only if there exist a faithfully flat $A \to A'$ s.t. $B' = \Pi A'$ (fin. pres.). //

Thm The rel. Frob. is an iso. for the following map of stacks over $\mathbb{Z}_p$.

$$\mathcal{A} = (\mathcal{M}_{Ell})^{\wedge}_{ss} \quad \text{— completion at supersing. locus}$$

$$\downarrow$$

$$\mathcal{B} = (\mathcal{M}_{FG})^{\wedge}_{h=2} \quad \text{— completion at "height = 2 locus".}$$

pf Let R be a local $\mathbb{Z}_p$-alg. with nilp. max'l ideal m. To give a map $\operatorname{Spec} R \to \mathcal{A}$ corresponds to an elliptic curve E over R s.t. E/m is super-sing. Similarly, $\operatorname{Spec} R \to \mathcal{B}$ corresponds to a f. grp. G over R s.t. G/m is height 2.

$$\begin{tikzcd}
\mathcal{A} \arrow[dr, "\text{iso?}"] \arrow[drr, bend left, "\varphi"] \arrow[ddr, bend right] & & \\
& P \arrow[r] \arrow[d] & \mathcal{A} \arrow[d] \\
& \mathcal{B} \arrow[r, "\Phi"] & \mathcal{B}
\end{tikzcd}$$

A map $\operatorname{Spec} R \to \mathcal{B}$ corresponds to an ell. curve J and a ∞ formal grp G as above together with an iso. $G^{\varphi} \approx \hat{J}$. If $\operatorname{Spec} R \to \mathcal{A}$ corresp. to the ell. curve C, then the comp. w. $\mathcal{A} \to \mathcal{B}$ corresponds to $J = C^{\varphi}$, $G = \hat{C}$. We must show that given $(J, G, G^{\varphi} \approx \hat{J})$, there exists C s.t. $C^{\varphi} = J$, $\hat{C} = G$. Now

Super-singular: An elliptic curve J over a field is super-singular if and only if

$$\left\{ \begin{matrix} p\text{-subgrps.} \\ \text{of } \hat{J} \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} p\text{-subgrps.} \\ \text{of } J \end{matrix} \right\}$$

$$\begin{tikzcd} J \arrow[r] \arrow[dr, "p"'] & J^{\varphi} \arrow[d] \\ & J \end{tikzcd} \qquad \begin{tikzcd} & K = \text{kernel} \arrow[d] \\ G \arrow[r] \arrow[dr, "p"'] & G^{\varphi} \arrow[d] \\ & G \end{tikzcd}$$

$\hat{J} = G^{\varphi}$, $K \subset J$; define $C = J/K$

$$
\begin{tikzcd}
K \arrow[r, equal] \arrow[d] & K \arrow[r, equal] \arrow[d] & K \arrow[d] \\
G^{\varphi} \arrow[r, "\simeq"] \arrow[d] & \hat{J} \arrow[r] \arrow[d] & J \arrow[d] \\
G \arrow[r] & \hat{J}/K \arrow[r] & J/K =: C.
\end{tikzcd}
$$

Similarly, construct $C^{\varphi} \xrightarrow{\simeq} J$. //

<u>Note</u> Maps Spec $\to \mathcal{A}$ etc. really give categories; we have only considered the objects.

Last time we discussed the construction of the completion $(\mathcal{M}_{Ell})^\wedge_{ss}$ along the super-singular locus. Today: ordinary part.

$$\begin{array}{l} \operatorname{Spec} R \\ \text{étale} \downarrow J \qquad \rightsquigarrow \quad E_J \text{ - homology theory} \\ \mathcal{M}_{Ell} \end{array}$$

Complete everything at a prime p. So K = p-adic K-theory.

Problem: Find an E_∞-ring spectrum R with $K_* R \approx K_* E_J$ as algebras of K_*K-comodules.

Recall that if R is an E_∞-ring spectrum, an elem.

$$S^n \xrightarrow{\alpha} R \quad \text{in } \pi_n R$$

induces

$$\begin{tikzcd} \operatorname{Sym}^k S^n \arrow[r] \arrow[rr, bend right, "P_k\alpha"'] & \operatorname{Sym}^k R \arrow[r] & R \end{tikzcd}$$

Similarly,

$$S^n \longrightarrow K \wedge R$$

induces

$$\begin{tikzcd} \mathrm{Sym}^k S^n \arrow[r] \arrow[d] & \mathrm{Sym}^k K \wedge R \arrow[d] \\ K \wedge \mathrm{Sym}^k S^n \arrow[r, dashed] & K \wedge R \end{tikzcd}$$

and $\pi_* K \wedge \mathrm{Sym}^k S^n$ are known by Atiyah's result that $(B\Sigma_k = \mathrm{Sym}^k S^0)$

$$K^*(B\Sigma_k) = \mathrm{Rep}(\Sigma_k)^\wedge_I$$

This implies that $\pi_0(K \wedge R)$ has an operation

$$\theta : \pi_0(K \wedge R) \longrightarrow \pi_0(K \wedge R)$$

s.t.

$$\psi(x) = x^p + p\,\theta(x)$$

is a ring-homomorphism. If $S(V)$ is the free E_∞-ring spectrum gen. by a spectrum V:

Prop If $\pi_{odd}(K \wedge V) = 0$, then $\pi_{odd}(K \wedge S(V)) = 0$, and $\pi_0(K \wedge S(V))$ is the free θ-alg. on $\pi_0(K \wedge V)$.

Can state problem more precisely:

Problem: Find E_∞-ring R with

$$\pi_*(K\wedge R) \cong \pi_*(K\wedge E_{\mathcal{J}})$$

as θ-algebras in K_*K-co-modules.

⌈Will discuss θ-algebra structure on $\pi_*(K\wedge E_{\mathcal{J}})$ in a moment...⌋

K_*K - co-module:

$$\begin{array}{c} \operatorname{Spec}\mathbb{Z}_p \\ \Big\downarrow \hat{\mathbb{G}}_m \\ \mathcal{M}_{FG} \end{array}$$

flat by Landweber's criterion

$$1-(1-x)(1-y) = x+y-xy$$

--- assoc. coh. th. is K-theory

$$\mathbb{Z}_p^\times = \mathrm{Iso}(\hat{\mathbb{G}}_m, \hat{\mathbb{G}}_m)$$

$$\begin{tikzcd}
\operatorname{Spec}\pi_0 K\wedge K \arrow[r] \arrow[d] & \operatorname{Spec}\mathbb{Z}_p \arrow[d] \\
\operatorname{Spec}\mathbb{Z}_p \arrow[r] & \mathcal{M}_{FG}
\end{tikzcd}$$

so a K_*K-co-module str. is equivalent to a cts. action by $\mathbb{Z}_p^*$. Moreover,

$$\begin{tikzcd}
\mathrm{Iso}(G, \hat{\mathbb{G}}_m) \arrow[d, phantom, "\cong" rotate=90] & \\
\mathrm{Spec}\, \pi_0 K \wedge R \arrow[r] \arrow[d] & \mathrm{Spec}\, \pi_0 R \arrow[d, "G"] \\
\mathrm{Spec}\, \mathbb{Z}_p \arrow[r] & \mathcal{M}_{FG}
\end{tikzcd}$$

—— as $\mathbb{Z}_p^*$-[illegible]

action of $\mathrm{Aut}(\hat{\mathbb{G}}_m)$ = action of $\mathbb{Z}_p^*$.

$$\begin{tikzcd}
\mathrm{Iso}(\hat{J}, \hat{\mathbb{G}}_m) \arrow[d, phantom, "\cong" rotate=90] & \\
\mathrm{Spec}\, D \arrow[r] \arrow[d] & \mathcal{M}_{Ell} \arrow[d] \\
\mathrm{Spec}\, \mathbb{Z}_p \arrow[r] & \mathcal{M}_{FG}
\end{tikzcd}$$

D = p-completion of Katz' ring of divided congruences.

Can explain θ-alg. str. on $\pi_*(K \wedge E_J)$.

R : p-tors. free ring ; then

~~Ψ~~-str. Θ-algebra structure $\longleftrightarrow$ $R \xrightarrow{\Psi} R$

$\Psi(x) \equiv x^p \pmod{pR}$

$$\Psi(x) = x^p + p\,\Theta(x).$$

Over $\pi_0(K \wedge E_J)$,

$$\begin{array}{ccccc} \hat{G}_m & \xrightarrow[\sim]{t} & \hat{J} & \subset & J \\ \cup & & \cup & & \\ \mu_p & \xrightarrow[\sim]{t} & t(\mu_p) & & \end{array}$$

so $J/t(\mu_p)$ is a new ell. curve over $\pi_0(K \wedge E_J)$. The map $\Psi : \pi_0(K \wedge E_J) \circlearrowleft$ is characterized by

$$\Psi^* J \xrightarrow{\sim} J/t(\mu_p).$$

First, consider the map Ψ for the ring D,

$$\operatorname{Spec} D = \operatorname{Iso}(\hat{J}_{univ}, \hat{G}_m)$$

$$\hat{G}_m \xrightarrow[\sim]{t} \hat{J} \longrightarrow J$$

$$\begin{tikzcd}
\hat{G}_m \arrow[r, "t", "\sim"'] \arrow[d, "{[p]}"] & \hat{J} \arrow[r] \arrow[d] & J \arrow[d] & D \\
\hat{G}_m \arrow[r, "t", "\sim"'] & \hat{J}/t(\mu_p) \arrow[r] & J/t(\mu_p) & D \arrow[u, "\psi"']
\end{tikzcd}$$

check: $\psi(x) \equiv x^p \ (p)$.

<u>Lemma</u> Let $R \to S$ be an étale map of p-complete rings, and let $\psi : R \to R$ be a ring-homom. with $\psi(x) \equiv x^p$ (mod pR). Then there exists a unique ring-homom. $\psi_S : S \to S$ with $\psi_S(x) \equiv x^p$ (mod pS) and which extends ψ [illegible]

This gives the Θ-algebra structure on $\pi_0(K \wedge E_{J_1})$.

E_2-model cat. obstr. theory.

$$\begin{array}{ccc}
\text{simpl. } E_\infty\text{-ring spectra} & \xrightarrow{K_*} & \text{simpl. } \Theta\text{-algebras in } K_*K\text{-co-modules}
\end{array}$$

(resolve problem with having odd htpy groups by using co-simplicial resolutions.)

when E is K-local.

$$\pi_* E_\infty(\mathrm{Sym}(V), E)$$

$$\approx \mathbb{Z}_p^*\text{-equivariant } \theta\text{-alg. maps}$$
$$K_*\mathrm{Sym}(V) \longrightarrow K_* E$$

Obstr. to existence are in

$$\mathrm{Der}^{n+2}(K_* E_J, \Omega^n K_* E_J)$$

and to uniqueness in

$$\mathrm{Der}^{n+1}(K_* E_J, \Omega^n K_* E_J).$$

Der in our category:

A = θ-algebra in K_*K co-modules.

What is an A-module? It is an θ-alg. in K_*K-co-modules of the form $A \oplus M$ with $M^2 = 0$ (Quillen).

$$A \oplus M \xrightarrow{\psi} A \oplus M$$

$$\psi(a, m) = (\psi(a), \psi(m))$$

so

$$\theta(a,m) = \frac{\psi(a,m) - (a,m)^p}{p}$$

$$= \frac{(\psi(a), \psi(m)) - (a^p, p a^{p-1} m)}{p}$$

$$= \left(\theta(a), \frac{\psi(m) - p a^{p-1} m}{p}\right).$$

Find that an A-module in θ-algebras in K_*K-comodules is: an ordinary A-mod. M with a compatible action of $\mathbb{Z}_p^*$ and a map $\theta : M \to M$ s.t.

$$\theta(am) = \psi(a)\,\theta(m).$$

Derivations:

$$\begin{tikzcd} A \arrow[r] \arrow[dr, equal] & A \oplus M \arrow[d] \\ & A \end{tikzcd} \qquad a \mapsto (a, Da)$$

$$D(ab) = Da.b + a.Db$$

$$D\theta(a) = \theta Da + a^{p-1} Da.$$

Can write

$$\mathrm{Der}_{\theta\text{-alg}}(A,M) = \mathrm{Hom}_{\mathbb{Z}_p[\theta]}(\mathbb{Z}_p, \mathrm{Der}(A,M))$$

where $\mathbb{Z}_p[\theta]$ acts on $\mathrm{Der}(A,M)$ by

$$(\theta \cdot D)(a) = D\theta(a) - \theta D(a) = a^{p-1} Da$$

Then Grothendieck sp' seq.

$$\mathrm{Ext}^s_{\mathbb{Z}_p[\theta]}(\mathbb{Z}_p, \mathrm{Der}^t(A,M))$$

$$\Rightarrow \mathrm{Der}^{s+t}_{\theta\text{-alg}}(A,M)$$

is really a l.e.s. since $\mathbb{Z}_p$ has a res. of length 1 as $\mathbb{Z}_p[\theta]$-module.

Fact: If A is formally smooth over $\mathbb{Z}_p$ (or whichever base we consider)

$$\mathrm{Der}^*(A,M) = \mathrm{Hom}_A(\Omega^1_A, M).$$

Our obstr. groups :

$$\mathrm{Der}^{s}_{\Theta\text{-alg}}(A, M) = 0 \qquad , \quad s > 1$$

$$H^{s}(\mathbb{Z}_p^{*}, \mathrm{Der}^{t}_{\Theta\text{-alg}}) = 0 \qquad , \quad s > 2.$$

The existence grps. $\mathrm{Der}^{n+2}_{\Theta\text{-alg}}$ will vanish --

We wish to construct a sheaf $\mathcal{O}^{top}$ of E_∞-ring spectra on $\mathcal{M}_{Ell}$ in the étale topology.

Step 1: Construct a functor

$$(\mathrm{Aff}/\mathcal{M}_{Ell})^{op}_{et} \longrightarrow E_\infty\text{-ring}$$

$$\begin{array}{c} \mathrm{Spec}\, R \\ \text{étale} \downarrow J \\ \mathcal{M}_{Ell} \end{array} \longmapsto \text{Landweber exact coh. theory.}$$

Step 2: Extend to non-affine

$$\begin{array}{c} \mathcal{N} \\ \downarrow \text{étale} \\ \mathcal{M}_{Ell} \end{array} \longmapsto \underset{\longleftarrow}{\mathrm{holim}}\ \mathcal{O}^{top}\left(\begin{array}{c} \mathrm{Spec}\, R \\ \downarrow J \\ \mathcal{M}_{Ell} \end{array} \right).$$

For step 1:

- complete at prime p
- complete at super-singular locus ✓
  — relative Frobenius.

- complete at ordinary locus
  - E_∞-obstructions groups.
- compare.

For the ordinary locus, we use the obstruction theory to form

$$L_{K(1)} tmf = \mathcal{O}^{top}(\mathcal{M}_{ord}) \to \mathcal{M}_{Ell}$$

Will define $\mathcal{O}^{top}(\mathcal{M}_{Ell})$ to be the homotopy pull-back

$$\begin{array}{ccc} \mathcal{O}^{top}(\mathcal{M}_{Ell}) & \longrightarrow & \mathcal{O}^{top}(\mathcal{M}_{ss}) \\ \downarrow & & \downarrow \\ \mathcal{O}^{top}(\mathcal{M}_{ord}) & \longrightarrow & L_{K(1)}\mathcal{O}^{top}(\mathcal{M}_{ss}) \end{array}$$

Show that the category of E_∞-algebras over $\mathcal{O}^{top}(\mathcal{M}_{Ell})$ which are even, periodic, and elliptic, and s.t.

$$\operatorname{Spec} \pi_0 E \longrightarrow \mathcal{M}_{Ell}$$

is étale is equivalent to the alg. cat.

$$(\mathrm{Aff}/\mathcal{M}_{Ell})_{et}.$$