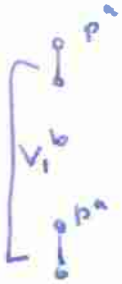
1

$L_{K(2)}\mathrm{tmf} = \varprojlim v_2^{-1}\mathrm{tmf}/p^a, v_1^b$

Mike 4/23/03

① $M(a,b) = (S^0 \cup_{p^a} e^1) \cup_{v_1^b} C\Sigma^{?}(S^0 \cup_{p^a} e^1)$



Claim 1:
~~for~~ any X, $L_{K(2)}X = \varprojlim_{a,b} L_{K(2)} X \wedge M(a,b)$

true, because

smashing w/ fin. spectra commutes with any localization

$$L_{K(2)}X \to \varprojlim_{a,b} L_{K(2)}X \wedge M(a,b)$$

& both sides are $K(2)$-local

need to show that the map is a ~~w.e.~~ $K(2)_*$-equiv. F = type 2 finite complex.

Künneth: suffices to show

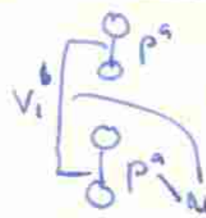
$$X \wedge F \to \left(\varprojlim \{L_{K(2)}X \wedge M(a,b)\}\right) \wedge F$$

a $K(2)$ iso.

$$\| \qquad \varprojlim\left(L_{K(2)}X \wedge M(a,b) \wedge F\right)$$

smashing w/ fin comm w/ $\varprojlim$

$0 \simeq F$



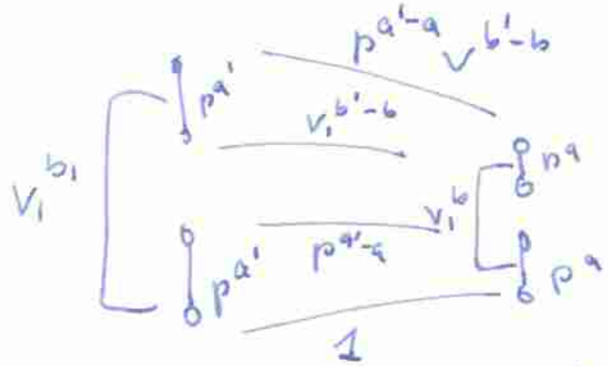
$M(a,b) \wedge F$

nilpotent maps in F $\Rightarrow$ for $a,b \gg 0$ both maps are null and

$$\Rightarrow M(a,b) \wedge F = F \vee \Sigma F \vee \Sigma^{|v_1^b|+1} F \vee \Sigma^{|v_1^b|+2} F$$

Now for $a,b \gg 0$, what do the maps look like

②



these will all become 0 for difference big enough

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so let's replace our system with one where they all are zero (apart of course from 1)

$\leadsto \varprojlim$ = just the bottom copy

(i.e. cofibre of ... $*$ = contractible).

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proves $L_{K(2)} X = \varprojlim L_{K(2)} X \wedge M(a,b)$ $\circlearrowleft v_2 = K(2)$-eq

$\Rightarrow \quad = \varprojlim L_{K(2)} X \wedge v_2^{-1} M(a,b)$

Now suppose R is a BP-module, then

$R \wedge v_2^{-1} M(a,b)$ is $K(2)$-local

reason

$\langle BP \wedge v_2^{-1} M(a,b) \rangle = \langle K(2) \rangle$

Module over ring spectrum is local over it b/c any acyclic mapping factors over a $R \wedge \ldots$

③

Suppose $R \wedge A$ = a BP-module, A fin type 0

$\Rightarrow R \wedge v_2^{-1} M(a_1, b)$ is $K(2)$-local

$\{A \mid$ with the property $\}$ = thick

$R \wedge v_2^{-1} M(a_1, b) \wedge A$ is $(K(2))$-local	$\Rightarrow L_{K(2)} R = \varprojlim_{M(a,b)} v_2^{-1} R \wedge$

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Now let's look at the case $R = tmf$

$A = D(A_1)$ then $R \wedge A = BP\langle 2\rangle$

$M_n \to L_n \to L_{n-1}$	GTM versus $K(n)$

this can be understood in terms of $K(n)$-local htpy theory

$L_{K(n)} X = \varprojlim L_n X \wedge M(a_0 \dots a_{n-1})$

$M_n X = \varinjlim L_n X \wedge D$ ———— n ————

Morava module

$A \quad \Gamma$	$\Gamma = A[r, s, t][\lambda^{\pm 1}]$
$x' = \lambda^3 (x + r)$	?
$y' = \lambda^2 (y + sx + t)$	

λ should go in here

(4) Let's take the following Hopf algebroid

$$\underbrace{\mathbb{Z} \rightrightarrows \mathbb{Z}[\lambda, \lambda^{-1}] \rightrightarrows}_{G_m}$$

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claim: G_m-comodules $\cong$ graded abelian groups

$$A \to A[\lambda, \lambda^{-1}] \qquad A = \bigoplus_{n \in \mathbb{Z}} A_n$$

$$a_n \otimes \lambda^n \longleftarrow a_n \in A_n$$

conversely, (exercise) If A is a comodule,

$$A_n := \{a \mid \psi(a) = a \otimes \lambda^n\}$$

$$\Rightarrow A = \bigoplus A_n$$

Consequences: no higher cohomology

$$A_n = \mathrm{Hom}_{\text{cot comod}}(\mathbb{Z}(n), A)$$

$$\mathbb{Z}(n) \longrightarrow \mathbb{Z}(n) \otimes \mathbb{Z}[\lambda^{\pm 1}]$$

$$1 \longmapsto 1 \otimes \lambda^n$$

$\Rightarrow$ no higher Ext groups

5

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Extension of Hopf algebras

$$A[r,s,t] \to A[r,s,t][\lambda^{\pm 1}] \to \mathbb{Z}[\lambda^{\pm 1}]$$

having λ in is equivalent to not having it in & having a grading.

$$\begin{tikzcd} & \mathcal{M}_{FGL} \arrow[d] \\ \mathcal{M}_{Ell} \arrow[r, "flat"'] & \mathcal{M}_{FG} \arrow[u, phantom, "\Uparrow"] \end{tikzcd}$$

flat $\longrightarrow \mathcal{M}_{FGL}$

$$tmf \longrightarrow tmf \wedge X(4) \rightrightarrows tmf \wedge X(4) \wedge X(4) \cdots$$

MU - injection

$$tmf \wedge MU \rightleftarrows tmf \wedge X(4) \wedge MU$$
$$\cong$$
$$tmf \wedge X(4)[x_5, x_6, \dots]$$

6

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$$tmf \wedge MU \longrightarrow tmf \wedge X(4) \wedge MU$$

exact: splits

b/c

injective:

but $tmf \wedge X(4)$ is cx orttd, $\Rightarrow$ MU-algebras

i.e. retracts of $MU \wedge \dots$

to show: this is MU-inj. resolution

i.e. each inj.

& exact after $\wedge MU$

Mike's email

(1)

Suppose that J is a generalised elliptic curve over a ring $U = \operatorname{spec} R$.

Then J is classified by a map

$$(*) \quad U \longrightarrow \mathcal{M}_{Ell},$$

where $\mathcal{M}_{Ell}$ is the moduli stack of elliptic curves.

If the map $(*)$ is flat then the formal completion of J is a Landweber exact formal group over R, and so gives a cohomology theory, $E(J)$.

<u>Theorem</u>: There is a unique (up to weak equivalence) lift of E to a sheaf of A_∞-ring spectra on the moduli stack of generalised elliptic curves in the "quasi-étale topology". (without additive life!)

Let's call this sheaf E. The spectrum tmf is the (-1) connected cover of $E(\mathcal{M}_{Ell})$:

$$\mathrm{tmf} = E(\mathcal{M}^{\circ}_{Ell}) \langle 0 .. \mathrm{infty} \rangle.$$

This produces tmf as an A_∞ ring spectrum. The notes on $K(1)$-local E_∞-ring spectra give this at the primes 2 and 3. I haven't worked out the details at larger primes (where I think they are easier)

Generalised elliptic curve: genus one curve with a marked smooth point, with at worst nodes as singularities.

Mike's email quasi-étale topology : $R \to S$ is quasi étale if
(2) it is flat and if the relative cotangent complex
is acyclic.

(2) Here are some other ~~more~~ things you might have heard : Let $X(4)$ denote the Thom spectrum of the map

$$\Omega SU(4) \to \Omega SU = BU.$$

Then

$$tmf \wedge X(4)$$

is complex orientable. In fact

$$\pi_* tmf \wedge X(4) = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6] = A,$$

and the formal group law is the one coming from the Weierstraß elliptic curve. Even more, the Hopf algebroid (A, Γ)

$$A = \pi_*(tmf \wedge X(4))$$
$$\Gamma = \pi_*(tmf \wedge X(4) \wedge X(4)) = A[r, s, t]$$

is just the Weierstraß Hopf algebroid, correspon-ding to the change of coordinates

$$x \mapsto x + r$$
$$y \mapsto y + sx + t$$

in the Weierstraß curve

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.$$

It follows that the $X(4)$ based Adams spectral sequence for tmf (which can easily be shown to be the MU-based ASS) has E^2-term the coho-

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(3)

mology of the Weierstraß Hopf-algebroid.

These facts follow easily from the fact that one can identify

Weierstraß curve = an elliptic curve with a coordinate modulo degree 5.

There is a proof in that long unfinished paper "Elliptic Curves and Stable Homotopy".

(3) One can compute

$$H^*(tmf;\mathbb{Z}/2) = A//A_2$$

where A_2 is the subalgebra of the mod 2 Steenrod algebra generated by Sq^1, Sq^2, and Sq^4 (you're supposed to be reminded of the computation

$$H^*(bo;\mathbb{Z}/2) = A//A_1.$$

It follows that, at the prime 2,

$$tmf \wedge D = BP\langle 2\rangle,$$

where D is a finite spectrum whose cohomology is "double of A_1" $= A_2//E[Q_0,Q_1]$.

This might characterize tmf, but I'm not sure.

I hope these comments help.

John:

$$L_{K(2)}\,tmf = \cancel{\;} \; EO_2$$

What I talked about was why (for $p=2$) Bolim of the elliptic spectra associated to elliptic curves in a formal neighborhood of the super-singular curve $C: y^2+y = x^3$, which should be $L_{K(2)}\,tmf$, is equivalent to the G_{48}-htpy fixed pts of E_2. Within the formal nghd of C there are the Deuring curves

$$y^2 + u_1 xy + y = x^3,$$

parametrized by $(E_2)_0 = W\mathbb{F}_4[[u_1]]$, and the quotient (D,Φ) of the elliptic curve Hopf algebroid (A,Λ) induced by the map $A \longrightarrow (E_2)_0$ that takes $a_1 \mapsto u_1$, $a_3 \mapsto 1$ and $a_2, a_4, a_6 \mapsto 0$ is a split Hopf algebroid, namely the one associated to the G_{48}-action on E_2.