Matthew Ando
13.10.03

The Witten genus

M^k Riemannian Spin manifold, D = Dirac operator, T = tangent bundle

Witten genus

$$w(M) = \operatorname{ind}\left(D \otimes \bigotimes_{n\geq 1} S_{q^n}(T^{\mathbb{C}} - \underline{r})\right)$$

where

$$S_t(V) = \sum_{k\geq 0} t^k \cdot S^k(V) \qquad \in \mathbb{Z}[\![q]\!]$$

note: $\quad S_t(V)^{-1} = \Lambda_{-t}(V)$

Characteristic series

$$\sigma(L,q) = (L^{1/2} - L^{-1/2}) \prod_{n\geq 1} \frac{(1-q^n L)(1-q^n L^{-1})}{(1-q^n)^2}$$

For $\left.\begin{array}{l} u = e^{2\pi i z} \\ q = e^{2\pi i \tau}\end{array}\right\}$ $\quad \sigma(u,q) = (u^{1/2} - u^{-1/2}) \prod_{n\geq 1} \frac{(1-q^n u)(1-q^n u^{-1})}{(1-q^n)^2}$

is essentially the Weierstrass σ-function for the lattice $\Lambda = 2\pi i \mathbb{Z} + 2\pi i \tau \mathbb{Z}$. I.e. σ defines a holomorphic function of $z \in \mathbb{C}$ which vanishes exactly on Λ.

Given points on the elliptic curve $C = \mathbb{C}/\Lambda$.

$P_1, \dots, P_n, Q_1, \dots, Q_n \in C$ represented by

$\tilde{P}_1, \dots, \tilde{P}_n, \tilde{Q}_1, \dots, \tilde{Q}_n \in \mathbb{C}$.

Suppose $\quad 0 = \sum P_i - \sum Q_i \iff \sum \tilde{P}_i - \sum \tilde{Q}_i \in \Lambda$

Then there exists a function, unique up to scalar multiple, f on C with divisor

$\sum (P_i) - \sum (Q_i)$, namely

$$\prod_{i=1}^{n} \frac{\sigma(z - \tilde{P}_i)}{\sigma(z - \tilde{Q}_i)}$$

Thm (Witten, Zagier). If $\frac{p_1}{2}(M)=0$, then $w(M)$ is the q-expansion of a modular form for $SL_2(\mathbb{Z})$.

Def An elliptic spectrum consists of

E : commutative ring spectrum, even periodic

C : elliptic curve over $\pi_0 E$

Then $E^0(\mathbb{C}P^\infty)$ is the ring of functions on a formal group G_E over $\pi_0 E$

$t : G_E \cong \hat{C}$ an iso of formal groups over $\pi_0 E$.

Thm (Ando-Hopkins-Strickland) If (E, C, t) is an elliptic spectrum, then there is a canonical map

$$\sigma(E, C, t) : MU\langle 6\rangle \longrightarrow E$$

natural in (E, C, t).

Example : Tate elliptic spectrum

$$\begin{tikzcd} \hat{\mathbb{G}}_m \arrow[r, phantom, "\underset{can.}{\cong}"] & \widehat{Tate} \arrow[r] \arrow[d] & Tate \arrow[dl] \\ & \mathbb{Z}[[q]] & \end{tikzcd}$$

Tate = Tate curve$/\mathbb{Z}[[q]]$

So get elliptic spectrum $K_{Tate} = (K[[q]], Tate, can.)$.

Moreover, the diagram commutes

$$\begin{tikzcd} MU\langle 6\rangle \arrow[r, "\sigma(K_{Tate})"] \arrow[d] & K[[q]] \\ MSU \arrow[r] & MSpin \arrow[u, "w"'] \end{tikzcd}$$

Now suppose that E is rational, i.e. there is a map $HQ \xrightarrow{z} E$.

We also have

$$MU\langle 6\rangle \to MU \xrightarrow{t} E\text{, so we get ratio}$$

$$\frac{z}{t} : \Sigma^\infty BU\langle 6\rangle_+ \longrightarrow E \quad \text{map of ring spectra.}$$

$$(\mathbb{C}P^\infty)^k \xrightarrow{\prod(1-L_i)} \Sigma^\infty BU\langle 2k\rangle_+ \longrightarrow E$$

corresponds to $\quad f \in E^0((\mathbb{C}P^\infty)^k) \cong E^0[\![z_1, \dots, z_k]\!]$

Then it is necessarily the case:

1) $f(0, \dots, 0) = 1$ $\qquad$ 2) $f(z_{\sigma(1)}, \dots, z_{\sigma(k)}) = f(z_1, \dots, z_k)$ for $\sigma \in \Sigma_k$

3) $$\frac{f(z_1, \dots, z_k)}{f(z_0 + z_1, z_2, \dots, z_k)} \cdot \frac{f(z_0, z_1 + z_2, \dots, z_k)}{f(z_0, z_1, z_3, \dots, z_k)} = 1 .$$

<u>Thm</u> For $k \le 3$,

$$\begin{aligned} &\mathrm{Ring\ Spectra}(\Sigma^\infty BU\langle 2k\rangle_+, E) \\ &\cong \mathrm{Rings}(E_* BU\langle 2k\rangle_+, E_0) \\ &\cong \{ f \text{ satisfying } 1), 2), 3) \} \end{aligned}$$

The natural maps $\Sigma^\infty BU\langle 2k+2\rangle_+ \to \Sigma^\infty BU\langle 2k\rangle_+$ correspond by precomposition to

$$f \longmapsto (\delta f)(z_1, \dots, z_{k+1}) = \frac{f(z_1, z_3, \dots, z_{k+1}) \cdot f(z_2, z_3, \dots, z_{k+1})}{f(z_1 + z_2, z_3, \dots, z_{k+1})}$$

So

$$\begin{tikzcd} \dfrac{\sigma(x+y+z)\sigma(x)\sigma(y)\sigma(z)}{\sigma(x+y)\sigma(x+z)\sigma(y+z)} & MU\langle 6\rangle \arrow[d] & \\ & MSU \arrow[d] & \\ \sigma(z) \arrow[uu, mapsto] & MU \arrow[r] & K_{Tate}, H\mathbb{Z} \end{tikzcd}$$

If C is an elliptic curve on $x, y \in C$, then there exists a function f on C with divisor

$$(0) + (-x-y) - (-x) - (-y), \text{ namely the above}$$

expression in $\sigma(x+y+z)$, ...

Suppose given a rig map $MU\langle 2k\rangle \longrightarrow E$

and the standard map $MU\langle 2k\rangle \to HQ \to E$ (since E is rational).

$$\left(\frac{f}{\frac{1}{z}}\right)^{\pm 1} : \Sigma^\infty BU\langle 6\rangle_+ \longrightarrow E.$$

What is the effect on π_{2n} of the adjoint $BU\langle 6\rangle \to \Omega^\infty E$?

Write $\frac{f}{\frac{1}{z}} \Leftrightarrow g(z_1, \ldots, z_k) = \delta^{k+1} h$ with

$$h(z) = \exp\left(\sum_{n=1}^{\infty} \frac{g_n}{n} z^n\right)$$

Proposition

$$\pi_{2n}\left(\tfrac{f}{\frac{1}{z}}\right) : \pi_{2n} BU\langle 6\rangle \longrightarrow \pi_{2n} E$$

is multiplication by $-\frac{g_n}{n} \cdot n!$

Proof:

$$\delta^{n-2}(h) = \exp\left(-\sum \frac{g_n}{n}\left((z_1 + \cdots + z_n) - \left\{\begin{matrix} n-1 \\ \text{terms}\end{matrix}\right\} + \left\{\begin{matrix} n-2 \\ \text{terms}\end{matrix}\right\} \ldots\right)\right)$$

Set $z_1 = \ldots = z_n$ $\qquad (\ldots)$ $\qquad$ (???)

$\square$

Note
$$\frac{\sigma(z)}{z} = \exp\left(-\sum_{k=2}^{\infty} \frac{G_{2k}}{2k} z^{2k}\right)$$

where G_{2k} = Eisenstein series
$= 2\zeta(2k) E_{2k}$.

$$\frac{e^{z/2} - e^{-z/2}}{z} = \exp\left(\sum_{k\geq 1} \frac{1}{2k} \frac{B_{2k}}{(2k)!} z^{2k}\right)$$

This implies that $MU \to MSpin \to K$ adjoint to

$BSU_+ \to K$ is multiplication by $\frac{B_{2n}}{2n}$ in π_{2n}

14. 10. 03
Matthew Ando

E_∞-orientations

Aim: Construct an orientation, i.e. map of ring spectra

$$\begin{tikzcd} MString \arrow[r] \arrow[d, equal] & tmf \\ MO\langle 8\rangle \arrow[ur] & \end{tikzcd}$$

Both are E_∞ ring spectra, and it is most convenient to construct a morphism of E_∞ ring spectra. For such maps there is an obstruction theory by May, Quinn and Ray. I will present this obstruction theory in a modern form.

$V \downarrow X$ vector bundle, E ring spectrum.

Get ring spectrum E^{X_+} (function spectrum) and a module spectrum over this $E^{(X^V)}$ (where X^V = Thom space of $V \downarrow X$).

The module structure comes from the relative diagonal $X^V \to X_+ \wedge X^V$.

A <u>Thom class</u> is a map $X^V \to E$ such that the adjoint $S^0 \to E^{X^V}$ makes this a free module over E^{X_+}.

Have two presheaves

$$\underline{E}(U \subseteq X \text{ open}) = E^{U_+}$$

$$\underline{E}_V(U) = E^{U^V} = E^{\text{Thom}(V \text{ restricted to } U)}$$

$\underline{E}_V$ is a presheaf of $\underline{E}$-modules which is locally free of rank 1.

The transition functions give a class

$$[V, E] \in H^1(X, \underline{E}^\times)$$

and V has a Thom-isomorphism in E-theory $\Leftrightarrow$ $[V, E]$ is the trivial cohomology class.

Units of an A_∞-ring spectrum defined by pullback

$$\begin{tikzcd}
GL_1 E \arrow[r] \arrow[d] & \Omega^\infty E \arrow[d] \\
(\pi_0 E)^\times \arrow[r] & \pi_0 E
\end{tikzcd}$$

If E is A_∞, $GL_1 E$ has a classifying space.

If E is A_∞, $GL_1 E$ is an infinite loop space.

In detail

$$\begin{tikzcd}
 & & \text{Spaces} \arrow[rr, shift left, "\Sigma^\infty_+"] & & \text{Spectra} \arrow[ll, shift left, "\Omega^\infty"] \\
\left(\begin{matrix}\text{grouplike} \\ E_\infty\text{-spaces}\end{matrix}\right) \arrow[rr, shift left] \arrow[d, shift right, "L"'] & & E_\infty\text{-spaces} \arrow[ll, shift left, "\text{grouplike components}"] \arrow[u] \arrow[rr, shift left, "\Sigma^\infty_+"] & & E_\infty\text{-ring spectra} \arrow[ll, shift left, "\Omega^\infty"] \arrow[u] \arrow[dllll, shift left, "gl_1"] \\
\left(\begin{matrix}\text{Connective} \\ \text{Spectra}\end{matrix}\right) \arrow[u, shift right, "\Omega^\infty"', ": \text{Quillen-equivalence}"] \arrow[urrrr, shift left, "\Sigma^\infty_+\Omega^\infty"] & & & &
\end{tikzcd}$$

Example: $E = S^0$, $GL_1(S^0) = G$ such that

$BGL_1(S^0) =$ classifying space for stable spherical fibration

Consider a map $bo\langle 8\rangle = b. \xrightarrow{\xi} bgl_1(S) = \Sigma\, gl_1(S)$

gives rise to

$$\begin{tikzcd}
 & GL_1 S^0 \arrow[d] \\
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & EGL_1 S^0 \arrow[d] \\
B \arrow[r, "\xi", "E_\infty \text{ map}"'] & BGL_1(S^0)
\end{tikzcd}$$

Gives pushout diagram of E_∞ ring spectra

$$\begin{tikzcd}
\Sigma^\infty_+ \Omega^\infty gl_1(S^0) \arrow[r] \arrow[d] & S^0 \arrow[d] \\
\Sigma^\infty_+ \Omega^\infty C \arrow[r] & \mathrm{Thom}(\xi) \arrow[ul, phantom, "T_{E_\infty}"]
\end{tikzcd}$$

where C is the pullback / ho fibre in spectra

$$\begin{tikzcd}
& gl_1 S \arrow[dl] & = & gl_1 S^0 \arrow[dl] \\
C \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & * \arrow[d] & \\
b \arrow[rr] & & bgl_1 S^0 &
\end{tikzcd}$$

From universal property of pushout and adjunctions we get for any E_∞ ring spectrum

$$\pi_* E_\infty(\mathrm{Thom}(\xi), R) = \pi_* \,\mathrm{Spectra}\left(\begin{tikzcd} gl_1 S \arrow[r, "gl_1(i)"] \arrow[d] & gl_1 R \\ C \arrow[ur, dashed] & \end{tikzcd} \right)$$

In particular, the obstruction is a map

$$b \longrightarrow bgl_1 S^0 \xrightarrow{bgl_1(i)} bgl_1 R$$

Example: $b = bo$, then $\Omega^\infty C = G/O$ and May-Quinn-Ray write the Thom space as

$$\mathrm{Thom}(\xi) = B(\Sigma^\infty_+ G/O, \Sigma^\infty_+ G, S^0).$$

Now $b = bo\langle 8\rangle$, so we study $MO\langle 8\rangle$-orientations. Suppose p is an odd prime.

$$\begin{tikzcd}
\Sigma^{-1} bo\langle 8\rangle \arrow[r, "\psi^c - 1"] & \Sigma^{-1} bo\langle 8\rangle \arrow[r] & j_c \arrow[r] \arrow[d] & gl_1 S \arrow[r] \arrow[d] & gl_1 R \\
& & bo\langle 8\rangle \arrow[r] & C \arrow[ur, dashed, "?"'] &
\end{tikzcd}$$

So the problem is equivalent to

$$\begin{tikzcd}
\Sigma^{-1} bo\langle 8\rangle \arrow[r, equal] \arrow[d] & \Sigma^{-1} bo\langle 8\rangle \arrow[d] & \\
j \arrow[r] \arrow[d] & gl_1 S \arrow[r, "gl_1(i)"] \arrow[d] & gl_1 R \\
bo\langle 8\rangle \arrow[r] & C \arrow[ur, dashed, "?"'] &
\end{tikzcd}$$

So extension to C correspond to extensions $bo\langle 8\rangle \longrightarrow gl_1 R$ restriction correctly to j.

For vector bundle $V \downarrow X$, E^{X^V} is a twisted form of $E(X_+)$.
The twists are classified by $H^1(X, E^\times) := \{X \longrightarrow BGL_1(E)\}$

Suppose E is A_∞ (but not nec. E_∞). Have principal fibration, pull back

$$\begin{tikzcd}
GL_1 E \arrow[r, dash] \arrow[d] & GL_1 E \arrow[d] \\
P_\xi \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & EGL_1 E \arrow[d] \\
X \arrow[r, "\xi"'] & BGL_1 E
\end{tikzcd}$$

this is an A_∞-GL_1E-space,
so $\Sigma^\infty_+ P_\xi$ is module over $\Sigma^\infty_+ GL_1 E$.

Thom(ξ) is the homotopy coequalizer in right E-modules

$$\Sigma^\infty_+ P_\xi \wedge \Sigma^\infty_+ GL_1 E \wedge E \rightrightarrows \Sigma^\infty_+ P_\xi \wedge E \longrightarrow X^\xi$$

Twisted cohomology: $E^*_\xi X = \pi_*(E\text{-mod}(X^\xi, E))$

$$E^\xi_* X = \pi_*(E\text{-mod}(E, X^\xi)) = \pi_* \,\mathrm{Spectra}(S^0, X^\xi)$$

16.10.03

Local string orientations

$$MString \longrightarrow tmf$$

Obstruction to orientation is the composite

$$\begin{tikzcd}
String \arrow[r, "\psi^\lambda - 1"] & string \arrow[r] \arrow[d, phantom, "\shortparallel"] & gl_1 S \arrow[r, "gl_1(c)"] \arrow[d] & gl_1(tmf) \\
& \Sigma^{-1} bo\langle 8\rangle \arrow[d] & gl_1 S / string \arrow[ur, dotted, "?"'] & \\
& j\lambda \arrow[uur, "\mathcal{F}"'] \arrow[d] & & \\
& bstring = bo\langle 8\rangle \arrow[uur] & &
\end{tikzcd}$$

$$\pi_* E_\infty(MString, tmf) = \pi_* gl_1 S/_{Spectra}\left(gl_1S/_{string}, gl_1 tmf\right)$$

$$= \pi_* j\lambda/Sp\,(bstring, gl_1 tmf)$$

Lemma: $\pi_i\left(\text{fiber}\left(gl_1 tmf \longrightarrow L_2\, gl_1 tmf\right)\right) = 0$ for $i > 3$

So for the obstruction problem we can replace $gl_1(tmf)$ by $L_2\, gl_1(tmf)$.

$$\begin{tikzcd}
& & Fb \arrow[d] & Fb \arrow[d] \\
j\lambda \arrow[r, "\mathcal{F}"] \arrow[d] & gl_1 S \arrow[r] \arrow[d] & L_{K(2)\vee K(1)}\, gl_1(tmf) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L_{K(2)}\, gl_1 tmf \arrow[d] \\
bo\langle 8\rangle = bstring \arrow[r] & gl_1S/string \arrow[ur, dashed, "?"] & L_{K(1)}\, gl_1 tmf \arrow[r] & L_{K(1)} L_{K(2)}\, gl_1 tmf
\end{tikzcd}$$

Since $K(2)_* bo\langle 8\rangle = 0$, there are no maps from $bo\langle 8\rangle$ to $L_{K(2)}\, gl_1\, tmf$..

So we only need a map into $L_{K(2)}\, gl_1\, tmf$. Now we'll use the logarithm

$$\ell_1 : L_{K(2)}\, gl_1(tmf) \xrightarrow{\simeq} L_{K(1)}\, tmf$$

Recall Bousfield-Kuhn functor

$$\begin{tikzcd}
\text{Spectra} \arrow[rr, "L_{K(n)}"] \arrow[dr, "\Omega^\infty"'] & & \text{Spectra} \\
& \text{pt. Spaces} \arrow[ur, "\Phi_n"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma^{-1} L_{K(1)} L_{K(2)}\, tmf \arrow[d] \\
\text{Fib} \arrow[d] \\
L_{K(2)}\, tmf
\end{tikzcd}$$

So we can exploit:

$$\begin{tikzcd}
L_{K(2)}\, gl_1(tmf) \arrow[r, "\ell_2", "\simeq"'] \arrow[d] & L_{K(2)}\, tmf \arrow[d, "\text{Mystery.}"] \\
L_{K(1)} L_{K(2)}\, gl_1(tmf) \arrow[r, "\simeq", "L_{K(1)}(\ell_2)"'] & L_{K(1)} L_{K(2)}\, tmf
\end{tikzcd}$$

Thus

$$\pi_0 E_\infty(MString, tmf) \xrightarrow[\text{iso away from 2}]{\text{onto}} \!\!\!\to$$

$\{ f \in \ker\left([bo\langle 8\rangle, L_{K(2)}\, tmf] \to [bo\langle 8\rangle, L_{K(1)} L_{K(2)}\, tmf] \right)$ such that f factors above diagram

$= \{ f \in \ker\left([KO^1_p, L_{K(2)}\, tmf] \to [KO^1_p, L_{K(1)} L_{K(2)}\, tmf] \right)$ such that... $\}$

<u>Lemma:</u> $\pi_*(L_{K(2)}\, tmf) / KO\text{-}2\text{-torsion}$ = ring of p-adic modular forms

p-adic modular form: $f = \sum a_n q^n$ with $a_n \in \mathbb{Q}_p$

$= \lim_i f_i$ with $f_i \in MF$.

(Recall that $\pi_{2n} L_{K(1)} tmf$ is related to a completion of
$\Gamma\left(\omega^{\otimes n} \text{ on elliptic curves/ adm with an iso } \hat{C} \cong \hat{\mathbb{G}}_m\right)$)
p-adic rings

Katz identified these completed sections with p-adic modular forms.

The kernel of the above surjection is 2-torsion.

Condition on the above f's is that they make the following diagram commute

$$\begin{tikzcd} L_{K(2)}\, \mathcal{J}\lambda \arrow[r] \arrow[d] & L_{K(1)}\, gl_1\, S \arrow[r] & L_{K(1)}\, gl_1\, tmf \arrow[r, "l_1"] & L_{K(1)}\, tmf \\ KO^1_p \arrow[urrr, dashed, "f"] & & & \end{tikzcd}$$

Define p-adic modular forms z_n by

$$z_n = \pi_{2n}(f)(u^n) \quad \in \quad \pi_{2n} L_{K(2)}\, tmf$$

Then:
0) z_n is a p-adic modular form of wt n. for $n \geq n_0$, never
1) z_n satisfy the "fancy Kummer congruences"
2) $z_0 = \lim_{k \to \infty} z_{(p-1)p^k} \stackrel{?}{=} \alpha(\lambda)$
3) $f \in \ker[KO^1_p, \text{mystery map}]$

σ-orientation $\quad \sigma : MU\langle 6\rangle \longrightarrow$ any elliptic spectrum

$$\frac{\sigma(t)}{t} = \exp\left(-\sum_{n\geq 4} \frac{G_n}{n!} t^n\right) \cdot \exp(\text{quadratic in } t)$$

ratio $\Leftrightarrow \Sigma^\infty BU\langle 6\rangle_+ \longrightarrow E$

with $\quad G_n = -\frac{B_n}{2n} + \sum_{r=1}^{\infty} \sigma_{n-1}(r)\, q^r$

The effect of the associated map $BU\langle 6\rangle \longrightarrow \Omega^\infty (K_{K(2)}\, tmf \wedge H\mathbb{Q})$ is

$z_n := (1 - \frac{1}{p}\psi) \cdot G_n \cdot (1 - \lambda^n)$.

Now the question is if the sequence of Z_n's satisfy the conditions 0) – 3) ?

Then they define a p-local string orientation !

Here ψ is a power operation which is "multiplication by p on the formal group".

If f is given by $f(q) = \sum a_r q^r \rightsquigarrow f(q) = \left(\sum a_r q^r\right)\left(\frac{du}{u}\right)^n$

then

$$\psi f(q) = \left(\sum_r a_r q^{pr}\right)\left(\frac{du^p}{u^p}\right)^n$$

$$= \sum a_r q^{pr} \cdot p^n \left(\frac{du}{u}\right)^n$$

Thus $\left(1 - \frac{1}{p}\psi\right) f(q) = f(q) - p^{n-1} f(q^p)$.

$$(1-\lambda^n)\left(G_n(q) - p^{n-1} G_n(q^p)\right) = \underbrace{(1-\lambda^n)(1-p^{n-1}V)\, G_n}_{G_n^*}$$

Serre proved the conditions 0) – 3) for the sequence G_n^*.