Unwinding the relative Tate diagonal

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Abstract

We show that a spectral sequence developed by Lipshitz and Treumann, for application to Heegaard–Floer theory, converges to a localized form of topological Hochschild homology with coefficients. This allows us to show that the target of their spectral sequence can be identified with Hochschild homology precisely when the topological Hochschild homology is torsion-free as a module over $\mathrm{THH}_*(\mathbb{F}_2)$, parallel to results of Mathew on degeneration of the Hodge-to-de Rham spectral sequence.

To carry this out, we apply work of Nikolaus and Scholze to develop a general Tate diagonal for Hochschild-like diagrams of spectra that respect a decomposition into tensor products. This allows us to discuss the extent to which there is a Tate diagonal relative to a base ring.

1 Introduction

Hochschild homology and cyclic covers

The primary goal of this paper is to understand a particular formality condition on Hochschild homology. Motivated by Heegaard–Floer homology of double covers, Lipshitz and Treumann developed a noncommutative version of the Hodge-to-de Rham spectral sequence with coefficients in [LT]. Given a homologically smooth and proper differential graded algebra over $\mathbb{F}_2$ and a bounded differential graded A-bimodule M, they give a spectral sequence with E_1-term

$$E^1_* = HH_*(A; M \otimes^L_A M).$$

The E_2-term is Tate cohomology for an action of the cyclic group C_2 [LT, Theorem 4]—this spectral sequence arises from the Tate construction for C_2 acting on the Hochschild complex $HH(A; M \otimes^L_A M)$. When $M = A$, the d_1-differential is trivial and the d_2-differential is the b-operator of Connes, and this spectral sequence is related to the Hodge to de Rham spectral sequence [Kal08]. Under an assumption called π-*formality*, they show that their spectral sequence converges to $HH_*(A; M)$. Without π-formality, the target of their spectral sequence is not easy to identify, and some

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of the steps in their identification make use of non-additive maps $x \mapsto x \otimes x$ on the homology level that do not lift to the chain level.

The main goal of this paper is to identify the target of Lipshitz–Treumann's spectral sequence with a periodic variant of Bökstedt's topological Hochschild homology with coefficients [Bök]. To begin stating our results, we recall the following result of Bhatt–Morrow–Scholze [BMS18] on topological Hochschild homology of perfect rings—for $\mathbb{F}_p$ this is due to Bökstedt [Bök], and for perfect fields, such as finite fields, this result was previously known by work of Hesselholt and Madsen [HM97]. If k is a perfect ring of characteristic p, the topological Hochschild homology $\mathrm{THH}_*(k)$ is a polynomial algebra $k[u]$ on a generator in degree 2. Moreover, the Tate cohomology ring $\widehat{H}^{-*}(C_p; k)$ is an algebra over $\mathrm{THH}_*(k)$ whose underlying module is

$$k[u^{\pm 1}] \cdot \{1, v\}.$$

Here 1, v, and u^{-1} are generators of $H^i(C_p; k)$ for $i = 0$, 1, and 2 respectively.

Theorem 1.1. *Suppose that k is a perfect ring of characteristic p, A is a (k-flat) homologically smooth differential graded k-algebra, and M is a (k-flat) bounded A-bimodule. Then there is a Tate cohomology spectral sequence with E_2-term*

$$\widehat{H}^*(C_p; HH_*^k(A, \underbrace{M \otimes_A^L \cdots \otimes_A^L M}_{p})) \Rightarrow \widehat{H}^*(C_p; k) \otimes_{k[u]} \mathrm{THH}_*(A; M).$$

Note that both sides of this spectral sequence are periodic: degree d is isomorphic to degree $(d+1)$ for all d. By combining this with base-change results for topological Hochschild homology, we will arrive at the following result.

Theorem 1.2. *The spectral sequence of Theorem 1.1 reduces to an ungraded spectral sequence of the form*

$$\widehat{H}^s(C_p; HH_s^k(A, \underbrace{M \otimes_A^L \cdots \otimes_A^L M}_{p})) \Rightarrow HH_*^k(A; M)$$

precisely when $\mathrm{THH}_*(A; M)$ *is torsion-free as a module over* $\mathrm{THH}_*(k) = k[u]$.

In particular, we can reinterpret Lipshitz–Treumann's development of π-formality as giving conditions under which topological Hochschild homology is torsion-free.

Algebra and homotopy theory

The connection between algebra and stable homotopy theory, and in particular topological Hochschild homology, arises through the following translation procedure.

- For a commutative ring k, there is an Eilenberg–Mac Lane spectrum Hk which has the structure of a commutative algebra.
- The category of differential graded k-modules is equivalent (in a derived sense) to the category of Hk-modules: this translation takes a complex V to a spectrum HV such that $H_*(V) \cong \pi_* HV$.[1]

[1]This is a high-powered version of the *Dold–Kan equivalence* between chain complexes and simplicial abelian groups.

- This equivalence is symmetric monoidal, in the sense of [Lur17], and it takes the (derived) tensor over Hk to the tensor over k.
- This equivalence preserves homotopy limits, homotopy colimits, and Tate constructions.

In particular, under this correspondence a differential graded k-algebra A lifts to an Hk-algebra HA, a k-linear differential graded A-bimodule lifts to an Hk-linear HA-bimodule, and the relative Hochschild complex $\mathrm{HH}^k(A;M)$ lifts to topological Hochschild homology $\mathrm{THH}^{Hk}(HA;HM)$. Moreover, there is a base-change formula

$$\mathrm{THH}^{Hk}(HA;HM) \simeq Hk \otimes_{\mathrm{THH}(Hk)} \mathrm{THH}(HA;HM),$$

and together these equivalences relate ordinary Hochschild homology and a base-change of THH [MM03, §5]. A sketched discussion of the translation procedure will occupy §2.[2]

The connection to equivariant stable theory was observed by Kaledin [Kal08], and our methods are very similar to those of Mathew [Mat17]: however, where Mathew makes use of the circle action on $\mathrm{THH}(A)$, we make use of actions of cyclic groups on THH with certain coefficients. In these terms, Theorem 1.1 is a consequence of the following.

Theorem 1.3. *Suppose that k is a commutative ring spectrum, A is a k-algebra, and M is a k-linear A-bimodule. Then there exists an action of C_p on the topological Hochschild homology* $\mathrm{THH}(A; M \otimes_A \cdots \otimes_A M)$ *and a natural* relative THH-diagonal

$$k^{tC_p} \otimes_{\mathrm{THH}(k)} \mathrm{THH}(A, M) \to \left[\mathrm{THH}^k(A, M \otimes_A M \otimes_A \cdots \otimes_A M)\right]^{tC_p}.$$

If A is a smooth k-algebra and the underlying k-module of M is perfect, this map is an equivalence.

Once the relative THH-diagonal is set up, the proof in §13 that this is an equivalence is relatively formal.

The relative Tate diagonal

The results of this paper rely on a piece of nonalgebraic structure: the *Tate diagonal*, which plays a prominent role in equivariant stable homotopy theory. For a spectrum X, the Tate diagonal is a natural map

$$\Delta\colon\ X \to (X^{\otimes p})^{tC_p}$$

that enjoys a great of structure. The Tate diagonal is lax symmetric monoidal, it is natural, and it is impervious to the action of the cyclic group C_p on $X^{\otimes p}$. These properties are concisely encoded by Nikolaus–Scholze's expression of functoriality on a category of finite free C_p-sets [NS17, III.3.8].

[2] This paper is vulnerable to a concern recently raised by [KKS19]: namely, the results in this paper rely heavily on machinery from [Lur09, Lur17], and it is very difficult for those outside the field to access it. We have done our best, but are aware that our efforts may prove inadequate.

The lax symmetric monoidality of the Tate diagonal allows us to construct a relative version, using the tensor product over a commutative ring spectrum k rather than over the sphere. For individual modules, the relative Tate diagonal behaves very similarly to the ordinary one. However, *the functoriality of the relative diagonal is less strong*: the lax symmetric monoidal compatibility does not play well with the cyclic group invariance. We would like to extend Nikolaus–Scholze's functoriality for the Tate diagonal in a way that exhibits the extent to which there can be a relative version.

The starting point is the observation that the cyclic bar construction $Z(A)$ that builds the Hochschild complex is not just a simplicial spectrum $\Delta^{op} \to \mathrm{Sp}$: there is a decomposition of each simplicial degree into a formal smash product of factors, and the structure maps respect this decomposition. This lifts it to a simplicial object in the *symmetric monoidal envelope* $\mathrm{Env}(\mathrm{Sp})$, which we will discuss in §3.

Associated to a diagram $X\colon K \to \mathrm{Env}(\mathcal{C})$ in a symmetric monoidal envelope, there is an underlying diagram of finite sets $K \to \mathit{Fin}$ representing the decomposition into tensor factors: we will define the *shape* $|X|$ to be the homotopy colimit, a simplicial set. The shape of $\mathrm{THH}(A)$ or $\mathrm{THH}(A; M)$ is the circle S^1. Given a diagram $X\colon K \to \mathrm{Env}(\mathcal{C})$ in a symmetric monoidal envelope with shape $|X|$ and a principal C_p-bundle classified by a map $f\colon |X| \to BC_p$, we will associate a new diagram *unwinding* X: a C_p-equivariant diagram $\psi^f X\colon K \to \mathrm{Env}(\mathcal{C})$. For any $i \in K$, $\psi^f(i)$ is isomorphic to the iterated formal tensor $X(i)^{\otimes p}$, but the maps in K go to maps between tensor powers that make use of the structure of the principal bundle.

Let $k^{\otimes Z}$ denote the *Loday construction* [Sch11] (sometimes called the *factorization homology* [Fra13]) of Z with coefficients in k, often also denoted by $\mathcal{L}_Z(k)$, $\int_Z k$, or $Z \otimes k$.

Proposition 1.4. *Let k be a commutative ring spectrum and K a sifted index category. For a diagram $X\colon K \to \mathrm{Env}(\mathrm{LMod}_k)$ in the symmetric monoidal envelope and a principal C_p-bundle $f\colon |X| \to BC_p$ over the shape of X, there is a natural map*

$$k^{tC_p} \underset{k^{\otimes |X|}}{\otimes} \left(\operatorname*{hocolim}_{i \in K} \bigotimes^{\mathrm{Sp}} X(i) \right) \to \left(\operatorname*{hocolim}_{i \in K} \bigotimes^{\mathrm{LMod}_k} \psi^f X(i) \right)^{tC_p}$$

called the relative Tate diagonal.

There is asymmetry between the tensor products in the source and target of the relative Tate diagonal—the source tensor takes place in spectra and the target tensor takes place in k-modules. We have traced several of our own misunderstandings, including an assertion that there are cyclotomic structures on relative THH and monoidality properties of a Tate diagonal on relative THH with coefficients, back to this root. The relative Tate diagonal *does not* imply that there is a k-module Tate diagonal

$$\operatorname*{hocolim}_{i \in K} \bigotimes^{\mathrm{LMod}_k} X(i) \to \left(\operatorname*{hocolim}_{i \in K} \bigotimes^{\mathrm{LMod}_k} \psi^f X(i)) \right)^{tC_p}$$

unless the map $k^{\otimes |X|} \to k^{tC_p}$ factors through the augmentation $k^{\otimes |X|} \to k$. This only holds in a few circumstances, such as when we can fix a trivialization of the bundle classified by f.[3] This is true, for example, when the index category K is a singleton, which allows one to construct a natural Tate diagonal $M \to (M \otimes_k \cdots \otimes_k M)^{tC_p}$ and a k-module version of the Hill–Hopkins–Ravenel norm [HHR16].

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2 Homological algebra and stable homotopy theory

Background

Sets (and topological spaces) have a natural diagonal map $\Delta\colon X \to (X^p)^{C_p}$. For an abelian group, we can compose with the universal multilinear map to get a natural transformation $A \to (A^{\otimes p})^{C_p}$, given by $a \mapsto a \otimes \cdots \otimes a$. This is a natural transformation of sets, but not a homomorphism: however, this problem vanishes modulo the image of the transfer homomorphism

$$(A^{\otimes p})_{C_p} \to (A^{\otimes p})^{C_p},$$

and hence determines a natural homomorphism from A to the Tate cohomology group $\widehat{H}^0(C_p; A^{\otimes p})$. This map is the (algebraic) Tate-valued Frobenius.

To get a chain-level or derived variant of this Frobenius, we must replace the Tate cohomology functor $\widehat{H}$ by a derived Tate construction; but now that we are no longer taking a quotient by transfers, this no longer strictly imposes the homomorphism property. As a result, in the construction of a derived version of the Tate-valued Frobenius we will lose the property of staying within algebra.

Before we introduce the Tate diagonal, we would like to translate the objects under consideration in [LT] to stable homotopy theory. In this section we will give some brief background on this translation process. The author claims no originality for the results in this chapter.

2.1 Module spectra and chain complexes

For any ordinary ring k, let $\mathrm{Ch}(k)$ be the category of chain complexes of k. For such chain complexes C and D, one can build a function space $\mathrm{Map}_k(C, D)$: start with a set of vertices given by chain maps $C \to D$, attach paths associated to chain homotopies, and so on. More concisely, using the Dold–Kan correspondence one can take the function complex $\mathrm{Hom}_k(C, D)$ and associate a simplicial set of maps $C \to D$. Because

[3] Thomas Nikolaus has pointed out to us that such a factorization through the augmentation is also possible when k is the spherical group algebra of a *discrete* abelian group.

$Ch(k)$ now has function spaces, one can speak of homotopy limits and colimits in the category of chain complexes, for example via an associated ∞-category which is made explicit in [Lur17, §1.3.1]. One can also form a localization $\mathrm{Ch}(k) \to \mathcal{D}(k)$ by inverting the quasi-isomorphisms; on the level of homotopy categories, this becomes the map from the classical chain homotopy category of k to the derived category.

Our starting point is the following theorem, which interprets the category of chain complexes as equivalent to a construction in stable homotopy theory.

Theorem 2.1 ([Lur17, 7.1.1.16, 7.1.2.13])**.** *Let k be a ring. Then there exists an equivalence of ∞-categories*

$$\theta\colon\ \mathcal{D}(k) \xrightarrow{\sim} \mathrm{LMod}_{Hk}$$

between the derived ∞-category of differential graded k-modules and the category of modules over the Eilenberg–Mac Lane spectrum Hk.

If k is commutative, this extends to an equivalence of symmetric monoidal ∞-categories, where the source carries the derived tensor product $\otimes_k^L$ and the target carries the relative smash product $\otimes_{Hk}$.

This result, in several strengths and several guises, has a long history in the literature and served as a motivation for many developments. It is present as an analogy in [Tho85, 5.32]; as an equivalence between the derived category of differential graded k-modules and the homotopy category of Hk-modules in [EKMM97, IV.2.4]; as an equivalence of model categories in [SS03, 5.1.6]; and an extension of this to a monoidal equivalence in [Shi07]. The above formulation is convenient because it allows us to apply the extensive machinery built in [Lur17].

In the description above, the equivalence θ of ∞-categories preserves all structure that can be expressed in a homotopy-invariant fashion.

Corollary 2.2. *Suppose that K is a simplicial set. Then composition with θ induces an equivalence of functor ∞-categories*

$$\mathrm{Fun}(K, \mathcal{D}(k)) \to \mathrm{Fun}(K, \mathrm{LMod}_{Hk}).$$

Example 2.3. If $K = BG$ is the classifying space of a finite group, maps $BG \to \mathcal{C}$ of ∞-categories are coherent actions of G on an object of the ∞-category $\mathcal{C}$. This shows that θ preserves coherent G-actions: chain complexes of k-modules with a coherent G-action are equivalent to Hk-modules with a coherent G-action. Chain complexes with strict G-action give rise to Hk-modules with coherent G-action under θ.

Corollary 2.4. *The functor θ preserves homotopy limit and colimit diagrams.*

Example 2.5. Given a chain complex C of $k[G]$-modules, the tensor product

$$W \otimes_G C$$

with a projective resolution W of $\mathbb{Z}$ over $\mathbb{Z}[G]$ is a representative for the homotopy colimit C_{hG} in $\mathcal{D}(k)$. Therefore, it is taken by θ to a homotopy colimit. Similarly, the function complex $\mathrm{Hom}_G(W, C)$ is a representative for the homotopy limit.

Example 2.6. Let $f\colon \Delta^{op} \to \mathrm{Ch}(k)$ represent a simplicial object in chain complexes of k-modules. Associated to this there is a double complex using the standard alternating sign boundary operators, and an associated totalization. This total complex is a representative for the homotopy colimit of the diagram f. As a result, θ takes this total complex to the homotopy colimit of the diagram $\theta \circ f$.

Corollary 2.7. *If k is commutative and $\mathcal{O}$ is an ∞-operad, θ induces an equivalence*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{D}(k)) \xrightarrow{\sim} \mathrm{Alg}_{\mathcal{O}}(\mathrm{LMod}_{Hk})$$

of ∞-categories of $\mathcal{O}$-algebras.

Example 2.8. Suppose $\mathcal{O}$ is an ordinary operad which is acted on freely by the symmetric groups. Then associated to $\mathcal{O}$ there is an ∞-operad such that objects with an action of $\mathcal{O}$ are equivalent to algebras over the associated ∞-operad. Since algebras over ∞-operads are invariant under equivalences, this allows us to translate A_∞ and E_∞ algebras between $\mathrm{Ch}(k)$ and LMod_{Hk}. For example, an associative differential graded k-algebra A gives rise to an A_∞-algebra θA in LMod_{Hk}. Similarly, differential graded modules and bimodules give rise to modules and bimodules over θA.

2.2 Tate constructions

The classical Tate cohomology of a group G with coefficients in a module was exported to the category of spectra by Greenlees and May [GM95] using equivariant stable homotopy theory, and generalized to the case of a stable ∞-category in [Lur17, 6.1.6.24]. In this section, we will recall some of the important properties satisfied by the Tate construction.

Proposition 2.9. *Let G be a finite group, $\mathcal{C}$ a stable ∞-category which admits countable homotopy limits and homotopy colimits, and M a G-equivariant object of $\mathcal{C}$. Then there is a natural* transfer map[4]

$$\mathrm{Tr}\colon M_{hG} \to M^{hG}$$

from the derived orbit object to the derived fixed-point object. If M is the free object $\oplus_{g\in G} M \simeq \prod_{g\in G} M$, this is equivalent to the natural composite

$$\left(\bigoplus_{g\in G} M\right)_{hG} \xrightarrow{\sim} M \xrightarrow{\sim} \left(\prod_{g\in G} M\right)^{hG}.$$

Definition 2.10. Let G be a finite group, $\mathcal{C}$ a stable ∞-category which admits countable limits and colimits, and M a G-equivariant object of $\mathcal{C}$. We write M^{tG} for the cofiber of the transfer $M_{hG} \to M^{hG}$, and refer to it as the *G-Tate construction on M* or simply the Tate construction.

Proposition 2.11. *The Tate construction has the following properties.*

- *It determines a functor $\mathcal{C}^{BG} \to \mathcal{C}$ from objects of $\mathcal{C}$ with G-action to $\mathcal{C}$.*

[4]Other authors refer to this as the *norm map*, which we prefer to reserve for multiplicative variants.

- *It preserves finite coproducts, finite products, homotopy pushouts, and homotopy pullbacks.*
- *Any functor $\mathcal{C} \to \mathcal{D}$ between stable ∞-categories that preserves countable homotopy limits and colimits also preserves Tate constructions. In particular, this is true of equivalences.*

A chain complex M of $k[G]$-modules determines an object in $\mathrm{Ch}(k)$ with G-action, and as such we can compare the Tate construction M^{tG} with more classical constructions.

Let E be a projective resolution of $\mathbb{Z}$ by finitely generated free $\mathbb{Z}[G]$-modules, and $E^\vee$ the dual complex $\mathrm{Hom}(E, \mathbb{Z})$. (For instance, we may take E to be the standard bar resolution.) Then there is a composite

$$E \to H_0(E) = \mathbb{Z} = H_0(E^\vee) \to E^\vee,$$

where we view $\mathbb{Z}$ as a complex concentrated in degree zero, and we can construct a mapping cone W. This complex W is an unbounded complex of finitely generated free $\mathbb{Z}[G]$-modules. Associated to any chain complex M with G-action, there is then a double complex $W \otimes_G M$ with three realizations.

Definition 2.12. Let M be a chain complex with G-action. We define the following chain-level Tate constructions as complexes:

$$\begin{aligned}
\mathrm{Tate}^{\oplus}(M)_n &= \bigoplus_{p+q=n} W_p \otimes_G M_q \\
\mathrm{Tate}^{\Pi}(M)_n &= \prod_{p+q=n} W_p \otimes_G M_q \\
\mathrm{Tate}(M)_n &= \bigcup_N \prod_{p+q=n, p \le N} W_p \otimes_G M_q
\end{aligned}$$

The boundary maps in these complexes are the standard boundary maps determined by the Leibniz rule $\partial(a \otimes b) = \partial a \otimes b + (-1)^{|a|} a \otimes \partial b$.

Proposition 2.13. *The chain-level Tate constructions for the action of G on M have the following properties.*

1. *All three Tate constructions preserve short exact sequences in M.*
2. *There are natural maps* $\mathrm{Tate}^{\oplus}(M) \to \mathrm{Tate}(M) \to \mathrm{Tate}^{\Pi}(M)$.
3. *The map* $\mathrm{Tate}^{\oplus}(M) \to \mathrm{Tate}(M)$ *is an isomorphism if M is bounded above.*
4. *The map* $\mathrm{Tate}(M) \to \mathrm{Tate}^{\Pi}(M)$ *is an isomorphism if M is bounded below.*
5. *There is a conditionally convergent Tate cohomology spectral sequence*

$$\widehat{H}^s(G; H_t(M)) \Rightarrow H_{t-s}(\mathrm{Tate}(M)).$$ [5]

[5] Cohomologically minded readers might prefer the indexing $\widehat{H}^p(G; H^q(M)) \Rightarrow H^{p+q}(\mathrm{Tate}(M))$.

6. *There is a natural short exact sequence*

$$0 \to M_{hG} \to M^{hG} \to \mathrm{Tate}(M) \to 0$$

of complexes, where the first complex is the complex of derived coinvariants and the second is the complex of derived invariants.

7. *The object* $\mathrm{Tate}(M)$ *is a representative for the homotopical Tate construction* M^{tG}*. In particular, an equivariant quasi-isomorphism* $M \to N$ *induces an equivalence* $\mathrm{Tate}(M) \to \mathrm{Tate}(N)$.

In particular, for bounded complexes there is no distinction between these three constructions. However, these three Tate constructions typically have quite different behavior for unbounded complexes.

2.3 Algebras and Hochschild complexes

The equivalence of symmetric monoidal ∞-categories between the category $\mathcal{D}(k)$ and the category LMod_{Hk} allows us to transport Hochschild complexes because they can be expressed diagrammatically. Given an associative differential graded k algebra A with k-linear bimodule M, the cyclic bar construction is a simplicial chain complex

$$Z^k(A, M)\colon\ \Delta^{op} \to \mathrm{Ch}(k)$$

whose associated total complex is the Hochschild complex. If all the tensor products in this complex are equivalent to the derived tensor products, then the functor θ from Theorem 2.1 preserves them, and takes this simplicial diagram to the cyclic bar construction

$$Z^{Hk}(\theta A, \theta M)\colon\ \Delta^{op} \to \mathrm{LMod}_{Hk}\,.$$

As in Example 2.6, θ takes the associated total complex to the homotopy colimit. The homotopy colimit geometric realization of this cyclic bar construction, which is the definition of topological Hochschild homology. As a result, we have an identification:

$$\theta(HH^k(A; M)) \simeq \mathrm{THH}^{Hk}(\theta A, \theta M).$$

We would now like to develop the interaction with the cyclic group.

Fix a projective resolution W of A as an A-bimodule with only finitely many generators. Since any two resolutions are equivalent, there is a quasi-isomorphism

$$B(A, A, A) \to W$$

of A-bimodules, where the source is the (total complex associated to the) two-sided bar resolution. This becomes a C_2-equivariant equivalence

$$[M \otimes_A B(A, A, A) \otimes_A M] \otimes_{A \otimes_k A^{op}} B(A, A, A) \to [M \otimes_A W \otimes_A M] \otimes_{A \otimes_k A^{op}} W,$$

where C_2 acts by rotational symmetry on the tensor products. The left-hand side is a bisimplicial object; its homotopy colimit realizes the Hochschild complex $HH^k(A; M \otimes^L_A M)$. Therefore, we have an equivalence of Tate constructions

$$\mathrm{Tate}(HH^k(A; M \otimes^L_A M)) \simeq \mathrm{Tate}\left([M \otimes_A W \otimes_A M] \otimes_{A \otimes_k A^{op}} W\right)$$

because Tate preserves quasi-isomorphisms. Moreover, if M is bounded and A is homologically smooth, we can choose W to be finitely generated; this makes the right-hand complex bounded, and so both become quasi-isomorphic to the direct-sum Tate complex

$$\mathrm{Tate}^{\oplus}\left([M \otimes_A W \otimes_A M] \otimes_{A \otimes_k A^{op}} W\right).$$

This last is the complex whose Tate spectral sequence was developed by Lipshitz and Treumann in [LT].

We now apply θ. We find that Lipshitz and Treumann's construction is carried to a model in stable homotopy theory: the a Tate construction

$$\left[\mathrm{THH}^{Hk}(\theta A; \theta M \otimes_{\theta A} \theta M)\right]^{tC_2}.$$

This translation now allows us to apply results in stable homotopy theory to understand Lipshitz and Treumann's Tate spectral sequence.

3 Envelopes

We recall that a *multicategory* (or colored operad) is a category where morphisms may have several inputs: maps are of the form $\{X_s\}_{s \in S} \to Y$ with S a finite index set. Every symmetric monoidal category $\mathcal{D}$ has an underlying multicategory $U\mathcal{D}$: we define maps $\{X_s\}_{s \in S} \to Y$ to be the same as maps $\bigotimes X_s \to Y$. In the other direction, associated to a multicategory C there is a symmetric monoidal category $\mathrm{Env}(C)$ called the *symmetric monoidal envelope*.

- The objects of $\mathrm{Env}(C)$ are formal tuples $(S, \{X_s\}_{s \in S})$ of a finite set and an S-indexed set of objects of C, representing a formal tensor $\bigotimes X_s$.
- The morphisms $(S, \{X_s\}) \to (T, \{Y_t\})$ in $\mathrm{Env}(C)$ are pairs of a map $f \colon S \to T$ and a collection of maps $g_t \colon \{X_s\}_{s \in p^{-1}(t)} \to Y_t$ in the multicategory C.

By construction, there is an equivalence between multifunctors $C \to U\mathcal{D}$ and symmetric monoidal functors $\mathrm{Env}(C) \to \mathcal{D}$; there is also a forgetful functor from $\mathrm{Env}(C)$ to the category of finite sets.

The coherent version of this construction is described in [Lur17, §2.2.4]. The analogues of multicategories are ∞-*operads*, and an ∞-operad $C^{\otimes}$ has an associated symmetric monoidal envelope $\mathrm{Env}(C)$. The universal property of the symmetric monoidal envelope is [Lur17, 2.2.4.9]: for any symmetric monoidal ∞-category $\mathcal{D}$, there is an equivalence between symmetric monoidal functors $\mathrm{Env}(C) \to \mathcal{D}$ and maps of ∞-operads $C^{\otimes} \to \mathcal{D}^{\otimes}$.

Here are some important properties of this construction.

- There is a natural symmetric monoidal functor $p \colon \mathrm{Env}(C) \to \mathrm{Fin}$ to the category of finite sets, whose fiber over S is equivalent to $\prod_{s \in S} C$, and under this correspondence the space of maps $\{X_s\}_{s \in S} \to \{Y_t\}_{t \in T}$ over a given map $f \colon S \to T$ is equivalent to a product $\prod_{t \in T} \mathrm{Map}_{C^{\otimes}}(\{X_s\}_{s \in f^{-1}(t)}, Y_t)$ of mapping spaces.

- The adjunction gives every symmetric monoidal ∞-category a natural symmetric monoidal functor $\bigotimes^{C}\colon \operatorname{Env}(C) \to C$, sending $\{X_s\}_{s\in S}$ to $\bigotimes_{s\in S} X_s$.

Example 3.1. For any associative algebra A in C with right module M and left module N, the two-sided bar construction $B(M, A, N)$ can be lifted from a simplicial object in C to a simplicial object in $\operatorname{Env}(C)$.

Example 3.2. Given an associative algebra A in a symmetric monoidal ∞-category C, there exists a lift of the cyclic bar construction $Z(A)$ from a simplical object in C to a simplicial object in $\operatorname{Env}(C)$. Similarly, for an algebra A with a bimodule M, the same is true for $Z(A, M)$, the cyclic bar construction with coefficients.

4 Pushforward

If C is symmetric monoidal, the tensor functor $\bigotimes^{C}\colon \operatorname{Env}(C) \to C$ has a relative version. Suppose that we have an S-indexed tuple $X = \{X_s\}_{s\in S}$ of objects of C and a map $\varphi\colon S \to T$ of finite sets. Then, associated to this, we will construct a T-indexed tuple $\varphi_!(X) = \{\bigotimes_{s\in f^{-1}(t)} X_s\}_{t\in T}$, which we refer to as the *fiberwise tensor*, together with a map $X \to \varphi_!(X)$. In the following we will exhibit some of the functoriality properties of this construction.

We first require some general intermediate results.

Lemma 4.1. *Suppose that $C^{\otimes} \to \mathcal{D}^{\otimes}$ is a coCartesian fibration of ∞-operads. Then the functor $\operatorname{Env}(C)^{\otimes} \to \operatorname{Env}(\mathcal{D})^{\otimes}$ is a coCartesian fibration of symmetric monoidal ∞-categories.*

Proof. The symmetric monoidal envelope $\operatorname{Env}(C)^{\otimes}$ is defined in [Lur17, 2.2.4.1] as the fiber product

$$C^{\otimes} \times_{\mathrm{Fin}_*} \operatorname{Act}(\mathrm{Fin}_*).$$

The result follows because fibration conditions are stable under base-change. □

Lemma 4.2. *Suppose that $C^{\otimes} \to \mathcal{D}^{\otimes}$ is a coCartesian fibration of symmetric monoidal ∞-categories and that $O^{\otimes}$ is an ∞-operad. Then the functor*

$$\operatorname{Alg}_O(C) \to \operatorname{Alg}_O(\mathcal{D})$$

lifts to a symmetric monoidal coCartesian fibration under the pointwise tensor product of O-algebras from [Lur17, 3.2.4.4].

Proof. For $\mathcal{D}^{\otimes}$ a symmetric monoidal ∞-category and $O^{\otimes}$ an ∞-operad, $\operatorname{Alg}_O(\mathcal{D})$ is the full subcategory of $\operatorname{Fun}(O^{\otimes}, \mathcal{D}^{\otimes})$ spanned by the maps of ∞-operads. There is a pointwise tensor product [Lur17, 3.2.4.4], defined so that maps $K \to \operatorname{Alg}_O(\mathcal{D})^{\otimes}$ over a fixed map $K \to \mathrm{Fin}_*$ are equivalent to commutative diagrams

$$\begin{tikzcd} K \times O^{\otimes} \arrow[r] \arrow[d] & \mathcal{D}^{\otimes} \arrow[d] \\ \mathrm{Fin}_* \times \mathrm{Fin}_* \arrow[r, "\wedge"] & \mathrm{Fin}_* \end{tikzcd}$$

where the top map restricts to a map of ∞-operads for any vertex of K.

By adjunction, then, the identity self-functor of $\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})^{\otimes}$ determines a commutative diagram

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{O}}(\mathcal{D})^{\otimes} \times \mathcal{O}^{\otimes} \arrow[r] \arrow[d] & \mathcal{D}^{\otimes} \arrow[d] \\ \mathrm{Fin}_* \times \mathrm{Fin}_* \arrow[r, "\wedge"] & \mathrm{Fin}_* . \end{tikzcd}$$

The topmost map sends pairs of inert morphisms in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})^{\otimes} \times \mathcal{O}^{\otimes}$ to inert morphisms in $\mathcal{D}^{\otimes}$ [Lur17, 3.2.4.3, (2)], and thus it is a *bifunctor* of ∞-operads in the sense of [Lur17, 2.2.5.3].

We now apply [Lur17, 3.2.4.3] to the bifunctor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{D})^{\otimes} \times \mathcal{O}^{\otimes} \to \mathcal{D}^{\otimes}$ and the coCartesian fibration $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$. This shows that, under the definition from [Lur17, 3.2.4.1], there is a coCartesian fibration

$$\mathrm{Alg}_{\mathcal{O}/\mathcal{D}}(\mathcal{C})^{\otimes} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{D})^{\otimes}.$$

However, unravelling the definition of the source we find that this is the natural functor

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{D})^{\otimes}$$

under the pointwise monoidal structure. In particular, the fiber over a map of ∞-operads $f\colon \mathcal{O}^{\otimes} \to \mathcal{D}^{\otimes}$ is the ∞-category of sections $\mathcal{O}^{\otimes} \to \mathcal{C}^{\otimes}$.

Moreover, by [Lur17, 3.2.4.3, (4)], a morphism $\alpha\colon A \to B$ in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ is coCartesian if and only if, for any $X \in \mathcal{O}$, the natural transformation the map $A(X) \to B(X)$ of $\mathcal{C}^{\otimes}$ is a coCartesian lift of its image in $\mathcal{D}^{\otimes}$. □

Proposition 4.3. *If $\mathcal{C}$ is a symmetric monoidal ∞-category, the map of functor categories* $\mathrm{Fun}(K, \mathrm{Env}(\mathcal{C})) \to \mathrm{Fun}(K, \mathrm{Fin})$ *extends, up to equivalence, to a symmetric monoidal coCartesian fibration.*

Proof. Because $\mathcal{C}$ is symmetric monoidal, we have a coCartesian fibration $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ and hence a coCartesian fibration

$$\mathrm{Env}(\mathcal{C})^{\otimes} \to \mathrm{Env}(\mathrm{Fin})^{\otimes} = \mathrm{Fin}^{\amalg}$$

by Lemma 4.1.

Given a simplicial set K, viewed as a simplicial set over Fin_* via $K \to \{1\} \subset \mathrm{Fin}_*$, let $K \to \mathcal{K}^{\otimes} \to \mathrm{Fin}_*$ be a fibrant replacement in the ∞-operadic model structure [Lur17, 2.1.4.6]; $\mathcal{K}^{\otimes}$ is an ∞-operad. This has the property that for any ∞-operad $\mathcal{C}^{\otimes}$, restricting maps of ∞-operads $\mathcal{K}^{\otimes} \to \mathcal{D}^{\otimes}$ to functors $K \to \mathcal{D}$ gives an equivalence of functor categories

$$\mathrm{Alg}_{\mathcal{K}}(\mathcal{D}) \simeq \mathrm{Fun}(K, \mathcal{D}).$$

For a symmetric monoidal ∞-category $\mathcal{C}$, we then get a commutative diagram

$$\begin{tikzcd} \mathrm{Alg}_{\mathcal{K}}(\mathrm{Env}(\mathcal{C})) \arrow[r] \arrow[d] & \mathrm{Fun}(K, \mathrm{Env}(\mathcal{C})) \arrow[d] \\ \mathrm{Alg}_{\mathcal{K}}(\mathrm{Fin}) \arrow[r] & \mathrm{Fun}(K, \mathrm{Fin}). \end{tikzcd}$$

The horizontal maps are equivalences, and the left-hand vertical map extends to a symmetric monoidal coCartesian fibration by Lemma 4.2. □

Remark 4.4. The straightening of this coCartesian fibration is a symmetric monoidal functor

$$\mathrm{Env}(C, -)\colon \ \mathrm{Fun}(K, \mathrm{Fin}) \to \mathrm{Cat}_\infty\,.$$

This sends $f\colon\ K \to \mathrm{Fin}$, up to equivalence, to the category $\mathrm{Env}(C, f)$ of lifts $F\colon\ K \to \mathrm{Env}(C)$ of f and sends a natural transformation $\varphi\colon\ f \to g$ to a functor $\varphi_!\colon\ \mathrm{Env}(C, f) \to \mathrm{Env}(C, g)$. Functoriality says that this respects composition in φ, and the coCartesian property means that every natural transformation $F \to G$ over φ factors essentially uniquely through a functor $\varphi_! F \to G$ over φ. Moreover, the description of coCartesian morphisms leads to the following: the map $\alpha\colon\ F \to \varphi_! F$ is characterized by the property that, for any object k of K, the map $\alpha(k)\colon\ F(k) \to (\varphi_! F)(k)$ of $C^{\otimes}$ is a coCartesian lift of the underlying map $\varphi(k)\colon\ f(k) \to g(k)$. In other words, if $F(k) = \{X_s\}_{s\in f(k)}$, then

$$(\varphi_! F)(k) = \left\{\bigotimes_{\varphi(s)=t} X_s\right\}_{t\in g(k)}.$$

The unit of $\mathrm{Env}(\mathcal{D}, f)$ is a functor $\mathbb{I}_f\colon\ K \to \mathrm{Env}(\mathcal{D})$, sending an object k to an indexed tuple $\{\mathbb{I}_{\mathcal{D}}\}_{s\in f(k)}$ of copies of the unit for the tensor product.

Example 4.5. The quotient map $\varphi\colon\ \Delta^1 \to S^1$ of simplicial sets is a natural transformation of functors $\Delta^{op} \to \mathrm{Fin}$. For any algebra A in C with right module M and left module N, the fiberwise tensor $\varphi_! B(M, A, N)$ of the two-sided bar construction is the cyclic bar construction $Z(A, N \otimes M)$ associated to the A-bimodule $N \otimes M$.

Example 4.6. For any K, there is a constant functor $*\colon\ K \to \mathrm{Fin}$ whose value is a singleton, and lifts of this to $\mathrm{Env}(C)$ are equivalent to functors $K \to C$. There always a natural transformation τ from any functor $f\colon\ K \to \mathrm{Fin}$ to the constant functor with value $*$, and under these identifications the functor $\tau_! F$ is the tensor product functor $\bigotimes^C \circ F\colon\ K \to C$. In particular, functoriality of fiberwise tensor tells us that we have natural equivalences

$$\bigotimes^{C} \circ \varphi_! F = \bigotimes^{C} \circ F.$$

5 Realization

We now consider homotopy colimits. A given diagram $K \to C$ may have a homotopy colimit, and because C is presentable these can be made into a functorial homotopy colimit $\mathrm{Fun}(K, C) \to C$. In general, there are only natural transformations

$$\operatorname*{hocolim}_i (F(i) \otimes G(i)) \to \operatorname*{hocolim}_{i,j} (F(i) \otimes G(j)) \to (\operatorname*{hocolim}_i F(i)) \otimes (\operatorname*{hocolim}_j G(j))$$

without making further assumptions. The first map is an equivalence if the index category is sifted (the diagonal $\Delta\colon\ K \to K \times K$ is cofinal [Lur09, 5.5.8.1]), and the second is an equivalence if the monoidal product of C preserves homotopy colimits in each variable separately. This allows us to conclude the following.

Proposition 5.1. *Suppose that K is sifted, that C is a symmetric monoidal ∞-category with K-indexed colimits, and that the symmetric monoidal structure preserves K-indexed colimits in each variable. Then there is a functor*

$$\operatorname*{hocolim}_K \colon \operatorname{Fun}(K, C) \to C$$

that is strong symmetric monoidal.

A commutative algebra object A in an ∞-operad determines a symmetric monoidal functor $A\colon \operatorname{Fin} \to \operatorname{Env}(C)$, where the image of a finite set S has a chosen equivalence with a constant indexed tuple $\{A\}_{s\in S}$.

Definition 5.2. Suppose that A is a commutative algebra object in C, that K is sifted, and that $X\colon K \to \operatorname{Fin}$ is a fixed diagram of finite sets. We define

$$A^{\otimes X} = \operatorname*{hocolim}_K \left(\bigotimes^{C} \circ A \circ X \right).$$

Example 5.3. Suppose that X is a simplicial finite set, viewed as a functor $X\colon \Delta^{op} \to \operatorname{Fin}$. Then $A^{\otimes X}$ can be identified with the Loday construction $A^{\otimes |X|}$.

Proposition 5.4. *Suppose that $\varphi\colon f \to g$ is a natural transformation of functors $K \to \operatorname{Fin}$ and that $F\colon K \to \operatorname{Env}(C)$ lifts f. Then there is a natural equivalence*

$$\operatorname*{hocolim}_K \left(\bigotimes^{C} \circ F \right) \to \operatorname*{hocolim}_K \left(\bigotimes^{C} \circ \varphi_! F \right).$$

Proof. This follows from the equivalence between $\bigotimes^C \circ \varphi_! F$ and $\bigotimes^C \circ F$ from Example 4.6. □

Example 5.5. If M is a right A-module N is a left A-module, then the identification between the pushforward of the two-sided bar construction and a cyclic bar construction from Example 4.5 gives us an equivalence

$$M \otimes_A N \simeq \operatorname{THH}(A; N \otimes M).$$

6 Adjoints and algebras

Envelopes are functorial: for a map of ∞-operads $R\colon \mathcal{D}^\otimes \to C^\otimes$, there is an induced symmetric monoidal functor $\operatorname{Env}(R)\colon \operatorname{Env}(\mathcal{D}) \to \operatorname{Env}(C)$. Further, if C and $\mathcal{D}$ are symmetric monoidal ∞-categories, a map of ∞-operads $\mathcal{D}^\otimes \to C^\otimes$ encodes a lax symmetric monoidal functor $R\colon \mathcal{D} \to C$. There is a resulting natural transformation

$$\bigotimes^{C} \circ \operatorname{Env}(R) \to R \circ \bigotimes^{\mathcal{D}},$$

and the functor R is strong symmetric monoidal precisely when this is a natural equivalence. Moreover, if R is the right adjoint to a strong symmetric monoidal functor then it is lax symmetric monoidal [Lur17, 7.3.2.7]. When we apply this to categories of lifts, we find the following result.

Proposition 6.1. *Suppose that $R\colon \mathcal{D} \to \mathcal{C}$ is a lax symmetric monoidal functor, and $f\colon K \to \mathrm{Fin}$ is a fixed functor. Then there are induced lax symmetric monoidal functors $\mathrm{Env}(R, f)\colon \mathrm{Env}(\mathcal{D}, f) \to \mathrm{Env}(\mathcal{C}, f)$ and $R\colon \mathrm{Fun}(K, \mathcal{D}) \to \mathrm{Fun}(K, \mathcal{C})$, together with a lax symmetric monoidal natural transformation*

$$\bigotimes^{\mathcal{D}} \circ \mathrm{Env}(R) \circ F \to R \circ \bigotimes^{\mathcal{C}} \circ F$$

for $F \in \mathrm{Env}(\mathcal{D}, f)$.

In general, a lax symmetric monoidal structure takes a commutative algebra to a commutative algebra and a module to a module; a lax symmetric monoidal natural transformation induces a natural map of commutative algebras and, on general objects, has the structure of a compatible map of modules. In the case of these envelope categories, this takes the following form.

Proposition 6.2. *Fix a functor $f\colon K \to \mathrm{Fin}$ and a lax symmetric monoidal functor $R\colon \mathcal{D} \to \mathcal{C}$. Then there is a natural map of commutative algebras*

$$\bigotimes^{\mathcal{C}} R(\mathbb{I}_f) \to R(\mathbb{I}_{\mathcal{D}})$$

in $\mathrm{Fun}(K, \mathcal{C})$. The functor $\bigotimes^{\mathcal{C}} \circ R\colon \mathrm{Env}(\mathcal{D}, f) \to \mathrm{Fun}(K, \mathcal{C})$ lifts to the category of $\bigotimes^{\mathcal{C}} R(\mathbb{I}_f)$-modules; the functor $R \circ \bigotimes^{\mathcal{D}}\colon \mathrm{Env}(\mathcal{D}, f) \to \mathrm{Fun}(K, \mathcal{C})$ lifts to the category of $R(\mathbb{I}_{\mathcal{D}})$-modules; the natural transformation of Proposition 6.1 lifts to natural a map of modules.

We now compose this with the natural transformation $\mathrm{hocolim}_K \circ R \to R \circ \mathrm{hocolim}_K$.

Proposition 6.3. *Suppose that K is sifted, $f\colon K \to \mathrm{Fin}$ is fixed, and that $R\colon \mathcal{D} \to \mathcal{C}$ is a lax symmetric monoidal functor between symmetric monoidal presentable ∞-categories. Then there is a natural map of commutative algebras*

$$R(\mathbb{I}_f)^{\otimes f} \to R(\mathbb{I}_{\mathcal{D}})$$

in $\mathcal{C}$. The functor $\mathrm{hocolim}_K \circ \bigotimes^{\mathcal{C}} \circ R\colon \mathrm{Env}(\mathcal{D}, f) \to \mathcal{C}$ lifts to the category of $R(\mathbb{I}_{\mathcal{D}})^{\otimes f}$–modules; the functor $R \circ \mathrm{hocolim}_K \circ \bigotimes^{\mathcal{D}}\colon \mathrm{Env}(\mathcal{D}, f) \to \mathcal{C}$ lifts to the category of $R(\mathbb{I}_{\mathcal{D}})$-modules; there is an induced transformation

$$\operatorname*{hocolim}_K \left(\bigotimes^{\mathcal{C}} \circ R \circ F \right) \to R \operatorname*{hocolim}_K \left(\bigotimes^{\mathcal{D}} \circ F \right)$$

of $R(\mathbb{I}_{\mathcal{D}})^{\otimes f}$-modules.

Definition 6.4. Suppose that K is sifted, $f\colon K \to \mathrm{Fin}$ is fixed, and that $R\colon \mathcal{D} \to \mathcal{C}$ is a lax symmetric monoidal functor between symmetric monoidal presentable ∞-categories. The *base-change map* is the natural transformation

$$R(\mathbb{I}_{\mathcal{D}}) \underset{R(\mathbb{I}_{\mathcal{D}})^{\otimes f}}{\otimes} \left(\operatorname*{hocolim}_K \bigotimes^{\mathcal{C}} \circ R \circ F \right) \to R \left(\operatorname*{hocolim}_K \bigotimes^{\mathcal{D}} \circ F \right)$$

of functors $\mathrm{Env}(\mathcal{D}, f) \to \mathrm{LMod}_{R(\mathbb{I}_{\mathcal{D}})}$, adjoint to the map of Proposition 6.3.

We now specialize this to the case where $\mathcal{D}$ is the category LMod_A of left modules over a fixed commutative algebra object.

Theorem 6.5. *Let A be a commutative algebra in a symmetric monoidal presentable ∞-category C. Suppose that K is sifted, $f\colon K \to \mathrm{Fin}$ is fixed, and that $R\colon \mathrm{LMod}_A \to C$ is the forgetful functor. Then R is lax symmetric monoidal, and the base-change map is a natural equivalence.*

Proof. The functor R is right adjoint to the strong symmetric monoidal functor $X \mapsto A \otimes X$, and hence is lax monoidal [Lur17, 7.3.2.7]. The functor R also preserves homotopy limits and colimits [Lur17, 4.2.3.3, 4.2.3.5].

We will first prove that the base-change map is an equivalence in the case where $K = *$ is the trivial category. In this case, without loss of generality, the map f is a choice of a finite set S and a lift F is equivalent to an S-indexed tuple $\{M_s\}$ of left A-modules. The base-change map is the map

$$A \underset{A^{\otimes S}}{\otimes} \left(\bigotimes M_s\right) \to \bigotimes^A M_s.$$

The base-change map is an equivalence whenever each M_s is an extended module of the form $A \otimes X_s$ for some X_s.

The natural augmentation of left A-modules $B(A, A, M) \to M$ from the two-sided bar construction gives rise to a diagram

$$\begin{tikzcd}
\mathrm{hocolim}_{\Delta^{op}} A \otimes_{A^{\otimes S}} \bigotimes B(A, A, M_s) \arrow[r] \arrow[d] & \mathrm{hocolim}_{\Delta^{op}} \bigotimes^A B(A, A, M_s) \arrow[d] \\
A \otimes_{A^{\otimes S}} \bigotimes M_s \arrow[r] & \bigotimes^A M_s.
\end{tikzcd}$$

The top map is a homotopy colimit of a diagram of equivalences because the bar construction levelwise consists of extended modules. Since C and LMod_A are presentable symmetric monoidal, by definition the tensor product preserves homotopy colimits in each variable and sifted homotopy colimits in general; therefore, the left and right maps are equivalences. The bottom map is then an equivalence.

Now suppose that K is a general sifted index category with map $f\colon K \to \mathrm{Fin}$. The base-change map is a map

$$A \underset{\mathrm{hocolim}_{k\in K} A^{\otimes f(k)}}{\otimes} \left(\underset{k\in K}{\mathrm{hocolim}} \bigotimes_{s\in f(k)} F(k)_s\right) \to \underset{k\in K}{\mathrm{hocolim}} \bigotimes^A_{k\in K, s\in f(k)} F(k)_s$$

Since K is sifted, and the forgetful functor preserves homotopy colimits, we can rewrite both sides as homotopy colimits indexed by $k \in K$. The base-change map is then equivalent to the homotopy colimit of the base-change maps indexed by $f(k)$, which we already showed to be equivalences. □

7 Shape

Definition 7.1. Let $C^{\otimes}$ be an ∞-operad with symmetric monoidal envelope $\mathrm{Env}(C)$. Given a functor $X\colon K \to \mathrm{Env}(C)$, the *shape* of X, denoted by $|X|$, is the homotopy colimit of the composite functor to the category $\mathcal{S}$ of spaces:

$$K \to \mathrm{Env}(C) \to \mathrm{Fin} \subset \mathcal{S}$$

Example 7.2. Suppose that $X\colon \Delta^{op} \to \mathrm{Env}(C)$ is a simplicial object. Then the composite $\Delta^{op} \to \mathrm{Env}(C) \to \mathrm{Fin}$ is a simplicial finite set, whose geometric realization is the shape $|X|$.

Example 7.3. Suppose A is an associative algebra in C. Then the functor $\mathrm{Env}(C) \to \mathrm{Fin}$ takes A to the associative algebra $*$ under coproduct. The cyclic bar construction $Z^{\otimes}(A)$ maps to the cyclic bar construction $Z^{\amalg}(*)$ and the associated shape is S^1.

Similarly, suppose A is an associative algebra with a left module N and a right module M. Then $\mathrm{Env}(C) \to \mathrm{Fin}$ takes the two-sided bar construction $\mathrm{Bar}^{\otimes}(M, A, N)$, whose homotopy colimit is $M \otimes_A N$, to the two-sided bar construction $\mathrm{Bar}^{\amalg}(*, *, *)$, which is isomorphic to the standard simplex Δ^1.

Definition 7.4. Let $C^{\otimes}$ be an ∞-operad and Y be a Kan complex. We define

$$\mathrm{Env}(C)_{/Y} = \mathrm{Env}(C) \times_{\mathcal{S}} \mathcal{S}_{/Y}\ .$$

Proposition 7.5. *Given an ∞-operad C and a functor $X\colon K \to \mathrm{Env}(C)$, the category of lifts of X to a functor $\widetilde{X}\colon K \to \mathrm{Env}(C)_{/Y}$ is equivalent to the space of maps $f\colon |X| \to Y$.*

Proof. By definition of the fiber product, lifts $\widetilde{X}$ are equivalent to lifts of the composite $K \to \mathcal{S}$ to $\mathcal{S}_{/Y}$; by definition of the slice category, these are equivalent to lifts of $K \to \mathcal{S}$ to natural maps from the diagram K to Y. However, the universal property of homotopy colimits precisely asserts that these extensions are equivalent to maps $|X| \to Y$. □

8 Free G-sets

In this section we will fix a finite group G and let BG be a Kan complex classifying principal G-bundles.

Definition 8.1. Let Fin be the category of finite sets, and $\mathrm{Free}(G)$ the category of finite free left G-sets and equivariant maps. Both categories are symmetric monoidal under disjoint union.

Remark 8.2. The category Fin is the symmetric monoidal envelope of the terminal multicategory $\{*\}$. In particular, any multicategory C has a canonical symmetric monoidal functor $\mathrm{Env}(C) \to Fin$, sending $\{x_s\}_{s \in S}$ to the indexing set S. Moreover, the one-point set $*$ is an algebra in Fin and as such is classified by a symmetric monoidal functor $\mathrm{Env}(Assoc) \to Fin$.

Similarly, the category $\mathrm{Free}(G)$ is the symmetric monoidal envelope of a one-object multicategory with underlying category BG.

Proposition 8.3. *Let $\mathcal{S}$ be the category of spaces, with Fin viewed as a full subcategory. The functor* $\mathrm{Free}(G) \to Fin_{/BG}$*, given by* $X \mapsto (p_X \colon EG \times_G X \to BG)$*, induces an equivalence of ∞-categories*

$$\mathrm{Free}(G) \to Fin_{/BG} = Fin \times_{\mathcal{S}} \mathcal{S}_{/BG}\,.$$

In particular, the space of lifts of a functor $X\colon I \to Fin$ to a functor $\widetilde{X}\colon I \to \mathrm{Free}(G)$ is equivalent to the space of maps $f\colon \operatorname{hocolim}_I X \to BG$, classifying principal G-bundles on the homotopy colimit.

Proof. Because BG is path-connected, an object $S \to BG$ is equivalent in $Fin_{/BG}$ to the image of $G \times S$. Therefore, this functor is essentially surjective, and so it suffices to show that it is fully faithful. This amounts to the assertion that for finite free G-sets X and Y, the diagram

$$\begin{tikzcd}
\mathrm{Map}_G(X, Y) \arrow[r] \arrow[d] & \mathrm{Map}(EG \times_G X, EG \times_G Y) \arrow[d] \\
\{p_X\} \arrow[r] & \mathrm{Map}(EG \times_G X, BG)
\end{tikzcd}$$

is a homotopy pullback diagram.

This diagram decomposes as a product diagram over the orbits of X, and so it suffices to take $X = G$. However, in this case we have the standard homotopy pullback diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & EG \times_G Y \arrow[d] \\
* \arrow[r] & BG.
\end{tikzcd}$$

Corollary 8.4. *There is an equivalence*

$$\mathrm{Free}(G) \times_{Fin} \mathrm{Env}(C) \simeq \mathrm{Env}(C)_{/BG}.$$

As a result, we write diagrams $K \to \mathrm{Env}(C)_{/BG}$ as pairs (X, f) of a functor $X\colon K \to \mathrm{Env}(C)$ and a classifying map $f\colon |X| \to BG$.

9 Unwinding

Proposition 9.1. *For an ∞-operad C, there is a symmetric monoidal* fiberwise tensor power *functor*

$$\psi\colon \mathrm{Env}(C)_{/BG} \to \mathrm{Env}(C)^{BG}.$$

Informally, the functor ψ sends a free G-set S and an $\bar{S}$-indexed family $\{c_{\bar{s}}\}_{\bar{s}\in\bar{S}}$ to the S-indexed family $\{c_s\}_{s\in S}$ with its G-action.

Proof. For $G = C_p$, this is [NS17, III.3.6]; we briefly recall their method.

The category $\mathrm{Env}(C)$ is symmetric monoidal, and the category $\mathrm{Fun}_\otimes(\mathrm{Env}(C), \mathrm{Env}(C))$ of symmetric monoidal functors inherits a pointwise symmetric monoidal structure. The inclusion of the identity functor id induces a symmetric monoidal functor from Fin, the free symmetric monoidal ∞-category on $\{id\}$, to $\mathrm{Fun}_\otimes(\mathrm{Env}(C), \mathrm{Env}(C))$; the value on S is the functor $X \mapsto X^{\otimes S}$. Composing with the symmetric monoidal functor $\mathrm{Free}_G \to \mathrm{Fin}^{BG}$ gives a symmetric monoidal functor $\mathrm{Free}_G \to \mathrm{Fun}_\otimes(\mathrm{Env}(C), \mathrm{Env}(C))^{BG}$. By [NS17, III.3.7], this structure is adjoint to a symmetric monoidal functor $\mathrm{Free}_G \times_{\mathrm{Fin}} \mathrm{Env}(C) \to \mathrm{Env}(C)^{BG}$. □

Definition 9.2. For a diagram $(X, f)\colon K \to \mathrm{Env}(C)_{/BG}$, represented by a map $X\colon K \to \mathrm{Env}(C)$ and a map $f\colon |X| \to BG$, we define the diagram obtained by *unwinding* X to be the composite

$$\psi^f X\colon K \xrightarrow{(X,f)} \mathrm{Env}(C)_{/BG} \xrightarrow{\psi} \mathrm{Env}(C)^{BG}.$$

Proposition 9.3. *The composite $\psi^f X\colon K \to \mathrm{Env}(C)^{BG} \to Fin^{BG}$ is the diagram of G-sets classified by the map $K \to \mathrm{Free}(G)$. In particular, on taking shapes there is a principal G-bundle $|\psi^f X| \to |X|$, classified by the map $f\colon |X| \to BG$.*

Example 9.4. There is a canonical prinicipal C_n-bundle $sd_n S^1 \to S^1 \xrightarrow{[n]} BC_n$ over the simplicial circle, and the unwinding $\psi^{[n]} Z(A)$ of the cyclic bar construction is the simplicial subdivision $sd_n Z(A)$ [BHM93]. More generally, if $f\colon P \to B$ is a principal G-bundle and A is a commutative algebra then $\psi^f(A^{\otimes B}) = A^{\otimes P}$.

By contrast, the unwinding $\psi^{[2]} Z(A; M)$ of the cyclic bar construction with coefficients is a simplicial object

$$M \otimes M \Leftarrow M \otimes A \otimes M \otimes A \Lleftarrow M \otimes A^{\otimes 2} \otimes M \otimes A^{\otimes 2} \cdots$$

Reorganizing terms, the above can be regarded as the cyclic bar construction of $A^{\otimes 2}$ with a particular bimodule structure on $M^{\otimes 2}$, as in the following definition.

Definition 9.5. Suppose that A is an algebra in a symmetric monoidal ∞-category C and that M is a k-linear A-bimodule. Fix an $n > 0$, and let $\tau\colon A^{\otimes n} \to A^{\otimes n}$ be a cyclic permutation generating an action of C_n. The *twisted tensor power* $M^{\circlearrowleft n}$ is the pullback of the ordinary $A^{\otimes n}$ bimodule $M^{\otimes n}$ along the map $1 \otimes \tau\colon (A^{\otimes n}) \otimes (A^{op})^{\otimes n} \to (A^{\otimes n}) \otimes (A^{op})^{\otimes n}$.

Remark 9.6. This twisted bimodule is C_n-equivariant with respect to the twist maps on $M^{\circlearrowleft n}$ and $A^{\otimes n}$.

Proposition 9.7. *There is a C_n-equivariant natural equivalence of simplicial objects*

$$\psi^{[n]} Z(A; M) \simeq Z(A^{\otimes n}; M^{\circlearrowleft n})$$

in $\mathrm{Env}(C)$.

Remark 9.8. These "cyclic" versions of THH with coefficients have also recently appeared in the work of Malkiewich–Ponto on traces [MP18].

10 The Tate diagonal

Fix a cyclic group C_p of prime order. For a based space W, there is a natural space-level diagonal map

$$W \to (W^{\wedge p})^{C_p}.$$

If X is a spectrum, then assembling the space-level diagonal maps gives a map called the *Tate diagonal*

$$X \to (X^{\otimes p})^{tC_p},$$

constructed by Greenlees–May in [GM95] and recently developed further in [NS17]. The Tate diagonal has a number of very useful properties: it is natural in X, it is impervious to the action of C_p on $X^{\otimes p}$, and it is lax symmetric monoidal. The compatibility between these properties is expressed as follows.

Theorem 10.1 ([NS17, III.3.8])**.** *For a finite free C_p-set T with quotient $\overline{T}$ and an indexed tuple $\{X_{\bar{t}}\}_{\bar{t}\in\overline{T}}$ of spectra, there is a Tate diagonal*

$$\bigotimes_{\bar{t}\in\overline{T}} X_{\bar{t}} \to \left(\bigotimes_{t\in T} X_{\bar{t}}\right)^{tC_p}.$$

The Tate diagonal is essentially unique as a BC_p-equivariant lax symmetric monoidal transformation between functors $\mathrm{Free}(C_p) \times_{\mathrm{Fin}} \mathrm{Sp}^{\otimes}_{act} \to \mathrm{Sp}$.

Our notation expresses this in the following way. The Tate diagonal is a lax symmetric monoidal natural transformation

$$\bigotimes^{\mathrm{Sp}} \circ X \to \left(\bigotimes^{\mathrm{Sp}} \circ \psi^f X\right)^{tC_p}$$

defined on (X, f) in $\mathrm{Env}(\mathrm{Sp})_{/BC_p}$.

Example 10.2. The lax symmetric monoidal structure then makes it possible for us to study the relationship with module structures. Given a commutative ring spectrum k, the iterated multiplication map $k^{\otimes p} \to k$ is C_p-equivariant and so there is a composite map

$$\phi\colon\ k \to (k^{\otimes p})^{tC_p} \to k^{tC_p}$$

called the *Tate-valued Frobenius* [NS17, IV.1.1].[6]

In these terms, we obtain the following indexed Tate diagonal.

Corollary 10.3. *Given a sifted index category K, there is a natural lax symmetric monoidal natural transformation*

$$\operatorname*{hocolim}_K \left(\bigotimes^{\mathrm{Sp}} X\right) \to \left(\operatorname*{hocolim}_K \bigotimes^{\mathrm{Sp}} \psi^f X\right)^{tC_p},$$

of functors $\mathrm{Fun}(K, \mathrm{Env}(\mathrm{Sp})_{/BC_p}) \to \mathrm{Sp}$.

[6]There are actually two maps $k \to k^{tC_p}$ of commutative algebras. One is the canonical unit $k \to k^{hC_p} \to k^{tC_p}$ because k has trivial C_p-action, and the other is ϕ.

Proof. When K is sifted, the functor $\mathrm{hocolim}_K$ is lax symmetric monoidal by Proposition 5.1. $\square$

Example 10.4. Suppose $E \to B$ is a principal C_p-bundle and that k is a commutative ring spectrum. Then the Loday constructions for B and E are related by a Tate diagonal:

$$k^{\otimes B} \to \left(k^{\otimes E}\right)^{tC_p}$$

Example 10.5. Let K be Δ^{op}, the simplicial index category. When applied to the cyclic bar construction $Z(A, M)$ in $\mathrm{Env}(\mathrm{Sp})$, the Tate diagonal becomes a natural transformation

$$\mathrm{THH}(A; M) \to \left[\mathrm{THH}(A^{\otimes p}; M^{\circlearrowleft p})\right]^{tC_p}.$$

on Hochschild homology with coefficients.

11 A relative Tate diagonal

The Tate diagonal from Corollary 10.3 takes place in the category of spectra. In this section we will examine the extent to which this admits a relative version, where X is a diagram of modules over a commutative ring spectrum k and we attempt to replace the monoidal structure of Sp with the monoidal structure in k-modules.

Theorem 11.1. *Suppose that k is a commutative ring spectrum, K is a sifted index category, and $(X, f)\colon K \to \mathrm{Env}(\mathrm{LMod}_k)_{/BC_p}$ is a diagram with shape $|X|$. Then there is a* relative Tate diagonal

$$k^{tC_p} \underset{k^{\otimes |X|}}{\otimes} \left(\underset{K}{\mathrm{hocolim}} \bigotimes^{\mathrm{Sp}} X\right) \to \left(\underset{K}{\mathrm{hocolim}} \bigotimes^{\mathrm{LMod}_k} \psi^f X\right)^{tC_p}.$$

Proof. Lax symmetric monoidality implies that the Tate diagonal

$$\varphi\colon\ k^{\otimes |X|} \to \left(k^{\otimes |\psi^f X|}\right)^{tC_p}$$

is a map of commutative ring spectra, and that the Tate diagonal

$$\left(\underset{K}{\mathrm{hocolim}} \bigotimes^{\mathrm{Sp}} X\right) \to \left(\underset{K}{\mathrm{hocolim}} \bigotimes^{\mathrm{Sp}} \psi^f X\right)^{tC_p}$$

is compatible with the $k^{\otimes |X|}$-module structure on the source and the $k^{\otimes |\psi^f X|}$-module structure on the target. Similarly, the augmentation map

$$k^{\otimes |\psi^f X|} \to k^{\otimes_k |\psi^f X|} \simeq k$$

is a C_p-equivarant map of commutative ring spectra, and the map

$$\underset{K}{\mathrm{hocolim}} \bigotimes^{\mathrm{Sp}} \psi^f X \to \underset{K}{\mathrm{hocolim}} \bigotimes^{\mathrm{LMod}_k} \psi^f X$$

is a C_p-equivariant map of $k^{\otimes|\psi^f X|}$-modules; we can then apply Tate spectra.

Putting these together, there is a composite map

$$\left(\operatorname*{hocolim}_K \bigotimes^{\mathrm{Sp}} X\right) \to \left(\operatorname*{hocolim}_K \bigotimes^{\mathrm{LMod}_k} \psi^f X\right)^{tC_p}.$$

This is a map of $k^{\otimes|X|}$-modules, with the target module pulled back from k^{tC_p}. The adjoint map is the desired relative Tate diagonal. □

Example 11.2. Let both K and the map $K \to \mathrm{Fin}$ be trivial. Then the composite map takes the form of a *k-module Tate diagonal*

$$M \to (M^{\otimes p})^{tC_p} \to (M^{\otimes_k p})^{tC_p}.$$

From the point of view of genuine-equivariant homotopy theory the Tate-valued Frobenius lifts k to a C_p-equivariant ring spectrum, and the k-module Tate diagonal lifts $M^{\otimes_k p}$ to a C_p-equivariant module called the relative norm of M.

Example 11.3. In the case of relative THH, this becomes a *relative* THH-*diagonal*

$$k^{tC_p} \otimes_{\mathrm{THH}(k)} \mathrm{THH}(A) \to \left[\mathrm{THH}^k(A)\right]^{tC_p}.$$

There is also

$$k^{tC_p} \otimes_{\mathrm{THH}(k)} \mathrm{THH}(A; M) \to \left[\mathrm{THH}^k(A^{\otimes p}; M^{\circlearrowleft p})\right]^{tC_p},$$

a relative THH-diagonal with coefficients.

12 Nonexistence of a true relative diagonal

We will use a calculation with topological Hochschild homology to illustrate the nonexistence of the k-module Tate diagonal. We learned this result from Lars Hesselholt.

Suppose there is a k-module Tate diagonal

$$\bigotimes_{s\in S}^{k} M_s \to \left[\bigotimes_{t\in T}^{k} M_{f(t)}\right]^{tC_p},$$

compatible with the one for spectra and functorial in pairs of a principal C_p-bundles $f\colon T \to S$ and an S-indexed tuple of k-modules. Then we could get a k-relative THH diagonal. We would get a diagram of the form

$$\begin{tikzcd} \mathrm{THH}(k) \arrow[r] \arrow[d] & \mathrm{THH}(k)^{tC_p} \arrow[d] \\ k \arrow[r, dotted] & k^{tC_p}. \end{tikzcd}$$

However, in the case of an Eilenberg–Mac Lane spectrum for $\mathbb{F}_p$ (or, more generally, for a perfectoid ring by calculations of Bhatt–Morrow–Scholze), Hesselholt–Madsen's calculations show that this would give a commutative diagram of graded rings

$$\begin{tikzcd}
\mathbb{F}_p[u] \arrow[r] \arrow[d] & \mathbb{F}_p[u^{\pm 1}] \arrow[d] \\
\mathbb{F}_p \arrow[r, dotted] & \mathbb{F}_p[u^{\pm 1}] \cdot \{1, v\}
\end{tikzcd}$$

upon taking coefficients.

13 Smooth algebras

In this section, we assume that k is a commutative ring spectrum.

Definition 13.1. Let A be a k-algebra and p a prime. We say that a k-linear A-bimodule *M satisfies Tate descent at p* if the relative THH diagonal

$$k^{tC_p} \otimes_{\mathrm{THH}(k)} \mathrm{THH}(A, M) \to (\mathrm{THH}^k(A^{\otimes_k p}, M^{\circlearrowleft_k p})^{tC_p}$$

is an equivalence. If M satisfies Tate descent at all primes, we simply say that M satisfies Tate descent.

Proposition 13.2. *The collection of k-linear A-bimodules satisfying Tate descent at p is a thick subcategory, and in particular is closed under finite limits and colimits.*

Proof. The k-module THH diagonal is a natural transformation of exact functors: it preserves cofiber sequences. In particular, the collection of objects for which it is an equivalence is a thick subcategory of A-bimodules. □

Proposition 13.3. *Any k-linear A-bimodule of the form $N \otimes_k A$, where N is a left A-module that is perfect as a k-module, satisfies Tate descent.*

Proof. The natural map $k \to A$ induces natural equivalences

$$\mathrm{THH}(k; N) \to \mathrm{THH}(A; N \otimes_k A)$$

and

$$\mathrm{THH}(k^{\otimes p}; N^{\circlearrowleft p}) \to \mathrm{THH}(A^{\otimes p}; (N \otimes_k A)^{\circlearrowleft p}).$$

Therefore, by naturality of the Tate diagonal it suffices to show this result when $A = k$. Because the collection of N-modules satisfying Tate descent is a thick subcategory, it suffices to show this result when $N = k$ in order to conclude it is true for all perfect k-modules.

In this case, we are considering the relative THH-diagonal

$$k^{tC_p} \otimes_{\mathrm{THH}(k)} \mathrm{THH}(k) \to \left[\mathrm{THH}^k(k^{\otimes_k p}; k^{\circlearrowleft_k p})\right]^{tC_p},$$

which simplifies to the natural transformation

$$k^{tC_p} \otimes_{\mathrm{THH}(k)} \mathrm{THH}(k) \to \left[\mathrm{THH}^k(k;k)\right]^{tC_p}.$$

Both sides are weakly equivalent to k^{tC_p}. Moreover, on both sides this equivalence induced by the map

$$k \to (k^{\otimes p})^{tC_p} \to k^{tC_p}$$

in degree 0 of the simplicial diagrams defining THH. ◻

Proposition 13.4. *If A is smooth, then all k-linear A-bimodules which are perfect as left k-modules satisfy Tate descent.*

Proof. Fix any A-bimodule M which is perfect over k, and let $\mathcal{T}$ be the full subcategory of k-linear A-bimodules B such that the bimodule $M \otimes_A B$ satisfies Tate descent. By Proposition 13.3, the bimodule $A \otimes_k A$ is in $\mathcal{T}$. The category $\mathcal{T}$ is a thick subcategory by Proposition 13.2. By definition, since A is smooth over k, A lies in the thick subcategory of k-linear A-bimodules generated by $A \otimes_k A$, and therefore A is in $\mathcal{T}$.

The equivalence of bimodules $M \simeq M \otimes_A A$ then shows that M satisfies Tate descent. ◻

Corollary 13.5. *If A is smooth and proper, then A satisfies Tate descent as a bimodule over itself.*

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