$-module, we obtain the following.

Proposition 5.4. *For any small category S and any functor $F\colon S^{op} \to \mathcal{S}p$, the functor F has a natural lift to the category of left* $\mathrm{End}(F)$*-modules.*

If S has pullbacks and a terminal object and F is representable by a spectrum object $Y \in \mathcal{S}p(S)$, we have a canonical identification

$$\mathrm{End}(F) \cong \mathrm{End}_{\mathcal{S}p(S)}(Y).$$

Proof. This is is a consequence of the fact that the enriched Yoneda functor is an embedding. □

In particular, although $\mathrm{End}(F)$ generally depends on the size of the source category when F is not representable, $\mathrm{End}(Y)$ does not depend on S. This has the following consequence.

Corollary 5.5. *Suppose that $\mathcal{C}$ is a large category with pullbacks and a terminal object and $F\colon \mathcal{C}^{op} \to \mathcal{S}p$ is a functor represented by an object $Y \in \mathcal{S}p(\mathcal{C})$. Then for any small subcategory $S \subset \mathcal{C}$ containing Y that is closed under pullbacks and the terminal object, the forgetful functor $F|_S$ has a natural lift to the category* $\mathrm{LMod}_{\mathrm{End}(Y)}$*; moreover, for $S \subset S'$ these lifts can be made compatible.*

Remark 5.6. Unless $\mathcal{C}^{op}$ is presentable, to argue from this that there is a lift to all of $\mathcal{C}^{op}$ we need more, such as access to a larger Grothendieck universe. In this paper we are mostly concerned with the induced module structure on objects or small diagrams of them, for which this corollary is sufficient.

Proposition 5.7. *Suppose that $\mathcal{C}$ and $\mathcal{D}$ are categories with pullbacks and a terminal object, and $g\colon \mathcal{C} \to \mathcal{D}$ is a functor with left adjoint f. Suppose that $F\colon \mathcal{C}^{op} \to \mathcal{S}p$ is represented by an object $Y \in \mathcal{S}p(\mathcal{C})$. Then there a map of associative algebras* $\mathrm{End}(Y) \to \mathrm{End}(gY)$ *and a commutative diagram*

$$\begin{tikzcd}
\mathcal{D}^{op} \arrow[r, "\Upsilon"] \arrow[d, "f^*"'] & \mathrm{LMod}_{\mathrm{End}(gY)} \arrow[d] \\
\mathcal{C}^{op} \arrow[r, "\Upsilon"'] & \mathrm{LMod}_{\mathrm{End}(Y)}.
\end{tikzcd}$$

Proof. The map $\mathrm{LMod}_A(\mathcal{S}p(\mathcal{C})) \to \mathrm{LMod}_A(\mathcal{S}p(\mathcal{D}))$ of Proposition 4.2 implies that the $\mathrm{End}(Y)$-module structure on Y induces an $\mathrm{End}(Y)$-module structure on gY, or equivalently a map $\mathrm{End}(Y) \to \mathrm{End}(gY)$ of rings. By Theorem 4.1 we have a commutative diagram as follows:

$$\begin{tikzcd}
\mathrm{LMod}_{\mathrm{End}(Y)}(\mathcal{S}p(\mathcal{C})) \arrow[r, "g"] \arrow[d, "\Upsilon"] & \mathrm{LMod}_{\mathrm{End}(Y)}(\mathcal{S}p(\mathcal{D})) \arrow[d, "\Upsilon"] & \mathrm{LMod}_{\mathrm{End}(gY)}(\mathcal{S}p(\mathcal{D})) \arrow[l] \arrow[d, "\Upsilon"] \\
\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{LMod}_{\mathrm{End}(Y)}) \arrow[r, "f^*"'] & \mathrm{Fun}(\mathcal{D}^{op}, \mathrm{LMod}_{\mathrm{End}(Y)}) & \mathrm{Fun}(\mathcal{D}^{op}, \mathrm{LMod}_{\mathrm{End}(gY)}) \arrow[l]
\end{tikzcd}$$

The objects Y and gY, with their actions, determine compatible objects in the top row. Their images in the bottom row determine the compatible functors desired. □

6 Algebras and operads

In this section, we fix an ∞-operad $\mathcal{O}^{\otimes}$ and a stable presentable $\mathcal{O}$-monoidal ∞-category $\mathcal{C}^{\otimes}$ [Lur17, 3.4.4.1]. In paticular, we have a coCartesian fibration $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$, and for all objects X of $\mathcal{O}$ the fiber $\mathcal{C}_X$ is stable and presentable. Moreover, for any objects X_i and Y of $\mathcal{O}$ and any active arrow $\{X_i\} \to Y$ in $\mathcal{O}^{\otimes}$, the induced functor $\prod \mathcal{C}_{X_i} \to \mathcal{C}_Y$ is colimit-preserving in each variable.

We further recall the *symmetric power* functors. For any objects X and Y in $\mathcal{O}$ and any $d \geq 0$, there is a functor

$$\mathrm{Sym}^d_{\mathcal{O}, X \to Y}\colon \mathcal{C}_X \to \mathcal{C}_Y$$

from [Lur17, 3.1.3.9] (where it is denoted $\mathrm{Sym}^d_{\mathcal{O},Y}$). More specifically, for each d there is a map of spaces $\mathcal{P}_{X \to Y}(d) \to B\Sigma_d$ whose fiber is the space $\mathrm{Mul}_{\mathcal{O}}(X \oplus X \oplus \cdots \oplus X, Y)$ of d-ary active morphisms $\alpha\colon (\mathcal{C}_X)^d \to \mathcal{C}_Y$ with its natural Σ_d-action; the space

$\mathcal{P}_{X\to Y}(d)$ parametrizes d-fold power functors. The symmetric power functor is the colimit:

$$\mathrm{Sym}^d_{\mathcal{O},X\to Y}(C) = \operatorname*{colim}_{\alpha\in\mathcal{P}_{X\to Y}(d)} \alpha(C\oplus C\oplus\cdots\oplus C).$$

The functors $\mathrm{Sym}^d_{\mathcal{O},X\to Y}$ preserve sifted colimits.

For any $\mathcal{O}$-algebra A in $\mathcal{C}$, the $\mathcal{O}$-algebra structure determines action maps

$$\mathrm{Sym}^d_{\mathcal{O},X\to Y}(A(X)) \to A(Y)$$

by [Lur17, 3.1.3.11]. Moreover, the evaluation functor $\mathrm{ev}_X\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}_X$ has a colimit-preserving left adjoint $\mathrm{Free}_{\mathcal{O},X}\colon \mathcal{C}_X \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ such that, for any Y, the natural map

$$\coprod_{d\geq 0} \mathrm{Sym}^d_{\mathcal{O},X\to Y}(M) \to \mathrm{Free}_{\mathcal{O},X}(M)(Y)$$

is an equivalence [Lur17, 3.1.3.13].

This free-forgetful adjunction gives rise to the following result.

Proposition 6.1. *For any $M\in\mathcal{C}_X$, there is a functor*

$$\Upsilon_M = F_{\mathcal{C}_X}(M,\mathrm{ev}_X(-))\colon\ \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{S}p$$

that is represented by a cospectrum object

$$U(M) = \{\mathrm{Free}_{\mathcal{O},X}(\Omega^n M)\} \in \mathcal{S}p(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{op}).$$

Let A be the initial object in $\mathcal{O}$-algebras. Because the categories $\mathcal{C}_X$ are pointed, there are equivalences $A\sim \mathrm{Free}_{\mathcal{O},X}(*)$ for all X, $A(Y)\simeq \mathrm{Sym}^0_{\mathcal{O},X\to Y}(M)$, and natural augmentations

$$\mathrm{Free}_{\mathcal{O},X}(M)\to A$$

for all $M\in\mathcal{C}_X$.

Definition 6.2. For $M\in\mathcal{C}_X$, define $\widetilde{\mathrm{Free}}_{\mathcal{O},X}(M)$ to be the fiber of the augmentation $\mathrm{Free}_{\mathcal{O},X}(M)\to A$.

In particular, there is an equivalence

$$\coprod_{d> 0} \mathrm{Sym}^d_{\mathcal{O},X\to Y}(M) \to \widetilde{\mathrm{Free}}_{\mathcal{O},X}(M)(Y)$$

by stability. Maps $\mathrm{Free}_{\mathcal{O},X}(M)\to\mathrm{Free}_{\mathcal{O},Y}(N)$ of objects augmented over A are the same as maps $M\to\widetilde{\mathrm{Free}}_{\mathcal{O},Y}(N)(X)$. This gives rise to the following description of stable natural transformations.

Proposition 6.3. *For $M\in\mathcal{C}_X$ and $N\in\mathcal{C}_Y$, the spectrum* $\mathrm{StNat}(\Upsilon_M,\Upsilon_N)$ *representing stable natural transformations from Υ_M to Υ_N is of the form*

$$F_{\mathcal{C}_Y}\left(N, \lim_n \Sigma^n(\widetilde{\mathrm{Free}}_{\mathcal{O},X}(\Omega^n M)(Y)\right).$$

Proof. By Proposition 5.1, in degree k the spectrum of stable natural transformations is

$$
\begin{aligned}
&F(\mathrm{Map}_{\mathcal{S}p(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{op})}(U(M),U(N))\\
&\quad\simeq \lim_n \mathrm{Map}_{(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{op})_*}(\mathrm{Free}_{\mathcal{O},X}(\Omega^{n-k}M),\mathrm{Free}_{\mathcal{O},Y}(\Omega^n N))\\
&\quad\simeq \lim_n \mathrm{Map}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})/A}(\mathrm{Free}_{\mathcal{O},Y}(\Omega^n N),\mathrm{Free}_{\mathcal{O},X}(\Omega^{n-k}M))\\
&\quad\simeq \lim_n \mathrm{Map}_{\mathcal{C}_Y}(\Omega^n N,\widetilde{\mathrm{Free}}_{\mathcal{O},X}(\Omega^{n-k}M)(Y))\\
&\quad\simeq \lim_n \mathrm{Map}_{\mathcal{C}_Y}(\Sigma^k N,\Sigma^n\widetilde{\mathrm{Free}}_{\mathcal{O},X}(\Omega^m M)(Y))\\
&\quad\simeq \mathrm{Map}_{\mathcal{C}_Y}(\Sigma^k N,\lim_n \Sigma^n\widetilde{\mathrm{Free}}_{\mathcal{O},X}(\Omega^n M)(Y)).
\end{aligned}
$$

As k varies these reassemble into a limit of function spectra

$$F_{\mathcal{C}_Y}(N,\lim_n \Sigma^n\widetilde{\mathrm{Free}}_{\mathcal{O},X}(\Omega^n M)(Y)),$$

as desired. □

In particular, the decomposition of free algebras into symmetric powers allows us to describe a decomposition of stable transformations according to the number of inputs.

Definition 6.4. For $d>0$, define the spectrum parametrizing natural operations of *fixed weight d* by

$$\mathrm{StNat}^{\langle d\rangle}(\Upsilon_M,\Upsilon_N)\simeq F_{\mathcal{C}_Y}\left(N,\lim_n \Sigma^n\,\mathrm{Sym}^d_{\mathcal{O},X\to Y}(\Omega^n M)\right).$$

There is a natural transformation

$$\coprod_{d>0}\mathrm{StNat}^{\langle d\rangle}(\Upsilon_M,\Upsilon_N)\to\mathrm{StNat}(\Upsilon_M,\Upsilon_N)$$

which is an equivalence if N is compact in $\mathcal{C}_Y$.

Proposition 6.5. *Composition multiplies weights in the following sense. For $X\in\mathcal{C}_X$, $Y\in\mathcal{C}_Y$, and $Z\in\mathcal{C}_Z$ and $d,e>0$ there are pairings*

$$\mathrm{StNat}^{\langle e\rangle}(\Upsilon_N,\Upsilon_P)\otimes\mathrm{StNat}^{\langle d\rangle}(\Upsilon_M,\Upsilon_N)\to\mathrm{StNat}^{\langle de\rangle}(\Upsilon_M,\Upsilon_P)$$

of spectra parametrizing stable natural transformations.

Proof. Composition determines a map

$$\mathrm{Mul}_{\mathcal{O}}(\oplus^e Y,Z)\times\mathrm{Mul}_{\mathcal{O}}(\oplus^d X,Y)^e\to\mathrm{Mul}_{\mathcal{O}}(\oplus^{de}X,Z)$$

that is equivariant with respect to the action of the group of block-preserving permutations $\Sigma_d\wr\Sigma_e\subset\Sigma_{de}$. Composing with the diagonal on $\mathrm{Mul}_{\mathcal{O}}(\oplus X^d,Y)$ makes this into a $\Sigma_e\times\Sigma_d$-equivariant map

$$\mathrm{Mul}_{\mathcal{O}}(\oplus^e Y,Z)\times\mathrm{Mul}_{\mathcal{O}}(\oplus^d X,Y)\to\mathrm{Mul}_{\mathcal{O}}(\oplus^{de}X,Z),$$

inducing a natural transformation of functors

$$\mathrm{Sym}^{e}_{\mathcal{O},Y\to Z}\circ \mathrm{Sym}^{d}_{\mathcal{O},X\to Y}\to \mathrm{Sym}^{de}_{\mathcal{O},X\to Z}$$

that preserve trivial objects. We get an induced natural transformation

$$\begin{aligned}
&\lim_n(\Sigma^n\circ \mathrm{Sym}^{e}_{\mathcal{O},Y\to Z}\circ\Omega^n)\circ \lim_m(\Sigma^m\circ \mathrm{Sym}^{d}_{\mathcal{O},X\to Y}\circ\Omega^m)\\
&\to \lim_{n,m}(\Sigma^n\circ \mathrm{Sym}^{e}_{\mathcal{O},Y\to Z}\circ\Omega^n)\circ(\Sigma^m\circ \mathrm{Sym}^{d}_{\mathcal{O},X\to Y}\circ\Omega^m)\\
&\to \lim_{n}(\Sigma^n\circ \mathrm{Sym}^{e}_{\mathcal{O},Y\to Z}\circ\Omega^n)\circ(\Sigma^n\circ \mathrm{Sym}^{d}_{\mathcal{O},X\to Y}\circ\Omega^n)\\
&\to \lim_{n}(\Sigma^n\circ \mathrm{Sym}^{e}_{\mathcal{O},Y\to Z}\circ \mathrm{Sym}^{d}_{\mathcal{O},X\to Y}\circ\Omega^n)\\
&\to \lim_{n}(\Sigma^n\circ \mathrm{Sym}^{de}_{\mathcal{O},X\to Z}\circ\Omega^n).
\end{aligned}$$

In addition, the functor $A\mapsto \lim_m \Sigma^m\,\mathrm{Sym}^{e}_{\mathcal{O},Y\to Z}(\Omega^m A)$ preserves cofiber sequences, and therefore determines a stable functor $\mathcal{C}_Y\to\mathcal{C}_Z$; by stability it determines a natural map of function spectra

$$F_{\mathcal{C}_Y}(A,B)\to F_{\mathcal{C}_Z}\left(\lim_m \Sigma^m\,\mathrm{Sym}^{e}_{\mathcal{O},Y\to Z}(\Omega^m A),\lim_p \Sigma^p\,\mathrm{Sym}^{e}_{\mathcal{O},Y\to Z}(\Omega^p B))\right).$$

Combining these with the composition pairing for function spectra gives a natural map

$$\begin{aligned}
&F_{\mathcal{C}_Z}(P,\lim_n \Sigma^n\,\mathrm{Sym}^{e}_{\mathcal{O},Y\to Z}\,\Omega^n N)\otimes F_{\mathcal{C}_Y}(N,\lim_m \Sigma^m\,\mathrm{Sym}^{e}_{\mathcal{O},Y\to Z}\,\Omega^m M)\\
&\to F_{\mathcal{C}_Z}(P,\lim_n \Sigma^n\,\mathrm{Sym}^{de}_{\mathcal{O},X\to Z}\,\Omega^m M),
\end{aligned}$$

which us our desired pairing $\mathrm{StNat}^{\langle e\rangle}(\Upsilon_N,\Upsilon_P)\circ\mathrm{StNat}^{\langle d\rangle}(\Upsilon_M,\Upsilon_N)\to\mathrm{StNat}^{\langle de\rangle}(\Upsilon_M,\Upsilon_P)$. □

Remark 6.6. The formulas from Proposition 6.3 and Definition 6.4, describing the spectra that parametrize stable natural transformations, bear an evident relation to Goodwillie's calculus of functors. We will briefly elaborate on this relationship.

The free functor $\widetilde{\mathrm{Free}}_{\mathcal{O},X}$ and its components $\mathrm{Sym}^{d}_{\mathcal{O},X\to Y}$ are functors between stable ∞-categories and do not preserve colimits in general. For a pointed functor F, the assignment

$$F\mapsto \lim_n \Sigma^n\circ F\circ\Omega^n$$

that appears is the formula for the first *coderivative* of the functor F [McC99, Lemma 2.16]: that is, it is the universal example of a functor $\mathbb{D}^1F$ with a natural transformation $\mathbb{D}^1F\to F$ such that $\mathbb{D}^1F$ preserves fiber sequences.

From this perspective, the fact that composition preserves weight is a reflection of the decomposition of the free functor $\widetilde{\mathrm{Free}}_{\mathcal{O},X}$ into gradings, together with the natural transformation

$$(\mathbb{D}^1F)\circ(\mathbb{D}^1G)\to\mathbb{D}^1(F\circ G)$$

obtained from the universal property of the first coderivative.

7 Spaces, modules, and algebras

We now specialize the results of the previous sections to our categories of interest.

Proposition 7.1. *Fix a commutative ring spectrum E.*

1. *There is a forgetful functor* $\mathrm{CAlg}_E \to \mathrm{LMod}_E$. *It has a left adjoint* Free_E, *given by the free commutative E-algebra on a left E-module.*
2. *There is a functor $\mathcal{S}^{op} \to \mathrm{CAlg}_E$ sending a space X to the commutative E-algebra E^X. It has a left adjoint* $\mathrm{Map}_{\mathrm{CAlg}_E}(-, E)$.
3. *The composite functor $\mathcal{S}^{op} \to \mathrm{CAlg}_E \to \mathrm{LMod}_E$ has a composite left adjoint $\mathrm{LMod}_E \to \mathrm{CAlg}_E \to \mathcal{S}^{op}$. This left adjoint is given by $M \mapsto Map_{\mathrm{LMod}_E}(M, E)$.*

Proof. The adjunction between E-modules and commutative algebras is the specialization of the adjunction of §6 to the case of the commutative ∞-operad. The adjunction between spaces and commutative algebras follows from the presentability of CAlg_E; the limit-preserving functor $\mathrm{Map}_{\mathcal{S}}(X, \mathrm{Map}_{\mathrm{CAlg}_E(-,E)})$ is representable by a commutative algebra E^X. The composite is the functor

$$\begin{aligned} M &\mapsto \mathrm{Free}_E(M) \\ &\mapsto \mathrm{Map}_{\mathrm{CAlg}_E}(\mathrm{Free}_E(M), E) \\ &\simeq \mathrm{Map}_{\mathrm{LMod}_E}(M, E). \end{aligned}$$ □

Corollary 7.2. *The spectrum object $E \in \mathcal{S}p$ lifts to a cospectrum object*

$$Y = \{\Omega^n E\} \in \mathcal{S}p(\mathrm{LMod}_E^{op}),$$

corepresenting the forgetful functor $\mathrm{LMod}_E \to \mathcal{S}p$.

Proof. The composite map $\mathcal{S}p(\mathrm{LMod}_E^{op}) \to \mathcal{S}p(\mathcal{S})$ is given by levelwise application of the right adjoint $M \mapsto \mathrm{Map}_{\mathrm{LMod}_E}(M, E)$. Applied to $Y = \{\Omega^n E\}$, we get the spectrum $\{\mathrm{Map}_{\mathrm{LMod}_E}(\Omega^n E, E)\} \simeq E$. □

Definition 7.3. Let $\mathcal{R}^E$ be the endomorphism ring of the cospectrum

$$\{\mathrm{Free}_E(\Omega^n E)\} \in \mathcal{S}p(\mathrm{CAlg}_E^{op})$$

from Corollary 7.2. We refer to $\mathcal{R}^E$ as the algebra of *stable power operations* on commutative E-algebras.

The underlying ring spectrum of $\mathcal{R}^E$ is

$$\operatorname*{holim}_n \Sigma^n \mathrm{Free}_E(\Omega^n E)$$

by Proposition 6.1.

We recall the statement of Theorem 1.1.

Theorem. *For any commutative ring spectrum E, the algebra of stable power operations fitting into a diagram of associative ring spectra*

$$E \to \mathcal{R}^E \to \mathrm{End}(E).$$

The ring $\mathcal{R}^E$ has a natural action on the underlying spectrum of an commutative E-algebra in a matter compatible with stable cohomology operations, in the sense that there is a canonical lift in the following diagram:

$$\begin{tikzcd}
\mathcal{S}p^{op} \arrow[r, "F(\Sigma^\infty_+(-),E)"] \arrow[dd, "E^{(-)}"'] & \mathrm{Mod}_{\mathrm{End}(E)} \arrow[d] \\
& \mathrm{Mod}_{\mathcal{R}^E} \arrow[d] \\
\mathrm{CAlg}_E \arrow[r] \arrow[ur, dotted] & \mathrm{Mod}_E
\end{tikzcd}$$

Proof. The spectra $\mathrm{End}(E)$, $\mathcal{R}^E$, and E are, respectively, the endomorphism rings of the functors $\mathcal{S}^{op} \to \mathcal{S}p$, $\mathrm{CAlg}_E \to \mathcal{S}p$, and $\mathrm{LMod}_E \to \mathcal{S}p$. Therefore, these functors automatically lift to the appropriate categories of modules, and thus we can apply Proposition 5.7. □

8 Fixed weight

Definition 8.1. Suppose that $\mathcal{O}$ is an operad which is levelwise free and that E is a commutative ring spectrum. Then we write $Op^E_{\mathcal{O}}\langle d\rangle = \mathrm{StNat}^{\langle d\rangle}(\mathrm{id})$ for the spectrum parametrizing stable power operations of weight d on the underlying spectra of $\mathcal{O}$-algebras in E-modules (Definition 6.4):

$$Op^E_{\mathcal{O}}\langle d\rangle = \lim_n \Sigma^n \mathrm{Sym}^d_{\mathcal{O}}(\Omega^n E)$$

If $\mathcal{O}$ is the commutative operad, we simply write $Op^E\langle d\rangle$.

Remark 8.2. An element $\theta \in \pi_k Op^E_{\mathcal{O}}\langle d\rangle$ induces compatible operations $\pi_m A \to \pi_{k+m} A$ as m varies; these operations are detected by the forgetful maps

$$\pi_k Op^E_{\mathcal{O}}\langle d\rangle \to \pi_k \Sigma^{-m} \mathrm{Sym}^d_{\mathcal{O}}(\Omega^{-m} E).$$

Our goal in this section is to analyze $Op^E_{\mathcal{O}}\langle d\rangle$, using results from Appendix A on equivariant stable homotopy theory. We recall from Definition A.16 that γ is the $(d-1)$-dimensional reduced permutation representation of the symmetric group Σ_d, and from Proposition A.17 that the colimit of $S^{n\gamma}$ is a model for the classifying space $\widetilde{E\mathcal{T}}$ for the family of $\mathcal{T}$ consisting of subgroups of Σ_d that do not act transitively on $\{1, \dots, d\}$.

Proposition 8.3. *There are equivalences*

$$\begin{aligned} Op^E_{\mathcal{O}}\langle d\rangle &\simeq \lim_n E\otimes [\mathcal{O}(d)/\Sigma_d]^{-n\gamma} \\ &\simeq \lim_n F_{\Sigma_d}(S^{n\gamma}, E\otimes \mathcal{O}(d)) \\ &\simeq F_{\Sigma_d}(\widetilde{E\mathcal{T}}, \mathcal{O}(d)_+\otimes E). \end{aligned}$$

Proof. By definition,

$$Op^E_{\mathcal{O}}\langle d\rangle \simeq \lim_n \Sigma^n \operatorname{Sym}^d_{\mathcal{O}}(E\otimes \Omega^n S^0).$$

Expanding the symmetric power functor on such a free E-module $E\otimes\Omega^n S^0$, we get

$$\Sigma^n E\otimes \Big[\mathcal{O}(d)_+\otimes_{\Sigma_d}(S^{-n})^{\otimes d}\Big] \simeq E\otimes (\mathcal{O}(d)_+\otimes_{\Sigma_d} S^{-n\gamma}),$$

which is homotopy equivalent to the Thom spectrum

$$E\otimes [\mathcal{O}(d)/\Sigma_d]^{-n\gamma}.$$

This recovers the first description.

To recover the second, we note that the freeness of the action of Σ_d on $\mathcal{O}(d)$ implies that we have the Adams isomorphism

$$[E\otimes S^{-n\gamma}\otimes \mathcal{O}(d)_+]_{\Sigma_d} \to [E\otimes S^{-n\gamma}\otimes\mathcal{O}(d)_+]^{\Sigma_d}.$$

given by the transfer. The right-hand side is equivalent to the function spectrum

$$F_{\Sigma_d}(S^{n\gamma}, E\otimes \mathcal{O}(d)_+).$$

Taking limits in n gives $F_{\Sigma_d}(\widetilde{E\mathcal{T}}, E\otimes\mathcal{O}(d)_+)$, as desired. □

In particular, the fact that $\mathcal{O}(d)$ is acted on freely identifies $(E\Sigma_d)_+\otimes\mathcal{O}(d)\otimes E$ with $\mathcal{O}(d)\otimes E$, and thus we arrive at the following.

Corollary 8.4. *The spectrum $Op^E_{\mathcal{O}}\langle d\rangle$ is naturally equivalent to the opposite $\mathcal{T}$-Tate spectrum $(E\otimes\mathcal{O}(d))^{t^{op}\mathcal{T}}$ (Definition A.1).*

Theorem 8.5. *The spectrum $Op^E_{\mathcal{O}}\langle d\rangle$ is p-local if $d = p^k$ for some prime p, and is trivial otherwise.*

For any prime number p, there is a p-local equivalence between the spectrum of stable power operations of weight p and a desuspended Tate spectrum:

$$Op^E_{\mathcal{O}}\langle p\rangle \simeq \Sigma^{-1}(\mathcal{O}(p)\otimes E)^{t\Sigma_p}_{(p)} \simeq \Sigma^{-1}\Big[(\mathcal{O}(p)\otimes E)^{tC_p}\Big]^{\mathbb{F}_p^\times}$$

Proof. The first result is is Corollary A.15, and the second is Proposition A.18. □

When $p = 2$, this is essentially [GM95, 16.1] and at odd primes it is closely related to [BMMS86, II.5.3].

Remark 8.6. We can rephrase this in terms of Goodwillie calculus: for any d, the first coderivative $\mathbb{D}^1(\mathrm{Sym}^d_{\mathcal{O}})$ is identified with the opposite Tate spectrum $(E \otimes \mathcal{O}(d)_+)^{t^{op}\mathcal{T}}$.

This allows us to calculate using spectral sequence methods, because $S^{n\gamma}$ is the $n(d-1)$-skeleton in a model for EC_p. The map from stable power operations of degree k and weight d to power operations $\pi_m \to \pi_{k+m}$ of weight d is induced by the map

$$\pi_k \lim_n F_{C_p}(S^{n\gamma}, E \otimes \mathcal{O}(d)) \to \pi_k F_{C_p}(S^{m\gamma}, E \otimes \mathcal{O}(d)).$$

Remark 8.7. It should be possible to assemble these into a collection of compatible group homology spectral sequences that eventually identify with the Tate spectral sequence. In the case $p = 2$, this is more immediate: there are natural cofiber sequences

$$(\Sigma_2)_+ \otimes S^{m\gamma} \to S^{m\gamma} \to S^{(m+1)\gamma}$$

that give rise to a cellular filtration and a spectral sequence with E_1-term

$$E^1_{p,q} = E_q(\mathcal{O}(2)).$$

The E_2-term recovers the Tate cohomology

$$\widehat{H}^{1-p}(\Sigma_2; E_q(\mathcal{O}(2))),$$

and the spectral sequence converges to $\pi_{p+q} Op^E_{\mathcal{O}}\langle 2\rangle$. If the E_1-term is truncated by sending those terms with $p < m$ to zero, the spectral sequence converges to $\Sigma^m \mathrm{Sym}^2_{\mathcal{O}}(\Omega^m E)$.

9 Weight p operations

In this section we will give examples of the results of the previous sections.

Example 9.1. Suppose H is the mod-p Eilenberg–Mac Lane spectrum and $\mathcal{O}$ is the commutative operad. The Tate spectral sequence collapses, and we find

$$\pi_k Op^{H\mathbb{F}_p}\langle p\rangle \cong \begin{cases} \mathbb{F}_p & \text{if } k \equiv 0, -1 \mod 2(p-1), \\ 0 & \text{otherwise.} \end{cases}$$

In particular, when $p = 2$ and k is arbitrary there is a unique nonzero stable weight-2 operation Q^k that increases degree by k—the Dyer–Lashof operation of the same name. This is represented by the nonzero element in the Tate cohomology spectral sequence in degree k, and goes to zero when we truncate the spectral sequence to recover operations on elements in any degree $m < k$.

At odd primes, there are generating weight-p operations P^s in degree $2s(p-1)$ and βP^s in degree $2s(p-1)-1$. This carries the usual warning that βP^s is just notation: β itself is not an operation, and βP^s is not determined by P^s.

Example 9.2. The case where $E = \mathbb{S}$ is the sphere spectrum is governed by the Segal conjecture. The spectrum of stable power operations of weight p is $\Sigma^{-1}\mathbb{S}_{(p)}^{t\Sigma_p}$, which is the desuspension $\Sigma^{-1}\mathbb{S}_p^\wedge$ of the p-complete sphere. In particular, there is a certain generating stable power operation of degree -1 which we denote by c, and all the stable power operations of weight p on commutative $\mathbb{S}$-algebras are of the form $x \mapsto \alpha c(x)$ for an element $\alpha \in \pi_*(\mathbb{S})_p^\wedge$. This operation lifts Q^{-1} at the prime 2, or βP^0 at odd primes, to an operation on homotopy groups.

At the prime 2, we can use the Tate spectral sequence to note that c has the following properties:

1. On elements in nonnegative degrees, the operation c acts trivially.
2. Consider the Tate cohomology spectral sequence
$$\widehat{H}^s(C_2; \pi_t\mathbb{S}) \Rightarrow \pi_{t-s}(\mathbb{S})_2^\wedge.$$
The generator of the E_1-term of the Tate spectral sequence in filtration s, which is nontrivial at E_2 if and only if s is even, maps to the squaring operation $\pi_{-(1+s)} \to \pi_{-2(1+s)}$.
3. In particular, the generator c is represented by the generator of $\widehat{H}^0(C_2; \pi_0\mathbb{S})$. For a commutative $\mathbb{S}$-algebra A and an element $x \in \pi_{-1}(A)$, we have $c(x) = x^2$.
4. Further information about c can be extracted from further information about the Tate cohomology spectral sequence: these are related to Mahowald's root invariants [MR93] and the stable homotopy groups of stunted projective spaces. For example, for a commutative $\mathbb{S}$-algebra A and an element $x \in \pi_{-2}(A)$, we have the identity $2c(x) = \eta x^2$ (where $\eta \in \pi_1(\mathbb{S})$ is the Hopf invariant element); for a commutative $\mathbb{S}$-algebra A and an element $x \in \pi_{-3}(A)$, we have the identities $4c(x) = \eta^2 x^2$ and $\eta c(x) = \nu x^2$.

Example 9.3. Suppose E is the complex K-theory spectrum K and $\mathcal{O}$ is the commutative operad. Then $\Sigma^{-1}E_{(p)}^{t\Sigma_p} \simeq \Sigma^{-1}K \otimes \mathbb{Q}_p$, the p-adic rationalization of K by [GM95, 19.1]. Even though this group of ***stable*** operations is torsion-free, the action on any ***particular*** homotopy group π_k factors through a torsion quotient isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p$, and so the operations always take torsion values.

More generally, McClure has given a formula for the p-adic completion of the free K-algebra [BMMS86, §IX], and this can be used to show that
$$Op^K\langle d\rangle \cong \begin{cases} K & \text{if } d = 1, \\ \Sigma^{-1}K \otimes \mathbb{Q}_p/\mathbb{Z}_p & \text{if } d = p^k, \\ 0 & \text{otherwise.} \end{cases}$$
If we replace K by its localization $K_{(p)}$, we p-localize the result, which eliminates the summands for $d \neq p^k$.

Example 9.4. In the case of the E_n-operads, $\mathcal{O}(p)_+$ is a finite free Σ_p-complex and so the Tate spectrum of $E \otimes \mathcal{O}(p)_+$ vanishes. There are no interesting stable operations for E_n-algebras.

Example 9.5. If E is a Lubin–Tate cohomology theory, we can carry out the above construction in the $K(n)$-local category. However, Kuhn has shown that Tate spectra vanish in the $K(n)$-local category, and so there are no stable $K(n)$-local weight-p operations.

10 Operations in mod-p homology

In the case of ordinary mod-p homology, we can make use of the following complete calculation of the mod-p homology of free algebras.

Definition 10.1. Let p be a prime. An *algebra with Dyer–Lashof operations* is a graded-commutative $\mathbb{F}_p$-algebra A equipped with operations of the following type.

Case I: $p = 2$. There are Dyer–Lashof operations

$$Q^s\colon \pi_k A \to \pi_{k+s} A$$

satisfying the following relations.

Additivity: $Q^s(x+y) = Q^s(x) + Q^s(y)$.

Instability: $Q^s(x) = 0$ if $s < |x|$.

Squaring: $Q^s(x) = x^2$ if $s = |x|$.

Unitality: $Q^s(1) = 0$ if $s > 0$.

Cartan formula: $Q^s(xy) = \sum_{i+j=s} Q^i(x) Q^j(y)$.

Adem relations: $Q^r Q^s(x) = \sum \binom{i-s-1}{2i-r} Q^{r+s-i} Q^i x$.

Case II: $p > 2$. There are operations

$$P^s\colon \pi_k A \to \pi_{k+2s(p-1)} A,$$
$$\beta P^s\colon \pi_k A \to \pi_{k+2s(p-1)-1} A$$

(written as $\beta^\epsilon P^s$ for $\epsilon \in \{0,1\}$) satisfying the following relations.

Additivity: $\beta^\epsilon P^s(x+y) = \beta^\epsilon P^s(x) + \beta^\epsilon Q^s(y)$.

Instability: $\beta^\epsilon P^s(x) = 0$ if $2s + \epsilon > |x|$.

Squaring: $P^s(x) = x^p$ if $2s = |x|$.

Unitality: $P^s(1) = 0$ if $s > 0$.

Cartan formula:

$$P^s(xy) = \sum_{i+j=s} P^i(x) P^j(y)$$
$$\beta P^s(xy) = \sum_{i+j=s} \beta P^i(x) P^j(y) + \sum_{i+j=s} P^i(x) \beta P^j(y)$$

Adem relations:

$$P^r P^s(x) = \sum (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r} P^{r+s-i} P^i(x)$$
$$P^r \beta P^s(x) = \sum (-1)^{r+i} \binom{(p-1)(i-s)}{pi-r} \beta P^{r+s-i} P^i(x)$$
$$- \sum (-1)^{r+i} \binom{(p-1)(i-s)-1}{pi-r-1} P^{r+s-i} \beta P^i(x)$$

These identities still hold after formally applying β on the left and eliminating terms involving $\beta\beta$.

Theorem 10.2 ([BMMS86, IX.2.1]). *Let $H = H\mathbb{F}_p$ be the mod-p Eilenberg–Mac Lane spectrum. Then for any spectrum X, there is a natural isomorphism between the homotopy of the free commutative H-algebra* $\mathrm{Free}(H \otimes X) \simeq H \otimes \mathrm{Free}(X)$ *and the free algebra $\mathbb{Q}(H_*X)$ in the category of algebras with Dyer–Lashof operations.*

If $\{e_i\}$ is a basis of a graded vector space V, then $\mathbb{Q}(V)$ is a free graded-commutative algebra on admissible monomials of excess $e(I) \geq |e_i|$*: those monomials $Q^{s_1} \dots Q^{s_r} e_i$ (resp. $\beta^{\epsilon_1} P^{s_1} \dots \beta^{\epsilon_r} P^{s_r} e_i$ if $p > 2$) to which the Adem relations and instability relations cannot be applied.*

Theorem 10.3. *The ring $\pi_* \mathcal{R}^H$ of stable power operations for commutative H-algebras is the completion $(\mathfrak{B}^*)^\wedge$ of Mandell's algebra of generalized Steenrod operations with respect to the excess filtration.*

Proof. Theorem 10.2 allows us to identify the homotopy groups of the sequence of spectra

$$\dots \to \Sigma^n \widetilde{\mathrm{Free}}(\Omega^n H) \to \Sigma^{n-1} \widetilde{\mathrm{Free}}(\Omega^{n-1} H) \to \dots$$

as an inverse system

$$\dots \to \Sigma^n \widetilde{\mathbb{Q}}(\Omega^n \mathbb{F}_p) \to \Sigma^{n-1} \widetilde{\mathbb{Q}}(\Omega^{n-1} \mathbb{F}_p) \to \dots$$

obtained by removing the unit. In addition, the Dyer–Lashof operations are stable and the products are unstable: the maps in this directed system annihilate all products, while preserving the operations Q^s or $\beta^\epsilon P^s$. Therefore, this inverse system is equivalent to the quotient inverse system

$$\dots \to \{Q^I e_n \mid e(I) \geq -n\} \to \{Q^I e_{n-1} \mid e(I) \geq -n+1\} \to \dots$$

(resp. the analogue at odd primes). The structure maps in this inverse system are surjective, and so there are no $\lim^1$-terms. The homotopy groups of $\mathcal{R}^H$ are then the completion of the group with basis

$$\{Q^I e_n \mid I \text{ admissible}\}$$

with respect to the excess filtration. □

Theorem 10.4. *There is a sequence of maps of algebras*

$$H \to \mathcal{R}^H \to F(H,H)$$

which, on taking homotopy groups, induces the composite

$$\mathbb{F}_p \to (\mathfrak{B}^*)^\wedge \to \mathfrak{A}^*$$

from the completed algebra of generalized Steenrod operations to the Steenrod algebra.

Proof. Since the algebra $(\mathfrak{B}^*)^\wedge$ is generated by the operations Q^s at $p = 2$, or $\beta^\epsilon P^s$ at odd primes, this follows from the fact that in the cohomology of spaces we have the identities

$$Q^s(x) = \mathrm{Sq}^{-s}(x)$$

at $p = 2$ and

$$\beta^\epsilon P^s(x) = \beta^\epsilon \mathrm{P}^{-s}(x)$$

at $p > 2$. This is the defining property of Steenrod's reduced power operations [Ste62]. □

A Equivariant results

In the following sections, we will need to assemble a number of results from equivariant stable homotopy theory. Our goal is to analyze Tate spectra constructed using families of subgroups and a version using their opposites.

A.1 Tate and opposite Tate spectra for families

Definition A.1. Suppose that G is a group with a family $\mathcal{F}$ of subgroups. For any G-spectrum Z, we define the *$\mathcal{F}$-Tate spectrum* to be

$$Z^{t\mathcal{F}} = \widetilde{E\mathcal{F}} \otimes F(EG_+, Z),$$

and the *opposite $\mathcal{F}$-Tate spectrum* to be

$$Z^{t^{op}\mathcal{F}} = F(\widetilde{E\mathcal{F}}, EG_+ \otimes Z).$$

If $\mathcal{F}$ consists entirely of the trivial subgroup, we simply write Z^{tG} and $Z^{t^{op}G}$.

Remark A.2. For any subgroup H of G, the H-fixed points might also be referred to as the $\mathcal{F}$-Tate spectrum.

Remark A.3. Note that the Tate and opposite Tate spectra depend only on the underlying spectrum Z with G-action, and not on any genuine-equivariant structure.

The following result of Greenlees is referred to as Warwick duality, and it extends [GM95, 16.1].

Theorem A.4 ([Gre01, 2.5, 4.1]). *For any group G with a family $\mathcal{F}$ of subgroups, and any G-spectrum Z, there is a natural equivalence*

$$\widetilde{E\mathcal{F}} \otimes F(E\mathcal{F}_+, Z) \simeq \Sigma F(\widetilde{E\mathcal{F}}, E\mathcal{F}_+ \otimes Z).$$

Corollary A.5. *We have a natural equivalence $Z^{tG} \simeq \Sigma Z^{t^{op}G}$ for any group G and any G-spectrum Z.*

A.2 Transfer splittings

In some cases, we can identify a Tate spectrum as a split summand of a Tate spectrum occuring for a smaller subgroup. In this section we will identify some conditions under which this holds.

Lemma A.6. *Suppose that Z is a G-spectrum, and that $n = [G : H]$ acts invertibly on Z. Then the composite $\mathrm{Tr}_H^G \mathrm{Res}_H^G$ acts on $\pi_*^G(Z^{t\mathcal{F}})$ and $\pi_*^G(Z^{t^{op}\mathcal{F}})$ by an isomorphism.*

Proof. Consider the map $\mathbb{S} \to \mathrm{End}(EG_+[1/n])$ of G-equivariant ring spectra. Applying π_0^G gives the map $A(G) \to \pi_0((\mathbb{S}[1/n])^{hG})$, where the former is the Burnside ring and the latter is complete in the topology defined by the augmentation I of the quotient $A(G) \to \mathbb{Z}$. The element $[G/H] \in \pi_0^G(S^0)$ maps in $A(G)/I$ to the unit $n \in \mathbb{Z}[1/n]$. Since $[G/H]$ reduces to a unit mod the augmentation ideal, and the ring is complete, $[G/H]$ is a unit in $\mathrm{End}(EG_+[1/n])$.

The term EG_+ in the formula for Tate spectra implies that map $\mathbb{S} \to \mathrm{End}(Z^{t\mathcal{F}})$ of equivariant ring spectra factors through the endomorphism ring $\mathrm{End}(EG_+[1/n])^{op}$, and similarly $\mathbb{S} \to \mathrm{End}(Z^{t^{op}\mathcal{F}})$ factors through $\mathrm{End}(EG_+[1/n])$. As a result, in both of these rings $[G/H] \in A(G)$ maps to a unit. However, this element represents the endomorphism $\mathrm{Tr}_H^G \mathrm{Res}_H^G$. □

Lemma A.7. *Suppose that Z is a G-spectrum and that H is a subgroup such that, for all x not in the normalizer $N_G(H)$, the group $H \cap {}^xH$ is in $\mathcal{F}$. Then the composite $\mathrm{Res}_H^G \mathrm{Tr}_H^G$ acts on $\pi_*^H(Z^{t\mathcal{F}})$ and $\pi_*^H(Z^{t^{op}\mathcal{F}})$ by*

$$\alpha \mapsto N(\alpha) = \sum_{w \in W_G(H)} w \cdot \alpha.$$

Proof. The objects $Z^{t\mathcal{F}}$ and $Z^{t^{op}\mathcal{F}}$ are modules over the smashing localization $\widetilde{E\mathcal{F}}$, and such modules are characterized by the property that their K-fixed points are trivial for all $K \in \mathcal{F}$. Therefore, $\pi_0^K(Z^{t\mathcal{F}}) = \pi_0^K(Z^{t^{op}\mathcal{F}}) = 0$ for all $K \in \mathcal{F}$.

The double coset formula says that

$$\mathrm{Res}_H^G \mathrm{Tr}_H^G \alpha = \sum_{[x] \in H \backslash G/H} \mathrm{Tr}^H_{H \cap {}^xH}(x \cdot \mathrm{Res}^H_{x^{-1}H \cap H} \alpha).$$

By assumption, the groups $H \cap {}^xH$ are in $\mathcal{F}$ for x not in the normalizer of H, and so those terms in the sum can be eliminated. We get a reduced formula

$$\mathrm{Res}_H^G \mathrm{Tr}_H^G \alpha = \sum_{x \in W_G(H)} x \cdot \alpha$$

as desired. □

Proposition A.8. *Suppose that G is a group with a family $\mathcal{F}$ of subgroups, that H is a subgroup such that $H \cap {}^xH$ is in $\mathcal{F}$ for all x not in the normalizer of H, and that Z is a G-spectrum that is acted on invertibly by $[G : H]$. Then the restriction from G to H induces an isomorphism*

$$\pi_*^G(Z^{t\mathcal{F}}) \to \left[\pi_*^H(Z^{t\mathcal{F}})\right]^{W_G(H)}$$

and similarly for opposite Tate spectra.

Proof. The restriction naturally maps into the $W_G(H)$-fixed points. Consider the triple composition

$$\pi_*^G(Z^{t\mathcal{F}}) \xrightarrow{\mathrm{Res}_H^G} \left[\pi_*^H(Z^{t\mathcal{F}})\right]^{W_G(H)} \xrightarrow{\mathrm{Tr}_H^G} \pi_*^G(Z^{t\mathcal{F}}) \xrightarrow{\mathrm{Res}_H^G} \left[\pi_*^H(Z^{t\mathcal{F}})\right]^{W_G(H)}$$

By Lemma A.6, the double composite $\pi_*^G(Z^{t\mathcal{F}}) \to \pi_*^G(Z^{t\mathcal{F}})$ is an isomorphism. By Lemma A.7, the double composite $\pi_*^H(Z^{t\mathcal{F}}) \to \pi_*^H(Z^{t\mathcal{F}})$ is the norm: however, on $W_G(H)$-fixed elements it is multiplication by $|W_G(H)|$, a divisor of $[G : H]$, and hence is an isomorphism. Therefore, both the restriction and transfer maps are isomorphisms. The same proof holds for opposite Tate spectra. $\square$

Corollary A.9. *Suppose that G is a group with an inclusion $\mathcal{F} \subset \mathcal{F}'$ of families of subgroups and that H is a subgroup such that the restrictions $\mathcal{F} \cap H$ and $\mathcal{F}' \cap H$ of these families to H are equal. Assume that $[G : H]$ acts invertibly on a G-spectrum Z and that $H \cap {}^xH$ is in $\mathcal{F}$ for all x not in the normalizer of H. Then the induced maps*

$$Z^{t\mathcal{F}} \to Z^{t\mathcal{F}'}$$

and

$$Z^{t^{op}\mathcal{F}'} \to Z^{t^{op}\mathcal{F}}$$

are equivalences.

Combining this with the Warwick duality of Theorem A.4, we obtain the following result.

Proposition A.10. *Suppose that G is a group with a family $\mathcal{F}$ of subgroups and that H is a subgroup such that the restriction $\mathcal{F} \cap H$ contains only the trivial group. Assume that $[G : H]$ acts invertibly on a G-spectrum Z and $H \cap {}^xH$ is trivial for all x not in the normalizer of H. Then there is a shifted equivalence between the Tate and opposite Tate spectra:*

$$(Z^{t\mathcal{F}})^G \simeq \left[Z^{tH}\right]^{W_G(H)} \simeq \Sigma(Z^{t^{op}\mathcal{F}})^G.$$

A.3 Localization properties

In this section, we will show that (opposite) Tate spectra naturally decompose as a finite direct sum over localizations.

Definition A.11. Suppose that G is a group with a family $\mathcal{F}$ of subgroups. We write $d(\mathcal{F})$ for the greatest common divisor of the indices $[G : H]$ for $H \in \mathcal{F}$. If $\mathcal{F}$ is understood, we simply write d.

Lemma A.12. *Suppose that G is a group with a family $\mathcal{F}$ of subgroups. If Z is a G-spectrum that is acted on invertibly by $d(\mathcal{F})$ then the spectra $Z^{t\mathcal{F}}$ and $Z^{t^{op}\mathcal{F}}$ are trivial.*

Proof. The map from the Burnside ring $A(G) = \pi_0^G(\mathbb{S})$ to the endomorphism ring of $Z^{t\mathcal{F}}$ factors through three rings:

1. $\pi_0^G \operatorname{End}(Z)$, where d acts by a unit;
2. $\pi_0^G \operatorname{End}(\widetilde{E\mathcal{F}}) \cong \pi_0^G \widetilde{E\mathcal{F}}$, where the ideal J generated by the basis elements $[G/H]$ for $H \in \mathcal{F}$ is sent to zero; and
3. $\pi_0^G \operatorname{End}(EG_+)^{op} \cong \pi_0 \mathbb{S}^{hG}$, which is complete in the topology generated by the augmentation ideal I.

Therefore, the result factors through the ring $(A_I^\wedge)[1/d]/J$. Since the ring is a finitely generated abelian group, this is the completion of the ring $A[1/d]/J$ with respect to the topology defined by the image of I; however, $A/(I+J) = \mathbb{Z}/d$, and so this completion is trivial. Therefore, $\operatorname{End}(Z^{t\mathcal{F}})$ must be the zero ring.

The same argument applies to $Z^{t^{op}\mathcal{F}}$, except that we must use the opposite rings $\operatorname{End}(\widetilde{E\mathcal{F}})^{op}$ and $\operatorname{End}(EG_+)$. □

Lemma A.13. *For any finite group G acting on a spectrum Z and any family $\mathcal{F}$, the natural localization maps induce decompositions*

$$Z^{t\mathcal{F}} \simeq \bigoplus_{q \mid d(\mathcal{F})} (Z_{(q)})^{t\mathcal{F}}$$

and

$$Z^{t^{op}\mathcal{F}} \simeq \bigoplus_{q \mid d(\mathcal{F})} (Z_{(q)})^{t^{op}\mathcal{F}}$$

where the sums range over the primes q dividing $d(\mathcal{F})$. In particular, localizing at a prime q commutes with taking Tate spectra or opposite Tate spectra.

Proof. First note that there is a natural G-equivariant fiber sequence

$$\bigoplus_{q \mid d} \Sigma^{-1} Z[1/q]/Z \to Z \to Z[1/d].$$

We also have a fiber sequence

$$\Sigma^{-1} Z[1/q]/Z \to Z_{(q)} \to Z_{\mathbb{Q}}.$$

The $\mathcal{F}$-Tate spectrum and opposite $\mathcal{F}$-Tate spectra for $Z[1/d]$ or $Z_{\mathbb{Q}}$ are trivial by Lemma A.12. Therefore,

$$Z^{t\mathcal{F}} \simeq \bigoplus_q \Sigma^{-1} (Z_{(q)}/Z)^{t\mathcal{F}} \simeq \bigoplus_q (Z_{(q)})^{t\mathcal{F}}$$

and similarly for opposite Tate spectra. □

A.4 Symmetric groups

We can now apply these results to the symmetric groups.

Proposition A.14. *Let $\mathcal{T}$ be the family of subgroups of Σ_n that do not act transitively on $\{1,\ldots,n\}$. If $n = p^k$ for some prime p, then $d(\mathcal{T})$ is p; otherwise $d(\mathcal{T}) = 1$.*

Proof. Any subgroup that does not act transtively is a subgroup of some conjugate of $\Sigma_k \times \Sigma_{n-k}$ for some k. The greatest common divisor of the indices $[\Sigma_n : \Sigma_k \times \Sigma_{n-k}]$ is the greatest common divisor of the binomial coefficients $\binom{n}{k}$ for $0 < k < n$. This is p if $n = p^k$ and 1 otherwise. □

Corollary A.15. *For any Σ_n-spectrum Z, $Z^{t\mathcal{T}}$ and $Z^{t^{op}\mathcal{T}}$ are p-local if $n = p^k$ for some prime p, and trivial otherwise. If Z is p-local, then these are trivial unless $n = p^k$.*

Definition A.16. For a natural number d, let γ be the $(d-1)$-dimensional reduced permutation representation of the symmetric group Σ_d, and S^γ the 1-point compactification viewed as a Σ_d-space. We will write $S^{n\gamma}$ for the n-fold smash power, and for $P \to X$ a principal Σ_d-bundle and $n \in \mathbb{Z}$ we write $X^{n\gamma}$ for the Thom spectrum of the associated virtual bundle $n\gamma$ on X.

In particular, as Σ_d-spaces there is an equivalence $S^1 \wedge S^\gamma \simeq (S^1)^{\wedge d}$. The spaces $S^{n\gamma}$ form a directed sequence of Σ_d-spaces

$$S^0 \to S^\gamma \to S^{2\gamma} \to S^{3\gamma} \to \cdots,$$

and the colimit turns out to be a classifying space for a family of subgroups as follows.

Proposition A.17. *Let $\mathcal{T}$ be the family of subgroups $H \subset \Sigma_d$ that do not act transitively on $\{1,\ldots,d\}$. Then there is a cofiber sequence*

$$E\mathcal{T}_+ \to S^0 \to \operatorname{colim} S^{n\gamma}$$

of based spaces. In particular, the colimit is a model for $\widetilde{E\mathcal{T}}$.

Proof. The inclusion of the unit sphere into the unit disc determine cofiber sequences

$$S(n\gamma)_+ \to S^0 \to S^{n\gamma}$$

that are compatible as n varies. We have

$$S(n\gamma)^H = S(n\gamma^H) \cong S^{n\dim(\gamma^H)-1}.$$

If $\gamma^H \neq 0$, the connectivity of this space grows in an unbounded fashion; if $\gamma^H = 0$, this space is empty for all n. Therefore, the colimit of the $S(n\gamma)$ is a model for $E\mathcal{F}$, where $\mathcal{F}$ is the family of subgroups H such that $\dim(\gamma^H) > 0$. However, by definition of γ, the dimension of γ^H is one less than the number of orbits for the action of H on $\{1,\ldots,d\}$, and so $\mathcal{F} = \mathcal{T}$. □

Proposition A.18. *Let W be the Weyl group $W_{\Sigma_p}(C_p) \cong \mathbb{F}_p^\times$. Then for any Σ_p-spectrum Z, the p-localization of the Tate spectrum for Σ_p splits off from the Tate spectrum for C_p, and both are identified with a shifted opposite Tate spectrum:*

$$Z_{(p)}^{t\Sigma_p} \simeq (Z^{tC_p})^W \simeq \Sigma Z_{(p)}^{t^{op}\Sigma_p}.$$

Proof. By Lemma A.13, it suffices to assume that Z is p-local. The subgroup C_p has index prime to p, and its intersection with the family $\mathcal{T}$ consists only of the trivial group. Therefore, the result follows by Proposition A.18. □

References

[Ber06] Julia E. Bergner, *Rigidification of algebras over multi-sorted theories*, Algebr. Geom. Topol. **6** (2006), 1925–1955. MR 2263055

[BM09] Hans-Joachim Baues and Fernando Muro, *Toda brackets and cup-one squares for ring spectra*, Comm. Algebra **37** (2009), no. 1, 56–82. MR 2482810

[BMMS86] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, *H_∞ ring spectra and their applications*, Lecture Notes in Mathematics, vol. 1176, Springer-Verlag, Berlin, 1986. MR 836132 (88e:55001)

[BW05] James Borger and Ben Wieland, *Plethystic algebra*, Adv. Math. **194** (2005), no. 2, 246–283. MR 2139914 (2006i:13044)

[GM95] J. P. C. Greenlees and J. P. May, *Generalized Tate cohomology*, Mem. Amer. Math. Soc. **113** (1995), no. 543, viii+178. MR 1230773 (96e:55006)

[Gre01] J. P. C. Greenlees, *Tate cohomology in axiomatic stable homotopy theory*, Cohomological methods in homotopy theory (Bellaterra, 1998), Progr. Math., vol. 196, Birkhäuser, Basel, 2001, pp. 149–176. MR 1851253

[Law18] Tyler Lawson, *Secondary power operations and the Brown–Peterson spectrum at the prime* 2, Ann. of Math. (2) **188** (2018), no. 2, 513–576. MR 3862946

[Lur07] Jacob Lurie, *18.917 Topics in Algebraic Topology: The Sullivan Conjecture*, MIT OpenCourseWare: Massachusetts Institute of Technology (http://ocw.mit.edu/), Fall 2007.

[Lur09] ———, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659 (2010j:18001)

[Lur17] ———, *Higher Algebra*, Draft version available at: http://www.math.harvard.edu/~lurie/papers/higheralgebra.pdf, 2017.

[Man01] Michael A. Mandell, *E_∞ algebras and p-adic homotopy theory*, Topology **40** (2001), no. 1, 43–94. MR 1791268 (2001m:55025)

[McC99] Randy McCarthy, *Dual calculus for functors to spectra*, Homotopy methods in algebraic topology, Boulder 1999, American Mathematical Society, 1999, pp. 183–216.

[MR93] Mark E. Mahowald and Douglas C. Ravenel, *The root invariant in homotopy theory*, Topology **32** (1993), no. 4, 865–898. MR 1241877 (94h:55025)

[Rez09] Charles Rezk, *The congruence criterion for power operations in Morava E-theory*, Homology, Homotopy Appl. **11** (2009), no. 2, 327–379. MR 2591924 (2011e:55021)

[Ste62] N. E. Steenrod, *Cohomology operations*, Lectures by N. E. STeenrod written and revised by D. B. A. Epstein. Annals of Mathematics Studies, No. 50, Princeton University Press, Princeton, N.J., 1962. MR 0145525