1. (10 points) Below are six statements about $n \times n$ **invertible** matrices A and B. Circle TRUE or FALSE. Provide a counterexample for each *false* statement. (You **do not need** to justify the *true* statements.) Each part is worth 2 points.

(a) TRUE FALSE $(AB)^{-1} = B^{-1}A^{-1}$.

(b) TRUE FALSE $\det((AB)^T) = \det A \det B$.

$\det((AB)^T) = \det(B^T A^T) = \det B^T \cdot \det A^T$

$= \det B \cdot \det A = \det A \cdot \det B$

(c) TRUE FALSE $A + B$ must be nonsingular.

$n=1 \quad A = (1), \quad B = (-1), \quad A+B = (0)$

(d) TRUE FALSE $\operatorname{rank}(AB) = n$.

A, B invertible $\Rightarrow AB$ invertible

$\Rightarrow \operatorname{rank}(AB) = n$

(e) TRUE FALSE $\ker A = \{0\}$.

$A\vec{z} = \vec{0}$ has a unique solution,

b/c A is nonsingular (invertible).

2. (15 points) Let

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & a & -1 \\ a & 1 & a \end{pmatrix},$$

for an arbitrary number $a \in \mathbb{R}$.

(a) (8 points) For what values of a is A regular? For these values of a, write down an LU factorization of A. Why doesn't your formula make sense for other values of a?

$$\begin{pmatrix} 1 & 0 & +1 \\ 1 & a & -1 \\ a & 1 & a \end{pmatrix} \xrightarrow[-a(1)+(3)]{-(+1)+(2)} \begin{pmatrix} 1 & 0 & +1 \\ 0 & a & -2 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow[a \neq 0]{-\frac{1}{a}(2)+(3)} \begin{pmatrix} 1 & 0 & 1 \\ 0 & a & -2 \\ 0 & 0 & \frac{2}{a} \end{pmatrix}$$

Thus, a is regular, iff $a \neq 0$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ a & 1/a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & a & -2 \\ 0 & 0 & \frac{2}{a} \end{pmatrix}$$

This formula does not make sense for $a=0$, because it ~~conto~~ would contain division by 0.

(b) (7 points) For what values of a is A nonsingular but not regular? For each of these values of a, write down a permuted LU factorization of A.

For $a=0$ A is not regular by (a).

For $a=0$ ~~$A = \begin{pmatrix} 1 & 0 & +1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{swap(2,3)} \begin{pmatrix} 1 & 0 \end{pmatrix}$~~

as above in (a)

$$A \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{swap(2,3)} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

(row-echelon form)

nonsingular, b/c diagonal entries are all nonzero

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{P} A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}}_{U}$$

3. (10 points) For a given $n \geq 2$, let $\mathcal{P}^{(n)}$ be the vector space of all polynomial functions of degree less than or equal to n, with addition and scalar multiplication defined in the standard way.

(a) (5 points) Let V be the set of polynomials $p(x)$ in $\mathcal{P}^{(n)}$ such that $p(2) = 0$. For example, $x - 2 \in V$, whereas $x^2 + 1$ is not. Is V a vector subspace of $\mathcal{P}^{(n)}$? Justify your answer.

$p_1, p_2 \in V \Rightarrow (p_1+p_2)(2) = p_1(2) + p_2(2) = 0 + 0 = 0$

$p \in V,\ c \in \mathbb{R} \Rightarrow (cp)(2) = c \cdot p(2) = c \cdot 0 = 0$

V is a subspace

(b) (5 points) Let W be the subset of $\mathcal{P}^{(n)}$ consisting of all polynomials of even degree and the zero polynomial. For instance, $1 + x + x^2$ is in W, whereas $1 + x$ is not. Is W a vector subspace of $\mathcal{P}^{(n)}$? Justify your answer.

No: $(1 + z + z^2) + (1 + z - z^2) = 2 + 2z$ is of odd degree

4. (15 points) Let A be a 4×5 matrix. Suppose another student, with whom you are working on the fundamental subspaces of A, claims to have found bases for the image and cokernel of A, namely

$$\text{img}\, A = \text{span}\left\{\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\2\end{pmatrix}\right\}$$

and

$$\text{coker}\, A = \text{span}\left\{\begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\0\\-1\end{pmatrix}\right\}.$$

Looking closer at these answers, you can tell that some things do not match.

(a) (7 points) Find **two** reasons why the other student is wrong, *i.e.*, identify **two** features of these bases that exclude them from being bases of the image and cokernel of one and the same matrix.

(1) dim img A = rank $A = r = 2$, A $m \times n$, $m=4, n=5$
dim coker $A = m - r = 4 - 2 \neq 3$ (Fund. Thm. of Linear Algebra)

(2) If $\vec{z} \in$ coker A, then $A^T\vec{z} = \vec{0}$, i.e., (every column of $A)^T \cdot \vec{z} = 0$. Then every ~~linear combination~~ vector from the column space ~~of A~~ img A of A multiplies by $\vec{z}$ to $\vec{0}$.
But $\begin{pmatrix}0 & 1 & -1 & 2\end{pmatrix}\begin{pmatrix}0\\1\\1\\1\end{pmatrix} = 2 \neq 0$

(b) (8 points) After taking up the problem yourself, you find that the original basis for the cokernel is correct, but the previously calculated image is wrong. Find a correct basis for the image. *Hint*: Figure out the expected dimension of img A first.

If dim coker $A = m - r = 3$, then
$r = 4 - 3 = 1$ and dim img $A = r = 1$.
Thus, all vectors of img A are scal~~ter~~ multiples of each other. ~~Ther~~ By above, they all multiply to 0 with all vectors, ~~including vector~~ from coker A.
Thus, we are solving the system $\begin{pmatrix}0&1&1&0\\0&1&1&1\\2&2&0&-1\end{pmatrix}$ or equivalently

$$\begin{pmatrix}2&2&0&-1\\0&1&1&0\\0&1&1&1\end{pmatrix} \longrightarrow \begin{pmatrix}2&2&0&-1\\0&1&1&0\\0&0&0&1\end{pmatrix}$$

x_3 free $\quad x_4 = 0 \quad x_3 = 1 \quad x_2 + x_3 = 0$

$x_2 = -1, \quad 2x_1 - 2 = 0, \quad x_1 = 1$

Or just note that $(1,-1,1,0)\cdot\vec{v} = 0$ $\forall \vec{v} \in$ coker A

Aswer: $\{(1,-1,1,0)\}$

5. (10 points)

(a) (8 points) Is the following matrix positive definite?

$$K = \begin{pmatrix} 1 & -1 & 3 \\ -1 & 3 & -1 \\ 3 & -1 & 12 \end{pmatrix}$$

(1) $K^T = K$

~~(2)~~

(2) $K \xrightarrow[-3(1)+(3)]{(1)+(2)} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix} \xrightarrow{-(2)+(3)} \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

$$K = LDL^T = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}}_{L^T}$$

The doagonal entries of D are $1, 2, 1 > 0$.
Thus K is positive definite.

(b) (2 points) Does the expression $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^T K \boldsymbol{y}$ define an inner product on $\mathbb{R}^3$? Why or why not?

$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T K \vec{y}$ defines an inner product, iff $K > 0$. Thus, $\vec{x}^T K \vec{y}$ defines an inner product on $\mathbb{R}^3$.