

The Witten genus

M^k Riemannian spin manifold, $D =$ Dirac operator, $T =$ tangent bundle

Witten genus

$$w(M) = \text{ind} \left(D \otimes \bigotimes_{n \geq 1} S_{q^n}(T^{\otimes -1}) \right)$$

where

$$S_t(V) = \sum_{k \geq 0} t^k \cdot S^k(V) \in \mathbb{Z}[[q]]$$

note: $S_t(V)^{-1} = \Lambda_{-t}(V)$

Characteristic series

$$\sigma(L, q) = (L^{1/2} - L^{-1/2}) \prod_{n \geq 1} \frac{(1 - q^n L)(1 - q^n L^{-1})}{(1 - q^n)^2}$$

For $u = e^{2\pi i \tau}$
 $q = e^{2\pi i \sigma}$

$$\sigma(u, q) = (u^{1/2} - u^{-1/2}) \prod_{n \geq 1} \frac{(1 - q^n u)(1 - q^n u^{-1})}{(1 - q^n)^2}$$

is essentially the Weierstrass σ -function for the lattice $\Lambda = 2\pi i \mathbb{Z} + 2\pi i \sigma \mathbb{Z}$. I.e. σ defines a holomorphic function f on \mathbb{C} which vanishes exactly on Λ .

Given points on the elliptic curve $C = \mathbb{C}/\Lambda$.

$P_1, \dots, P_n, Q_1, \dots, Q_n \in C$ represented by

$\bar{P}_1, \dots, \bar{P}_n, \bar{Q}_1, \dots, \bar{Q}_n \in \mathbb{C}$.

Suppose $0 = \sum P_i - \sum Q_i \Leftrightarrow \sum \bar{P}_i - \sum \bar{Q}_i \in \Lambda$

Then there exists a function, unique up to scalar multiple, f on C with divisor

$\sum (P_i) - \sum (Q_i)$, namely

$$\prod_{i=1}^n \frac{\sigma(z - \bar{P}_i)}{\sigma(z - \bar{Q}_i)}$$

Thm (Witten, Zagier) : If $\hat{P}_2(M) = 0$, then $w(M)$ is the q -expansion of a modular form for $SL_2(\mathbb{Z})$.

Def An elliptic spectrum consists of

E : commutative ring spectrum, even periodic

C : elliptic curve over $\pi_0 E$

Then $E^0(\mathbb{C}P^\infty)$ is the rep of points on a formal group G_E over $\pi_0 E$

$t : G_E \cong \hat{C}$ an iso of formal groups over $\pi_0 E$.

Thm (Ando-Hopkins-Strickland) If (E, C, t) is an elliptic spectrum,

then there is a canonical map

$$\sigma(E, C, t) : MV\langle 6 \rangle \rightarrow E$$

natural in (E, C, t) .

Example : Tate elliptic spectrum

$$\begin{array}{ccc} \hat{G}_m & \xrightarrow[\text{can.}]{\cong} & \hat{\text{Tate}}(\mathbb{C}P^\infty) \longrightarrow \text{Tate} \\ & & \downarrow \swarrow \\ & & \mathbb{Z}\langle q \rangle \end{array}$$

$$\text{Tate} = \text{Tate curve} / \mathbb{Z}\langle q \rangle$$

So get elliptic spectrum $K_{\text{Tate}} = (K\langle q \rangle, \text{Tate}, \text{can.})$.

Moreover, the diagram commutes

$$\begin{array}{ccc} MV\langle 6 \rangle & \xrightarrow{\sigma(K_{\text{Tate}})} & K\langle q \rangle \\ \downarrow & & \uparrow w \\ MSU & \longrightarrow & MSpin \end{array}$$

Now suppose that E is rational, i.e. there is a map $H\mathbb{Q} \xrightarrow{\cong} E$

We also have $MU\langle 6 \rangle \rightarrow MU \xrightarrow{t} E$, so we get ratio

$$\frac{\cong}{t} : \Sigma^\infty BU\langle 6 \rangle_+ \rightarrow E \text{ map of ring spectra.}$$

$$(\mathbb{C}P^\infty)^k \xrightarrow{\Pi(1-L_i)} \Sigma^\infty BU\langle 2k \rangle_+ \rightarrow E$$

corresponds to $f \in E^0(\mathbb{C}P^\infty)^k \cong E^0(\mathbb{Z}_{21}, \dots, \mathbb{Z}_{2k})$

Then it is necessarily the case:

$$1) f(0, \dots, 0) = 1 \quad 2) f(\sigma(0), \dots, \sigma(n)) = f(z_1, \dots, z_n) \text{ for } \sigma \in \Sigma_k$$

$$3) \frac{f(z_1, \dots, z_n)}{f(z_1+z_2, z_2, \dots, z_n)} \cdot \frac{f(z_0, z_1+z_2, \dots, z_n)}{f(z_0, z_1, z_3, \dots, z_n)} = 1$$

Thm For $k \leq 3$,

Ring Spectra $(\Sigma^\infty BU\langle 2k \rangle_+, E)$

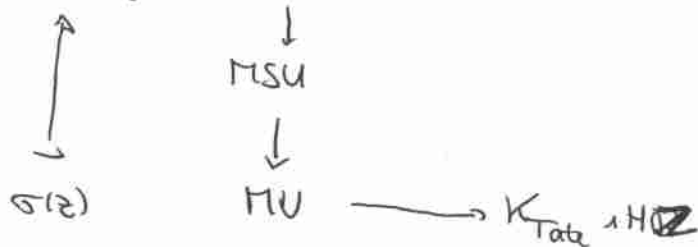
$$\cong \text{Rings } (E \otimes BU\langle 2k \rangle_+, E_0)$$

$$\cong \{f \text{ satisfying } 1), 2), 3)\}$$

The natural maps $\Sigma^\infty BU\langle 2k+2 \rangle_+ \rightarrow \Sigma^\infty BU\langle 2k \rangle_+$ correspond by precomposition to

$$f \mapsto (\sigma f)(z_1, \dots, z_{k+1}) = \frac{f(z_1, z_3, \dots, z_{k+2}) \cdot f(z_2, z_3, \dots, z_{k+2})}{f(z_1+z_2, z_3, \dots, z_{k+2})}$$

So

$$\frac{\sigma(x+y, z) \sigma(x) \sigma(y) \sigma(z)}{\sigma(x+y) \sigma(x+z) \sigma(y+z)} MU\langle 6 \rangle$$


If C is an elliptic curve on $x, y \in \mathbb{C}$, then there exists a function f on C with divisor

$$(0) + (-x-y) - (-x) - (-y), \text{ namely the above}$$

expression in $\mathcal{O}(Ary+z) / \dots$

Suppose given a rig map $MU \langle 2k \rangle \rightarrow E$

and the standard map $\mathcal{O}MU \langle 2k \rangle \rightarrow H\mathbb{Q} \rightarrow E$ (since E is rational).

$$\left(\frac{f}{z}\right)^{\pm 1} : \Sigma^{\infty} BU \langle 6 \rangle_+ \rightarrow E.$$

What is the effect on π_{2n} of the adjoint $BU \langle 6 \rangle \rightarrow \mathcal{O}E$?

Write $\frac{f}{z} \Leftrightarrow g(z_1, \dots, z_k) = \delta^{k+1} h$ with

$$h(z) = \exp\left(\sum_{n=1}^{\infty} \frac{g_n}{n} z^n\right)$$

Proposition $\pi_{2n}\left(\frac{f}{z}\right) : \pi_{2n} BU \langle 6 \rangle \rightarrow \pi_{2n} E$
is multiplication by $-\frac{g_n}{n} \cdot n!$

Proof: $\delta^{n-2}(h) = \exp\left(-\sum \frac{g_n}{n} \left(\binom{n-1}{k} z_1^{k+1} \dots z_n^{k+1} - \dots\right)\right)$
Set $z_1 = \dots = z_n$ (acc.) \square (???)

Note $\frac{\sigma(z)}{z} = \exp\left(-\sum_{k=2}^{\infty} \frac{G_{2k}}{2k} z^{2k}\right)$

where G_{2k} = Eisenstein series
 $= z \zeta(2k) E_{2k}$

$$\frac{e^{z/2} - e^{-z/2}}{z} = \exp\left(\sum_{k=2}^{\infty} \frac{1}{2k} \frac{B_{2k}}{(2k)!} z^{2k}\right)$$

This implies that $MU \rightarrow \pi_{Spin} \rightarrow K$ adjoint to

~~BSU_+~~ $\rightarrow K$ is multiplication by $\frac{B_{2n}}{2n}$ in π_{2n}

Eoo-orientations

Aim: Construct an orientation, i.e. map of ring spectra

$$\begin{array}{ccc} \text{MStr.} & \longrightarrow & \text{tmf} \\ \parallel & & \nearrow \\ \text{Ho} \langle 8 \rangle & & \end{array}$$

Both are Eoo ring spectra, and it is most convenient to construct a morphism of Eoo ring spectra. For such maps there is an obstruction theory by May, Quinn and Ray. I will present this obstruction theory in a modern form.

$V \downarrow X$ vector bundle, E ring spectrum.

Get ring spectrum E^{X+} (fraction spectrum) and a module spectrum over this E^{X^V} (where $X^V = \text{Thom space of } V \downarrow X$).

The module structure comes from the relative diagonal $X^V \rightarrow X_{+1} \times X^V$.

A Thom class is a map $X^V \rightarrow E$ such that the adjoint $S^0 \rightarrow E^{X^V}$ makes this a free module over E^{X+} .

Have two presheaves

$$\underline{E}(U \subseteq X \text{ open}) = E^{U+}$$

$$\underline{E}_V(U) = E^{U^V} = E^{\text{Thom}(V \text{ restricted to } U)}$$

\underline{E}_V is a presheaf of \underline{E} -modules which is locally free of rank 1.

The transition functions give a class

$$[V, E] \in H^1(X, \underline{E}^X)$$

and V has a Thom-isomorphism in E -theory $\Leftrightarrow [V, E]$ is the trivial cohomology class.

Units of an A_{∞} -ring spectrum defined by pullback

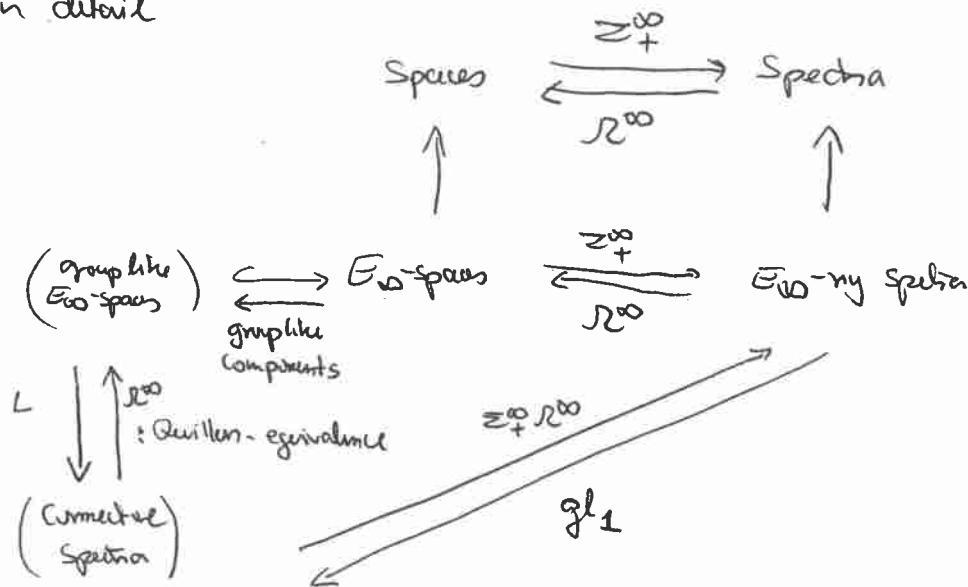
$$GL_1 E \longrightarrow \mathcal{R}^{\infty} E$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ (\pi_0 E)^{\times} & \longrightarrow & \pi_0 E \end{array}$$

If E is A_{∞} , $GL_1 E$ has a classifying space.

If E is A_{∞} , $GL_1 E$ is an infinite loop space.

In detail



Example: $E = S^0$, $GL_1(S^0) = G$ such that

$BGL_1(S^0) =$ classifying space for stable spherical fibration

Consider a map

gives rise to

$$bo \langle 8 \rangle = b. \xrightarrow{\xi} bgl_1(S) = \Sigma gl_1(S)$$

$$\begin{array}{ccc} & & GL_1 S^0 \\ & & \downarrow \\ P \lrcorner & \longrightarrow & E GL_1 S^0 \\ \downarrow & & \downarrow \\ B & \xrightarrow[\text{E}_0 \text{ map}]{\xi} & BGL_1(S^0) \end{array}$$

Give pushout diagram of E_{∞} ring spectra

$$\begin{array}{ccc} \Sigma_+^{\infty} \mathcal{R}^{\infty} gl_2(S^0) & \longrightarrow & S^0 \\ \downarrow & & \downarrow \\ \Sigma_+^{\infty} \mathcal{R}^{\infty} C & \longrightarrow & Thom(\xi) \end{array}$$

where C is the pullback / homotopy fibre in spectra

$$\begin{array}{ccc} C & \longrightarrow & * \\ \downarrow & & \downarrow \\ b & \longrightarrow & bgl_2 S^0 \end{array}$$

From universal property of pushout and adjunctions we get for any E_{∞} ring spectrum

$$\pi_* E_{\infty}(Thom(\xi), R) = \pi_* \text{Spectra} \left(\begin{array}{ccc} gl_2 S^0 & \xrightarrow{gl_2(i)} & gl_2 R \\ \downarrow & \nearrow & \\ C & & \end{array} \right)$$

In particular, the obstruction is a map

$$b \longrightarrow bgl_2 S^0 \xrightarrow{bgl_2(i)} bgl_2 R$$

Example: $b = b_0$, then $\mathcal{R}^{\infty} C = G/O$ and May-Quinn-Ray write the Thom space as

$$Thom(\xi) = B(\Sigma_+^{\infty} G/O, \Sigma_+^{\infty} G, S^0)$$

Now $b = b_0 \langle \mathcal{R} \rangle$, so we study $\pi_0 \langle \mathcal{R} \rangle$ -orientations. Suppose p is an odd prime.

$$\begin{array}{ccccccc} \Sigma^{-2} b_0 \langle \mathcal{R} \rangle \xrightarrow{\psi^2} \Sigma^{-4} b_0 \langle \mathcal{R} \rangle & \longrightarrow & j_* C & \longrightarrow & gl_2 S & \longrightarrow & gl_2 R \\ & & \downarrow & & \downarrow & \nearrow & \\ & & b_0 \langle \mathcal{R} \rangle & \longrightarrow & C & & \end{array}$$

So the problem is equivalent to

$$\begin{array}{ccc}
 \mathbb{Z}^{-2} \text{bo}\langle \mathcal{R} \rangle & \cong & \mathbb{Z}^{-2} \text{bo}\langle \mathcal{R} \rangle \\
 \downarrow & & \downarrow \\
 j & \longrightarrow & \text{gl}_1 S \xrightarrow{\text{gl}_2(i)} \text{gl}_1 R \\
 \downarrow & & \downarrow \\
 \text{bo}\langle \mathcal{R} \rangle & \longrightarrow & \mathbb{C} \text{ --- } \overset{?}{\dashrightarrow} \text{?}
 \end{array}$$

So extension to \mathbb{C} compared to extensions $\text{bo}\langle \mathcal{R} \rangle \rightarrow \text{gl}_1 R$ restricts correctly to j .

For vector bundle $V \rightarrow X$, E^{X^V} is a twisted form of $E(X_+)$.

The twists are classified by $H^1(X, E^X) := \{X \rightarrow \text{BGL}_2(E)\}$

Suppose E is Ave (but not nec. E_∞). Have principal fibrations, pull back

$$\begin{array}{ccc}
 & \text{GL}_2 E & \longrightarrow & \text{GL}_1 E \\
 & \downarrow & & \downarrow \\
 \text{this is an} & & & \\
 \text{Ave-GL}_1 E\text{-space,} & \longrightarrow & \text{P}_f & \longrightarrow & E\text{GL}_1 E \\
 & & \downarrow & & \downarrow \\
 \text{so } \mathbb{Z}_+^\infty \text{P}_f \text{ is module over} & & X & \longrightarrow & \text{BGL}_2 E \\
 \mathbb{Z}_+^\infty \text{GL}_2 E & & & & \{
 \end{array}$$

$\text{Thom}(\xi)$ is the homotopy cogenerator in right E -modules

$$\mathbb{Z}_+^\infty \text{P}_f \wedge \mathbb{Z}_+^\infty \text{GL}_2 E \wedge E \implies \mathbb{Z}_+^\infty \text{P}_f \wedge E \longrightarrow X^\xi$$

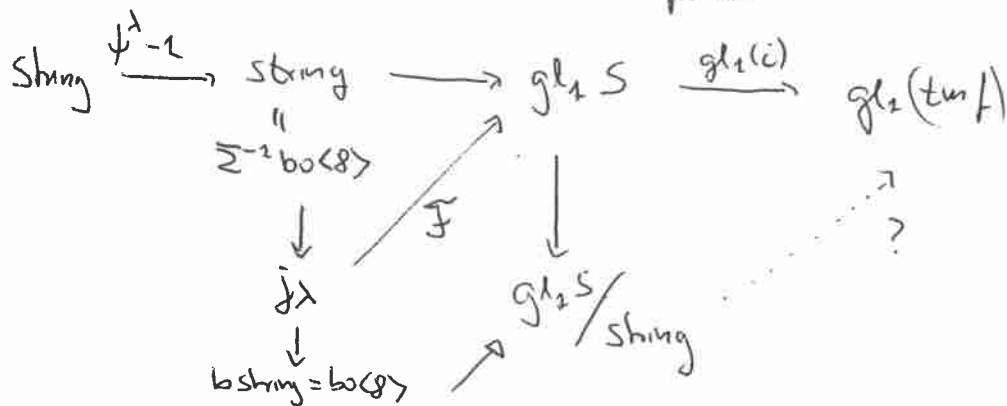
Twisted cohomology: $E_f^a X = \pi_{a*}(E\text{-mod}(X^f, E))$

$$E_*^f X = \pi_* (E\text{-mod}(E, X^f)) = \pi_* \text{Spectra}(S^0, X^f)$$

Local string orientations

$$MString \longrightarrow tmf$$

Obstruction to orientation is the composite

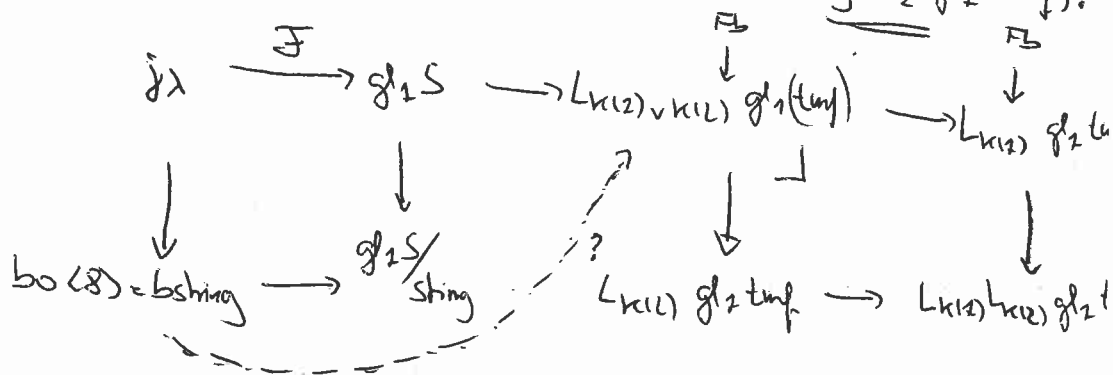


$$\begin{aligned} \pi_* E_{\infty}(MString, tmf) &= \pi_* gl_2 S / Spreda (gl_2 S / string, gl_2 tmf) \\ &= \pi_* j\lambda / Sp (bostring, gl_2 tmf) \end{aligned}$$

Lemma:

$$\pi_i (\text{fiber } (gl_2 tmf \rightarrow L_2 gl_2 tmf)) = 0 \text{ for } i > 3$$

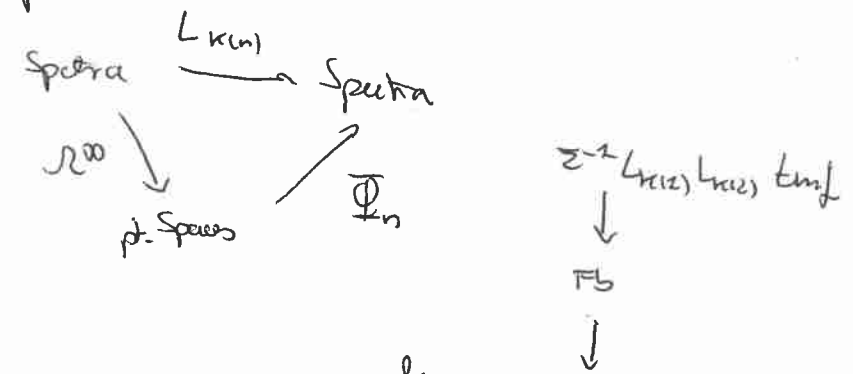
So for the obstruction problem we can replace $gl_2(tmf)$ by $L_2 gl_2(tmf)$.



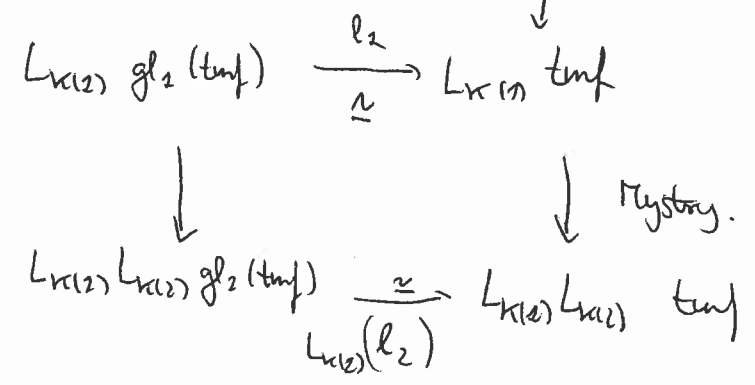
Since $K(2)_* b\mathbb{O}\langle 8 \rangle = 0$, there are no maps from $b\mathbb{O}\langle 8 \rangle$ to $L_{K(2)} gl_2 tmf$.
 So we may need a map into $L_{K(2)} gl_2 tmf$. Now we'll use the
 logarithm

$$l_1: L_{K(2)} gl_2 (tmf) \xrightarrow{\cong} L_{K(1)} tmf$$

Recall Bousfield-Kuhn functor



So we can exploit:



Thus

$$\pi_0 E_{\infty}(\text{Mystery}, tmf) \xrightarrow[\text{iso away from 2}]{\text{onto}} \dots$$

$\exists f \in \ker(\bar{c}\mathbb{O}\langle 8 \rangle, L_{K(2)} tmf) \rightarrow [b\mathbb{O}\langle 8 \rangle, L_{K(1)} L_{K(2)} tmf]$ such that
 \exists factors above
 $\Rightarrow \exists f \in \ker([KO^1_p, L_{K(2)} tmf] \rightarrow [KO^1_p, L_{K(2)} L_{K(2)} tmf])$ such that...
 diagram

Lemma: $\pi_{\mathbb{F}_2}(L_{K(2)} tmf) /_{KO-2\text{-torsion}} = \text{ring of p-adic modular forms}$

p-adic modular form: $f = \sum a_n q^n$ with $a_n \in \mathbb{Q}_p$
 $= \lim_i f_i$ with $f_i \in \mathbb{F}_2$.

(Recall that $\pi_{2n} L_{K(12)}$ is related to a completion of $\Gamma(\omega^{\otimes n}$ on elliptic curves / adicity an iso $\hat{C} \cong \hat{G}_m$)
 p-adic rings

Katz identified these completed sections with p-adic modular forms.

The kernel of the above surjection is 2-torsion.

Condition on the above f 's is that they make the following diagram commute

$$\begin{array}{ccccccc}
 L_{K(12)} \hat{\mathcal{A}} & \longrightarrow & L_{K(12)} \mathfrak{gl}_2 S & \longrightarrow & L_{K(12)} \mathfrak{gl}_2 \text{tmf} & \xrightarrow{l_2} & L_{K(12)} \text{tmf} \\
 \downarrow & & & & & & \downarrow \\
 KO_p^1 & \dashrightarrow & \dashrightarrow & \dashrightarrow & \dashrightarrow & \dashrightarrow & \dashrightarrow
 \end{array}$$

Define p-adic modular forms z_n by

$$z_n = \pi_{2n}(f)(u^n) \in \pi_{2n} L_{K(12)} \text{tmf}$$

Then: 0) z_n is a p-adic modular form of wt n . for $n \geq n_0$, new

1) z_n satisfy the "fancy Kummer congruences"

$$2) z_0 = \lim_{k \rightarrow \infty} z_{(p-1)p^k} \stackrel{?}{=} \alpha(x)$$

3) $f \in \ker [KO_p^1, \text{mystery map}]$

σ -orientation $\sigma: MU\langle 6 \rangle \longrightarrow \text{any elliptic spectrum}$

$$\frac{\sigma(t)}{t} = \exp\left(-\sum_{n \geq 4} \frac{G_n}{n!} t^n\right) \cdot \exp(\text{quadratic in } t)$$

ratio $\Leftrightarrow Z^{\infty} BU\langle 6 \rangle_+ \rightarrow E$

with $G_n = -\frac{B_n}{2n} + \sum_{r=1}^{\infty} \sigma_{n-2}(r) q^r$

The effect of the associated map $BU\langle 6 \rangle \rightarrow \text{supp}(\pi_{K(12)} \text{tmf} \wedge HD)$ is

$$z_n := \left(1 - \frac{1}{p} t\right) \cdot G_n \cdot (1 - t^n)$$

Now the question is if the sequence of Z_n 's satisfy the conditions 0) - 3) ?

Then they define a special Shrij operation !

Here ψ is a power operation which is "multiplication by p on the formal exp".

If f is given by $f(q) = \sum a_r q^r \rightsquigarrow \psi f(q) = (\sum a_r q^r) \left(\frac{du}{u}\right)^n$

$$\begin{aligned} \text{then} \\ \psi f(q) &= \left(\sum_r a_r q^{pr} \right) \left(\frac{du^p}{u^p} \right)^n \\ &= \sum a_r q^{pr} \cdot p^n \left(\frac{du}{u} \right)^n \end{aligned}$$

$$\text{Thus } \left(1 - \frac{1}{p} \psi\right) f(q) = f(q) - p^{n-1} f(q^p).$$

$$(1 - \lambda^n) (G_n(q) - p^{n-1} G_n(q^p)) = (1 - \lambda^n) \underbrace{(1 - p^{n-1} V)}_{G_n^*} G_n$$

Some proved the conditions 0) - 3) for the sequence G_n^* .