HOMOTOPY TOPOI AND EQUIVARIANT ELLIPTIC COHOMOLOGY

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Abstract

We use the language of homotopy topoi, as developed by Lurie [17], Rezk [21], Simpson [23], and Töen-Vezossi [24], in order to provide a common foundation for equivariant homotopy theory and derived algebraic geometry. In particular, we obtain the categories of G-spaces, for a topological group G, and E-schemes, for an E_∞-ring spectrum E, as full topological subcategories of the homotopy topoi associated to sheaves of spaces on certain small topological sites. This allows for a particularly elegant construction of the equivariant elliptic cohomology associated to an oriented elliptic curve A and a compact abelian Lie group G as an essential geometric morphism of homotopy topoi. It follows that our definition satisfies a conceptually simpler homotopy-theoretic analogue of the Ginzburg-Kapranov-Vasserot axioms [8], which allows us to calculate the cohomology of the equivariant G-spectra S^V associated to representations V of G.

To my parents.

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Chapter 1

Introduction

1.1 The Equivariant Witten Genus

1.1.1 Genera

Classically, a genus is defined to be a map of graded rings

$$\sigma : M_* \to E_* \tag{1.1}$$

where M_* is the coefficient ring of some bordism theory. Typically, E_* is also the coefficient ring of a generalized homology theory.

The most general means of producing genera comes from the theory of formal groups. In [20], Quillen proved that the coefficient ring MU_* of complex cobordism is isomorphic to the Lazard ring, the ring (co)representing (commutative, one-dimensional) formal group laws. In other words, the set of formal group laws over a ring R is in one-to-one correspondence with the set of ring maps $MU_* \to R$. Moreover, the MU_*-coalgebra MU_*MU classifies strict isomorphisms of formal group laws.

In the language of schemes this says that the stack $\mathcal{M}$ associated to the groupoid scheme

$$\operatorname{Spec}(MU_*, MU_*MU) \tag{1.2}$$

classifies commutative, one-dimensional formal groups. In practice, one often uses the Landweber exact functor theorem to check whether or not the functor on spaces defined by the formula

$$E_*(X) := MU_*(X) \otimes_{MU_*} E_*, \tag{1.3}$$

classified by a formal group law $f : \operatorname{Spec} E_* \to \mathcal{M}$ on E_* is a homology theory. The formal group may then be recovered from the homology theory $E_*(-)$ as its value on $\mathbb{CP}^\infty$, for $MU_*(\mathbb{CP}^\infty)$ is the universal formal group law.

Given a homology theory E_* as above, evaluating on a point determines a genus. Conversely, if we restrict to genera of the form $\sigma : MU_* \to E_*$ where σ is Landweber exact, we see that a genus determines a morphism of generalized homology theories $MU_*(X) \to E_*(X)$.

1.1.2 Elliptic genera and rigidity

Elliptic cohomology has its origins in the study of certain genera that were known to take values in various rings of modular forms. In light of the above discussion, given such a genus $\sigma : M_* \to E_*$, one is naturally led to the question of whether or not there exists a homology theory E with $E_*(*) \cong E_*$. Such a homology theory is called an elliptic homology theory, for a variety of overlapping reasons, perhaps the simplest of which stems from the fact that, roughly speaking, modular forms correspond to sections of tensor powers of the line bundle

$$\omega := e^*\Omega^1_{A/S}, \tag{1.4}$$

where $e : S \to A$ is the identity section of the universal elliptic curve A over the moduli stack of elliptic curves S.

The Brown Representability Theorem gives a means of representing generalized homology theories in the homotopy category of spectra, i.e. the stabilization of the homotopy theory of spaces. In particular, an elliptic spectrum is defined to be an even-periodic ring spectrum E together with an isomorphism of formal groups

$$\operatorname{Spec} \pi_0 E^{\mathbb{CP}^\infty} \to \widehat{A} \tag{1.5}$$

where $\widehat{A}$ is the formal group of an elliptic curve A. Not only do elliptic spectra exist, but there is a sort-of universal elliptic spectrum called the spectrum of "topological modular forms" (or "tmf" for short) which, though not technically an elliptic spectrum, can be thought of as a sheaf of elliptic spectra over the moduli stack of Weierstrass forms (which contains the moduli stack of elliptic curves as a dense open subspace).

Given a morphism $\sigma : M \to E$ of ring spectra, the corresponding genus $\sigma^* : M^* \to E^*$ on cohomology rings is said to be rigid if for any compact, connected Lie group G, the associated Borel-equivariant genus

$$\sigma^*_G : M^*_G := M^*(BG) \to E^*(BG) =: E^*_G \tag{1.6}$$

factors through the map $E^* \to E^*(BG)$ in cohomology induced by the projection $BG \to *$.

It turns out that rigidity is a very restrictive property; for example, rigidity implies strong multiplicativity, which is to say that the genus of the total space of a principal fibration with structure group G is equal to the product of the genus of the base with the genus of the fiber [?]. It has a number of other important consequences as well.

The following heuristic argument for establishing the rigidity of the Witten genus gives some motivation for the construction of a string-orientable equivariant elliptic cohomology theory. Suppose there exists a genuine G-equivariant elliptic cohomology theory Ell_G. By this we mean an equivariant cohomology theory with the following features:

(i) Ell_G is string-orientable, i.e. if $V \to X$ is a String-orientable real (or $\mathrm{String}_{\mathbb{C}}$-orientable complex) G-equivariant vector bundle over a G-space X, then there is a Thom isomorphism $\mathrm{Ell}^*_G(X^V) \cong \mathrm{Ell}^*_G(X_+)$;

(ii) For any G-space X there is a completion isomorphism $\mathrm{Ell}^*_G(X)_{\hat{I}} \to \mathrm{Ell}^*(X_G)$, where I is the augmentation ideal and $X_G := EG \times_G X$ is the Borel construction on X;

(iii) $\mathrm{Ell}^0_G(*) \cong \Gamma(\mathcal{O}_{A_G})$ for an elliptic curve A over $\operatorname{Spec}\mathrm{Ell}^0$; here A_G is scheme

$$A_G = \mathrm{Hom}(\mathrm{Hom}(\mathbb{T}, T_G), A)/W_G, \tag{1.7}$$

where T_G is a maximal torus of G and $W_G = N_G T_G/T_G$ is the Weyl group of G.

Let M be a string manifold, and consider the commutative diagram

$$\begin{tikzcd} \mathrm{Ell}_G(M) \arrow[r] \arrow[d, "p_!"'] & \mathrm{Ell}(M_G) \arrow[d, "p_!"'] \\ \mathrm{Ell}_G(*) \arrow[r] & \mathrm{Ell}(*_G) \end{tikzcd} \tag{1.8}$$

in which the horizontal maps are the completion homomorphisms and the vertical maps are the Gysin maps obtained from the Pontrjagin-Thom construction.

By definition, the Borel-equivariant Witten genus of M is $p_!(1) \in \mathrm{Ell}(*_G) = \mathrm{Ell}(BG)$. The horizontal maps are ring maps, so $p_!(1)$ comes from a class in $E_G(*)$. But the G-equivariant elliptic cohomology of a point is isomorphic to the ring of regular functions on A_G, and if r denotes the rank of G, then A_G is a quotient of A^r by a subgroup of Σ_r. Since A itself is projective it is easy to show that each A_G is projective, so the only regular functions are the constants. Hence $\mathrm{Ell}^0_G \cong \mathrm{Ell}^0$, so the Witten genus is rigid.

1.1.3 The axiomatic approach of Ginzburg-Kapranov-Vasserot

We briefly review the axiomatic approach of Ginzburg-Kapranov-Vasserot [8]. Let $A \to S$ be an elliptic curve, i.e. an abelian group S-scheme whose fiber at every geometric point is an elliptic curve. Let G be a connected compact Lie group with maximal torus T and Weyl group W, and let A_G be the S-scheme

$$A_G := \mathrm{Hom}(\mathrm{Hom}(T, \mathbb{T}), A)/W. \tag{1.9}$$

Write $\mathrm{Coh}(A_G)$ for the category of coherent sheaves of $\mathcal{O}_{A_G}$-modules.

According to Ginzburg-Kapranov-Vasserot, a G-equivariant elliptic cohomology theory associated to A consists contravariant functors Ell^i_G from the category of pairs of finite G-complexes into $\mathrm{Coh}(A_G)$, together with natural sheaf morphisms

$$\partial : \mathrm{Ell}^i_G(X) \to \mathrm{Ell}^{i+1}_G(X, U) \tag{1.10}$$

for any pair of finite G-complexes (X, U) and multiplication maps

$$\mathrm{Ell}^i_G(X, U) \otimes \mathrm{Ell}^j_G(Y, V) \to \mathrm{Ell}^{i+j}_G(X \times Y, X \times V \coprod_{U \times V} U \times Y) \tag{1.11}$$

which are associative, graded commutative and functorial.

The functors Ell^i_G must have the property that any two G-homotopic maps of pairs induce the same maps on Ell^i_G, and for every pair (X, U) the natural sequence

$$\cdots \to \mathrm{Ell}^i_G(X, U) \to \mathrm{Ell}^i_G(X) \to \mathrm{Ell}^i_G(U) \to \mathrm{Ell}^{i+1} \to \cdots \tag{1.12}$$

is exact. Moreover, the Ell^* should be even and periodic in the sense that there are natural isomorphisms

$$\mathrm{Ell}^{i-1}_G(X, U) \cong \mathrm{Ell}^{i+1}_G(X, U) \otimes \omega, \tag{1.13}$$

where ω is the pullback to A_G of the normal bundle for the inclusion of the identity section $S \to A$, $\mathrm{Ell}^{2i}_G(*) \cong \omega^{\otimes(-i)}$ and $\mathrm{Ell}^{2i-1}_G(*) = 0$.

Furthermore, there are axioms relating the functors $\mathrm{Ell}^i_{(-)}$ for different groups. Let $\varphi : H \to G$ be a homomorphism of compact connected Lie groups. Then φ induces a restriction map

$$\varphi^* : \mathrm{Coh}(A_G) \to \mathrm{Coh}(A_H) \tag{1.14}$$

with a right adjoint

$$\varphi_* : \mathrm{Coh}(A_H) \to \mathrm{Coh}(A_G); \tag{1.15}$$

the fact that φ_* takes coherent $\mathcal{O}_{A_H}$-modules to coherent $\mathcal{O}_{A_G}$-modules follows from the fact that the induced map $A_H \to A_G$ is proper. On the level of spaces, φ induces an induction map $\varphi_! : \mathcal{S}^H \to \mathcal{S}^G$ given by $(-) \times_H G$ which is left adjoint to the restriction map $\varphi^* : \mathcal{S}^G \to \mathcal{S}^H$. The $\mathrm{Ell}^*_{(-)}$ should satisfy the following change-of-groups axioms.

(i) *Restriction*: There exists an Eilenberg-Moore spectral sequence

$$E_1^{i,j} = L_i\varphi^* \circ \mathrm{Ell}^j_G \implies \mathrm{Ell}^{j-i}_H \circ \varphi^*, \tag{1.16}$$

where $L_i\varphi^*$ is the i^{th} left derived functor of the inverse image; in particular, if φ is flat (which is the case when $\varphi : H \to G$ is a fibration) then $\varphi^* \circ \mathrm{Ell}^i_G \cong \mathrm{Ell}^i_H \circ \varphi^*$.

(ii) *Induction*: There exists a natural multiplicative morphism of functors

$$T_\varphi : \mathrm{Ell}_G \circ \varphi_! \implies \varphi_* \circ \mathrm{Ell}_H \tag{1.17}$$

from the category of pairs of finite G-complexes to $\mathrm{Coh}(A_G)$ such that if φ is a surjection with kernel K and Y is a K-free H-complex with quotient map $q : Y \to Y/K$ then the composite

$$p^* \circ T_\varphi : \mathrm{Ell}_G(\varphi_! Y/K) \to \varphi_*\mathrm{Ell}_H(Y/K) \to \varphi_*\mathrm{Ell}_H(Y) \tag{1.18}$$

is an isomorphism.

(iii) *Künneth*: For any G-complex X and H-complex Y, the natural map

$$\pi_G^*\mathrm{Ell}^i_G(X) \otimes \pi_H^*\mathrm{Ell}^j_H(Y) \to \mathrm{Ell}^{i+j}_{G\times H}(X \times Y), \tag{1.19}$$

where $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$ are the projections, is an isomorphism.

Ginzburg-Kapranov-Vasserot go on to conjecture that any elliptic curve A gives rise to a unique equivariant elliptic cohomology theory, natural in A. They add that better formulation of this conjecture would involve the Lewis-May-Steinberger category of G-spectra. In other words, there should exist a sheaf $\mathcal{F}_G$ of G-spectra on A_G such that for any homomorphism $\varphi : H \to G$ there is an equivalence

$$\mathbb{L}\varphi^* \mathcal{F}_G \simeq \mathcal{F}_H, \tag{1.20}$$

where $\mathbb{L}\varphi^*$ is the total derived functor of the inverse image.

We shall see that these axioms are really statements about geometric morphisms of homotopy topoi [17], [21]. In particular, it is possible to define elliptic cohomology as a map of homotopy topoi and derive the change-of-group formulas from relations among the geometric morphisms defined by the induction-restriction adjunction of a group homomorphism.

1.1.4 Equivariant elliptic cohomology via derived algebraic geometry

To make the Ginzburg-Kapranov-Vasserot conjecture precise entails rather a lot of machinery, much of which we set up in the next two chapters. We give a quick summary here.

Recently there has been quite a bit of work on the subject of derived algebraic geometry. The general idea is that one ought to be able to construct a category of "derived schemes" by gluing together "affine derived schemes". A category of affine derived affine schemes is roughly the opposite category of the category of monoids in a closed symmetric monoidal topological (or simplicial) model category together with a Grothendieck topology on these affine objects. Continuous functors from such a category to another topological category then admit derived analogues in the usual way. Examples include topological abelian groups or spectra, in which case the monoids are the topological commutative rings or E_∞ ring spectra, respectively, equipped with one of the standarn topologies (i.e. Zariski, ètale, etc.). The classical example is that of discrete abelian groups. In this case the monoids are the discrete commutative rings endowed with the Zariski topology, and one recovers classical scheme theory.

Let us suppose for the moment that we have an algebraic geometry of E_∞-ring spectra in which we can define the notion of a derived elliptic curve A. In particular, we require A to be an abelian group derived S-scheme, say, where $S = \operatorname{Spec} E$ for some elliptic spectrum E. Then it is reasonable to suppose that we can define A_G as above as a starting point in the construction of equivariant elliptic cohomology. The main point here is that in order to specify a continuous homotopy-colimit preserving functor from G-spaces to an arbitrary topological category $\mathcal{T}$, it is enough to specify a continuous functor on the full topological subcategory of G-orbits. This then gives a good candidate for the equivariant elliptic cohomology associated to the derived elliptic curve A, at least for abelian compact Lie groups G, as follows.

Any G-orbit is isomorphic to one of the form G/H for a closed subgroup H of G. The functor that sends a compact connected Lie group G to the derived scheme A_G should associate to the closed inclusion $i : H \to G$ the affine derived A_G-scheme A_H and therefore a coherent $\mathcal{O}_{A_G}$-algebra $i_*\mathcal{O}_{A_H}$. We might hope that there is a natural topological model structure on the category of quasicoherent $\mathcal{O}_{A_G}$-algebras and consequently on its opposite category of affine A_G-schemes. Then the assignment which takes the closed subgroup H of G to the affine A_G-scheme A_H should provide the object function of a covariant functor from G-orbits to affine A_G-schemes.

Provided we can extend this to a continuous functor on the topological category of G-orbits, the theorem tell us that this extends to a functor on all G-spaces by forming a derived analogue of the left Kan extension construction. Passing from affine A_G-schemes to the opposite category of quasicoherent $\mathcal{O}_{A_G}$-modules, we obtain a continuous contravariant functor from G-spaces to quasicoherent $\mathcal{O}_{A_G}$-algebras. This extends in the obvious way to a functor from pairs of G-spaces to quasicoherent $\mathcal{O}_{A_G}$-modules, yielding a candidate for equivariant elliptic cohomology.

This is essentially the construction we will give. It turns out that one needs an extra bit of information in order to make this work, namely, Lurie's notion of an "orientation" on a derived elliptic curve, or, in the rational case, Greenlees' notion of a "coordinate divisor" on a rational elliptic curve. Given this, it is possible to construct equivariant elliptic cohomology theories satisfying a derived version of the Ginzburg-Kapranov-Vasserot axioms which are actually much simpler to state and not too difficult to prove, provided the necessary machinery is in place.

1.2 Summary of Results and Preliminary Remarks

1.2.1 Notations and coventions

Throughout this paper we let $\mathcal{S}$ denote the usual topological model category of compactly generated Hausdorff spaces. We think of $\mathcal{S}$ as providing an ambient universe in which to carry out our various constructions, playing the same role in topological category theory as the category of sets does in the ordinary category theory. This point of view is consistent with the fact that $\mathcal{S}$ is the basic homotopy topos, in the sense that all other homotopy topoi are left-exact localizations of categories of presheaves of spaces on small topological categories.

We want to model spaces in such a way that the mapping spaces have the correct homotopy type. One way to do this it to define $\mathrm{Map}(X, Y)$ to be the geometric realization of the simplicial set of maps between the singular complexes of X and Y. Since the singular complex of a space is always cofibrant and fibrant, our mapping spaces are homotopy invariant. Note that composition is strictly associative in $\mathcal{S}$, at least up to coherent isomorphism, and that $\mathcal{S}$ is cartesian closed with respect to this mapping space.

If $\mathcal{T}$ is a topological model category then tensors and cotensors of spaces K with objects X of $\mathcal{T}$ will be written $K \otimes X$ and X^K, respectively. The space of maps between any pair of objects X, Y of $\mathcal{T}$ will be written $\mathrm{Map}(X, Y)$, and $*$ will denote a terminal object in $\mathcal{T}$.

We remind the reader that there are canonical equivalences

$$\mathrm{Map}(K \otimes X, Y) \cong \mathrm{Map}(K, \mathrm{Map}(X, Y)) \cong \mathrm{Map}(X, Y^K), \tag{1.21}$$

so that in particular the functor $K \otimes (-) : \mathcal{T} \to \mathcal{S}$ commutes with colimits while the functor $(-)^K : \mathcal{T} \to \mathcal{S}$ commutes with limits. Moreover, we have $* \otimes X \cong X \cong X^*$, since $\mathrm{Map}(*, \mathrm{Map}(X, Y)) \cong \mathrm{Map}(X, Y)$.

If in addition $\mathcal{T}$ is closed symmetric monoidal with respect to a product $\otimes$ (not to be confused with the tensor operation above), then we write $\mathbb{F}_\otimes(Y, -)$, or simply $\mathbb{F}(Y, -)$ when the operation $\otimes$ is understood, for the right adjoint of $(-) \otimes Y$. In other words, each Y determines an isomorphim

$$\mathrm{Map}(X \otimes Y, Z) \cong \mathrm{Map}(X, \mathbb{F}(Y, Z)), \tag{1.22}$$

natural in X, Y and Z.

Note that any topological category $\mathcal{T}$ has an underlying ordinary category $\pi_0\mathcal{T}$ with the same class of objects but whose morphism set

$$\pi_0\mathcal{T}(X, Y) := \pi_0\mathrm{Map}(X, Y) \tag{1.23}$$

is defined as the set of path components of the mapping space $\mathrm{Map}(X, Y)$. Of course, since $\mathrm{Map}(X, Y)$ will normally be a cofibrant space this is the same as the set of connected components of $\mathrm{Map}(X, Y)$. If $\mathcal{T}$ is a topological model category and $\mathrm{Map}(X, Y)$ is defined properly as to be homotopy invariant, then $\pi_0\mathcal{T}$ is equivalent to the homotopy category of $\mathcal{T}$.

1.2.2 Outline of this paper

It is clear from the derived version of the Ginzburg-Kapranov-Vasserot axioms that the theory of derived schemes provides a natural framework for the consideration of certain equivariant cohomology theories. Just as in the classical context, the category of derived schemes may be regarded as the full topological subcategory of the derived Zariski site consisting of the locally representable objects. In other words, derived schemes are a full topological subcategory of a homotopy topos. But G-spaces are also a full topological subcategory of the a homotopy topos of $\mathrm{Orb}(G)$-spaces. Our construction of equivariant elliptic cohomology amounts to specifying a morphism of homotopy topoi.

We begin with a discussion of topological model categories with emphasis on the Bousfield-Kan model categories, i.e. those that are Quillen equivalent to the category of presheaves of spaces on a small topological category. The homotopy topoi, then, are precisely the topological model categories which are topologically Quillen equivalent to left-exact localizations of Bousfield-Kan model categories.

In the next chapter we give the introduce the necessary derived algebro-geometric machinery, following recent work of Töen-Vezossi [24]. We state and prove some basic results necessary for the definition of a derived elliptic curve. When equipped with Lurie's notion of an orientation, a derived elliptic curve encodes the exact data necessary to specify the geometric morphism defining equivariant elliptic cohomology.

In the fourth chapter we present our construction of the equivariant elliptic cohomology associated to an oriented elliptic curve A and a compact abelian Lie group G. In particular, we show that the theory in questions satisfies a derived version of the Ginzburg-Kapranov-Vasserot axioms.

The fifth and final chapter is devoted to calculating the equivariant elliptic cohomology of the Thom space of an equivariant vector bundle as a line bundle over the equivariant elliptic cohomology of the base. We show that the cohomology of the sphere associated to a virtual complex string representation is a trivial line bundle.

1.2.3 Statement of Results

The precise statements of our main results are as follows.

Theorem 1 *The category of $\mathcal{S}^G$ of G-spaces for a topological group G and Sch_E of E-schemes for a commutative $\mathbb{S}$-algebra E are full topological subcategories of the homotopy topoi of sheaves of spaces on the topological sites $\mathrm{Orb}(G)$ of G-orbits and Aff_E of affine E-schemes.*

This observation allows us to define the equivariant elliptic cohomology associate to an oriented elliptic curve, which is done in two steps.

Theorem 2 *To specify a homotopy colimit preserving functor from the homotopy topos of sheaves on $\mathrm{Orb}(G)$ to the homotopy topos of sheaves on Aff_E it suffices to specify a continuous orb(G)-diagram of sheaves on Aff_E.*

We then consider the special case of the $\mathrm{Orb}(G)$-diagrams of E-schemes induced by an oriented elliptic curve A over $\mathrm{Spec}\, E$.

Theorem 3 *An elliptic curve A over* $\operatorname{Spec} E$ *equipped with an orientation* $\operatorname{Spf} E^{B\mathbb{T}} \simeq \widehat{A}$ *determines for each compact abelian Lie group G an* $\operatorname{Orb}(G)$*-diagram of abelian group E-schemes such that the induced Quillen map $\mathcal{S}^{\operatorname{Orb}(G)} \to \mathcal{S}^{\operatorname{Aff}_E}/A_G$ is an essential geometric morphism of homotopy topoi.*

We write $X \otimes_G A$ for the left adjoint of the inverse image functor $\mathcal{S}^{\operatorname{Aff}_E}/A_G \to \mathcal{S}^{\operatorname{Orb}(G)}$ and call this the G-equivariant elliptic cohomology functor. In order to justify this terminology, let Γ denote the composite functor $\mathcal{S}^{\operatorname{Aff}_E} \to \operatorname{Aff}_E \to \operatorname{Alg}_E$ obtained by composing the left adjoint of the Yoneda embedding $\operatorname{Aff}_E \to \mathcal{S}^{\operatorname{Aff}_E}$ with the opposite category functor $\operatorname{Aff}_E \to \operatorname{Alg}_E$.

Theorem 4 *The assignment which associates to a finite G-space X the $\pi_* E$-algebra $\pi_*\Gamma(X \otimes_G A)$ satisfies the G-equivariant suspension and cofibration axioms and therefore determines a cohomology theory on the category of finite G-spaces.*

In general, however, we do not apply the functor $\pi_*\Gamma(-)$; rather, following Ginzburg-Kapranov-Vasserot [8], we simply regard the sheaf $X \otimes_G A$ as the elliptic cohomology of the G-space X.

Theorem 5 *If $V \to X$ is a virtual G-equivariant complex vector bundle on X then the G-equivariant elliptic cohomology of V is naturally a line bundle over the scheme $X \otimes_G A$.*

Chapter 2

Homotopy Theory of Diagrams and Homotopy Topoi

2.1 Continuous diagrams of spaces

2.1.1 Algebras and coalgebras

Let I be a small topological category, which we regard as a category object in $\mathcal{S}$ by writing I_0 for the discrete space of objects of I and I_1 for the space of arrows of I. Since I_0 is a discrete space, any space X over I_0 is canonically isomorphic to the disjoint union of its fibers X_i over the points i of I_0. In particular, the category $\mathcal{S}^{I_0}$ inherits the structure of a closed symmetric monoidal category from that of $\mathcal{S}$, since any space Y over I_0 determines an adjunction

$$\mathrm{Map}(X \times Y, Z) \cong \prod_i \mathrm{Map}(X_i \times Y_i, Z_i) \cong \prod_i (X_i, Z_i^{Y_i}) \cong \mathrm{Map}(X, \mathbb{F}(Y, Z)) \tag{2.1}$$

which is evidently natural in Y. Moreover, $\mathcal{S}^{I_0}$ is topologically complete and cocomplete, with limits, colimits, tensors and cotensors taken fiberwise over I_0.

This rather trivial observation will ultimately imply that the category $\mathcal{S}^I$ of continuous functors from I to $\mathcal{S}$, or I-diagrams in $\mathcal{S}$, is itself a closed symmetric monoidal, topologically complete and cocomplete category. To begin, define endomorphisms $\mathbb{E}_I$ and $\mathbb{F}_I$ of $\mathcal{S}^{I_0}$ by the formulas

$$\begin{aligned} (\mathbb{E}_I X)_j &:= \coprod_i X_i \times \mathrm{Map}(i, j) \\ (\mathbb{F}_I Y)_i &:= \prod_j \mathbb{F}(\mathrm{Map}(i, j), Y_j), \end{aligned} \tag{2.2}$$

and note that $\mathbb{E}_I$ and $\mathbb{F}_I$ fit into an adjunction

$$\mathrm{Map}(\mathbb{E}_I X, Y) \cong \mathrm{Map}(X, \mathbb{F}_I Y). \tag{2.3}$$

Proposition 1 *The endomorphisms $\mathbb{E}_I$ and $\mathbb{F}_I$ extend to a monad and comonad, repspectively, on $\mathcal{S}^{I_0}$, such* *that the category* Alg_I *of* $\mathbb{E}_I$*-algebras is isomorphic to the category* Coalg_I *of* $\mathbb{F}_I$*-coalgebras, each of which are isomorphic to the topological category* $\mathcal{S}^I$ *of continuous functors* $I \to \mathcal{S}$*.*

Proof. Define natural transformations $\eta_I : 1 \to \mathbb{E}_I \leftarrow \mathbb{E}_I^2 : \mu_I$ by the formulas

$$\begin{tikzcd}
X_k \arrow[r, "(\eta_I X)_k"] \arrow[d, "\cong"] & (\mathbb{E}_I X)_k & & (\mathbb{E}_I^2 X)_k \arrow[ll, "(\mu_I X)_k"'] \arrow[d, "\cong"] \\
X_k \arrow[r] & \coprod_j X_j \times \mathrm{Map}(j,k) \arrow[u, "\cong"] & & \coprod_{i,j} X_i \times \mathrm{Map}(i,j) \times \mathrm{Map}(j,k) \arrow[ll]
\end{tikzcd} \tag{2.4}$$

and, dually, natural transformations $\epsilon_i : 1 \leftarrow \mathbb{F}_i \to \mathbb{F}_i^2 : \delta_I$ by the formulas

$$\begin{tikzcd}
Z_i & (\mathbb{F}_I Z)_i \arrow[l, "(\epsilon_I Z)_i"'] \arrow[rr, "(\delta_I Z)_i"] \arrow[d, "\cong"] & & (\mathbb{F}_I^2 Z)_i \\
Z_i \arrow[u, "\cong"] & \prod_j \mathbb{F}(\mathrm{Map}(i,j), Z_j) \arrow[l] \arrow[rr] & & \prod_{i,j} \mathbb{F}(\mathrm{Map}(i,j) \times \mathrm{Map}(j,k), Z_k). \arrow[u, "\cong"]
\end{tikzcd} \tag{2.5}$$

The fact that I is a topological category forces the requisite relations among the natural transformations.

Now a continuous functor $I \to \mathcal{S}$ consists of a space $Y = \coprod_i Y_i$ over I_0 along with an associative, unital action

$$\coprod_i Y_i \times \mathrm{Map}(i,k) \to Y_k \tag{2.6}$$

or, equivalently, a coassciative, counital coaction

$$Y_i \to \prod_k \mathbb{F}(\mathrm{Map}(i,k), Y_k) \tag{2.7}$$

of I. The former is the same as an $\mathbb{E}_I$-algebra structure on Y while the latter is the same as an $\mathbb{F}_I$-coalgebra structure on Y. □

2.1.2 Free and cofree diagrams

Let Y be a space over I_0. It is well known that the I_0-space $\mathbb{E}_I Y$ has a canonical $\mathbb{E}_I$-algebra structure, called the free $\mathbb{E}_I$-algebra on Y, and dually that the I_0-space $\mathbb{F}_I Y$ has a canonical $\mathbb{F}_I$-coalgebra structure, called the cofree $\mathbb{F}_I$-coalgebra on Y. The terminology reflects the fact that $\mathbb{E}_I$ is left-adjoint to the forgetful functor $\mathrm{Alg}_I \to \mathcal{S}^{I_0}$ and $\mathbb{F}_I$ is right-adjoint to the forgetful functor $\mathrm{Coalg}_I \to \mathcal{S}^{I_0}$.

Thus if X is an $\mathbb{E}_I$-algebra and Z is an $\mathbb{F}_I$-coalgebra, we may form the free $\mathbb{E}_I$-algebra $\mathbb{E}_I X$ and the cofree $\mathbb{F}_I$-coalgebra $\mathbb{F}_I Z$ on the underlying I_0-spaces of X and Z, respectively, in order to obtain an endofunctor $\mathbb{E}_I$ on Alg_I and an endofunctor $\mathbb{F}_I$ on Coalg_I. Iterating this process yields an augmented simplicial $\mathbb{E}_I$-algebra

$$\cdots \begin{smallmatrix}\longrightarrow\\\longrightarrow\\\longrightarrow\end{smallmatrix} \mathbb{E}_I^2 X_0 \rightrightarrows \mathbb{E}_I X_0 \longrightarrow X \tag{2.8}$$

and a coaugmented cosimplicial $\mathbb{F}_I$-coalgebra

$$Z \longrightarrow \mathbb{F}_I Z_0 \rightrightarrows \mathbb{F}_I^2 Z_0 \begin{smallmatrix}\longrightarrow\\\longrightarrow\\\longrightarrow\end{smallmatrix} \cdots, \tag{2.9}$$

where the augmentation map is the $\mathbb{E}_I$-algebra structure map φ_X of X and the face and degeneracy maps are various composites of μ_I, η_I and $\mathbb{E}_I^p\varphi$, just as the coaugmentation map is the $\mathbb{F}_I$-coalgebra structure map ψ_Z of Z and the coface and codegneracy maps are various composites of δ_I, ϵ_I and $\mathbb{F}_I^q\psi$.

Lemma 1 *Any $\mathbb{E}_I$-algebra X is the colimit of the simplicial $\mathbb{E}_I$-algebra $\mathbb{E}_I^* X$, and any $\mathbb{F}_I$-coalgebra Z is the limit of the cosimplicial $\mathbb{F}_I$-coalgebra $\mathbb{F}_I^* Z$.*

Proof. It is enough to show that X is the coequalizer of the pair

$$\mathbb{E}_I^2 X_0 \underset{\mu_I X}{\overset{\mathbb{E}_I\varphi_X}{\rightrightarrows}} \mathbb{E}_I X_0 \tag{2.10}$$

and that Z is the equalizer of the pair

$$\mathbb{F}_I Z_0 \underset{\delta_I Z}{\overset{\mathbb{F}_I\psi_Z}{\rightrightarrows}} \mathbb{F}_I^2 Z_0, \tag{2.11}$$

which is immediate. Indeed, any $\mathbb{E}_I$-algebra Y fits into the diagram

$$\begin{tikzcd}
\mathrm{Map}(X_0, Y_0) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & \mathrm{Map}(\mathbb{E}_I X_0, Y_0) \arrow[d, "\cong"] \\
\mathrm{Alg}_I(\mathbb{E}_I X_0, Y) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Alg}_I(\mathbb{E}_I^2 X_0, Y),
\end{tikzcd} \tag{2.12}$$

the equalizer of which is $\mathrm{Map}(X, Y)$; likewise, any $\mathbb{F}_I$-coalgebra Y fits into the diagram

$$\begin{tikzcd}
\mathrm{Map}(Y_0, Z_0) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & \mathrm{Map}(Y_0, \mathbb{F}_I Z_0) \arrow[d, "\cong"] \\
\mathrm{Coalg}_I(Y, \mathbb{F}_I Z_0) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Coalg}_I(Y, \mathbb{F}_I^2 Z_0),
\end{tikzcd} \tag{2.13}$$

the equalizer of which is $\mathrm{Map}(Y, Z)$. □

2.1.3 Topological bicompleteness and cartesian closedness

The ordinary notions of completeness and cocompleteness are not adequate in the topologically enriched setting. The root of the problem lies in what is meant by an adjunction. Since functors between topological categories are assumed continuous, it is natural to assume that adjunctions are continuous as well, i.e. that they determine natural homeomorphisms on mapping spaces.

In particular, if I is topological category and X is a continuous I-diagram in a topological category $\mathcal{T}$, the statement that X has a limit L must mean that the space of morphisms from the constant I-diagram on a space K to X is naturally equivalent to the space of maps from K to L. We adopt the corresponding convention concerning colimits, and that the topological category $\mathcal{T}$ is bicomplete if for every small topological category I, every I-diagram X in $\mathcal{T}$ has a limit and a colimit.

There is also the important notion of tensors and cotensors of objects of $\mathcal{T}$ with spaces. Given an object X of $\mathcal{T}$, the tensor functor $(-) \otimes X$ and the cotensor functor $Y^{(-)}$ can be defined as adjoints to the mapping space functor, and hence are essentially unique if they exist. A bicomplete topological category $\mathcal{T}$ will be said to be topologically bicomplete if in addition it is tensored and cotensored over spaces. This is the convention adopted in EKMM [7], and is equivalent to the statement that $\mathcal{T}$ has all indexed limits and colimits in the sense of Kelly [16].

Recall that a topological category $\mathcal{T}$ is said to be topologically bicomplete if it is complete and cocomplete as well as tensored and cotensored over spaces. Note that tensors and cotensors, when defined, are unique up to natural isomorphism, since for any object Y of $\mathcal{T}$, the tensor $(-) \otimes Y : \mathcal{S} \to \mathcal{T}$ is by definition left-adjoint to the mapping space $\mathrm{Map}(Y, -) : \mathcal{T} \to \mathcal{S}$ while the cotensor $Y^{(-)} : \mathcal{S} \to \mathcal{T}^{\mathrm{op}}$ is left adjoint to the mapping space $\mathrm{Map}(-, Y) : \mathcal{T}^{\mathrm{op}} \to \mathcal{S}$.

The following fact is a topological generalization of well-known results in ordinary category theory. See EKMM [7] for similar results along these lines.

Proposition 2 *Let $\mathcal{T}$ be a cartesian-closed topologically bicomplete category equipped with a continuous monad $\mathbb{E}$ and a continuous comonad $\mathbb{F}$ that fit into a continuous adjunction*

$$\mathrm{Map}(\mathbb{E}X, Z) \cong \mathrm{Map}(X, \mathbb{F}Z). \tag{2.14}$$

Then the categories $\mathrm{Alg}_{\mathbb{E}}$ of $\mathbb{E}$-algebras and $\mathrm{Coalg}_{\mathbb{F}}$ of $\mathbb{F}$-coalgebras are isomorphic cartesian-closed topologically bicomplete categories.

Proof. Any $\mathbb{E}$-algebra $\mathbb{E}Y \to Y$ is adjoint to an $\mathbb{F}$-coalgebra $Y \to \mathbb{F}Y$ and conversely, giving the isomorphism $\mathrm{Alg}_{\mathbb{E}} \cong \mathrm{Coalg}_{\mathbb{F}}$. Given a diagram of $\mathbb{E}$-algebras X^α, let $\lim X^\alpha$ be the limit of the underlying diagram in $\mathcal{T}$. The natural map

$$\mathbb{E} \lim X^\alpha \to \lim \mathbb{E} X^\alpha \to X^\alpha \tag{2.15}$$

endows $\lim X^\alpha$ with a canonical $\mathbb{E}$-algebra structure in such a way that $\mathbb{E} \lim X^\alpha \to \lim X^\alpha$ is easily seen to be the limit of the X^α in the category of $\mathbb{E}$-algebras. Similarly, given a diagram Z^α of $\mathbb{F}$-coalgebras, let $\operatorname{colim} Z^\alpha$ be the colimit of the underlying diagram in $\mathcal{T}$. Again, then natural map

$$\operatorname{colim} Z^\alpha \to \operatorname{colim} \mathbb{F} Z^\alpha \to \mathbb{F} \operatorname{colim} Z^\alpha \tag{2.16}$$

endows $\operatorname{colim} Z^\alpha$ with a canonical $\mathbb{F}$-coalgebra structure such that $\operatorname{colim} Z^\alpha \to \mathbb{F} \operatorname{colim} Z^\alpha$ is the $\mathbb{F}$-coalgebra colimit of the Z^α. Therefore $\mathrm{Alg}_{\mathbb{E}} \cong \mathrm{Coalg}_{\mathbb{F}}$ are both bicomplete.

Next we construct tensors and cotensors with spaces. To do so, note that $\mathbb{E}$ commutes with tensors and $\mathbb{F}$ commutes with cotensors, since

$$\begin{aligned} \mathrm{Map}(\mathbb{E}(K \otimes X), Y) &\cong \mathrm{Map}(K \otimes X, \mathbb{F}Y) \cong \mathrm{Map}(K, \mathrm{Map}(X, \mathbb{F}Y)) \\ &\cong \mathrm{Map}(K, \mathrm{Map}(\mathbb{E}X, Y)) \cong \mathrm{Map}(K \otimes \mathbb{E}X, Y) \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \mathrm{Map}(Y, \mathbb{F}(Z^K)) &\cong \mathrm{Map}(\mathbb{E}Y, Z^K) \cong \mathrm{Map}(K, \mathrm{Map}(\mathbb{E}Y, Z)) \\ &\cong \mathrm{Map}(K, \mathrm{Map}(Y, \mathbb{F}Z)) \cong \mathrm{Map}(Y, \mathbb{F}Z^K). \end{aligned} \tag{2.18}$$

Hence the $\mathbb{E}$-algebras are naturally tensored over $\mathcal{S}$ and the $\mathbb{F}$-coalgebras are naturally cotensored over $\mathcal{S}$. It follows that $\mathrm{Alg}_{\mathbb{E}} \cong \mathrm{Coalg}_{\mathbb{F}}$ is naturally tensored and cotensored over $\mathcal{S}$.

It remains to show that both categories are cartesian closed. Write Z^Y for the $\mathcal{T}$-object of maps from Y to Z. Then the structure map of an $\mathbb{E}$-algebra Y and the costructure map of an $\mathbb{F}$-coalgebra Z induce a pair of maps

$$Z^Y \to Z^{\mathbb{E}Y} \cong (\mathbb{F}Z)^Y \tag{2.19}$$

and

$$Z^Y \to (\mathbb{F}Z)^Y \tag{2.20}$$

from Z^Y to $(\mathbb{F}Z)^Y$, the second being $\mathbb{F}$ applied to the functor $(-)^Y$. Apply $\mathbb{F}$ to this pair of maps and define $Z^Y_{\mathbb{F}}$ to be the equalizer

$$\mathbb{F}(Y^Z) \rightrightarrows \mathbb{F}((\mathbb{F}Z)^Y) \tag{2.21}$$

in the category of $\mathbb{F}$-coalgebras. Then if X is another $\mathbb{F}$-coalgebra, we obtain a morphism of equalizer diagrams

$$\begin{tikzcd}
\mathrm{Map}(X, Z^Y_{\mathbb{F}}) \arrow[r] \arrow[d] & \mathrm{Map}(X, \mathbb{F}(Z^Y)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & \mathrm{Map}(X, \mathbb{F}((\mathbb{F}Z)^Y)) \arrow[d, "\cong"] \\
\mathrm{Map}(X \times Y, Z) \arrow[r] & \mathrm{Map}_{\mathcal{J}}(X \times Y, Z) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Map}_{\mathcal{J}}(X \times Y, \mathbb{F}Z)
\end{tikzcd} \tag{2.22}$$

in which the second and third vertical maps are isomorphisms. Hence the first vertical maps is an isomorphism as well, so the categories are cartesian closed. □

Corollary 1 *For any small topological category I the category $\mathcal{S}^I$ of continuous I-diagrams in $\mathcal{S}$ is a cartesian-closed topologically bicomplete category.* □

2.1.4 Continuous Kan extensions

Given small topological categories I and J and a functor $\Pi : J \to I$ we obtain a functor $\Pi^* : \mathcal{S}^I \to \mathcal{S}^J$ by pulling back an I-space along Π. Note that Π^* necessarily preserves limits and colimits in $\mathcal{S}^I$, so it is natural to ask whether or not Π^* has left and right adjoints $\Pi_! : \mathcal{S}^J \to \mathcal{S}^I$ and $\Pi_* : \mathcal{S}^J \to \mathcal{S}^I$, respectively.

It is automatic that Π^* has a right adjoint Π_*, defined by the formula

$$(\Pi_* Y)_i := \mathrm{Map}(\Pi^* \mathbb{E}_i, Y). \tag{2.23}$$

The left adjoint is a topologically enriched notion of the left Kan extension of $\mathcal{S}^J$ along Π.

Proposition 3 *The functor $\Pi^* : \mathcal{S}^I \to \mathcal{S}^J$ has a left adjoint $\Pi_! : \mathcal{S}^J \to \mathcal{S}^I$.*

Proof. Recall that any J-space Y is canonically isomorphic to the coequalizer of the pair of maps

$$\mathbb{E}_J^2 Y_0 \rightrightarrows \mathbb{E}_J Y_0, \tag{2.24}$$

one of which is induced by the monadic structure map $\mu_J : \mathbb{E}_J^2 \to \mathbb{E}_J$ applied to Y_0 and the other of which is the composite of $\mathbb{E}_J$ with the J-space structure map $\mathbb{E}_J Y_0 \to Y_0$.

Define a functor $\Pi_{0!} : \mathcal{S}^{J_0} \to \mathcal{S}^{I_0}$ by composition with $\Pi_0 : J_0 \to I_0$, and observe that this yields a continuous adjunction

$$\mathcal{S}^{I_0}(\Pi_{0!}Y_0, X_0) \cong \mathcal{S}^{J_0}(Y_0, J_0 \times_{I_0} X_0). \tag{2.25}$$

Since $\Pi : J \to I$ is a functor, there is a natural transformation

$$\Pi_{1!} : \Pi_{0!} \circ \mathbb{E}_J \to \mathbb{E}_I \circ \Pi_{0!} \tag{2.26}$$

given by the evident map

$$\Pi_{0!}(J_1 \times_{J_0} (-)) \to I_1 \times_{I_0} \Pi_{0!}(-) \tag{2.27}$$

induced by $\Pi_1 : J_1 \to I_1$. Since $\mathcal{S}^I$ has coequalizers, we may extend $\Pi_{0!}$ to a functor $\Pi_! : \mathcal{S}^J \to \mathcal{S}^I$ by defining $\Pi_! Y$ to be the coequalizer of the pair of I-space maps

$$\mathbb{E}_I \Pi_{0!} \mathbb{E}_J Y_0 \rightrightarrows \mathbb{E}_I \Pi_{0!} Y_0, \tag{2.28}$$

where one of the maps is the composite $\mu_I \circ \mathbb{E}_I \circ \Pi_{1!}$ applied to Y_0 and the other is $\mathbb{E}_I \circ \Pi_{0!}$ applied to the J-space structure map $\mathbb{E}_J Y_0 \to Y_0$.

Now it is straightforward to check that $\Pi_!$ is left adjoint to Π^*. Indeed, we have a chain of adjunctions

$$\mathcal{S}^I(\mathbb{E}_I \Pi_{0!} Y_0, X) \cong \mathcal{S}^{I_0}(\Pi_! Y_0, X_0) \cong \mathcal{S}^{J_0}(Y_0, J_0 \times_{I_0} X_0) \cong \mathcal{S}^J(\mathbb{E}_J Y_0, \Pi^* X); \tag{2.29}$$

it follows that the middle and bottom horizontal maps in the commutative diagram

$$\begin{tikzcd}
\mathcal{S}^I(\Pi_! Y, X) \arrow[r] \arrow[d] & \mathcal{S}^J(Y, \Pi^* X) \arrow[d] \\
\mathcal{S}^I(\mathbb{E}_I \Pi_{0!} Y_0, X) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{S}^J(\mathbb{E}_J Y_0, \Pi^* X) \arrow[d, shift left] \arrow[d, shift right] \\
\mathcal{S}^I(\mathbb{E}_I \Pi_{0!} \mathbb{E}_J Y_0, X) \arrow[r] & \mathcal{S}^J(\mathbb{E}_J^2 Y_0, \Pi^* X)
\end{tikzcd} \tag{2.30}$$

are isomorphisms. But the vertical forks are equalizers, so the top horizontal map is an isomorphism as well. □

Example 1 Let $\Pi : H \to G$ be a homomorphism of topological monoids. Thinking of H and G as topological categories (with a single object) and Π as a continuous functor, we see that Π^* takes a G-space X and regards it as an H-space X with structure map

$$H \times X \to G \times X \to X. \tag{2.31}$$

The right adjoint Π_* takes an H-space to the coinduced G-space $\mathrm{Map}^H(G,Y)$, i.e. equalizer of the evident pair of G-space maps

$$\mathrm{Map}(G,Y) \rightrightarrows \mathrm{Map}(H \times G, Y), \tag{2.32}$$

while the left adjoint $\Pi_!$ takes an H-space Y is the induced G-space $G \times_H Y$, the coequalizer of the evident pair of G-space maps

$$G \times H \times Y \rightrightarrows G \times Y, \tag{2.33}$$

and is precisely dual to Π_*.

2.2 Homotopy Topoi

2.2.1 The topological model category of I-spaces

We briefly sketch the basic steps involved in providing $\mathcal{S}^{I_0}$ with the structure of a cofibrantly generated topological model category. Recall [9] that the cofibrations

$$\partial\Delta^p \to \Delta^p, \tag{2.34}$$

for each $p \geq 0$, generate the cofibrations in $\mathcal{S}$, while the trivial cofibrations

$$\Lambda^q \to \Delta^q, \tag{2.35}$$

for each $q > 0$, generate the trivial cofibrations. Here, the space Λ^q may be taken to be the geometric realization of any horn Λ^q_n.

Observe that for each object i of I we have a functor from spaces to I_0-spaces that takes a space S to the I_0-space $S_i \to \{i\} \to I_0$, and is left adjoint to the functor from I_0-spaces to spaces which restricts an I_0-space $X \to I_0$ to its fiber X_i over i. We claim that the set of maps

$$\partial\Delta^p_i \to \Delta^p_i, \tag{2.36}$$

for each $i \in I_0$ and $p \geq 0$, together with the maps

$$\Lambda_j^q \to \Delta_j^q, \tag{2.37}$$

for each $j \in I_0$ and $q > 0$, form, respectively, sets of generating cofibrations and trivial cofibrations for a topological model structure on $\mathcal{S}^{I_0}$ in which a map $X \to Y$ is a fibration or equivalence if and only if each fiber $X_i \to Y_i$ is a fibration or equivalence of spaces.

Proposition 4 *A map $f : V \to W$ has the left lifting property with respect to the class of trivial fibrations in $\mathcal{S}^{I_0}$ if and only if for each i the fiber $g_i : V_i \to W_i$ is a cofibration in $\mathcal{S}$.*

Proof. Let $g : X \to Y$ be a trivial fibration. Then

$$\mathrm{Map}(W, X) \to \mathrm{Map}(V, X) \times_{\mathrm{Map}(V,Y)} \mathrm{Map}(W, Y) \tag{2.38}$$

is a trivial fibration if and only if each component

$$\mathrm{Map}(W_i, X_i) \to \mathrm{Map}(V_i, X_i) \times_{\mathrm{Map}(V_i,Y_i)} \mathrm{Map}(W_i, Y_i) \tag{2.39}$$

is a trivial fibration. Since the maps $X_i \to Y_i$ are trivial fibrations, we see that f has the left lifting property with respect to g if each $V_i \to W_i$ is a cofibration.

Conversely, suppose for some i the map $f_i : V_i \to W_i$ is not a cofibration. Then there exists a trivial fibration $g_i : X_i \to Y_i$ and maps $V_i \to X_i$, $W_i \to Y_i$, such that the composites $V_i \to Y_i$ are equal and there is no lift of W_i to X_i. Let g be a trivial fibration in $\mathcal{S}^{I_0}$ whose fiber over i is g_i. Then there is no lift of W to X. □

This shows that a map $f : X \to Y$ of I_0-spaces is therefore a fibration or cofibration if and only if f is a fibration or cofibration of spaces. Hence, in order to factor a map of I_0-spaces, we need only factor it in the underlying category $\mathcal{S}$, since any such factorization extends to a factorization in $\mathcal{S}^{I_0}$. Thus we see that $\mathcal{S}^{I_0}$ has a topological model category structure induced by that of $\mathcal{S}$, in which $\mathbb{E}_i$ of the sets of generating (trivial) cofibrations are generating (trivial) cofibrations in $\mathcal{S}^{I_0}$. The lifting axioms are immediate.

Similar arguments work to provide a cofibrantly generated topological model structure on $\mathcal{S}^I$, except it is no longer true in general that the forgetful functor detects cofibrations. In this case, the model structure is induced from that on $\mathcal{S}^{I_0}$ by the adjunction $\mathrm{Map}(\mathbb{E}_I X_0, Y) \cong \mathrm{Map}(X_0, Y_0)$; in other words, a map $f : X \to Y$ of I-spaces is defined to be a fibration or equivalence if the underlying map of I_0-spaces $f_0 : X_0 \to Y_0$ is a fibration or equivalence.

We have already shown that $\mathcal{S}^I$ is topologically complete and cocomplete. It therefore suffices to prove the factorization and lifting axioms. To do so, let $\mathbb{E}_i$ be the functor from spaces to I-spaces that associates to a space S the free I-space on the I_0-space $S_i \to \{i\} \to I_0$, and observe that $\mathbb{E}_i$ is left adjoint to the functor which associates to an I-space X its fiber X_i over I_0. Thus if S is a small object in $\mathcal{S}$ it follows that $\mathbb{E}_i S$ is a small object in $\mathcal{S}^I$.

It is clear that the set of maps

$$\mathbb{E}_i \partial\Delta^p \to \mathbb{E}_i \Delta^p, \tag{2.40}$$

for each i in I_0 and $p \geq 0$, detect the trivial fibrations in $\mathcal{S}^I$, and similarly that the set of maps

$$\mathbb{E}_j \Lambda^q \to \mathbb{E}_j \Delta^q, \tag{2.41}$$

for each j in I_0 and $q > 0$, detect the fibrations in $\mathcal{S}^I$. Hence these sets of maps generate the cofibrations and trivial cofibrations in $\mathcal{S}^I$, where by definition a map of I-spaces is a cofibration (respectively, trivial cofibration) if it has the left lifting property with respect to the class of trivial fibrations (respectively, the class of fibrations) in $\mathcal{S}^I$. Since each of these maps are maps of small objects, we may use the small object argument to prove the factorization axioms. We will need the following fact.

Proposition 5 *Let $f : V \to W$ be a trivial cofibration. Then f is a cofibration and an equivalence.*

Proof. Clearly f is a cofibration, for if f has the left lifting property with respect to the class of fibrations in $\mathcal{S}^I$ then f also has the left lifting property with respect to the subclass of trivial fibrations in $\mathcal{S}^I$.

To show that f is an equivalence, let $g_0 : X_0 \to Y_0$ be a fibration in $\mathcal{S}^{I_0}$. Then $g = I \times_{I_0} g_0$ is a fibration in $\mathcal{S}^I$, so there is a lift of W to $X = I \times_{I_0} X_0$. Since the underlying I_0-space map of g is g_0, this determines a lift of W_0 to X_0. Hence f_0 is a trivial cofibration in $\mathcal{S}^{I_0}$, so in particular it is an equivalence. Therefore, by definition, f is an equivalence in $\mathcal{S}^I$. □

Let $f : X \to Y$ be a map of I-spaces. To factor f as a trivial cofibration followed by a fibration, set $X^0 = X$ and inductively define X^n to be the pushout

$$\begin{array}{ccc} \coprod_j \coprod_{\mathcal{D}_j} \mathbb{E}_j \Lambda^q & \longrightarrow & X^{n-1} \\ \downarrow & & \downarrow \\ \coprod_j \coprod_{\mathcal{D}_j} \mathbb{E}_j \Delta^q & \longrightarrow & X^n \end{array} \tag{2.42}$$

where $\mathcal{D}_j$ is the set of commutative diagrams of the form

$$\begin{tikzcd} \Lambda^q \arrow[r] \arrow[d] & X_j^{n-1} \arrow[d] \\ \Delta^q \arrow[r] & Y_j \end{tikzcd} \tag{2.43}$$

in $\mathcal{S}$. Let X^∞ be the colimit of the sequence $X^0 \overset{\sim}{\rightarrowtail} X^1 \overset{\sim}{\rightarrowtail} \cdots \overset{\sim}{\rightarrowtail} X^n \overset{\sim}{\rightarrowtail} \cdots$ of trivial cofibrations in $\mathcal{S}^I$. By construction, $X \to X^\infty$ has the left lifting property with respect to the class of fibrations in $\mathcal{S}^I$, so $X \to X^\infty$ is a trivial cofibration, and in particular an equivalence.

Thus it suffices to show that the map $X^\infty \to Y$ is a fibration. Fix an object j of I and a $q > 0$, and suppose given a commutative diagram

$$\begin{tikzcd} \mathbb{E}_j\Lambda^q \arrow[r] \arrow[d] & X^\infty \arrow[d] \\ \mathbb{E}_j\Delta^q \arrow[r] & Y. \end{tikzcd} \tag{2.44}$$

Since $\mathbb{E}_j\Lambda^q$ is small, the map $\mathbb{E}_j\Lambda^q \to X^\infty$ factors through some X^{n-1}, and we obtain a commutative diagram

$$\begin{tikzcd} \mathbb{E}_j\Lambda^q \arrow[r] \arrow[d] & X^{n-1} \arrow[r] \arrow[d] & X^\infty \arrow[d] \\ \mathbb{E}_j\Delta^q \arrow[r] & X^n \arrow[r] & Y. \end{tikzcd} \tag{2.45}$$

By construction of X^n there is a lift of X^n to X^∞ such that the whole diagram commutes. This gives a lift of $\mathbb{E}_j\Delta^q$ to X^∞ and shows that $X^\infty \to Y$ is a fibration.

The same argument (with the generating trivial cofibrations replaced by the generating cofibrations) works to factor a map as a cofibration followed by a trivial fibration. Thus it only remains to verify the lifting axioms.

Since a cofibration is defined to be a map of I-space that has the left lifting property with respect to the class of trivial fibrations, we only need to show that a cofibration that is also an equivalence has the left lifting property with respect to the class of all fibrations. Let $V \to W$ be such a cofibration and let $X \to Y$ be a fibration. We use the small object argument to factor $V \to W$ as the composite of the trivial cofibration and equivalence $V \to V^\infty$ followed by the fibration $V^\infty \to W$, and note that since $V \to W$ and $V \to V^\infty$

are equivalences, the map $V^\infty \to W$ is actually a trivial fibration. Consider the commutative diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & X \arrow[d] \\ V \arrow[r] \arrow[d] & V^\infty \arrow[r] \arrow[d] & Y \arrow[d] \\ W \arrow[r] & W \arrow[r] & Y, \end{tikzcd} \tag{2.46}$$

and note that there is lift of W to V^∞ in the lower left-hand square since $V \to W$ is a cofibration and $V^\infty \to W$ is a trivial fibration, and that there is also a lift of V^∞ to X in the upper right-hand square since $V \to V^\infty$ is a trivial cofibration and $X \to Y$ is a fibration. Hence the composite of the two lifts gives the desired lift of W to X.

We record our results in the following theorem.

Theorem 6 *Let I be a small topological category. Then the category $\mathcal{S}^I$ of I-spaces admits a topological model category structure such that the set of maps*

$$\mathbb{E}_i \partial \Delta^p \to \mathbb{E}_i \Delta^p, \tag{2.47}$$

for each $i \in I_0$ and $p \geq 0$, generate the class of cofibrations, and the set of maps

$$\mathbb{E}_j \Lambda^q \to \mathbb{E}_j \Delta^q, \tag{2.48}$$

for each $j \in I_0$ and $q > 0$, generate the class of trivial cofibrations. Moreover, a map of I-spaces is a fibration or equivalence if and only if the underlying I_0-space map is a fibration or equivalence in $\mathcal{S}^{I_0}$. □

2.2.2 Presentable topological model categories

The notion of a "universal homotopy theory" first appeared in [6] in the context of simplicial sets and was used implicitly in [18] in the topological context. Since we will make extensive use of the topological analogue of [6] we adapt the formalism introduced there to the category of spaces.

A particularly nice class of topological model categories are the presheaf categories, by which we mean those of the form $\mathcal{S}^I$ for some small topological category I. The utility of these topological presheaf categories lies in their following universal property, a direct homotopy-theoretic analogue of the usual universal property enjoyed by presheaf categories.

Theorem 7 *Let I be an essentially small topological category and let $\mathcal{T}$ be a topological model category. Then an I-diagram Δ in $\mathcal{T}$ determines a morphism of topological model categories $\mathcal{T} \to \mathcal{S}^I$.*

Proof. The argument is a slight generalization of the one given in [6]. The basic idea is simple enough. The freeness property of presheaf categories allows one to canonically express a presheaf of spaces X on I as a weighted colimit of free presheaves $\mathbb{E}_i$. Specifically, the presheaf X is naturally isomorphic to the coequalizer of pair of maps of representable presheaves

$$\coprod_{i,j} X_i \otimes \mathrm{Map}(\mathbb{E}_i, \mathbb{E}_j) \otimes \mathbb{E}_j \rightrightarrows \coprod_i X_i \otimes \mathbb{E}_i, \tag{2.49}$$

and the I-diagram Δ allows us to define a $\mathcal{T}$-object $\Delta^* X$ as the coequalizer of the pair of maps

$$\coprod_{i,j} X_i \otimes \mathrm{Map}(\mathbb{E}_i, \mathbb{E}_j) \otimes \Delta^*\mathbb{E}_j \rightrightarrows \coprod_i X_i \otimes \Delta^*\mathbb{E}_i, \tag{2.50}$$

where

$$\Delta^*\mathbb{E}_i := \Delta_i \tag{2.51}$$

is the evaluation of the I-diagram Δ at i. Note that Δ^* is evidently left-adjoint to the functor Δ_* which associates to an object Y in $\mathcal{T}$ the presheaf of spaces $\mathrm{Map}(\Delta, Y)$ on I, so we obtain a continuous adjunction

$$\mathcal{T}(Delta^* X, Y) \cong \mathcal{S}^I(X, \Delta_* Y) \tag{2.52}$$

which is easily seen to be a Quillen adjunction. □

Proposition 6 *Let I be a small topological category and let $\mathcal{J}$ be a set of maps in $\mathcal{S}^I$. Then the left localization $\mathcal{S}^I_{\mathcal{J}}$ exists in the sense that there is an essentially unique topological model category $\mathcal{S}^I_{\mathcal{J}}$ together with a morphism of topological model categories $\mathcal{S}^I \to \mathcal{S}^I_{\mathcal{J}}$ such that any morphism of topological model categories $\mathcal{S}^I \to \mathcal{T}$ which sends the maps in $\mathcal{J}$ to equivalences in $\mathcal{T}$ factors through $\mathcal{S}^I_{\mathcal{J}}$.*

Proof. This is the usual construction of a $\mathcal{J}$-local model category structure on $\mathcal{S}^I$. A good reference in Hirschhorn's book [14]. □

Definition 1 (Dugger [6]) *A topological model category $\mathcal{T}$ is presentable if there exists a small topological category I and a morphism of topological model categories $\mathcal{S}^I \to \mathcal{T}$ such that for some set of maps $\mathcal{J}$ in $\mathcal{S}^I$*

that are taken to equivalences in $\mathcal{T}$, the induced morphism $\mathcal{S}^I_{\mathcal{J}} \to \mathcal{T}$ is an equivalence of topological model categories. The data $(\mathcal{S}^I \to \mathcal{T}, \mathcal{J})$ will be called a presentation of $\mathcal{T}$.

2.2.3 Topological sites and sheaves

Following the approach of Toën-Vezzosi, we define the notion of a small topological site I and the associated homotopy topos of sheaves of spaces on I.

Definition 2 *An object K of a topological model category $\mathcal{T}$ is compact if for all filtered categories Λ and all Λ-diagrams X_λ in $\mathcal{T}$, the natural map*

$$\operatorname{colim}_\lambda \operatorname{Map}(K, X_\lambda) \to \operatorname{Map}(K, \operatorname{colim}_\lambda X_\lambda) \tag{2.53}$$

is an equivalence.

The notion of a covering family generalizes to the topological setting without change.

Definition 3 *A topological site $(I, \mathcal{J})$ is a small topological category I with finite homotopy limits such that for each object i of I there is a specified collection $\mathcal{J}_i$ of families of objects $i_\alpha \to i$ over i, called covering families, such that:*

(i) *any collection of equivalences $\{i_\alpha \to i\}$ is a covering of i;*

(ii) *if $\{i_\alpha \to i\}$ is a covering of i and $f : i' \to i$ is any morphism in I then $\{i_\alpha \times_i i' \to i'\}$ is a covering of i';*

(iii) *if $\{i_\alpha \to i\}$ is a covering of i and for each α, $\{i_{\alpha\beta} \to i_\alpha\}$ is a covering of i_α, then $\{i_{\alpha\beta} \to i\}$ is a covering of i.*

Let X be a presheaf of spaces on a topological site $(I, \mathcal{J})$. Since the nerve of a covering family of an object i is an augmented simplicial object over i, evaluation gives a coaugmented cosimplicial space under X_i. We say that X is a sheaf if for all objects i and all coverings of i, the coaugmentation map induces an equivalence between X_i and the total space of the cosimplicial space associated to the covering.

Let $\mathcal{S}^I_{\mathcal{J}}$ denote the full subcategory of the presheaf category $\mathcal{S}^I$ consisting of the sheaves.

Theorem 8 *The category $\mathcal{S}^I_{\mathcal{J}}$ of sheaves of spaces on I is a presentable topological model category over $\mathcal{S}^I$.*

Proof. The idea is to identify $\mathcal{S}^I_{\mathcal{J}}$ with the left Bousfield localization of $\mathcal{S}^I$ with respect to the set of maps

$$\operatorname{hocolim}_\alpha i_\alpha \to i \tag{2.54}$$

for each cover $\{i_\alpha \to i\}$ of i. Here, the homotopy colimit is meant to denote the geometric realization of the simplicial nerve of the cover. Note that since $\mathcal{S}^I$ is a presentable topological model category, it is known how to localize $\mathcal{S}^I$ with respect to an arbitrary set of maps.

But the $\mathcal{J}$-local objects of $\mathcal{S}^I$ are precisely the sheaves, and the localization construction provides a left adjoint to the inclusion of the full subcategory of $\mathcal{J}$-local objects. The resulting Quillen map $\mathcal{S}^I_{\mathcal{J}} \to \mathcal{S}^I$ gives $\mathcal{S}^I_{\mathcal{J}}$ the structure of a presentable topological model category over $\mathcal{S}^I$. □

Given a site structure $\mathcal{J}$ on a small topological category I, we refer to the model structure on $\mathcal{S}^I_{\mathcal{J}}$ as the $\mathcal{J}$-local model structure on $\mathcal{S}^I$.

2.2.4 Homotopy topoi

Just as a Grothendieck topos is defined to be a left-exact localization of a presheaf category, a homotopy topos may be defined as a left-exact localization of a topological model category of presheaves of spaces on a small topological category. While this is certainly the most concise definition of a homotopy topos, there are other more geometric approaches.

The following definition is due to C. Rezk [21]. For ease of notation, all limits and colimits is the following definition are assumed to be derived.

Definition 4 *A category with patching is a topological model category such that:*

(1a) *if X_α is a set of objects and Y is an object over $X := \coprod_\alpha X_\alpha$ then the natural map $\coprod_\alpha Y \times_X X_\alpha \to Y$ is an equivalence;*

(1b) *if $Y_\alpha \to X_\alpha$ if a set of maps, giving a map $Y := \coprod_\alpha Y_\alpha \to \coprod_\alpha X_\alpha =: X$, then for each α the natural map $Y_\alpha \to Y \times_X X_\alpha$ is an equivalence;*

(2a) *if $X \cong X_1 \coprod_{X_0} X_2$ and for each i we define $Y_i := Y \times_X X_i$ then the natural map $Y \to Y_1 \coprod_{Y_0} Y_2$ is an equivalence;*

(2b) *if $Y_1 \times_{X_1} X_0 \cong Y_0 \cong X_0 \times_{X_2} Y_2$ and $X := X_1 \coprod_{X_0} X_2$ then for each i the natural map $Y_i \to Y \times_X X_i$ is an equivalence.*

The following theorem of Rezk [21] is a homotopical analogue of Giraud's theorem.

Theorem 9 (Rezk [21]) *A presentable topological model category is a homotopy topos if and only if it has patching.* □

It is easy to generalize the notion of a geometric morphism of topoi to the topological context.

Definition 5 *A geometric morphism of homotopy topoi* $\Gamma : \mathcal{U} \to \mathcal{T}$ *consists of a continuous Quillen adjunction*

$$\mathcal{U}(\Gamma^* X, Y) \cong \mathcal{T}(X, \Gamma_* Y) \tag{2.55}$$

such that $\mathbb{L}\Gamma^*$ *preserves finite homotopy limits.*

Since we will not consider analogues of the so-called "logical" morphisms, a morphism of homotopy topoi will always refer to a geometric morphism.

Definition 6 *A geometric morphism* $\Gamma : \mathcal{U} \to \mathcal{T}$ *is essential if* $\Gamma^* : \mathcal{T} \to \mathcal{U}$ *has a left adjoint* $\Gamma_! : \mathcal{U} \to \mathcal{T}$.

For the most part, we will only be concerned will essential geometric morphisms. These frequently arise via the induction-restriction adjunction on sheaves that results from a morphism of topological sites. Of course, we still have to check that our categories of sheaves of spaces on topological sites are actually homotopy topoi, which is to say, that the $\mathcal{J}$-localization functor is left exact.

The basic idea is that the category of coverings of an object is filtered in a suitable sense [15], and that quite generally filtered colimits commute with finite limits.

Corollary 2 *If* $(I, \mathcal{J})$ *is a small topological site then the topological model category* $\mathcal{S}^I_{\mathcal{J}}$ *of continuous sheaves of spaces on* I *with respect to* $\mathcal{J}$ *is a homotopy topos.* $\square$

2.3 Homotopy Theory of G-Spaces

2.3.1 The topological model structure

Let $\mathrm{Orb}(G)$ be the small, full topological subcategory of the category $\mathcal{S}^G$ of G-spaces consisting of those G-orbits of the form G/H for some closed subgroup H of G, and let $\mathcal{S}^{\mathrm{Orb}(G)}$ denote the homotopy topos of presheaves of spaces on $\mathrm{Orb}(G)$. The inclusion of the subcategory $\mathrm{Orb}(G) \to \mathcal{S}^G$ determines a continuous adjunction

$$\mathcal{S}^G(Y \otimes \mathrm{Orb}(G), X) \to \mathcal{S}^{\mathrm{Orb}(G)}(Y, \mathrm{Map}(\mathrm{Orb}(G), X)) \tag{2.56}$$

by left Kan extension and therefore a morphism of topological model categories $\mathcal{S}^G \to \mathcal{S}^{\mathrm{Orb}(G)}$.

Proposition 7 *The functor* $(-) \otimes \mathrm{Orb}(G) : \mathcal{S}^{\mathrm{Orb}(G)} \to \mathcal{S}^G$ *is a left-exact localization functor.*

Proof. We must show that $(-) \otimes \mathrm{Orb}(G)$ is homotopy left-exact and that the derived counit of the adjunction

$$\mathbb{L}\mathrm{Map}(\mathrm{Orb}(G), \mathbb{R}X) \otimes \mathrm{Orb}(G) \to X \tag{2.57}$$

is a weak equivalence of G-spaces for all G-spaces X.

Left-exactness is a consequence of the fact that geometric realization of simplicial G-spaces is left-exact, a direct corollary of the analogous fact for simplicial spaces. Note that this implies in particular that the cofibrant replacement functor on G-spaces is left-exact.

To see that the derived counit is a weak equivalence of G-spaces, first observe that any G-space X is already fibrant so it suffices to consider the natural transformation

$$\mathbb{L}\mathrm{Map}(\mathrm{Orb}(G), X) \otimes \mathrm{Orb}(G) \to X. \tag{2.58}$$

If X is a G-cell complex, then this is just Elmendorf's theorem [19]. Otherwise, take a cofibrant replacement $\mathbb{L}X$ and observe that since both $\mathbb{L}\mathrm{Map}(\mathrm{Orb}(G), \mathbb{L}X)$ and $\mathbb{L}\mathrm{Map}(\mathrm{Orb}(G), X)$ are weakly equivalent to $\mathrm{Map}(\mathrm{Orb}(G), \mathbb{L}X)$, they are in particular weakly equivalent to each other, and are therefore cofibrant replacements of $\mathrm{Map}(\mathrm{Orb}(G), X)$. Hence me may reduce to the case in which X is cofibrant. □

In particular we see that G-spaces form a homotopy topos, and also embed into the larger homotopy topos of $\mathrm{Orb}(G)$-spaces.

2.3.2 The geometric morphism associated to a group homomorphism

Let $\varphi : H \to G$ be a group homomorphism. Then the restriction functor

$$\varphi^* : \mathcal{S}^G \to \mathcal{S}^H \tag{2.59}$$

preserves limits and colimits, so it is natural to ask whether or not φ^* is the inverse image of an essential geometric morphism $\varphi_* : \mathcal{S}^H \to \mathcal{S}^G$. Of course the left and right adjoints of φ^* exist and are nothing more than induction and coinduction, respectively.

Explicitely, the essential geometric morphism $\varphi : \mathcal{S}^H \to \mathcal{S}^G$ is given by the adjunction

$$\mathcal{S}^H(\varphi^* X, Y) \cong \mathcal{S}^G(X, \varphi_* Y) \tag{2.60}$$

in which $\varphi_* Y$ is the G-space

$$\varphi_* Y = \mathrm{Map}^H(G, Y) \tag{2.61}$$

of H-equivariant maps from G to Y, where G is regarded as an H-space via φ. The left adjoint $\varphi_!$ of φ^* is the induced G-space

$$\varphi_! Y := Y \times_H G; \tag{2.62}$$

its definition is precisely dual to φ_*.

Although the functors φ_* and $\varphi_!$ are easier to define in $\mathcal{S}^G$, it is instructive to compute their analogues on $\mathcal{S}^{\mathrm{Orb}(G)}$. The homomorphism $\varphi : H \to G$ defines a continuous functor $\varphi : \mathrm{Orb}(H) \to \mathrm{Orb}(G)$ by the rule

$$\varphi(H/L) := G/\varphi(L). \tag{2.63}$$

Hence the restriction functor φ^* may be regarded as precomposition with φ.

Having determined φ^*, we can say something about φ_*. Taking $X = G/K$ for some closed subgroup K of G, we have that

$$\varphi_* Y(G/K) \cong \mathcal{S}^G(G/K, \varphi_* Y) \cong \mathcal{S}^H(\varphi^* G/K, Y), \tag{2.64}$$

so φ_* is completely determined by restriction of G-orbits. Unfortunately restriction is rather poorly behaved unless $\mathrm{Orb}(H) \to \mathrm{Orb}(G)$ is essentially surjective; that is, unless $\varphi : H \to G$ is surjective, in which case

$$\varphi^* G/K \cong H/\varphi^{-1} K. \tag{2.65}$$

In general, however, $\varphi^* G/K$ need not be an H-orbit.

2.3.3 Fibrations of topological abelian groups

Let $i : H \to G$ be a closed immersion of compact abelian Lie groups with cokernel G/H. Then for each n, the sequence

$$* \to H^n \to G^n \to (G/H)^n \to * \tag{2.66}$$

is a short exact sequence of compact abelian Lie groups, and so the induced map of classifying spaces

$$* \to BH \to BG \to B(G/H) \to * \tag{2.67}$$

is a short exact sequence of topological abelian groups.

Since the map $BG \to B(G/H)$ is in fact a fibration, its kernel BH is therefore also the homotopy fiber of the map $BG \to B(G/H)$. If instead we use the canonical homotopy fiber $BG \times_{B(G/H)} B(G/H)^{\Delta^1}$, we obtain a short exact sequence of short exact sequences of topological abelian groups

$$\begin{tikzcd}
 & * \arrow[d] & * \arrow[d] & * \arrow[d] & \\
* \arrow[r] & * \arrow[r] \arrow[d] & \Omega B(G/H) \arrow[r] \arrow[d] & \Omega B(G/H) \arrow[r] \arrow[d] & * \\
* \arrow[r] & BH \arrow[r] \arrow[d] & BG \times_{B(G/H)} B(G/H)^{\Delta^1} \arrow[r] \arrow[d] & B(G/H)^{\Delta^1} \arrow[r] \arrow[d] & * \\
* \arrow[r] & BH \arrow[r] \arrow[d] & BG \arrow[r] \arrow[d] & B(G/H) \arrow[r] \arrow[d] & * \\
 & * & * & * &
\end{tikzcd} \tag{2.68}$$

in which the map $BH \to BG \times_{B(G/H)} B(G/H)^{\Delta^1}$ is an equivalence.

Lemma 2 *The map $i_* : G/H \to \Omega B(G/H)$ adjoint to the inclusion of the subspace $i^* : \Sigma(G/H) \to B(G/H)$ is a homomorphism of topological abelian groups.*

Proof. Clearly i_* commutes with products and respects the terminal object. □

Corollary 3 *The composite*

$$G/H \to \Omega B(G/H) \to BG \times_{B(G/H)} B(G/H)^{\Delta^1} \tag{2.69}$$

is a homomorphism of topological abelian groups. □

Corollary 4 *A morphism of short exact sequences of topological abelian groups of the form*

$$\begin{tikzcd}
* \arrow[r] & K \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & G/K \arrow[r] \arrow[d] & * \\
* \arrow[r] & H \arrow[r] & G \arrow[r] & G/H \arrow[r] & *
\end{tikzcd} \tag{2.70}$$

induces a morphism of fiber sequences of topological abelian groups

$$
\begin{tikzcd}
G/K \arrow[r] \arrow[d] & BG \times_{B(G/K)} B(G/K)^{\Delta^1} \arrow[r] \arrow[d] & BG \arrow[r] \arrow[d] & B(G/K) \arrow[d] \\
G/H \arrow[r] & BG \times_{B(G/H)} B(G/H)^{\Delta^1} \arrow[r] & BG \arrow[r] & B(G/H)
\end{tikzcd}
\tag{2.71}
$$

such that all the maps are group homomorphisms.

2.3.4 Topological abelian group valued functors on the orbit category

In this section we present an important construction fundamental to our definition of equivariant elliptic cohomology. As usual, fix a compact abelian Lie group G, let $\mathrm{Sub}(G)$ denote the category of closed subgroups of G, and define $\mathrm{Orb}(G)$ to be the full topological subcategory of the category $\mathcal{S}^G$ of G-spaces on the objects of the form G/H for some H in $\mathrm{Sub}(G)$.

Let $B : \mathrm{Sub}(G) \to \mathrm{Ab}(\mathcal{S})$ be the classifying space functor, taking a subgroup $H \leq G$ to its classifying space BH, which (if we model BH correctly) is itself a topological abelian group with multiplication, inversion, and identity induced from that on H.

Proposition 8 *Let $K \leq H \leq G$ be a chain of closed subgroups of G. Then there is a canonical isomorphism of topological abelian groups*

$$G/H \cong \mathrm{Map}^G(G/K, G/H) \tag{2.72}$$

given by sending gH to the G-map which takes $g'K$ to $g'gH$.

Proof. In general, a G-orbit P has the property that a G-map $f : P \to X$ is completely determined by its value on a single point $p \in P$, since any other $q \in P$ is of the form gp for some $g \in G$ and $f(gp) = gf(p)$. In particular, we see that if $P = G/K$ and $X = G/H$, then $f(K) = gH$ for some $g \in G$, giving the inverse $\mathrm{Map}^G(G/K, G/H) \to G/H$. $\square$

Now $\mathrm{Sub}(G)$ may be regarded as a topological subcategory of $\mathrm{Orb}(G)$ via the embedding which takes $K \leq H$ to the unique map $f : G/K \to G/H$ characterized by $f(K) = H$, and the question arises as to whether or not one may extend B to all of $\mathrm{Orb}(G)$ in a homotopy coherent manner. Specifically, we ask for a continuous functor $A : \mathrm{Orb}(G) \to \mathrm{Ab}_{\mathcal{S}}$ together with a natural equivalence $B \to A|_{\mathrm{Sub}(G)}$.

We present one possible solution here. Define $A(G/H)$ to be the homotopy fiber of the map

$$A(G/H) \to B(G) \to B(G/H), \tag{2.73}$$

and note that the universal property of $A(G/H)$ together with the natural fiber sequence

$$B(H) \to B(G) \to B(G/H) \tag{2.74}$$

gives a natural equivalence $B(H) \to A(G/H)$. The results from the previous section imply that G/H naturally acts on $A(G/H)$, so we get a natural map

$$G/H \cong \mathrm{Aut}(G/H) \to \mathrm{Aut}(A(G/H)) \tag{2.75}$$

which gives the value of A on the mapping spaces $\mathrm{Map}^G(G/H, G/H)$.

Proposition 9 *To specify a continuous functor* $A : \mathrm{Orb}(G) \to \mathrm{Ab}_{\mathbb{S}}$ *it suffices to specify* A *on the subcategory* $\mathrm{Sub}(G)$ *together with a natural action of* $\mathrm{Aut}(G/H)$ *on* $A(G/H)$ *for each* H *in* $\mathrm{Sub}(G)$.

Proof. If $K \leq H$, then any map $G/K \to G/H$ factors as the composite of the based map $G/K \to G/H$ followed by an automorphism of G/H. □

Chapter 3

Algebraic Geometry over the Sphere Spectrum

3.1 Topoi Associated to $\mathbb{S}$

3.1.1 Affine $\mathbb{S}$-schemes

We outline the Toën-Vezzosi approach [24] to algebraic geometry over the sphere spectrum. Given a Grothendieck topology on the (opposite) category of commutative $\mathbb{Z}$-algebras, such as the Zariski or étale, it is possible to lift this topology to the (opposite) category of commutative $\mathbb{S}$-algebras in a standard way. The associated topos is then the resulting homotopy topos of sheaves of spaces on this site.

Let $\mathrm{Alg}_{\mathbb{S}}$ denote the topological model category of commutative $\mathbb{S}$-algebras. We will refer to an object R of $\mathrm{Alg}_{\mathbb{S}}$ simply as an $\mathbb{S}$-algebra and use the convention that by an $\mathbb{S}$-algebra we always mean a commutative $\mathbb{S}$-algebra. We ignore the issue of universes entirely and simply note that when necessary, we will freely restrict (without change of notation) to the full subcategory of κ-presentable $\mathbb{S}$-algebra for some cardinal κ.

The opposite topological model category $\mathrm{Aff}_{\mathbb{S}} := \mathrm{Alg}_{\mathbb{S}}^{\circ}$ is called the category of affine $\mathbb{S}$-schemes. If E is an $\mathbb{S}$-algebra, we write $\operatorname{Spec} E$ for the corresponding affine $\mathbb{S}$-scheme, so that

$$\mathrm{Map}(\operatorname{Spec} F, \operatorname{Spec} E) \cong \mathrm{Map}(E, F) \tag{3.1}$$

by definition.

It is important to note that there is a continuous functor $\pi_0 : \mathrm{Alg}_{\mathbb{S}} \to \mathrm{Alg}_{\mathbb{Z}}$ which associates to an $\mathbb{S}$-algebra E its "underlying" $\pi_0\mathbb{S} = \mathbb{Z}$-algebra $\pi_0 E$, and that (as the notation would suggest) this functor factors through the homotopy category $\pi_0\mathrm{Alg}_{\mathbb{S}}$ of $\mathrm{Alg}_{\mathbb{S}}$. However, the resulting functor (of discrete topological categories) $\pi_0\mathrm{Alg}_{\mathbb{S}} \to \mathrm{Alg}_{\mathbb{Z}}$ need not be full, faithful, or essentially surjective. The question of determining its essential image is an interesting one, which we will not elaborate upon here, save for remarking that the restriction of π_0 to the category $\mathrm{Alg}_{\mathbb{S}}^{+}$ of connective $\mathbb{S}$-algebras has a continuous right adjoint $H : \mathrm{Alg}_{\mathbb{Z}} \to \mathrm{Alg}_{\mathbb{S}}^{+}$ which associates to a $\mathbb{Z}$-algebra R the Eilenberg-Mac Lane $\mathbb{S}$-algebra HR. In other words, there is a natural map

$$\mathrm{Alg}^+_{\mathbb{S}}(E, HR) \cong \mathrm{Alg}_{\mathbb{Z}}(\pi_0 E, R) \tag{3.2}$$

which is an equivalence of spaces (in particular, this implies that the space of algebra maps $E \to HR$ is homotopy discrete).

We suggestively abuse notation and generally write $\pi_0 \operatorname{Spec}$ and $H \operatorname{Spec}$, instead of $\operatorname{Spec} \pi_0$ and $\operatorname{Spec} H$, for the corresponding functors of affine schemes. Hence

$$\mathrm{Aff}_{\mathbb{Z}}(\pi_0 \operatorname{Spec} E, \operatorname{Spec} R) \cong \mathrm{Aff}^+_{\mathbb{S}}(\operatorname{Spec} E, H \operatorname{Spec} R), \tag{3.3}$$

and one might naturally ask whether or not we can put a topology on $\mathrm{Aff}_{\mathbb{S}}$ such that the functor $\pi_0 : \mathrm{Aff}_{\mathbb{S}} \to \mathrm{Aff}_{\mathbb{Z}}$, where $\mathrm{Aff}_{\mathbb{Z}}$ is given one of the standard topologies, such that H takes covers to covers.

We shall see that such a topology does in fact exist and is based upon the following notion of flatness.

Definition 7 *An E-algebra $f : E \to F$ is flat if $\pi_0 f : \pi_0 E \to \pi_0 F$ is a flat $\pi_0 E$-algebra and the natural map*

$$\pi_* E \otimes_{\pi_0 E} \pi_0 F \cong \pi_* F \tag{3.4}$$

is an isomorphism of $\pi_* E$-algebras.

3.1.2 Schemes and stacks over $\mathbb{S}$

Let $\mathcal{S}^{\mathrm{Aff}_{\mathbb{S}}}$ denote the topos of presheaves of spaces on $\mathrm{Aff}_{\mathbb{S}}$. Recall that $\mathcal{S}^{\mathrm{Aff}_{\mathbb{S}}}$ is a presentable topological model category in which the fibrations and equivalences are defined levelwise, so in particular every object is fibrant. Given a topology τ on $\mathrm{Aff}_{\mathbb{S}}$, we want to localize $\mathcal{S}^{\mathrm{Aff}_{\mathbb{S}}}$ so that the fibrant objects are the τ-sheaves.

If $f : \operatorname{Spec} F \to \operatorname{Spec} E$ is a cover of E then we may form the geometric realization Bf of the groupoid associated to the cover. We claim that the natural map $Bf \to B\mathbf{1}_E \cong \operatorname{Spec} E$ is a homotopy monomorphism, and should therefore be regarded as a sieve on $\operatorname{Spec} E$.

Definition 8 *The homotopy topos $\mathcal{S}^{\mathrm{Aff}_{\mathbb{S}}}_{\tau}$ is the (left-exact) localization of $\mathcal{S}^{\mathrm{Aff}_{\mathbb{S}}}$ with respect to the sieves $Bf \to \operatorname{Spec} E$ induced by the τ-covers $f : \operatorname{Spec} F \to \operatorname{Spec} E$.*

3.1.3 The underlying ordinary scheme of a derived scheme

By definition, as $\mathbb{S}$-scheme is a Zariski-locally affine sheaf on $\mathrm{Aff}_{\mathbb{S}}$. In particular, and $\mathbb{S}$-scheme X admits an atlas, i.e. an affine cover $\operatorname{Spec} E \to X$. Write $\operatorname{Spec} F$ for the homotopy pullback $\operatorname{Spec} E \times_X \operatorname{Spec} E$, so X is equivalent to the realization of the nerve of the groupoid scheme

$$\operatorname{Spec} F \rightrightarrows \operatorname{Spec} E. \tag{3.5}$$

Define the underlying ordinary scheme $\pi_0 X$ to be the coequalizer of the pair of maps

$$\operatorname{Spec} \pi_0 F \rightrightarrows \operatorname{Spec} \pi_0 E, \tag{3.6}$$

which necessarily exists by the gluing lemma [12], which is to say that schemes are stable under Zariski-descent.

In particular, we see that any $\mathbb{S}$-scheme scheme X defines a topological space $\operatorname{Spc} X := \operatorname{Spc} \pi_0 X$, patched together from the topological spaces $\operatorname{Spc} \operatorname{Spec} E_i$ constituting an affine cover of X. We also obtain the structure sheaf $\mathcal{O}_X$ of X as a sheaf of E_∞-ring spectra on $\operatorname{Spc} X$. Of course, we have that $\pi_0 \mathcal{O}_X \cong \mathcal{O}_{\pi_0 X}$ as sheaves of rings on $\operatorname{Spc} X$.

Associated to this structure sheaf $\mathcal{O}_X$ is the category $\operatorname{Mod}_{\mathcal{O}_X}$ of $\mathcal{O}_X$-modules, and the subcategories of (quasi)coherent sheaves of $\mathcal{O}_X$-modules. If $X \cong \operatorname{Spec} E$ is affine, then a quasicoherent $\mathcal{O}_X$-module is any $\mathcal{O}_X$-module equivalent to one induced from an E-module.

Definition 9 *A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is quasicoherent if and only if for each map of the form $f : \operatorname{Spec} E \to X$, the induced $\mathcal{O}_{\operatorname{Spec} E}$-module $f^*\mathcal{F}$ is quasicoherent.*

Of course it is sufficient to check this on any collection of affine schemes which cover of X.

3.2 The Derived Category of a Derived Ringed Space

3.2.1 Derived Ringed Spaces

A derived ringed space $(X, \mathcal{O}_X)$ is simply a topological space X equipped with a sheaf $\mathcal{O}_X$ of E_∞-ring spectra on X. A morphism $(f, \mathcal{O}_f) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map of topological spaces $f : X \to Y$ together with a map $\mathcal{O}_f : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of E_∞-ring spectra on Y.

Given a derived ringed space $(X, \mathcal{O}_X)$ we have the category $\operatorname{Mod}_X := \operatorname{Mod}_{\mathcal{O}_X}$ of $\mathcal{O}_X$-modules. It is a topological model category with objects the presheaves of $\mathcal{O}_X$-modules and morphisms natural transformations of presheaves of $\mathcal{O}_X$-modules, where a map is a weak equivalence or a fibration if it is a local weak equivalence or local fibration. In particular, the fibrant objects are the local objects, i.e. the sheaves of $\mathcal{O}_X$-modules. The homotopy category of Mod_X is called the derived category of X.

3.2.2 The adjuction associated to a map of derived ringed spaces

Given a map $(f, \mathcal{O}_f) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of derived ringed spaces, we obtain a Quillen adjunction

$$\mathrm{Mod}_X(f^*N, M) \to \mathrm{Mod}_Y(N, f_*M) \tag{3.7}$$

of topological model categories. As usual, f_*M is the presheaf of $\mathcal{O}_Y$-modules defined by

$$f_*M(V) := M(f^{-1}V), \tag{3.8}$$

while f^*N is the presheaf of $\mathcal{O}_X$-modules $f^*N := f^{-1}N \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$, where $f^{-1}N$ is the presheaf on X defined by

$$f^{-1}N(U) := \mathrm{colim}_{V \supset f(U)} N(V). \tag{3.9}$$

Let us first check that this a topologically enriched adjunction. To see this, it suffices to show that

$$\mathrm{Mod}_{f^{-1}\mathcal{O}_Y}(f^{-1}N, M) \cong \mathrm{Mod}_{\mathcal{O}_Y}(N, f_*M), \tag{3.10}$$

naturally in M and N. But this is clear since f defines a continuous functor $f^{-1} : \mathrm{Open}(Y) \to \mathrm{Open}(X)$ and therefore the restriction functor

$$f_* : \mathrm{Mod}_{f^{-1}\mathcal{O}_Y} \to \mathrm{Mod}_{\mathcal{O}_Y} \tag{3.11}$$

admits a continuous left Kan extension which is clearly

$$f^{-1} : \mathrm{Mod}_{\mathcal{O}_Y} \to \mathrm{Mod}_{f^{-1}\mathcal{O}_Y}. \tag{3.12}$$

Since f_* evidently preserves local fibrations and local weak equivalences, we see that in fact f determines a continuous Quillen adjunction.

Proposition 10 *A map $(f, \mathcal{O}_f) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of derived ringed spaces determines an adjunction*

$$\mathrm{Map}(\mathbb{L}f^*N, M) \cong \mathrm{Map}(N, \mathbb{R}f_*M) \tag{3.13}$$

on the level of derived categories.

It will also be convenient to have a theory of derived local-ringed spaces.

Definition 10 *A derived local-ringed space is a topological space X equipped with a sheaf of E_∞-ring spectra*

$\mathcal{O}_X$ on X such that at each point $x \in X$ the stalk $\mathcal{O}_x$ of $\mathcal{O}_X$ at x is a local E_∞-ring spectrum (i.e. $\pi_0\mathcal{O}_x$ is a local ring). A morphism of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of map $f : X \to Y$ of spaces together with a morphism $\mathcal{O}_f : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of ring spectra on $\mathcal{O}_Y$ such that $\pi_0\mathcal{O}_f : \pi_0\mathcal{O}_Y \to \pi_0 f_*\mathcal{O}_X$ is a morphism of local rings.

There are two main examples to keep in mind. The first is the case of a derived elliptic curve A over $\operatorname{Spec} K\mathbb{C} \cong \operatorname{Spec} HP\mathbb{C}$. The underlying classical elliptic curve A_0 is then a complex projective variety and therefore admits the structure of a complex analytic manifold $\widetilde{A}_0$ together with a morphism of (classical) locally ringed spaces $(\widetilde{A}_0, \mathcal{O}_{\widetilde{A}_0}) \to (A_0, \mathcal{O}_{A_0})$. The analytic topology on $\widetilde{A}_0$ induces an analytic topology on the underlying space of A, and we denote the resulting derived locally ringed space by $\widetilde{A}$. This gives rise to the circle-equivariant theories originally defined by Grojnowski [11] and studied by a number of others.

The second example of interest is the case of a derived elliptic curve A over $\operatorname{Spec} K\mathbb{Q} \cong \operatorname{Spec} HP\mathbb{Q}$. The underlying classical elliptic curve A_0 admits a coarser topology known as the torsion-point topology, which has a basis of open sets consisting of the compliments of the torsion points. This is the site underlying Greenlees' construction of rational circle-equivariant elliptic cohomology [10].

3.2.3 $\mathbb{S}$-schemes as locally affine local-ringed spaces

Recall that an ordinary scheme X is defined to be a local ringed space $(\operatorname{Spc} X, \mathcal{O}_X)$ that is locally of the form $(\operatorname{Spc}\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some commutative $\mathbb{Z}$-algebra R. $\mathbb{S}$-schemes may be characterized similarly.

Proposition 11 *The category $\mathrm{Sch}_{\mathbb{S}}$ of $\mathbb{S}$-schemes is the full topological subcategory of the category of spaces equipped with sheaves of $\mathbb{S}$-algebras that are locally affine.* □

In other words, an $\mathbb{S}$-scheme X is a ringed space $(\operatorname{Spc} X, \mathcal{O}_X)$ that is locally equivalent to an affine $\mathbb{S}$-scheme $(\operatorname{Spc}\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some commutative $\mathbb{S}$-slgebra R, and a morphism $f : Y \to X$ of $\mathbb{S}$-schemes is just a morphism $(\operatorname{Spc} f, \mathcal{O}_f) : (\operatorname{Spc} Y, \mathcal{O}_Y) \to (\operatorname{Spc} X, \mathcal{O}_X)$ of ringed spaces.

Because of this close relationship between derived schemes and ordinary schemes, there are many properties of ordinary schemes which generalize nicely to the homotopical setting.

3.2.4 Fibered products of $\mathbb{S}$-schemes

In this section we show that fibered products, defined by a suitable universal property, exist in the category of $\mathbb{S}$-schemes and are unique up to equivalence.

Definition 11 *Let X be an $\mathbb{S}$-scheme and $Y \to X$, $Z \to X$ a pair of $\mathbb{S}$-schemes over X. Then the fibered* *product of Y and Z over X is defined to be any $\mathbb{S}$-scheme $Y \times_X Z$ such that the natural map*

$$(Y \times_X Z)(E) \to Y(E) \times_{X(E)} Z(E) \tag{3.14}$$

is a homotopy equivalence for all $\mathbb{S}$-algebras E.

Note that since X, Y, Z are $\mathbb{S}$-schemes, they are in particular fibrant objects of the Zariski topos, while $\operatorname{Spec} E$, being representable, is cofibrant, so the mapping spaces have the correct homotopy type.

We construct a model for the fibered product $Y \times_X Z$ using atlases (i.e., coverings by affine $\mathbb{S}$-schemes) and a model for the fibered product of affine derived schemes.

Proposition 12 *If $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and $Z = \operatorname{Spec} C$ are affine $\mathbb{S}$-schemes, then $\operatorname{Spec} B \wedge_A C$ is a model for the fibered product $Y \times_X Z$.*

Proof. Affine $\mathbb{S}$-schemes are dual to commutative $\mathbb{S}$-algebras, and

$$\mathrm{Alg}_{\mathbb{S}}(B \wedge_A C, E) \cong \mathrm{Alg}_{\mathbb{S}}(B, E) \times_{\mathrm{Alg}_{\mathbb{S}}(A,E)} \mathrm{Alg}_{\mathbb{S}}(C, E) \tag{3.15}$$

as required. □

Next we consider the case where X is affine and Y, Z are arbitrary.

Proposition 13 *If $X = \operatorname{Spec} A$ is an affine $\mathbb{S}$-scheme, then for any pair of X-schemes Y and Z the fibered product $Y \times_X Z$ exists.*

Proof. Let $V \to Y$ and $W \to Z$ be atlases for Y and Z, respectively. By the previous proposition, the fibered product $V \times_X W$ is a coproduct of affine X-schemes and it is easy to check that $V \times_X W$ is an atlas for $Y \times_X Z$. □

Combining the two propositions we obtain the following theorem.

Theorem 10 *In the category of $\mathbb{S}$-schemes, fibered products exist and are unique up to homotopy.*

Proof. Let $U \to X$ be an atlas for X. It is easy to check that the restrictions $Y \times_X U$ and $W = Z \times_X U$ of Y and Z to U are $\mathbb{S}$-schemes, as they are disjoint unions of open subschemes. Hence there are atlases $V \to Y \times_X U \to Y$ and $W \to Z \times_X U \to Z$ for Y and Z such that the maps $Y \to X$ and $Z \to X$ factor through U. We know that the fibered product $V \times_U W$ exists as a scheme and gives an atlas for $Y \times_X Z$, so we see that $Y \times_X Z$ is an $\mathbb{S}$-scheme. □

Corollary 5 *Fibered products exist in the category of S-schemes for any $\mathbb{S}$-scheme S.*

Proof. By definition we have a fiber sequence

$$\mathrm{Map}_S(T, Y \times_X Z) \to \mathrm{Map}(T, Y \times_X Z) \to \mathrm{Map}(T, S); \tag{3.16}$$

since homotopy limits commute with each other and $\mathrm{Map}(T, Y \times_X Z)$ is equivalent to the homotopy pullback $\mathrm{Map}(T, Y) \times_{\mathrm{Map}(T,X)} \mathrm{Map}(T, Z)$, it follows that $\mathrm{Map}_S(T, Y \times_X Z)$ is equivalent to the homotopy pullback $\mathrm{Map}_S(T, Y) \times_{\mathrm{Map}_S(T,X)} \mathrm{Map}_S(T, Z)$. Moreover, the universal property of homotopy limits shows that $Y \times_X Z$ is unique up to homotopy. □

3.3 Line Bundles, Divisors and Invertible Sheaves

3.3.1 The strict additive group

One of the first issues to arise when one tries to actually use derived algebraic geometry is the question of what is the proper analogue of the affine line. In ordinary algebraic geometry, the affine line represents the functor which associates to a scheme its ring of regular functions. Restricted to affine schemes, this is of course the functor which associates to a ring its set of elements.

In the derived setting, the first nontrivial continuous functor from ring spectra to spaces that comes to mind is the one which associates to a ring spectrum R its zero-space $\Omega^\infty R$. This is represented by the ring $\mathrm{Sym}\,\mathbb{S}$, the free E_∞ ring on the sphere spectrum. The problem with this is that the homotopy ring $\pi_*\Omega^\infty \mathrm{Sym}\,\mathbb{S}$ is nowhere close to being the symmetric algebra on the ring $\pi_*\Omega^\infty\mathbb{S}$.

An obvious remedy for this is to use the ring $\Sigma^\infty_+\mathbb{N} \cong \bigvee_{\mathbb{N}} \mathbb{S}$ instead; it clearly has the feature that

$$\pi_*\Sigma^\infty_+\mathbb{N} \cong \pi_* \bigvee_{\mathbb{N}} \mathbb{S} \cong \bigvee_{\mathbb{N}} \pi_*\mathbb{S} \cong \pi_*\mathbb{S}[t], \tag{3.17}$$

the polynomial algebra on the graded ring $\pi_*\mathbb{S}$. The disadvantage here is that $\bigvee_{\mathbb{N}} \mathbb{S}$ is not free, or even necessarily cofibrant, as an E_∞-ring spectrum. Hence the (derived) functor it represents is difficult to compute. We can say the following, however.

Proposition 14 *The functor*

$$\mathrm{Map}(\Sigma^\infty_+\mathbb{N}, -) \cong E_\infty(\mathbb{N}, \Omega^\infty -) \tag{3.18}$$

from E_∞-ring spectra to E_∞-ring spaces factors, up to homotopy, through the category of strictly commutative topological rings.

For this reason, we regard the derived scheme

$$\mathbb{A} := \mathbb{A}^1 := \operatorname{Spec} \Sigma^\infty_+ \mathbb{N} \tag{3.19}$$

as a strictly commutative derived analogue of the affine line, and refer to it as the strict affine line. The associated additive group is therefore called the strict additive group.

3.3.2 The strict multiplicative group

The strict multiplicative group ought to be the group of linear automorphisms of the strict additive group. If we regard the ring $\Sigma^\infty_+ \mathbb{N}$ representing the strict additive group as the group ring on the commutative monoid $\mathbb{N}$, the strict multiplicative group is analogously the group ring on the commutative monoid $\mathbb{Z}$. As in algebra, the coaction of $\mathbb{Z}$ on $\mathbb{N}$ induces a coaction of $\Sigma^\infty_+ \mathbb{Z}$ on $\Sigma^\infty_+ \mathbb{N}$ and therefore an action of $\operatorname{Spec} \Sigma^\infty_+ \mathbb{Z}$ on $\operatorname{Spec} \Sigma^\infty_+ \mathbb{N}$. In this we obtain the analogue

$$\mathbb{G} := \mathbb{G}_m := \operatorname{Spec} \Sigma^\infty_+ \mathbb{Z} \tag{3.20}$$

of the multiplicative group.

3.3.3 Derived principal $\mathbb{G}$-bundles and derived line bundles

Having defined the derived abelian group scheme $\mathbb{G}$, we immediately obtain the notion of a principal $\mathbb{G}$-bundle.

Proposition 15 *Let $B\mathbb{G}$ denote the geometric realization of the derived abelian group scheme $\mathbb{G}$, regarded as a derived Zariski sheaf. Then $B\mathbb{G}$ represents the functor which associates to a derived scheme X the moduli space of principal $\mathbb{G}$-bundles on X.*

Similarly, we define a line bundle on a derived scheme X to be a scheme of the form

$$P \times_{\mathbb{G}} \mathbb{A} \to X \tag{3.21}$$

for some principal $\mathbb{G}$-bundle $P \to X$.

3.4 Elliptic Spectra and Oriented Elliptic Curves

3.4.1 Abelian group $\mathbb{S}$-schemes

Recent unpublished work of J. Lurie on topological modular forms seems to indicate that elliptic spectra E are really just shadows of the much more general notion of a derived elliptic curve. In order to make this precise we need the notion of an abelian group object in the category of S-schemes.

Let A be a derived scheme over a base scheme $\operatorname{Spec} E$ for some commutative $\mathbb{S}$-algebra E. If T is another derived E-scheme, let $A(T)$ denote the space of T-valued points of A.

Definition 12 *An abelian group structure on the derived scheme A is a factorization of its functor of points $A(-) : \mathrm{Aff}_E^{\mathrm{op}} \to \mathcal{S}$ through the category $\mathrm{Ab}(\mathcal{S})$ of topological abelian groups.*

Of course, such a factorization will in general only exist up to coherent homotopy.

There is a slight simplification of the definition of a derived abelian group scheme that is useful in practice. It requires the following well-known result, for which the author is unable to find a suitable reference.

Proposition 16 *The category of topological abelian groups is equivalent to the category of connective $H\mathbb{Z}$-module spectra as topological model categories.* □

Oftentimes it may be easier to factor the functor of points of a derived scheme through the category of $H\mathbb{Z}$-modules, which maps to the category of spaces via the zero-space functor Ω^∞. This automatically factors through the category of connective $H\mathbb{Z}$-module spectra.

We shall be primarily concerned with the case of derived elliptic curves over a derived scheme S.

Definition 13 *A derived elliptic curve over S is a flat abelian S-scheme A such that $\pi_0 A$ is an elliptic curve over $\pi_0 S$.*

This definition, however, is a bit too general, for if S is the spectrum of an even-periodic R-algebra E we must ensure that the formal group of A is equivalent to the formal group $\operatorname{Spf} E^{\mathbb{CP}^\infty_+}$.

We shall also require formal analogues of our abelian group S-schemes. Fortunately we will not need these concept in general, but only in the particular case of in which our formal group schemes are isomorphic to the formal affine line. Let $p : \mathbb{A} \to \mathbb{A}$ be the p^{th}-power map and let $0 : S \to \mathbb{A} = \mathbb{A}^1_S$ denote the zero section. Define the S-scheme $\mathrm{Nil}^p = \mathrm{Nil}^p_S$ to be the pullback

$$\begin{tikzcd} \mathrm{Nil}^p \arrow[r] \arrow[d] & S \arrow[d, "0"'] \\ \mathbb{A} \arrow[r, "p+1"] & \mathbb{A} \end{tikzcd} \tag{3.22}$$

in the category of S-schemes (if $S = \operatorname{Spec} R$ is affine then this corresponds to a homotopy pushout in the category of R-algebras). Clearly Nil^p has the universal property that $\mathrm{Nil}^p(R)$ is the subspace of the space of E_∞-ring maps from $\Sigma^\infty_+\mathbb{N}$ to R consisting of the $p+1$-nilpotent points. Also define Nil^{-1} to be the empty S-scheme and

$$\mathrm{Nil} := \mathrm{Nil}^\infty = \operatorname{colim}_p \mathrm{Nil}^p\,. \tag{3.23}$$

Note that Nil_S carries a number of important abelian structures, most notably the additive structure $\widehat{\mathbb{A}}_S$ and the multiplicative structure $\widehat{\mathbb{G}}_S$. We write Nil_S when we are referring to the underlying scheme without a choice of abelian structure.

For our purposes we will only need strictly commutative, one-dimensional formal group schemes over a derived scheme S.

Definition 14 *A one-dimensional abelian formal S-scheme is an S-scheme $\widehat{A}$, isomorphic to* Nil_S*, together with a strictly commutative group structure on $\widehat{A}$. A choice of isomorphism $\gamma : \widehat{A} \to \mathrm{Nil}_S$ is called a coordinate on $\widehat{A}$.*

Hence, as a functor to spaces, the R-valued points of $\widehat{A}$ are identical to the R-valued points of Nil, namely the subspace of $E_\infty(\Sigma^\infty_+\mathbb{N}, R)$ consisting of the nilpotents. However, the group structure may be quite different.

3.4.2 Elliptic spectra

We begin with the definition of an elliptic spectrum.

Definition 15 *An elliptic spectrum (E, A, φ) consists of an even periodic R-algebra E together with an elliptic curve A_0 over E_0 and an isomorphism of formal groups*

$$\varphi : \operatorname{Spf} \pi_0 E^{B\mathbb{T}} \to \widehat{A}_0, \tag{3.24}$$

where $\widehat{A}_0$ as usual denotes the formal completion of A_0 along the identity section.

When the context is clear we will usually omit A and φ from the notation and simply refer to E as an elliptic spectrum with elliptic curve A_0 and formal group $\widehat{A}_0$. Actually this ambiguity is rather suggestive, since the only thing missing seems to be a derived elliptic curve.

Given an elliptic spectrum E, one might ask whether or not there exists a derived elliptic curve A over E such that A_0 is isomorphic to the elliptic curve of E. If such an A does exist, one is naturally led to the question of whether or not the classical formal group underlying the derived formal group $\widehat{A}$ is isomorphic to the formal group of E, or better if perhaps $\widehat{A}$ is equivalent to $\operatorname{Spf} E^{B\mathbb{T}}$ as derived formal groups.

3.4.3 Oriented S-schemes

In order to compare the two derived formal groups we require that the derived elliptic curve comes equipped with an orientation.

Definition 16 *An orientation of an abelian S-scheme A is a homomorphism of strictly commutative topological groups*

$$\theta : B\mathbb{T} \to A(S), \tag{3.25}$$

where $A(S)$ denotes the space of sections of A over S, i.e. the group of S-valued points of A. An oriented elliptic curve A over S is a derived elliptic curve A over S together with an orientation.

Suppose that $S = \operatorname{Spec} E$ for some even-periodic R-algebra E. It is easy to see that certain oriented elliptic curves over S give rise to elliptic structures on E. Indeed, since $B\mathbb{T}$ is connected,

$$\operatorname{Hom}(B\mathbb{T}, A(S)) \cong \operatorname{Hom}_S(\operatorname{Spf} E^{B\mathbb{T}}, A), \tag{3.26}$$

and therefore any orientation adjoint to an equivalence $\widehat{\eta} : \operatorname{Spf} E^{B\mathbb{T}} \to \widehat{A}$ will do. The much harder question is determining the moduli space of oriented elliptic curves over a given elliptic spectrum.

There is a considerable amount of current research in this direction. The Goerss-Hopkins-Miller approach, which has apparently succeeded although not all the details have yet appeared in print, covers a certain compactification of the moduli space of elliptic curves (obtained by adding nodal curves at infinity) with a sheaf of commutative $\mathbb{S}$-algebras $\mathcal{O}_{tmf}$. Recently Lurie has announced that the moduli space of oriented elliptic curves is representable by a Delign-Mumford $\mathbb{S}$-stack $\mathcal{M}_{\mathrm{Ell}}$, and while the precise claim in not known to the author, the suspicion is that the space of global sections of a suitable compactification of $\mathcal{M}_{\mathrm{Ell}}$ is equivalent to the space of global sections of $\mathcal{O}_{tmf}$.

Chapter 4

Equivariant Elliptic Cohomology

4.1 Equivariant Cohomology Theories from Oriented Abelian Schemes

4.1.1 A general construction

Fix a topological group G and let $\mathrm{Orb}(G)$ denote the category of G-orbits, by which we mean the full topological subcategory of the topological category of G-spaces consisting of those objects of the form G/H for some closed subgroup H of G. It is a consequence of Elmendorf's theorem that the homotopy theory of G-spaces is equivalent to the homotopy theory of $\mathrm{Orb}(G)$-spaces, i.e. continuous presheaves of spaces on $\mathrm{Orb}(G)$, equipped with the topological analogue of the Bousfield-Kan model structure.

Now the topological model category $\mathcal{S}^{\mathrm{Orb}(G)}$ of $\mathrm{Orb}(G)$-spaces may be regarded as the free topological model category generated by the small topological category $\mathrm{Orb}(G)$, in the sense that any continuous $\mathrm{Orb}(G)$-diagram Γ in an arbitrary topological model category $\mathcal{T}$ determines a topological Quillen adjunction

$$\mathcal{T}(\Gamma^* X, Y) \cong \mathcal{S}_{\mathrm{Orb}(G)}(X, \Gamma_* Y) \tag{4.1}$$

where

$$\Gamma^* X := \Gamma \otimes_{\mathrm{Orb}(G)} X \tag{4.2}$$

is the categorical tensor product of the $\mathrm{Orb}(G)$-diagram Γ with the $\mathrm{Orb}(G)$-space X, and

$$\Gamma_* Y := \mathrm{Map}_{\mathrm{Orb}(G)}(\Gamma, Y) \tag{4.3}$$

is the $\mathrm{Orb}(G)$-space obtained by mapping the $\mathrm{Orb}(G)$-diagram Γ into Y.

4.1.2 The derived scheme $\mathrm{Hom}(G^\circ, A)$

Let A be a derived abelian group scheme over some base scheme S and let F be a finitely generated abelian group. We wish to define a derived abelian group scheme $\mathrm{Hom}(F, A)$ of homomorphisms from F to A.

To specify a limit-preserving functor $\mathrm{Ab}^{\mathrm{op}} \to \mathrm{Ab}(\mathrm{Sch}_S)$ it suffices to specify the functor on a generator $\mathbb{Z}$ of Ab. Thus we define

$$\mathrm{Hom}(\mathbb{Z}, A) := A. \tag{4.4}$$

We formally extend this to a functor on all of $\mathrm{Ab}^{\mathrm{op}}$ be requiring that $\mathrm{Hom}(-, A)$ preserves finite homotopy limits. Any finitely-generated abelian group admits a cofibrant replacement in the derived category of $\mathbb{Z}$-modules by a cellular $\mathbb{Z}$-module of the form

$$\mathbb{Z}^{\oplus p} \to \mathbb{Z}^{\oplus q} \tag{4.5}$$

for some nonnegative integers p and q. Hence we define $\mathrm{Hom}(F, A)$ to be the fiber of the induced map $A^q \to A^p$; this specifies $\mathrm{Hom}(F, A)$ up to natural equivalence.

Precomposing with the functor $\mathrm{Hom}(-, \mathbb{T})$ from compact abelian Lie groups to finitely generated abelian groups gives us a continous covariant functor from the category of compact abelian Lie groups to the category of abelian S-schemes. If G is a compact abelian Lie group then we write A_G for the scheme $\mathrm{Hom}(\mathrm{Hom}(G, \mathbb{T}), A)$.

4.1.3 A functor from G-orbits to A_G-schemes

Let $i : H \to G$ be the inclusion of a closed subgroup and let $K = G/H$ be the associated G-orbit. Then corresponding to the quotient homomorphism $q : G \to K$ we obtain a map of abelian S-schemes

$$A_q : A_G \to A_K. \tag{4.6}$$

We define the A_G-scheme $A_G(K)$ to be the fiber of A_q, so that $A_G(K)$ fits into the fiber sequence

$$A_G(K) \to A_G \to A_K \tag{4.7}$$

of abelian S-schemes.

If A is oriented, then taking S-valued points and gives a morphism of fiber sequences of topological abelian groups

$$\begin{tikzcd}
BH \arrow[r] \arrow[d] & BG \arrow[r] \arrow[d] & BK \arrow[d] \\
\mathrm{Map}(S, A_G(K)) \arrow[r] & \mathrm{Map}(S, A_G) \arrow[r] & \mathrm{Map}(S, A_K)
\end{tikzcd} \tag{4.8}$$

and therefore induces a homomorphism

$$K \to \mathrm{Aut}_{A_G} A_G(K) \tag{4.9}$$

in which K acts on the A_G-scheme $A_G(K)$ by translation. That is, the fiber of the homomorphism $\mathrm{Map}(S, A_G(K)) \to \mathrm{Map}(S, A_G)$ is precisely the space of translation automorphisms of $A_G(K)$ which act trivially on A_G.

Proposition 17 *Let A be an oriented abelian group scheme over a derived scheme S. Then A determines a continuous functor $A_G : \mathrm{Orb}(G) \to \mathrm{Sch}_{A_G}$ by the rule $A_G(G/H) := A_H$ on objects and, if K is a subgroup of H, the natural map $\mathrm{Map}_G(G/K, G/H) \cong G/H \to \mathrm{Map}_{A_G}(A_K, A_H)$ on morphisms.*

4.2 From finite G-complexes to A_G-schemes

4.2.1 A functor from G-spaces to affine A_G-schemes

Recall that the category Aff_{A_G} of affine A_G-schemes is topologically cocomplete and that the topological model category of G-spaces is topologically equivalent to the topological model category of $\mathrm{Orb}(G)$-spaces. Hence, to specify and homotopy colimit preserving functor from G-spaces to A_G-schemes, it is enough to specify a continuous $\mathrm{Orb}(G)$-diagram of A_G-schemes.

Theorem 11 (Elmendorf [19]) *Let X be a G-space. Then the natural map from the geometric realization of the simplicial G-space*

$$\coprod_{K \le H \le G} G/K \times \mathrm{Map}(G/K, G/H) \times \mathrm{Map}(G/H, X) \rightrightarrows \coprod_{H \le G} G/H \times \mathrm{Map}(G/H, X) \tag{4.10}$$

to X is a weak equivalence of G-spaces.

Therefore we may define the A_G-scheme $A_G(X)$ to be the geometric realization of the simplicial A_G-scheme

$$\coprod_{K \le H \le G} A_K \otimes \mathrm{Map}(G/K, G/H) \otimes \mathrm{Map}(G/H, X) \rightrightarrows \coprod_{H \le G} A_H \otimes \mathrm{Map}(G/H, X). \tag{4.11}$$

In practice, however, one would use a cellular decomposition of a G-space X in order to obtain a cellular decomposition of the A_G-scheme $A_G(X)$.

We will also write $X \otimes A_G := X \otimes_{\mathrm{Orb}(G)} A_G$ for the A_G-scheme $A_G(X)$ since it is the categorical tensor product of the $\mathrm{Orb}(G)^{\mathrm{op}}$-diagram $\mathrm{Map}(G/H, X)$ with the $\mathrm{Orb}(G)$-scheme $A_G(G/H)$.

4.2.2 A scheme-theoretic formulation of the axioms

In practice it is usually easier to regard equivariant elliptic cohomology as a covariant functor from finite G-spaces to A_G-schemes. The geometric nature of the category of A_G-schemes reflects the geometry of the category of finite G-spaces, or, better, compact G-manifolds. For example, principal fibrations of G-manifolds go to principal fibrations of A_G-schemes, etc. Moreover, we have seen that a covariant functor from finite G-spaces to A_G-schemes determines a contravariant functor from finite G-spectra to coherent $\mathcal{O}_{A_G}$-modules, so the former construction is perhaps more fundamental.

Recall that a homomorphism of compact abelian Lie groups $\varphi : H \to G$ induces a homomorphism of derived abelian group schemes $\varphi : A_H \to A_G$. This in turn induces a geometric morphism

$$\mathcal{S}^{\mathrm{Aff}_{A_H}}(\varphi^* X, Y) \simeq \mathcal{S}^{\mathrm{Aff}_{A_G}}(X, \varphi_* Y) \tag{4.12}$$

of homotopy topoi in which, if $X \to A_G$ is an A_G-scheme, then $\varphi^* X$ is the base-change $X \times_{A_G} A_H \to A_H$; the right adjoint φ_* is harder to describe. Note that φ^* not only preserves schemes but affine schemes as well.

It turns out that φ^* has a left adjoint $\varphi_!$, which as a functor on schemes takes an A_H-scheme $Y \to A_H$ to the A_G-scheme $Y \to A_H \to A_G$ obtained by composition with φ. In other words, φ^* is the inverse image of an essential geometric morphism $\mathcal{S}^{\mathrm{Aff}(A_G)} \to \mathcal{S}^{\mathrm{Aff}(A_H)}$, totally analogous to the essential geometric morphism $\mathcal{S}^{\mathrm{Orb}(G)} \to \mathcal{S}^{\mathrm{Orb}(H)}$. Note that $\varphi_!$ preserves affine schemes if and only if $\varphi : A_H \to A_G$ is an affine map. This will be the case when $\varphi : H \to G$ is a finite map, such as a closed immersion.

More specifically, an equivariant elliptic cohomology theory in this sense should consist of a covariant functor $(-) \otimes_G A$ from the topos $\mathcal{S}^G$ of G-spaces to the topos $\mathcal{S}^{\mathrm{Aff}(A_G)}$ of Zariski sheaves on A_G satisfying the following change-of-group axioms:

(i) Induction: if $\varphi : H \to G$ is the inclusion of a closed subgroup then for any finite H-space Y there is a natural equivalence $\varphi_!(Y \otimes_H A) \to (\varphi_! Y) \otimes_G A$;

(ii) Restriction: if $\varphi : H \to G$ is any homomorphism then for any finite G-space X there is a natural equivalence $(\varphi^* X) \otimes_H A \to \varphi^*(X \otimes_G A)$;

(iii) Completion: if $i : \widehat{A}_G \to A_G$ denotes the formal completion then for any finite G-space X there is a natural equivalence $i_! i^*(X \otimes_G A) \to (X \times EG) \otimes_G A$;

(iv) Kunneth: if X is a finite G-space and Y is a finite H-space then there is a natural equivalence $(X \times Y) \otimes_{G \times H} A \to \pi_G^*(X \otimes_G A) \times \pi_H^*(Y \otimes_H A)$.

For infinite G-spaces we adopt the usual convention; that is, a general G-complex X is equivalent to the (filtered) homotopy colimit of its finite G-subcomplexes, so we define

$$X \otimes_G A = (\operatorname{colim}_\lambda X_\lambda) \otimes_G A := \operatorname{colim}_\lambda X_\lambda \otimes_G A, \tag{4.13}$$

to be the formal filtered homotopy colimit of A_G-schemes, which is to say a formal A_G-scheme.

4.2.3 Verification of the axioms

We start with the induction axiom. Let $\varphi : H \to G$ be the inclusion of a closed subgroup. Consider the special case of an H-orbit H/K, so that $H/K \otimes_H A \simeq A_K$. On the other hand, the induced G-space $\varphi_!(H/K) \simeq * \times_K H \times_H G \simeq * \times_K G \simeq G/K$ has cohomology $G/K \otimes_G A \simeq A_K$, so the statement holds for orbits. The general case follows from the fact that $\varphi_!$ is left-adjoint to φ^* and therefore preserves continuous homotopy colimits.

For the restriction axiom, let $\varphi : H \to G$ be any homomorphism of compact Lie groups, let K be a closed subgroup of G, and let $L := \varphi^{-1}K$ be the inverse image of K in H. Since φ^* has a right adjoint φ_*, it necessarily preserves continuous homotopy colimits. It therefore suffices to check the restriction axiom on orbits, which amounts to verifying that as A_H-schemes, $G/K \otimes_H A_L \simeq A_K \times_{A_G} A_H$, where G/K is acted on by H via φ with stabilizer L.

This is immediate if $\varphi : H \to G$ is a fibration, which is to say, surjective onto the components of G containing a point in the image of H. For then $\varphi : A_H \to A_G$ is also a fibration so the homotopy pullback is equivalent to the ordinary pullback A_L, and $\varphi^* G/K \simeq H/L$ is the orbit which cohomology A_L.

Since any homomorphism $\varphi : H \to G$ factors as the composite of a surjection followed by an injection, we may reduce to the case in which $\varphi : H \to G$ is the inclusion of a closed subgroup. Then a G-orbit G/K

corresponding to the inclusion $i : K \to G$ of a closed subgroup determines a short exact sequence of short exact sequences of compact abelian Lie groups

$$\begin{tikzcd}
 & * \arrow[d] & * \arrow[d] & * \arrow[d] & \\
* \arrow[r] & L \arrow[r] \arrow[d, "j"'] & K \arrow[r] \arrow[d, "i"'] & K/L \arrow[r] \arrow[d] & * \\
* \arrow[r] & H \arrow[r, "\varphi"] \arrow[d, "q"'] & G \arrow[r] \arrow[d, "p"'] & G/H \arrow[r] \arrow[d] & * \\
* \arrow[r] & H/L \arrow[r, "\psi"] \arrow[d] & G/K \arrow[r] \arrow[d] & (G/K)/(H/L) \arrow[r] \arrow[d] & * \\
 & * & * & * &
\end{tikzcd} \tag{4.14}$$

such that L is the intersection of H with K in G.

Consider the very special case in which $K = *$.

Proposition 18 *The natural map $G \otimes_H \operatorname{Spec} E \to \operatorname{Spec} E \times_{A_G} A_H$ is an equivalence.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
G \otimes_H \operatorname{Spec} E \arrow[r] \arrow[d] & BH \otimes \operatorname{Spec} E \arrow[r] \arrow[d] & A_H \arrow[d] \\
\operatorname{Spec} E \arrow[r] & BG \otimes \operatorname{Spec} E \arrow[r] & A_G
\end{tikzcd} ; \tag{4.15}$$

since both squares are evidently homotopy cartesian it follows the the rectangle itself is homotopy cartesian. □

The case of a general orbit G/K now follows. Indeed, since $p : G \to G/K$ and $q : H \to H/L$ are surjective, we have that

$$\begin{aligned}(\varphi^* G/K) \otimes_H A \simeq (q^* \psi^* G/K) \otimes_H A \\ \simeq q^*((\psi^* G/K) \otimes_{H/L} A) \simeq q^* \psi^* (G/K \otimes_G A) \\ \simeq \varphi^* p^* (G/K \otimes_G A) \simeq \varphi^* (G/K \otimes_G A),\end{aligned} \tag{4.16}$$

as claimed.

The proof of the completion axiom follows along similar lines. The functor

$$i^* : \mathrm{Sch}_{A_G} \to \mathrm{Sch}_{\widehat{A}_G} \tag{4.17}$$

is the pullback $(-) \times_{A_G} \widehat{A}_G$, and similarly the cohomology of the functor

$$(-) \times EG : \mathcal{S}^G \to \mathcal{S}^G_{EG} \tag{4.18}$$

is the pullback $(-) \times_{A_G} EG \otimes A_G$.

We claim that the cohomology of EG, as an A_G-scheme, is precisely the formal completion $i : \widehat{A}_G \to A_G$. To see this, note that since EG is a free G-space, its cohomology is calculated as the realization of the simplicial A_G-scheme

$$\cdots \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; EG \otimes G \otimes A_G(G) \rightrightarrows EG \otimes A_G(G), \tag{4.19}$$

which is clearly the same as $EG \otimes_G A_G(G)$.

Proposition 19 *The natural map*

$$EG \otimes_G A_G(G) \to BG \otimes S \tag{4.20}$$

is an equivalence.

Proof. Since G acts trivially on S, the natural equivalence $A_G(G) \to S$ is G-equivariant and therefore induces an equivalence of coequalizer diagrams

$$\begin{tikzcd} EG \otimes G \otimes A_G(G) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & EG \otimes G \otimes S \arrow[d, shift left] \arrow[d, shift right] \\ EG \otimes A_G(G) \arrow[r] & EG \otimes S. \end{tikzcd} \tag{4.21}$$

The desired equivalence $EG \otimes_G A_G(G) \to EG \otimes_G S \simeq BG \otimes S$ follows by passing to coequalizers. □

Hence we are reduced to giving equivalences $BG \otimes S \to \widehat{A}_G$, natural in compact abelian Lie groups G. Recall that the natural map

$$\mathbb{Z}\mathbb{CP}^1 \otimes S \to \widehat{A} \tag{4.22}$$

adjoint to the inclusion

$$\mathbb{CP}^1 \otimes S \simeq \operatorname{Spec} \mathcal{O}_A/\mathcal{I}^2 \to \operatorname{colim}_p \operatorname{Spec} \mathcal{O}_A/\mathcal{I}^p \simeq \widehat{A} \tag{4.23}$$

is an equivalence of derived formal group schemes, and that, just as A_G is defined to be the scheme $\mathrm{Hom}(G^\circ, A)$ of G°-order points of A, $\widehat{A}_G$ may also be taken to be the formal scheme $\mathrm{Hom}(G^\circ, \widehat{A})$ of G°-order points of $\widehat{A}$. Since BG may be regarded as the space $\mathrm{Hom}(G^\circ, B\mathbb{T})$ of G°-order points of $B\mathbb{T}$ and the map $\mathbb{Z}\mathbb{C}\mathbb{P}^1 \to B\mathbb{T}$ is an equivalence, it follows that $BG \otimes S$ and $\widehat{A}_G$ are naturally equivalent as derived formal group schemes.

It remains only to verify the Kunneth formula, which proceeds along the same lines. It boils down to the fact that the natural map

$$A_{G\times H} \to A_G \times A_H \tag{4.24}$$

is an equivalence for any pair of abelian compact Lie groups G and H.

4.3 Equivariant Elliptic Cohomology and Homotopy Topoi

4.3.1 Suitable sites of definition

It is well known how different sites can give rise to equivalent topoi. For example, the topos of a topological space X, i.e. the category of sheaves on the site $\mathrm{Open}(X)$ of open subsets of X, with the usual notion of covering, is equivalent to the category of sheaves on the site determined by any basis for the topology of X. This is just another way of saying that sheaves are determined locally.

For us, it will be convenient to model the derived Zariski topos of an $\mathbb{S}$-scheme X as the topological model category $\mathcal{S}^{\mathrm{Sch}(X)}$ of sheaves of spaces on a small, full subcategory of the Zariski site $\mathrm{Sch}(X)$ of X-schemes. This is equivalent to the topos $\mathcal{S}^{\mathrm{Aff}(X)}$ via the inclusion of the full topological subcategory $\mathrm{Aff}(X) \to \mathrm{Sch}(X)$, since a general X-scheme is necessarily patched together from affine X-schemes.

Nevertheless, this point of view has a distinct advantage when considering geometric morphisms induced by maps of $\mathbb{S}$-schemes $f : T \to S$. Although the essential geometric morphism associated to the S-scheme structure map $p : T \to S$ is always induced by a morphism of sites $p^*_{\mathrm{Aff}} : \mathrm{Aff}(S) \to \mathrm{Aff}(T)$, unless p itself is affine, p^*_{Aff} will not admit a left adjoint until we extend p^*_{Aff} to the functor $p^*_{\mathrm{Sch}} : \mathrm{Sch}(S) \to \mathrm{Sch}(T)$. Then the left adjoint $p_!$ is simply the functor which takes an T-scheme and regards it as an S-scheme via p.

The example of interest to us is the morphism of $\mathrm{Spec}\, E$-schemes associated to an oriented elliptic curve $A \to \mathrm{Spec}\, E$ and a homomorphism of compact abelian Lie groups $\varphi : H \to G$. Since the associated homomorphism of abelian group schemes $A_H \to A_G$ is affine if and only if φ is finite (which is to say that the fibers are finite discrete spaces), the pushforward $p_!$ of an affine A_H-schemes is not in general an affine A_G-scheme.

The ideal situation is one in which all morphisms of homotopy topoi of interest to us are induced by morphisms of sites. We have already considered the algebro-geometric side of the story, so it remains to consider the topological side. Given a homomorphism of compact Lie groups $\varphi : H \to G$, it is not in general the case (unless φ is surjective) that the restriction of a G-orbit is an H-orbit. However, it is always the case that the induced G-space of an H-orbit is a G-orbit. The is precisely opposite the case of schemes, and in particular we see that p^* does not restrict to a morphism of sites $\mathrm{Orb}(G) \to \mathrm{Orb}(H)$. The solution to this is to enlarge the orbit category.

Let G be a compact Lie group, and define a generalized G-orbit to be the total G-space of the fibre bundle $P \times_G G/H \to M$ associated to a principal G-bundle P on a compact manifold M. This is the natural generalization to manifolds of the notion of a finite G-set with stabilizer H for a finite group G and subgroup H. We write $\mathrm{Orb}(G)$ for the category of generalized G-orbits, since the particular site of definition can be chosen as a matter of convenience. The advantage of considering generalized G-orbits is that the restriction of a generalized G-orbit P to H via a homomorphism $H \to G$ is a generalized H-orbit.

Proposition 20 *If $\varphi : H \to G$ is a homomorphism of compact Lie groups then the restriction functor $\varphi^* : \mathrm{Orb}(G) \to \mathrm{Orb}(H)$ admits a left adjoint $\varphi_! : \mathrm{Orb}(H) \to \mathrm{Orb}(G)$.*

Proof. By definition, $\varphi_!$, if it exists, would satisfy the formula

$$\begin{aligned} \mathrm{Map}^G(\varphi_! Q \times_H H/L, P \times_G G/K) &\cong \mathrm{Map}^H(Q \times_H H/L, \varphi^* P \times_G G/K) \\ \cong \mathrm{Map}^G(Q \times_H H/L \times_H G, P) &\cong \mathrm{Map}^G(Q \times_H G \times_G G/\varphi(L), P \times_G G/K); \end{aligned} \tag{4.25}$$

since $Q \times_H G \times_G G/\varphi(L)$ is a fiber bundle with structure group G and fiber $G/\varphi(L)$, we see that $\varphi_! Q$ is a generalized G-orbit. □

The next thing to do is define the cohomology of generalized orbits. To this end, let $A \to \operatorname{Spec} E$ be an oriented elliptic curve and let G be a compact abelian Lie group. Define a functor $p^* : \mathrm{Orb}(G) \to \mathrm{Sch}(A_G)$ by the rule

$$(P \otimes_G G/H) \otimes_G A := P \otimes_G (G/H \otimes_G A) \cong P \otimes_G A_H, \tag{4.26}$$

where G acts on the A_G-scheme A_H via the translation action of G/H on A_H. Note that this is homotopy invariant since G acts freely on P.

Recall that the terminal object $*_{\mathcal{T}}$ of a homotopy topos $\mathcal{T}$ determines a canonical geometric morphism $\mathcal{T} \to \mathcal{S}$ defined by the adjunction

$$\mathcal{T}(X \otimes *_{\mathcal{T}}, Y) \cong \mathcal{S}(X, \mathrm{Map}(*_{\mathcal{T}}, Y)) \tag{4.27}$$

in which the space of points $\mathrm{Map}(*, Y)$ of a $\mathcal{T}$-object Y should be regarded as a $\mathcal{T}$-externalization functor, right adjoint to the $\mathcal{T}$-internalization functor $X_{\mathcal{T}} := X \otimes *_{\mathcal{T}}$. Note that since $(-)_{\mathcal{T}}$ preserves finite homotopy limits, structured objects such as $\mathcal{S}$-categories internalize to $\mathcal{T}$-categories.

Now we already know that in order to specify a continuous colimit-preserving functor $\mathcal{T} \to \mathcal{S}^{\mathrm{Orb}(G)}$ it is enough to give a continuous $\mathrm{Orb}(G)$-diagram $\mathcal{F}(G)$ in $\mathcal{T}$. The question arises, then, as to when this functor will be left exact. As might be expected, this has to do with when the $\mathcal{T}$-category $\mathcal{F}(G)_{\mathcal{T}}$ associated to the $\mathrm{Orb}(G)$-diagram $\mathcal{F}(G)$ is filtered.

Note that the $\mathcal{T}$-internalization $\mathrm{Orb}(G)_{\mathcal{T}}$ of the orbit category is filtered since $\mathrm{Orb}(G)$ has a terminal object, the trivial orbit.

The functor $\mathrm{Orb}(G) \to \mathcal{S}^{\mathrm{Aff}(A_G)}$ has the property that it factors through the Yoneda embedding $\mathrm{Aff}(A_G) \to \mathcal{S}^{\mathrm{Aff}(A_G)}$. In other words, it is determined by a morphism of sites $f^* : \mathrm{Orb}(G) \to \mathrm{Aff}(A_G)$. It follows that the restriction functor $f^* : \mathcal{S}^{\mathrm{Aff}(A_G)} \to \mathcal{S}^{\mathrm{Orb}(G)}$ has both left and right adjoints $f_!$ and f_*, respectively, given by the usual formulas for left and right Kan extension. Hence $f^* : \mathcal{S}^{\mathrm{Aff}(A_G)} \to \mathcal{S}^{\mathrm{Orb}(G)}$ is the inverse image of an essential geometric morphism $f : \mathcal{S}^{\mathrm{Orb}(G)} \to \mathcal{S}^{\mathrm{Aff}(A_G)}$.

4.3.2 $\mathbb{S}$-algebras from sheaves on the Zariski topos of $\mathbb{S}$

We need a general method, analogous to taking rings of regular functions, of recovering $\mathbb{S}$-algebras from $\mathbb{S}$-schemes or, more generally, sheaves on the Zariski site associated to $\mathbb{S}$.

Proposition 21 *The Yondea embedding $y_* : \mathrm{Aff}_{\mathbb{S}} \to \mathcal{S}^{\mathrm{Aff}_{\mathbb{S}}}$ has a left adjoint y^*.* □

Proof. Any sheaf X may be canonically presented as a colimit of representable sheaves $\mathrm{Spec}\, E$ for some commutative $\mathbb{S}$-algebras E. That is, X is the coequalizer of the pair of maps

$$\coprod_{E,F} \mathrm{Spec}\, F \otimes \mathrm{Map}(\mathrm{Spec}\, F, \mathrm{Spec}\, E) \otimes X(E) \rightrightarrows \coprod_{E} \mathrm{Spec}\, E \otimes X(E), \tag{4.28}$$

where the objects E range over some essentially small full subcategory of commutative $\mathbb{S}$-algebras. But the category $\mathrm{Aff}_{\mathbb{S}}$ itself is topologically bicomplete, so we may define y^*X to be the colimit of the same diagram in the category of affine $\mathbb{S}$-schemes. □

Composed with the contravariant opposite-category functor $\mathrm{Aff}_{\mathbb{S}} \to \mathrm{Alg}_{\mathbb{S}}$, we obtain a contravariant functor $\Gamma : \mathcal{S}^{\mathrm{Aff}_{\mathbb{S}}} \to \mathrm{Alg}_{\mathbb{S}}$ which takes homotopy colimits to homotopy limits. In other words, $\Gamma(X)$ is the commutative $\mathbb{S}$-algebra obtained as the equalizer of the pair of maps

$$\prod_{E} E^{X(E)} \rightrightarrows \prod_{E,F} E^{X(E) \times \mathrm{Map}(E,F)}. \tag{4.29}$$

In practice, of course, we require that X is both cofibrant and fibrant in order to ensure that $\Gamma(X)$ has the correct homotopy type. The notation is meant to be suggestive, since Γ_* (restricted to the category of $\mathbb{S}$-schemes) can also be regarded as the pushforward to $\operatorname{Spec}\mathbb{S}$ of the structure sheaf, i.e. the derived functor of global sections.

4.3.3 The equivariant elliptic cohomology groups E_G^*

Replacing the sphere spectrum by an elliptic spectrum E, we apply the procedure of the last section to an oriented elliptic curve $A \to \operatorname{Spec} E$ in order to obtain, for each G-space X, actual E-algebras $\Gamma(X \otimes_G A)$. We write $\Gamma_*(X \otimes_G A)$ for the graded ring $\pi_*\Gamma(X \otimes_G A)$.

If $* \to X$ is a pointed G-space, define the E-module E_G^X to be the fibre of the map of E-algebras

$$\Gamma(X \otimes_G A) \to \Gamma(* \otimes_G A) \cong \Gamma(A_G). \tag{4.30}$$

We shall write $E_G^{-*}(X)$ for the graded π_*E-module $\pi_*E_G^X$. The abbreviation $E_G^*(X)$ is somewhat of an abuse of notation, since the G-equivariant theories $E_G^*(-)$ depend on the particular oriented elliptic curve $A \to \operatorname{Spec} E$. On the other hand, this notation is suggestive of the fact that the E_G are equivariant extensions of E-theory. It remains to show that $E_G^*(-)$ is actually a cohomology theory, although this is formal. We sketch the argument.

A cohomology theory on G-spaces is a (weak) homotopy-invariant contravariant functor from the category of pointed G-spaces to $\mathbb{Z}$-graded abelian groups that satisfies the suspension and cofibre axioms. In other words, if X is a pointed G-space then for each $n \in \mathbb{Z}$ there is an isomorphism

$$E_G^{n+1}(S^1 \wedge X) \cong E_G^n(X), \tag{4.31}$$

and if $X \to Y$ is a G-equivariant map of pointed G-spaces with cofibre Z, then for each n the induced sequence of abelian groups

$$E_G^n(Z) \to E_G^n(Y) \to E_G^n(X) \tag{4.32}$$

is exact.

Now for G a compact abelian Lie group and A an oriented elliptic curve over an $\mathbb{S}$-algebra E, we claim that the functor $E_G^*(-)$ is a G-equivariant cohomology theory. The homotopy invariance is immediate, since

$\pi_*\Gamma((-) \otimes_G A)$ is a homotopy-invariant functor. To verify the suspension axiom, it suffices to observe that

$$E_G^{S^1 \wedge X} \cong \Omega E_G^X \tag{4.33}$$

and that for any spectrum F, $\pi_*\Omega F \cong \pi_{*-1}F$.

Lastly, for the cofibre axiom, it suffices to observe that the functor $\Gamma((-) \otimes_G A)$ takes homotopy colimits to homotopy limits. Thus if $X \to Y$ is a map of G-spaces with cofibre Z, then

$$\Gamma(Y \otimes_G A) \to \Gamma(X \otimes_G A) \tag{4.34}$$

is a fibre sequence of E-algebras, so that

$$E_G^Z \to E_G^Y \to E_G^X \tag{4.35}$$

is a fibre sequence of E-modules. Applying π_n, it follows that the sequence

$$E_G^{-n}(Z) \to E_G^{-n}(Y) \to E_G^{-n}(X) \tag{4.36}$$

is exact.

Chapter 5

Equivariant Elliptic Cohomology of Complex Representation Sphere Spectra

5.1 Equivariant Thom Spaces

5.1.1 Borel-equivariant Chern classes

We shall see that the obstruction to equivariant elliptic orientability lies in the Borel-equivariant Chern classes. Throughout this section G will denote a fixed compact abelian Lie group and

$$U := \mathrm{colim}_n\, U(n) \tag{5.1}$$

will denote the infinite unitary group.

Recall that the monoid $\mathrm{Rep}(G)$ of (isomorphism classes of) complex representations of G is naturally isomorphic to $\mathbb{N}G^\circ$, the free monoid on the group $G^\circ := \mathrm{Hom}(G, \mathbb{T})$ of irreducible complex representations of G. Let $\mathbb{U} := \mathbb{U}_G$ be the complete complex G-universe

$$\mathbb{U} := \mathbb{C}^{\oplus \mathbb{N}G^\circ}. \tag{5.2}$$

Note that the classifying space $B_G U(n)$ of principal G-equivariant $U(n)$-bundles is modeled by the Grassmannian $\mathbb{G}(\mathbb{U}, n)$ of n-dimensional subspaces of $\mathbb{U}$.

We record the following well-known fact concerning the moduli G-space $B_G K$ of principal G-equivariant K-bundles.

Proposition 22 *Let G/H be a G-orbit. Then there is a natural map of G/H-spaces*

$$\mathrm{Map}_G(G/H, B_G K) \to \mathrm{Map}(EG \times_G G/H, BK), \tag{5.3}$$

where G/H acts on $EG \times_G G/H$ via the map

$$G/H \cong \mathrm{Aut}_G(G/H) \to \mathrm{Aut}_{BG}(EG \times_G G/H) \to \mathrm{Aut}(EG \times_G G/H) \tag{5.4}$$

which translates by gH.

Proof. A convenient model for $B_G K$ is the G-space $\mathrm{Map}(EG, EK)/K$. Since BK has a trivial K-action, the map of K-spaces

$$\mathrm{Map}(EG, EK) \to \mathrm{Map}(EG, BK) \tag{5.5}$$

induced by the projection $EK \to BK$ factors through the quotient

$$\mathrm{Map}(EG, EK) \to \mathrm{Map}(EG, EK)/K. \tag{5.6}$$

Taking H-fixed points gives the result. □

Define the Borel-equivariant Eilenberg-Mac Lane space $K_G(\mathbb{Z}, n)$ to be the G-space

$$K_G(\mathbb{Z}, n) := \mathrm{Map}(EG, K(\mathbb{Z}, n)). \tag{5.7}$$

Note that the non-equivariant Chern classes $c^n : BU \to K(\mathbb{Z}, 2n)$ induce G-maps $\mathrm{Map}(EG, BU) \to \mathrm{Map}(EG, K(\mathbb{Z}, 2n))$.

Definition 17 *The Borel-equivariant Chern classes*

$$c_G^n : B_G U \to K_G(\mathbb{Z}, 2n) \tag{5.8}$$

are defined as the composite of the natural G-map $B_G U \to \mathrm{Map}(EG, BU)$ with the G-map $\mathrm{Map}(EG, BU) \to \mathrm{Map}(EG, K(\mathbb{Z}, 2n))$ induced by the non-equivariant Chern classes.

The inclusions $B_G U(k) \to B_G U$ give Borel-equivariant Chern classes c_G^i for $i \leq k$.

5.1.2 The Thom spaces $M_G U(n)\langle 2k \rangle$

We can now generalize to the equivariant world the definition of the connective covers $BU(n)\langle 2k \rangle$ of $BU(n)$. Recall that the $BU(n)\langle 2k+2 \rangle$ are defined inductively as the fibers of the Chern classes

$$c^k : BU(n)\langle 2k \rangle \to K(\mathbb{Z}, 2k). \tag{5.9}$$

Following this recipe, we define spaces $B_GU(n)$ in precisely the same way. That is, we set

$$B_GU(n)\langle 2\rangle := B_GU(n) \tag{5.10}$$

and inductively define $B_GU(n)\langle 2k+2\rangle$ to be the fiber of the Borel-equivariant Chern class

$$c_G^k : B_GU(n)\langle 2k\rangle \to K_G(\mathbb{Z}, 2k). \tag{5.11}$$

In particular, we obtain a tower of fibrations

$$B_GU(n)\langle 2n\rangle \to \cdots \to B_GU(n)\langle 2k\rangle \to \cdots \to B_GU(n). \tag{5.12}$$

Now the tautological G-equivariant complex n-plane bundle

$$V(n) \to B_GU(n) \tag{5.13}$$

pulls back to give canonical bundles

$$V(n)\langle 2k\rangle \to B_GU(n)\langle 2k\rangle \tag{5.14}$$

for each $k \leq n$ (we needn't worry about the $B_GU(n)\langle 2k\rangle$ for $k > n$ since they are equivariantly contractible).

Definition 18 *The G-equivariant Thom space $M_GU(n)\langle 2k\rangle$ is the Thom space of the G-equivariant complex vector bundle $V(n)\langle 2k\rangle$; i.e., it fits into the cofiber sequence of pointed G-spaces*

$$S(V(n)\langle 2k\rangle)_+ \to B_GU(n)\langle 2k\rangle_+ \to M_GU(n)\langle 2k\rangle \tag{5.15}$$

where $S(V(n)\langle 2k\rangle)$ denotes the sphere bundle of $V(n)\langle 2k\rangle$.

5.1.3 The string group and its complex analogue

We review the usual homotopy theoretic construction of the string group. Since for n sufficiently large

$$\pi_1 SO(n) \cong \mathbb{Z}/2\mathbb{Z}, \tag{5.16}$$

$SO(n)$ has a double cover $\mathrm{Spin}(n)$, a connected and simply connected compact Lie group. Now

$$H^3(\mathrm{Spin}(n)) \cong \mathbb{Z}, \tag{5.17}$$

and it can be shown that the $K(\mathbb{Z},2)$-bundle over $\mathrm{Spin}(n)$ corresponding to the choice of a generator $\gamma : \mathrm{Spin}(n) \to K(\mathbb{Z},3)$ has the structure of an infinite-dimensional Lie group called $\mathrm{String}(n)$. In particular, we obtain a short exact sequence of topological groups

$$K(\mathbb{Z},2) \to \mathrm{String}(n) \to \mathrm{Spin}(n) \tag{5.18}$$

which is also a fibration sequence. Taking the colimit over all n, the resulting sequence

$$K(\mathbb{Z},2) \to \mathrm{String} \to \mathrm{Spin} \tag{5.19}$$

shows that String is an extension of Spin by $K(\mathbb{Z},2)$.

The analogue for complex Lie groups is slightly simpler. Again, for n sufficiently large,

$$H^3(SU(n)) \cong \mathbb{Z} \tag{5.20}$$

and the choice of a generator $\gamma : SU(n) \to K(\mathbb{Z},3)$ gives a fibration

$$K(\mathbb{Z},2) \to \mathrm{String}^{\mathbb{C}} \to SU(n). \tag{5.21}$$

Taking the colimit over all n, we obtain a fibration

$$K(\mathbb{Z},2) \to \mathrm{String}^{\mathbb{C}} \to \ SU \tag{5.22}$$

which gives $\mathrm{String}^{\mathbb{C}}$ the structure of an extension of SU by $K(\mathbb{Z},2)$. In particular, we see that $B\,\mathrm{String}^{\mathbb{C}}$ is homotopy equivalent to the space $BU\langle 6\rangle$.

5.1.4 The fixed point spaces $B_G U\langle 2k\rangle^H$

In order to calculate the equivariant elliptic cohomology of the $B_G U\langle 2k\rangle$ we must first analyze the associated $\mathrm{Orb}(G)$-space of fixed points $B_G U\langle 2k\rangle^H$.

Proposition 23 *The realization of the* $\mathrm{Orb}(G)$*-space*

$$\mathbb{Z}H^\circ \times BU \tag{5.23}$$

is a model for $B_G U\langle 0\rangle$*.*

Proof. Since $U \cong \mathrm{colim}_{n\in\mathbb{N}}\, U(n)$, we have that $BU \cong \mathrm{colim}_n\, BU(n)$. Since G is abelian, we know from the discussion of equivariant classifying spaces in [19] that for any compact Lie group L, there is an equivalence of G/H-spaces

$$B_G L^H \simeq \coprod_{\rho\in\mathrm{Rep}(H,L)} BZ(\rho), \tag{5.24}$$

where $\mathrm{Rep}(H,L)$ is the set of conjugacy classes of homomorphisms $\rho : H \to L$ and $Z(\rho)$ is the centralizer of ρ in L, i.e. the group of $l \in L$ such that

$$\rho(h) = l\rho(h)l^{-1} \tag{5.25}$$

for all $h \in H$. Taking the colimit over $n \in \mathbb{N}$ and forming the group completion $\mathbb{Z}H^o \times BU$ of the monoid $\mathbb{N}H^o \times BU$, it follows that

$$B_G U\langle 0\rangle^H \simeq \mathbb{Z}H^o \times BU \tag{5.26}$$

as G/H-spaces. □

Let $\mathbb{I}H^o$ denote the augmentation ideal in the group ring $\mathbb{Z}H^o$.

Proposition 24 *For each* $n \in \mathbb{N}$ *there is a short exact sequence of abelian groups*

$$* \to \mathbb{I}^{n+1}H^\circ \to \mathbb{I}^n H^\circ \to \mathrm{Sym}^n_{\mathbb{Z}}\, H^\circ \to * \tag{5.27}$$

where $\mathbb{I}^n H^\circ$ *is the image of the natural map*

$$\mathrm{Sym}^n_{\mathbb{Z}H^o}\, \mathbb{I}H^\circ \to \mathbb{I}^{n-1}H^\circ \tag{5.28}$$

from the n^{th} *symmetric power of the augmentation ideal* $\mathbb{I}H^o$ *over* $\mathbb{Z}H^o$ *to* $\mathbb{I}^{n-1}H^\circ$*.*

Proof. This is an easy consequence of the universal properties of the functors Sym^n, $\mathbb{I}$, $\mathbb{Z}$, and the image. □

Proposition 25 *For* $k \leq 2$*, the realization of the* $\mathrm{Orb}(G)$*-space*

$$\mathbb{I}^k H^o \times BU\langle 2k\rangle \tag{5.29}$$

is a model for $B_G U\langle 2k\rangle$.

Proof. We have already taken care of the case $k = 0$. For $k = 1$, it suffices to observe that $K_G(\mathbb{Z}, 0) \cong \mathbb{Z} \cong K(\mathbb{Z}, 0)$ and $\mathrm{Sym}^0_{\mathbb{Z}} H^\circ \cong \mathbb{Z}$. For $k = 2$, we have

$$K_G(\mathbb{Z}, 2)^H \simeq \mathrm{Map}(BH, B\mathbb{T}) \simeq \mathrm{Hom}(H, \mathbb{T}) \times B\mathbb{T} \simeq H^\circ \times K(\mathbb{Z}, 2), \tag{5.30}$$

and therefore a map of fiber sequences

$$\begin{tikzcd} B_G U\langle 4\rangle^H \arrow[r] \arrow[d] & B_G U\langle 2\rangle^H \arrow[r] \arrow[d] & K_G(\mathbb{Z}, 2) \arrow[d] \\ \mathbb{I}^2 H^\circ \times BU\langle 4\rangle \arrow[r] & \mathbb{I} H^\circ \times BU\langle 2\rangle \arrow[r] & H^\circ \times K(\mathbb{Z}, 2) \end{tikzcd} \tag{5.31}$$

such that the maps on the total spaces and base spaces are equivalences. Hence the fibers are equivalences as well. □

For $k = 3$, however, we get an extra factor of $K(H^\circ, 1)$ from the fibration

$$\mathbb{I}^3 H^\circ \times K(H^\circ, 1) \times BU\langle 6\rangle \to \mathbb{I}^2 H^\circ \times BU\langle 4\rangle \to \mathrm{Sym}^2_{\mathbb{Z}} H^o \times K(H^\circ, 2) \times K(\mathbb{Z}, 4), \tag{5.32}$$

where of course the base space is really $K_G(\mathbb{Z}, 4)^H$. But the map

$$K(H^o, 1) \times \mathbb{I}^3 H^\circ \times BU\langle 6\rangle \to \mathbb{I}^2 H^o \times BU\langle 4\rangle \tag{5.33}$$

clearly factors through the map

$$\mathbb{I}^3 H^\circ \times BU\langle 6\rangle \to \mathbb{I}^2 H^o \times BU\langle 4\rangle, \tag{5.34}$$

and this induces a similar factorization on Thom spaces.

It is useful to consider the group ring $\mathbb{Z}A$ and the powers $\mathbb{I}^k A$ of the augmentation ideal for an arbitrary abelian group object in a homotopy topos. Ideally, we wish to define objects $\mathbb{J}^n A$ representing the groups

$$\mathrm{Hom}(\mathbb{J}^n A, B) \simeq C^n(A, B) \tag{5.35}$$

of B-valued n-cocycles on A. We state some facts regarding these objects without proof.

Since $C^0(A, B) = \mathrm{Map}(A, B)$, we have that $\mathbb{J}^0 A \simeq \mathbb{Z}A$, the free abelian group object on A. Similarly, $C^1(A, B) = \mathrm{Map}_+(A, B)$, so $\mathbb{J}^1 A \simeq \mathbb{I}A$, the augmentation ideal in the group ring $\mathbb{Z}A$.

More generally, we have that

$$\mathbb{J}^n A \simeq \mathrm{Sym}^n_{\mathbb{Z}A} \mathbb{I}A, \tag{5.36}$$

the n^{th} symmetric power of the $\mathbb{Z}A$-module $\mathbb{I}A$. This naturally maps to $\mathbb{I}^n A$, the n^{th} power of the augmentation ideal $\mathbb{I}A$. We define $\mathbb{K}^n A$ to be the kernel of this map, giving a short exact sequence of functors

$$* \to \mathbb{K}^n \to \mathbb{J}^n \to \mathbb{I}^n \to * \tag{5.37}$$

of abelian group objects in $\mathcal{T}$.

Now if $\widehat{A} \to \operatorname{Spec} E$ is the derived formal group of an even-periodic $\mathbb{S}$-algebra E, the results of Ando-Hopkins-Strickland [4] imply that the derived formal group scheme Spf $E^{BU\langle 2n\rangle}$ (at least for $n \leq 3$) can be identified with the derived formal group scheme $\mathbb{J}^n \widehat{A}$. In particular, the cohomology of $BU\langle 2n\rangle$ represents the functor $C^n(\widehat{A}, (-))$, and they use this to show that the set of ring spectrum maps $\sigma : MU\langle 2n\rangle E$ is naturally isomorphic to the set of Θ^n-structures on the invertible $\mathcal{O}_{\widehat{A}}$-module $\widehat{\mathcal{I}(e)}$ (again for $n \leq 3$).

5.2 Orientable Complex Representations

5.2.1 Cohomology of representation spheres

Let $\rho : G \to \mathbb{T}$ be a one-dimensional complex representation of the compact abelian Lie group G. The representation sphere S^ρ is defined to be the cofiber of the sequence of pointed G-spaces

$$S(\rho)_+ \to *_+ \to S^\rho, \tag{5.38}$$

where the sphere $S(\rho)$ of ρ is just the circle $\mathbb{T}$ with G acting via ρ. Clearly this cofiber sequence is equivalent to the restriction to G of the cofiber sequence of $\mathbb{T}$-spaces

$$\mathbb{T}_+ \to *_+ \to S^{\mathbb{T}} \tag{5.39}$$

where $S^{\mathbb{T}}$ is the representation sphere associated to the identity representation $\mathbb{T} \to \mathbb{T}$. Since each of these $\mathbb{T}$-spaces are finite $\mathbb{T}$-cell complexes, it follows from the change-of-groups formula that

$$\rho^* \mathcal{O}^{\mathbb{T}}_A(S^{\mathbb{T}}) \simeq \mathcal{O}^G_A(\rho^* S^{\mathbb{T}}) \cong \mathcal{O}^G_A(S^\rho). \tag{5.40}$$

On the other hand, the cohomology of the cofiber sequence gives a fiber sequence of $\mathcal{O}_A^{\mathbb{T}}$-modules

$$\mathcal{O}_A^{\mathbb{T}}(S^{\mathbb{T}}) \to \mathcal{O}_A^{\mathbb{T}}(*_+) \to \mathcal{O}_A^{\mathbb{T}}(\mathbb{T}_+); \tag{5.41}$$

since $\mathcal{O}_A^{\mathbb{T}}(*_+) \simeq \mathcal{O}_A$ and $\mathcal{O}_A^{\mathbb{T}}(\mathbb{T}_+) \simeq e_* \mathcal{O}_{\operatorname{Spec} E}$, we see that $\mathcal{O}_A^{\mathbb{T}}(S^{\mathbb{T}})$ is equivalent to the ideal sheaf $\mathcal{I}_A(e)$ of the identity section $e : \operatorname{Spec} E \to A$.

Proposition 26 *Let $\rho : G \to \mathbb{T}$ be an irreducible complex representation of the compact abelian Lie group G. Then*

$$\mathcal{O}_A^G(S^\rho) \simeq \rho^* \mathcal{I}_A(e) \tag{5.42}$$

as coherent $\mathcal{O}_A^G$-modules. □

Since $\mathcal{O}_A^G$ is a tensor-triangulated functor, this determines the cohomology of S^ρ for any virtual complex representation ρ of G.

Corollary 6 *Let $\rho \in \mathbb{Z}G^\circ$ be a virtual complex representation of G, say*

$$\rho = \sum_{\alpha \in G^\circ} n_\alpha \cdot \alpha. \tag{5.43}$$

Then

$$\mathcal{O}_A^G(S^\rho) \simeq \bigotimes_\alpha \alpha^* \mathcal{I}_A(e)^{\otimes n_\alpha} \tag{5.44}$$

as coherent $\mathcal{O}_A^G$-modules.

Proof. The natural map $\bigwedge_\alpha S^{n_\alpha \cdot \alpha} \to S^\rho$ is an equivalence of finite G-spectra. □

5.2.2 The Thom space of a trivial bundle

The representation spheres are the simplest case of equivariant Thom spaces. Slightly more complicated are those arising from equivariant bundles over G-orbits.

Let $i : H \to G$ be the inclusion of a closed subgroup. We know that isomorphism classes of G-equivariant vector bundles on G/H correspond to representations of H. Thus if

$$\rho : H \to \mathbb{T} \tag{5.45}$$

is a one-dimensional complex representation of H, the cofiber sequence

$$S(\rho)_+ \to *_+ \to S^\rho \tag{5.46}$$

of H-spaces induces a cofiber sequence

$$i_! S(\rho)_+ \to i_! *_+ \to i_! S^\rho \tag{5.47}$$

of G-spaces such that $i_! S^\rho \cong S^\rho \wedge_H G_+$ is the Thom space of the trivial bundle $\mathbb{C} \times_H G$ over $* \times_H G = G/H$.

By the induction axiom for the inclusion of H into G it follows that $\mathcal{O}_A^G(i_! S^\rho)$ is equivalent to the fiber in the fiber sequence

$$i_* \mathcal{O}_A^H(S^\rho) \to i_* \mathcal{O}_A^H(*_+) \to i_* \mathcal{O}_A^H(S(\rho)_+). \tag{5.48}$$

But $\mathcal{O}_A^H(S^\rho) \simeq \rho^* \mathcal{I}_A(e)$, so we obtain the formula

$$\mathcal{O}_A^G(i_! S^\rho) \simeq i_* \rho^* \mathcal{I}_A(e). \tag{5.49}$$

Let $j : \mathbb{T} \to \mathbb{T} \times_H G$ be the cobase change of i along ρ and let $\sigma : G \to \mathbb{T} \times_H G$ be the induced homomorphism.

Proposition 27 *There is a natural equivalence*

$$\sigma^* j_* \mathcal{I}_A(e) \to i_* \rho^* \mathcal{I}_A(e) \tag{5.50}$$

of coherent $\mathcal{O}_A^G$-modules.

Proof. This is a formal consequence of the fact that the natural transformation

$$j_! \circ \rho^* \implies \sigma^* \circ i_! \tag{5.51}$$

of functors $\mathcal{S}_+^{\mathbb{T}} \to \mathcal{S}_+^G$ is an equivalence. Certainly they agree on underlying spaces, as they are both given by the formula $(-) \wedge_H G_+$, so it suffices to check that the G-actions agree. But this is clear since the action of G on the $\mathbb{T} \times_H G$-space $j_! X$ is induced via i from the action of H. □

It follows that the cohomology of the Thom space of a G-equivariant virtual vector bundle over an orbit G/H is an invertible $\mathcal{O}_A^G(G/H)$-module.

5.2.3 The local Thom isomorphism

In this section we show that the cohomology of any G-equivariant complex vector bundle is an invertible module over the cohomology of the base. In other words, there is always a local Thom isomorphism in a suitable sense.

Theorem 12 *Let $p : V \to X$ be a G-equivariant complex vector bundle over a G-space X. Then $\mathcal{O}_A^G(X^V)$ is an invertible $\mathcal{O}_A^G(X_+)$-module.*

Proof. Let $Y \to X$ be a trivializing cover for V; in other words, X is the realization of the simplicial G-space

$$Y_n := Y \times_X \times \cdots \times_X Y \tag{5.52}$$

and for each n, the pullback

$$W_n := V \times_X Y_n \tag{5.53}$$

is a trivial G-equivariant vector bundle.

Similarly, for each n, we have Thom spaces $Y_n^{W_n}$, and the Thom space X^V is the realization of the simplicial Thom space $Y_n^{W_n}$. Likewise, in cohomology, $X \otimes_G A$ is the realization of the simplicial A_G-scheme

$$Y_n \otimes_G A \simeq Y \otimes_G A \times_{X \otimes_G A} \times \cdots \times_{X \otimes_G A} Y \otimes_G A, \tag{5.54}$$

the equivalence following from the fact that $(-) \otimes_G A$, as a functor from G-spaces to A_G-schemes, commutes with homotopy pullbacks.

Since $Y \to X$ is a covering map and the functor $(-) \otimes_G A$ preserves covers, the map $Y \otimes_G A \to X \otimes_G A$ is a covering map and is therefore faithfully flat. It follows from the theory of faithfully flat descent that the simplicial line bundle over the simplicial scheme $Y_n \otimes_G A$ associated to the cohomology of the simplicial Thom space $Y_n^{W_n}$ descends to a line bundle on $X \otimes_G A$. By exactness, this line bundle is necessarily equivalent to the scheme corresponding to the coherent $\mathcal{O}_A^G(X_+)$-module $\mathrm{Sym}_{\mathcal{O}_A^G(X_+)} \mathcal{O}_A^G(X^V)$. Hence $\mathcal{O}_A^G(X^V)$ is an invertible $\mathcal{O}_A^G(X_+)$-module. □

Chapter 6

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