$(\mathcal{M}_{Ell})_{ord}$: $c_4 \not\equiv 0$.

$\mathcal{O}^{top}(\mathcal{M}_{Ell})^{\wedge}_{p}$ — only involves curves with either $c_4 \not\equiv 0$ or $c_6 \not\equiv 0$.

(This is true for every prime $p \geq 5$.)

After localizing at any prime p,

$$\mathcal{M}_{Ell} = \Delta^{-1}\mathcal{M}_{Weier} \cup v_1^{-1}\mathcal{M}_{Weier}$$

$$= \Delta^{-1}\mathcal{M}_{Weier} \cup c_4^{-1}\mathcal{M}_{Weier}$$

$$\left(p \geq 3 : \; = \Delta^{-1}\mathcal{M}_{Weier} \cup c_6^{-1}\mathcal{M}_{Weier} \right).$$

To calc. $\pi_* \mathcal{O}^{top}(\mathcal{M}_{Ell})$:

1) Find an étale cover

$$\operatorname{Spec} R \to \mathcal{M}_{Ell}.$$

2) Write down the nerve

$$\cdots \operatorname{Spec} R_2 \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \operatorname{Spec} R_1 \rightrightarrows \operatorname{Spec} R_0 \to \mathcal{M}_{Ell}$$

$$R_0 = R$$

$$\mathcal{O}_n := \mathcal{O}^{top}(\operatorname{Spec} R_n)$$

3) htpy spectral sequence for co-simpl. spectrum

$$\cdots \ \pi_* \mathcal{O}_2 \ \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} \ \pi_* \mathcal{O}_1 \ \leftleftarrows \ \pi_* \mathcal{O}_0 \ .$$

We have

$$\pi_0 \mathcal{O}_n = R_n = H^0(\operatorname{Spec} R_n; \omega^{\otimes 0}),$$

$$\pi_{odd} \mathcal{O}_n = 0$$

$$\pi_{2t} \mathcal{O}_n = H^0(\operatorname{Spec} R_n, \omega^{\otimes t})$$

where ω is the sheaf of invariant 1-forms on

$$\operatorname{Spec} R_n \xrightarrow{J_n} \mathcal{M}_{Ell} .$$

So E_2-term is the coh. of the cx.

$$\cdots \ \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} H^0(\operatorname{Spec} R_1, \omega^{\otimes t}) \leftleftarrows H^0(\operatorname{Spec} R_0, \omega^{\otimes t}) \ .$$

This is the Cech cx. for calculating $H^*(\mathcal{M}_{Ell}, \omega^{\otimes t})$, so sp. seq. is:

$$E_2^{s,t} = H^{-s}(\mathcal{M}_{Ell}, \omega^{\otimes \frac{t}{2}}) \Rightarrow \pi_{s+t} \mathcal{O}^{top}(\mathcal{M}_{Ell}) .$$

The E_2-term is coh. of a coherent module, so we can calc. it by using the flat (but not étale) cover

$$\operatorname{Spec} \mathbb{Z}_p[c_4^{\pm 1}, c_6] \amalg \operatorname{Spec} \mathbb{Z}_p[c_4, c_6^{\pm 1}]$$

$$\downarrow \qquad (p \geq 5)$$

$$\mathcal{M}_{Ell}$$

Let $A = \mathbb{Z}_p[c_4^{\pm 1}, c_6]$. Then

$$\operatorname{Spec} A \times_{\mathcal{M}_{Ell}} \operatorname{Spec} A = \operatorname{Spec} A[\lambda^{\pm 1}]$$
$$=: \operatorname{Spec} \Gamma.$$

$\mathbb{Z}_p[\lambda^{\pm 1}]$ - Hopf algebra

$\lambda \mapsto \lambda \otimes \lambda$

A = co-module over $\mathbb{Z}_p[\lambda^{\pm 1}]$.

Cech complex = complex for calc.

$$\operatorname{Ext}_{\mathbb{Z}_p[\lambda^{\pm 1}]\text{-comod.}}(\mathbb{Z}_p, A)$$

$$\mathbb{Z}_p \to \mathbb{Z}_p[\lambda^{\pm 1}] \otimes \mathbb{Z}_p \qquad 1 \mapsto 1 \otimes 1$$

Prop The category of $\mathbb{Z}[\lambda^{\pm 1}]$ - co-modules is equivalent to the category of graded abelian groups.

pf $A = \bigoplus A_n \longmapsto A \longrightarrow A \otimes \mathbb{Z}[\lambda^{\pm 1}]$

$$a \in A_n \longmapsto a \otimes \lambda^n .$$

$A =$ co-module $\rightsquigarrow A = \bigoplus A_n$

$$A_n = \{ a \in A \mid a \mapsto a \otimes \lambda^n \} .$$

$$a \in A, \quad a \mapsto \sum \alpha_k \otimes \lambda^k \Rightarrow a = \sum \alpha_k .$$

$$\begin{tikzcd} A \arrow[r] \arrow[dr, equal] & A \otimes \mathbb{Z}[\lambda^{\pm 1}] \arrow[d, "\varepsilon"] \\ & A \end{tikzcd} \qquad \begin{array}{c} \lambda \\ \downarrow \\ 1 \end{array}$$

By co-associativity

$$A \xrightarrow{\psi} A \otimes \mathbb{Z}[\lambda^{\pm 1}] \rightrightarrows A \otimes \mathbb{Z}[\lambda^{\pm 1}] \otimes \mathbb{Z}[\lambda^{\pm 1}]$$

$$a \longmapsto \sum \alpha_k \otimes \lambda^k \longmapsto \sum \psi(\alpha_k) \otimes \lambda^k$$

$$\longmapsto \sum \alpha_k \otimes \lambda^k \otimes \lambda^k$$

so $\psi(\alpha_k) = \alpha_k \otimes \lambda^k .$ //

Now

$$\mathrm{Hom}_{\mathbb{Z}_p[\lambda^{\pm 1}]\text{-comod}}(\mathbb{Z}_p, A) = A_0$$

which (by prop.) is clearly an exact functor, so for $s > 0$,

$$\mathrm{Ext}^s_{\mathbb{Z}_p[\lambda^{\pm 1}]\text{-comod}}(\mathbb{Z}_p, A) = 0 .$$

$$\begin{array}{ccc} H^0(\mathrm{Spec}\, A, \omega^{\otimes 0}) & = & \mathbb{Z}_p[c_4^{\pm 1}, c_6] \\ \cup & & \cup \\ H^0(c_4^{-1} \mathcal{M}_{Ell}, \omega^{\otimes 0}) & = & \lambda^0\text{-eigenspace} \end{array}$$

$$= \text{degree zero part} = \mathbb{Z}_p\left[\frac{c_6^2}{c_4^3}\right].$$

$$\begin{array}{ccc} H^0(\mathrm{Spec}\, A, \omega^{\otimes n}) & = & \mathbb{Z}_p[c_4^{\pm 1}, c_6] \cdot \left(\frac{dx}{y}\right)^n \\ \cup & & \cup \\ H^0(c_4^{-1} \mathcal{M}_{Ell}, \omega^{\otimes n}) & = & \text{degree } n \text{ part} \end{array}$$

$$\bigoplus_n H^s(\mathcal{M}_{Ell}, \omega^{\otimes n}) = \begin{cases} \mathbb{Z}_p[c_4^{\pm 1}, c_6] & s = 0 \\ 0 & s > 0 . \end{cases}$$

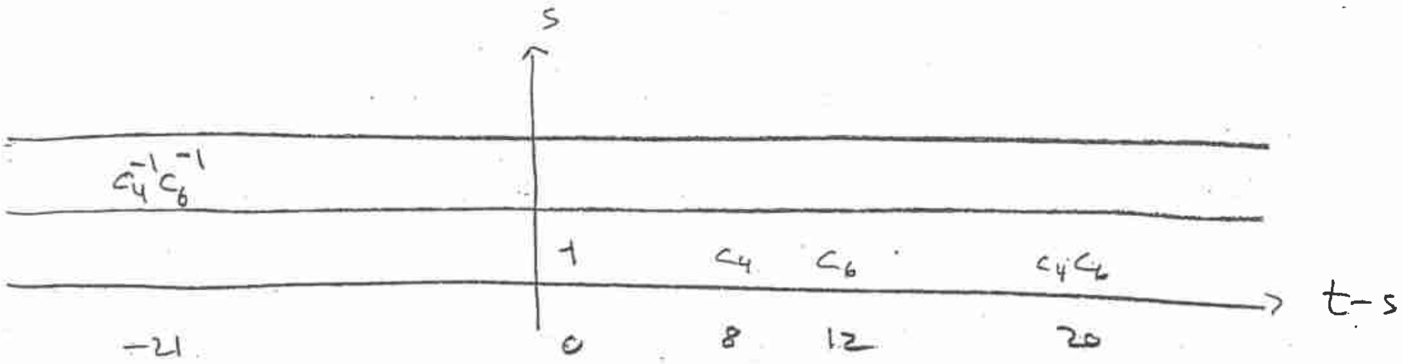
Mayer-Vietoris

$$
\begin{array}{c}
\vdots \\
\downarrow \\
H^s(\mathcal{M}_{Ell}, \omega^{\otimes n}) \\
\downarrow \\
H^s(c_4^{\pm 1}\mathcal{M}_{Ell}, \omega^{\otimes n}) \oplus H^s(c_6^{\pm 1}\mathcal{M}_{Ell}, \omega^{\otimes n}) \\
\downarrow \\
H^s((c_4 c_6)^{-1}\mathcal{M}_{Ell}, \omega^{\otimes n}) \\
\downarrow \\
\vdots
\end{array}
$$

becomes

$$
\begin{array}{rcl}
 & & 0 \\
 & & \downarrow \\
H^0 & = & \mathbb{Z}_p[c_4, c_6] \\
 & & \downarrow \\
 & & \mathbb{Z}_p[c_4^{\pm 1}, c_6] \oplus \mathbb{Z}_p[c_4, c_6^{\pm 1}] \\
 & & \downarrow \\
 & & \mathbb{Z}_p[c_4^{\pm 1}, c_6^{\pm 1}] \\
 & & \downarrow \partial \\
H^1 & = & \{ \text{basis } c_4^{-i} c_6^{-j} \quad i, j > 0 \} \\
 & & \downarrow \\
 & & 0
\end{array}
$$

Spectral sequence for $\pi_* \mathcal{O}^{top}(\mathcal{M}_{Ell})^\wedge_p$, $p \geq 5$:

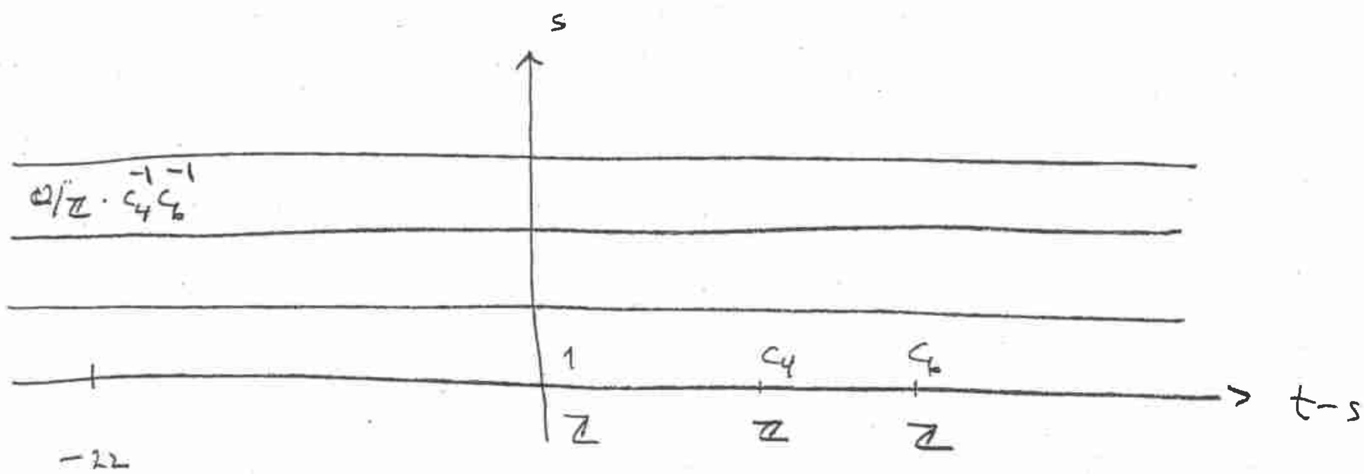


If we invert 6,

$$\begin{tikzcd}
\mathcal{O}^{top}(\mathcal{M}_{Ell}) \arrow[r] \arrow[d] & \prod_p \mathcal{O}^{top}(\mathcal{M}_{Ell})^\wedge_p \arrow[d] \\
\mathbb{Q} \otimes \mathcal{O}^{top}(\mathcal{M}_{Ell}) \arrow[r] & \mathbb{Q} \otimes \prod_p \mathcal{O}^{top}(\mathcal{M}_{Ell})^\wedge_p
\end{tikzcd}$$

$$\pi_* = \mathbb{Q}[c_4, c_6]$$

So $\pi_* \mathcal{O}^{top}(\mathcal{M}_{Ell})[\frac{1}{6}]$ is:



If we work with

$$tmf = \mathcal{O}^{top}(\mathcal{M}_{Weier}),$$

then the negative htpy grps disappear, i.e.

$$tmf = \tau_{\geq 0}\, \mathcal{O}^{top}(\mathcal{M}_{Ell}).$$

$p=3$: $H^*(\mathcal{M}_{Weier}, \omega^{\otimes n})$

<u>Claim</u> $\operatorname{Spec} \mathbb{Z}[b_2, b_4]$ $\qquad y^2 = x^3 + b_2 x^2 + b_4 x$

cover: $\downarrow$ $\qquad x \mapsto x + r$

$\mathcal{M}_{Weier}$ $\qquad r^3 + b_2 r^2 + b_4 r = 0$

$$b_2 \mapsto \lambda^2 b_2$$

$$b_4 \mapsto \lambda^4 b_4.$$

$$\bigoplus_n H^*(\mathcal{M}_{Weier}, \omega^{\otimes n}) = \operatorname{Ext}^{**}_{(A,\Gamma)\text{-comod}}(A, A)$$

$A = \mathbb{Z}[b_2, b_4]$ $\qquad \deg b_i = 2i$

$\Gamma = A[r]/(r^3 + b_2 r^2 + b_4 r).$

Hopf algebroid

(This is written up in Tilman Bauer: Computation of the homotopy of the spectrum tmf.)

Injective resolution (similar to the standard res. of $\mathbb{Z}$ by $\mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$-mod.)

$$A \to \Gamma \to \Gamma \to \Gamma \to \Gamma \to \cdots$$

$$\begin{aligned} 1 &\longmapsto 0 \\ r &\longmapsto 1 \\ r^2 &\longmapsto 2r \end{aligned}$$

periodic of period two

$$\begin{aligned} 1 &\longmapsto 0 \\ r &\longmapsto 0 \\ r^2 &\longmapsto 2 \end{aligned}$$

Find

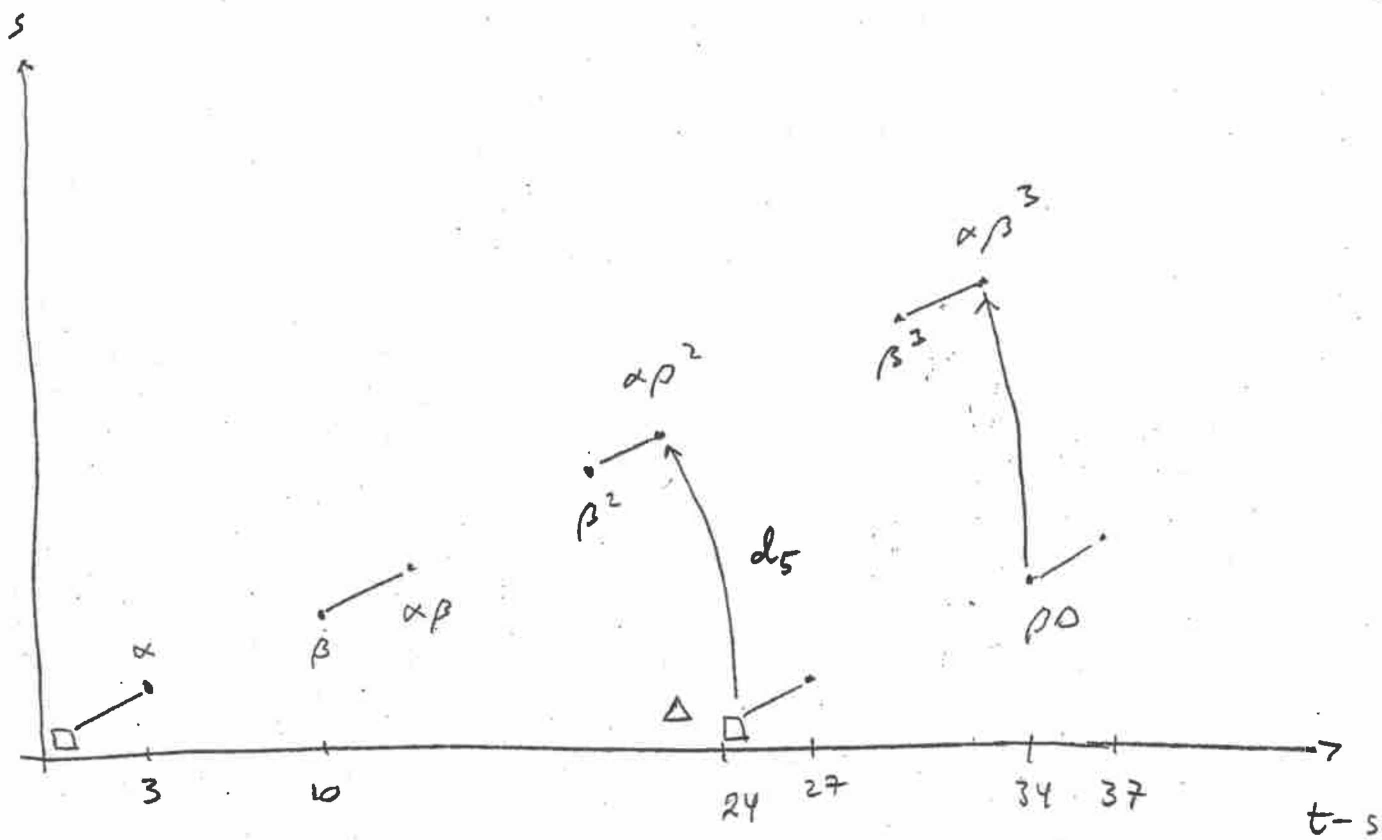
$$\bigoplus_n H^*(\mathcal{M}_{Weier}, \omega^{\otimes n})$$

$$= \mathbb{Z}[c_4, c_6, \Delta] \big/ (c_4^3 - c_6^2 - 1728\Delta) \otimes \Lambda\{\alpha\} \otimes S\{\beta\}$$

$$(c_4\alpha, c_6\alpha, c_4\beta, c_6\beta, (3, c_4, c_6)\alpha, (3, c_4, c_6)\beta)$$

$\alpha \in H^1(\mathcal{M}_{Weier}, \omega^{\otimes 2})$

$\beta \in H^2(\mathcal{M}_{Weier}, \omega^{\otimes 6})$ $\qquad\qquad \beta = \langle \alpha, \alpha, \alpha \rangle$



$\square = \mathbb{Z}_{(3)}$

$\bullet = \mathbb{Z}/3\mathbb{Z}$

$d_5(\beta\Delta) = \beta\, d_5(\Delta) = \beta^2$

Massey product arg. shows that every class in degr. higher than β^4 is annihilated by diff., and $E_{?} = E_\infty$.