Say height of $G \geq h$. If $g'(0)$ is a unit, say height $G = h$.

Suppose ht $G = h$

$$[p](x) = u \cdot x^{p^h} + \cdots$$

$$= u \cdot x^{p^h} \underbrace{(1 + O(x))}_{\text{unit in } R[[x]]}$$

so $R[[x]] / [p](x)$ free R-mod. rk. h.

Start with G over general ring R.

$v_0 = p$

G over $R/(p)$: $[p](x) = g(x^p) = v_1 x^p + \cdots$

$v_1 \in R/(p)$.

G over $R/(p, v_1)$: $[p](x) = v_2 x^{p^2} + \cdots$

$\vdots$

$v_n \in R/(p, v_1, \ldots, v_{n-1})$: $[p](x) = v_n x^{p^n} + \cdots$

Rem The ideal gen. by $v_n \in R/(p, v_1, \dots, v_{n-1})$ is independent of x, i.e. an invariant of the formal group.

pf $y = \lambda x + \cdots \qquad v_n \mapsto \lambda^{p^n-1} v_n$ . . //
$\lambda \in R^*$

Spectra and homology theories

Def A spectrum is a collection of spaces

$$E = \{E_n\}_{n \in \mathbb{Z}}$$

together with homeomorphisms

$$E_n \xrightarrow[\approx]{t_n} \Omega E_{n+1}.$$

A map of spectra is

$$E = \{E_n, t_n^E\} \xrightarrow{f} F = \{F_n, t_n^F\}$$

$$\begin{array}{ccc} E_n & \xrightarrow{f_n} & F_n \\ \approx \downarrow t_n^E & & \approx \downarrow t_n^F \\ \Omega E_{n+1} & \xrightarrow{\Omega f_{n+1}} & \Omega F_{n+1} \end{array}$$

Rem $E = \{E_n, t_n^E\}$ a spectrum, then

$$X \mapsto E^n(X) = [X, E_n] \quad \text{htpy classes of maps}$$

is a cohomology theory.

Thm (Brown representability)

$$\begin{pmatrix} \text{spectra} \\ + \\ \text{htpy cl.} \\ \text{of maps} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{cohomology th.} \\ + \\ \text{nat. transf.} \end{pmatrix}$$

is an equivalence of categories. //

We really wish to associate functorially to each elliptic curve a spectrum and maps between spectra rather than just homotopy classes of maps.

$$\Sigma E_n \xrightarrow{s_n} E_{n+1} \quad \text{adjoint to}$$

$$E_n \xrightarrow{t_n} \Omega E_{n+1}.$$

Prop (Whitehead)

$$X \mapsto E_n(X) = \lim_{k\to\infty} \pi_{n+k}(E_k \wedge X)$$

is a homology theory. //

Thm The functor

$$\begin{pmatrix} \text{spectra} \\ + \\ \text{htpy cl.} \\ \text{of maps} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{homology th.} \\ + \\ \text{nat. transf.} \end{pmatrix}$$

is a bijection on objects and surjective on maps. (Maps in the kernel — phantom maps — are fairly well understood.) //

ex A ab. group ; Eilenberg-MacLane sp.

$$E_n = K(A,n)$$

$$\pi_i K(A,n) = \begin{cases} A & i = n \\ 0 & i \neq n \end{cases}$$

Possible to get $K(A,n) \xrightarrow{\approx} \Omega K(A,n+1)$.

This spectrum $HA = \{K(A,n)\}$ represents singular (co)homology w. coeff. in A. //

ex Let $Q(X) = \lim \Omega^n \Sigma^n X$, then

Prop $\{Q(\Sigma^n X)\}$ "is" a spectrum:

$\Sigma^\infty X$ — the suspension spectrum of X.

Prop ~~Htpy cl. of spectrum~~ maps $\Sigma^\infty X \to E$ $= E^0(X)$.

Complex cobordism:

$BU(n)$ = classifying sp. of $U(n)$ $= \lim_{N\to\infty} \mathrm{Grass}_n(\mathbb{C}^N)$

$MU(n)$ = Thom space of univ. n-plane bdle. $= \lim_{N\to\infty} \overline{V_n}$ ← one-pt comp.

$BU(n) \subset BU(n+1)$ MU a spectrum

$\Sigma^2 MU(n) \to MU(n+1)$. $MU_{2k} = \lim_{N\to\infty} \Omega^{2N} MU(N+k)$

$MU_k(S^0)$ = cobordism grp. of k-dim. stably almost cx. mfds.

$$\begin{array}{ll} V & \text{complex} \\ \downarrow & \text{v.b. of} \\ X & \dim_{\mathbb{R}} m \end{array} : \quad MU^{*+m}(\bar{V}) \approx MU^*(\bar{X}).$$

Thm (Milnor) $MU_*(S^0) = \mathbb{Z}[a_1, a_2, \ldots]$. //

$|a_i| = 2i$

$MP_*(X) := \mathbb{Z}[u^{\pm 1}] \otimes MU_*(X) \qquad |u| = +2.$

$MP_0(X) = \mathbb{Z}\left[\frac{a_1}{u}, \frac{a_2}{u^2}, \ldots\right]$.

$$\begin{tikzcd} MU(0) \arrow[d] & MU(1) \arrow[d, "\tilde{x}"] \approx BU(1) = \mathbb{C}P^\infty \\ MU_0 & MU_1 \end{tikzcd}$$

$$\tilde{x} \in MU^2(\mathbb{C}P^\infty) \to MU^2(\mathbb{C}P^1) \approx MU^0(S^0) \ni 1$$

$x = u \cdot \tilde{x} \in MP^0(\mathbb{C}P^\infty)$ a coordinate on G_{MU}, so G_{MU} a formal group law.

Thm (Quillen) G_{MU} over $L = \pi_0 MP$ is the universal formal grp. law, i.e. given any formal grp. law G / R, there exists a unique map $f : L \to R$ s.t. $f^* G_{MU} = G$. //

G — a formal grp. law over R $\leftrightarrow$

$\pi_0 MP = L \longrightarrow R$

$$X \longmapsto MP_*(X) \otimes_{MP_0(pt)} R$$

When is this a homology theory?

Thm (Landweber) If for every prime p,

$$(v_0, v_1, v_2, \ldots)$$

is a regular sequence, then

$$MP_*(X) \otimes_{\pi_0 MP} R$$

is a homology theory.

Recall that $v_0, v_1, \ldots \in R$ is <u>regular</u> if v_i is a non-zero-divisor mod $(v_0, \ldots, v_{i-1})$.
Cond. only depends on v_i mod $(v_0, \ldots, v_{i-1})$ and in fact only on (v_i) mod $(v_0, \ldots, v_{i-1})$.
In part., if $(v_0, \ldots, v_{i-1}) = R$, then v_i is a non-zero-divisor.

<u>ex</u> $\mathbb{G}_m$ over $\mathbb{Z}$.

$p, v_1, v_2, \ldots$

p non-zero-div. in $\mathbb{Z}$

$v_1 = 1 \in \mathbb{Z}/p\mathbb{Z}$ non-zero-div.

so by Landweber get a homology th. (K-homology).

9/24/03

Stacks

A groupoid is a cat. in which all morphisms are isomorphisms.

$$X_0 \leftleftarrows X_1 \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}\; X_1 \times_{X_0} X_1$$

"objects" "morphisms"

(A category is a simpl. set where the n-simplices is the fibered prod. of the 1-simpl. over the 0-simpl.; a groupoid is a cyclic set where the n-simplices is the fib. prod. of X_1 over X_0.)

A stack is a sheaf G of groupoids with the property: If $\mathcal{U} = \{U_i \to X\}$ is a covering then

$$G(X) \to \mathrm{desc}(G;\mathcal{U})$$

is an equivalence of cat.

Here $\mathrm{desc}(G;\mathcal{U})$ is the category of descent data. The objects are

are pairs $(\{a_i\}, \{a_i \xrightarrow{\lambda_{ij}} a_j\})$ with $a_i \in G(U_i)$ and λ_{ij} a morphism in $G(U_i \cap U_j)$ s.t. on $U_i \cap U_j \cap U_k$

$$\begin{tikzcd} a_i \arrow[r, "\lambda_{ij}"] \arrow[dr, "\lambda_{ik}"'] & a_j \arrow[d, "\lambda_{jk}"] \\ & a_k \end{tikzcd}$$

commutes.

ex G = a group,

$X \longmapsto$ cat. of principal G-bundles over X + isomorphisms

is a stack. //

Suppose (X_0, X_1) is a groupoid, X_0, X_1 spaces. Then

$$U \longmapsto (C(U, X_0), C(U, X_1))$$

is a sheaf of groupoids. But this is usually not a stack.

ex $X_0 = pt$, $X_1 = G$ = group

$U \longmapsto (pt, C(U,G))$

cat. of trivial
= principal G-bdls
over U

But trivial bdls glue to give a bundle, but not a trivial bdl. in general. "

Associated stack:

Suppose (F_0, F_1) is a sheaf of groupoids. Then

$ass(F_0, F_1)(X)$ = limit of descent data for open covers of X

is a stack. To give an object of $ass(F_0, F_1)(X)$ is to give a cover $\{U_i \to X\}$, objects $a_i \in F(U_i)$ and iso's $a_i \xrightarrow{\lambda_{ij}} a_j \in F(U_i \cap U_j)$ s.t. the

cocycle cond.
$$\begin{tikzcd} a_i \arrow[rr, "\lambda_{ij}"] \arrow[dr, "\lambda_{ik}"'] & & a_j \arrow[dl, "\lambda_{jk}"] \\ & a_k & \end{tikzcd}$$
holds on $U_i \cap U_j \cap U_k$.

The assoc. stack of the sheaf of groupoids in the second example above is iso. to the stack in the first ex.

A space X determines a stack (F_0, F_1) with

$$F_0(U) = C(U, X)$$
$$\uparrow \downarrow \uparrow \quad \text{identity maps}$$
$$F_1(U) = C(U, X)$$

So Spaces $\subset$ Stacks.

Algebraic examples:

Aff = opposite cat. of comm. rings

$\operatorname{Spec} R \longleftarrow\!\shortmid \; R$

open: $U \longrightarrow X$
$\operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ flat

cover: $U \longrightarrow X$
$\operatorname{Spec} S \to \operatorname{Spec} R$ faithfully flat

Faithfully flat : S flat over R and for every R-mod. M

$$M \otimes_R S = 0 \iff M = 0,$$

or equivalently,

$$\{\text{prime ideals in } S\} \twoheadrightarrow \{\text{prime ideals in } R\}$$

If $R \to S$ is faithfully flat, then

$$R \to S \rightrightarrows S \otimes_R S$$

is an equalizer.

$$A = \mathbb{Z}[a_1, \ldots, a_6]$$

$$\Gamma = A[\lambda^{\pm 1}, r, s, t]$$

$(\operatorname{Spec} A, \operatorname{Spec} \Gamma)$ a groupoid in Aff

(A, Γ) a Hopf algebroid.

This gives a sheaf of groupoids on Aff for the flat topology:

$$R \mapsto (\operatorname{Ring}(A, R), \operatorname{Ring}(\Gamma, R)).$$

This is a sheaf of groupoids with

obj : $y^2 + a_1xy + a_3y = x^3 + \cdots$ over R

mor : changes of coord.

— not a stack. The assoc. stack

$$\text{ass}(\operatorname{Spec} A, \operatorname{Spec} \Gamma) = \mathcal{M}_{\text{Weier}}$$

has

$$\mathcal{M}_{\text{Weier}}(\operatorname{Spec} R) = \left\{ \begin{array}{ll} E & \text{proper of} \\ & \text{rel. dim. 1 with} \\ e \uparrow\downarrow & \text{all geometric fibers} \\ & \text{reduced irreducible} \\ \operatorname{Spec} R & \text{arithmetic genus 1} \\ & e \rightarrow \text{sm. point} \\ & \\ & + \text{ ISO.'s} \end{array} \right\}$$

Ex (topological) G acting on X $\rightsquigarrow$ groupoid with

$$\text{ob} = X$$

$$\text{map}(x,y) = \{ g \in G \mid gx = y \}.$$

— not a stack on spaces over X.

Assoc. stack is

$$\mathrm{ass}(X_0, X_1)(U) = \begin{array}{l}\text{cat. of principal}\\ G\text{-bdls } E \to U\\ \text{together with a}\\ \text{section}\end{array}$$

$$\begin{tikzcd} E \times_U E \\ \\ U \arrow[uu, hook, bend left, "s"] \arrow[uu, leftarrow] \end{tikzcd}$$

//

Formal groups:

$\mathcal{M}_{FG}(R)$ — formal groups over R + iso's.

$L = \pi_0 MP$ — ring represents universal formal group law.

$\Gamma = L[t_0^{\pm 1}, t_1, t_2, \ldots]$ — universal iso.

$$x \longmapsto t_0 x + t_1 x^2 + t_2 x^3 + \cdots$$

$= \pi_0 (MP \wedge MP)$

$(\mathrm{Spec}\, L, \mathrm{Spec}\, \Gamma)$ — groupoid

$\mathrm{ass}(\mathrm{Spec}\, L, \mathrm{Spec}\, \Gamma) = \mathcal{M}_{FG}$.