1. (15 points) In the following problem, use the standard Euclidean dot product.

(a) (5 points) Find a basis for the orthogonal complement of $W \subset \mathbb{R}^4$ where

$$W = \text{span}\left\{ \begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}2\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\2\\2\\2\end{pmatrix} \right\}.$$

These are linearly independent, so $\dim W = 3$ and $\dim W^\perp = 4-3=1$. Thus, we are looking for just one basis vector, $\vec{v}$:

$$\begin{pmatrix}1&0&0&-1\\2&0&0&0\\1&2&2&2\end{pmatrix}\vec{v}=0 \qquad \begin{pmatrix}1&0&0&-1\\2&0&0&0\\1&2&2&2\end{pmatrix} \xrightarrow[-(1)+(3)]{-2(1)+(2)} \begin{pmatrix}1&0&0&-1\\0&0&0&2\\0&2&2&3\end{pmatrix} \xrightarrow{\text{swap}(2,3)} \begin{pmatrix}1&0&0&-1\\0&2&2&3\\0&0&0&2\end{pmatrix}$$

$x_3 = 1$ (free) $\quad x_4 = 0 \quad 2x_2 + 2x_3 = 0 \quad x_2 = -1 \quad x_1 - x_4 = 0 \quad x_1 = 0$

$\vec{v} = (0,-1,1,0)^T$

(b) (10 points) Find an orthonormal basis for the space $W \subset \mathbb{R}^4$. Gram-Schmidt

$$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} \qquad \begin{pmatrix}2\\0\\0\\0\end{pmatrix} - \frac{1}{\sqrt{2}}\left\langle \begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}2\\0\\0\\0\end{pmatrix}\right\rangle \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

$$= \begin{pmatrix}2\\0\\0\\0\end{pmatrix} - \frac{1}{2}\cdot 2\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\2\\2\\2\end{pmatrix} - \frac{1}{2}\left\langle \begin{pmatrix}1\\2\\2\\2\end{pmatrix}, \begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\right\rangle \begin{pmatrix}1\\0\\0\\-1\end{pmatrix} - \frac{1}{2}\left\langle \begin{pmatrix}1\\2\\2\\2\end{pmatrix}, \begin{pmatrix}1\\0\\0\\+1\end{pmatrix}\right\rangle \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\2\\2\\2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} - \frac{3}{2}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\2\\2\\0\end{pmatrix}$$

$$\vec{v}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

2. (10 points) For the following problems, circle TRUE or FALSE. **If false, state why.**

(a) (3 points) TRUE FALSE Let Q be an orthogonal matrix. Then $\det(Q) = \pm 1$.

$(QQ^T = I \Rightarrow \det Q \det Q^T = 1$, but $\det Q^T = \det Q)$

(b) (3 points) TRUE FALSE Every linearly independent basis is an orthogonal basis.

$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^2$

(c) (4 points) TRUE FALSE The quadratic function

$$f(x, y) = x^2 + 6xy - 2y^2 - 8x + 5y + 12$$

has a unique global minimum.

$K = \begin{pmatrix} 1 & 3 \\ 3 & -2 \end{pmatrix}$ $\det K = -2 - 9 < 0 \Rightarrow$ The form

$\vec{x}^T K \vec{x} = x^2 + 6xy - 2y^2$ is not positive definite

Look for values that go to $-\infty$!

$x = 0$: ~~f(0,y)~~ $f(0, y) = -2y^2 + 5y + 12$ will become unboundedly negative as y gets larger

3. (15 points) Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation whose matrix representation with respect to the standard basis is given by

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 4 & -1 & 1 \\ -2 & 0 & 0 \end{pmatrix}.$$

Find the matrix representation of L with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

$$A\vec{v}_1 = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix} = 2\vec{v}_1$$

$$L\vec{v}_j = \sum_{i=1}^{3} b_{ij} \vec{v}_i$$

$$A\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -\vec{v}_2$$

$$B = (b_{ij})$$

$$A\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \vec{v}_1 + \vec{v}_3$$

$$\cancel{B4} \quad B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

5. (15 points) Let A be a symmetric 3×3 matrix with eigenvalue and eigenvector pairs as follows:

$$\lambda_1 = 2, \qquad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$\lambda_2 = -1, \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$

$$\lambda_3 = -1, \qquad \mathbf{v}_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

(a) (3 points) What is the determinant of A?

$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 2$

(b) (3 points) Is A positive definite? Why or why not?

No, because not all eigenvalues are > 0.

(c) (3 points) How many Jordan blocks for the Jordan canonical form of A have? Briefly justify your answer. (You do **not** need to find the Jordan canonical form to answer this question!)

Three, because every symmetric matrix is diagonalizable.

(d) (6 points) Write out the spectral factorization of A if possible. If not, state why.

$A = Q\Lambda Q^T$ $\quad \Lambda = \begin{pmatrix} 2 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix}$

The columns of Q will be elements of an orthonormal basis in which A diagonalizes to Λ. $\vec{v}_1 \perp \vec{v}_2$ & $\vec{v}_3$, so we need to normalize $\vec{v}_1$ and apply Gram-Schmidt to $\{\vec{v}_2, \vec{v}_3\}$:

$\vec{u}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$

$\vec{u}_2 = \dfrac{\vec{v}_2}{\|\vec{v}_2\|} = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$; $\vec{w}_3 := \vec{v}_3 - \langle \vec{v}_3, \vec{u}_2 \rangle \vec{u}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1 \\ -1/2 \end{pmatrix}$;

$\vec{u}_3 = \dfrac{\vec{w}_3}{\|\vec{w}_3\|} = \dfrac{\sqrt{2}}{\sqrt{3}} \begin{pmatrix} -1/2 \\ 1 \\ -1/2 \end{pmatrix}$; $\quad Q = \begin{pmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & \sqrt{2}/\sqrt{3} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix}$

$\|\vec{w}_3\| = \sqrt{\frac{1}{4} + 1 + \frac{1}{4}} = \dfrac{\sqrt{3}}{\sqrt{2}}$