MATH 4242: APPLIED LINEAR ALGEBRA
SELECTED SOLUTIONS TO SAMPLE TEST 1

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Problem 4. For the digraph 2.6.3(a) on p. 127 of the text,

(1) Find the incidence matrix;
(2) Find a basis of the cokernel of the incidence matrix.
(3) What is the dimension of the cokernel and what does it tell you about the number of independent circuits in the digraph.

Solution: Answers to (1) and (2) will depend on your choice of labeling the vertices and the edges. You have to choose a labeling first. I am choosing labeling of vertices that starts with the lower left vertex and goes around the graph clockwise. I am also labeling the edge which is not part of the triangle first, the longer side of the triangle as my edge #2 and the other sides of the triangle clockwise from there.

(1) This gives us the following incidence matrix:

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

(2) By definition, $\operatorname{coker} A = \ker A^T = \{\boldsymbol{z} \in \mathbb{R}^4 \mid A^T\boldsymbol{z} = \mathbf{0}\}$, and all we need to do is to find a basis in the space of solutions of the homogeneous linear system $A^T\boldsymbol{z} = \mathbf{0}$. Using Gaussian elimination over A^T, we then reduce it to row echelon form

$$\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

[*Comment not needed for a complete solution, but useful for double-checking of whether we are on the right track.*: This is not surprising, as $\operatorname{rank} A = \operatorname{rank} A^T$ is supposed to be # vertices $-1 = 4 - 1 = 3$, given that the graph is connected. Thus, $\dim \ker A^T =$ # edges $- \operatorname{rank} A = 1$, and we just have to be looking for one basis vector.]

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We have got one free variable z_4, which we may set to 1 to get a basis: $z_4 = 1$. Back substitution gives us $z_3 = z_2 = -1$ and $z_1 = 0$. And here is our basis for $\operatorname{coker} A$:

$$\left\{ \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

(3) Since the basis in (2) had just one element, $\dim \operatorname{coker} A = 1$. This is also known to be the number of independent circuits of the digraph.

Problem 5. Which of the following formulas define a norm on $\mathbb{R}^2$? Briefly justify your answer.

(1) $\|(x, y)\| = \min\{|x|, |y|\}$.

Solution: Do not be trapped by this looking like the ∞ norm, as I actually did when I was solving it. It is not, as it uses minimum rather than maximum. You figure, the max in the ∞ norm should have been there for a reason and produce an educated guess: No, it is not a norm. The rest is to find a counterexample to one of the norm axioms. For instance, $\|(1, 0)\| = \min\{1, 0\} = 0$, and this violates the positivity property $\|(x, y)\| > 0$ for all $(x, y) \neq \mathbf{0}$.

(2) $\|(x, y)\| = |x + y| + |x - y|$.

Solution: Here the answer is yes. Explanation consists in checking all the three axioms of a norm.

$$\|c(x, y)\| = |cx + cy| + |cx - cy| = |c| \, \|(x, y)\| \, ;$$
$$\|(x, y)\| = |x + y| + |x - y| \geq 0; \text{ with } = 0 \text{ iff } (x, y) = \mathbf{0}.$$

Indeed, $|x + y| + |x - y| = 0$ if and only if $x + y = 0 = x - y$, which happens exactly when $x = y = 0$. Finally,

$$\|(x_1, y_1) + (x_2, y_2)\| = |x_1 + x_2 + y_1 + y_2| + |x_1 + x_2 - y_1 - y_2|$$
$$\leq |x_1 + y_1| + |x_2 + y_2| + |x_1 - y_1| + |x_2 - y_2| = \|(x_1, y_1)\| + \|(x_2, y_2)\| \, .$$

The inequality in the middle is the standard inequality $|a + b| \leq |a| + |b|$ for real numbers (which happens to be the triangle inequality for the 1 norm on $\mathbb{R}^1$).

(3) $\|(x, y)\| = |x|^3 + |y|^3$.

Solution: This is just like the 3 norm, but without the cube root over the right-hand side. Again, you figure, there must be a reason for the cube root there and come up with an educated guess : No. Indeed, $\|2(1, 0)\| = \|(2, 0)\| = 8 \neq 2 = 2 \, \|(1, 0)\|$.