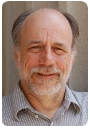
Unfinished Business

Dennis E. White

November 30, 2012

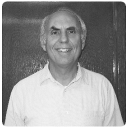
Outline

Heads



Peter Olver

Gene Fabes



Larry Gray

Dick McGehee

Naresh Jain

Willard Miller

My Head

Johannes C. C. Nitsche

Deans


Steven Crouch


Francis Kulacki

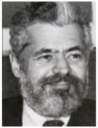

Rama Murthy


Ted Davis


Gordon Beavers


Roger Staehle


Willard Miller


Ettore Infante


Walter Johnson

My Dean

Richard Swalin

Presidents

Eric Kaler

Nils Hasselmo

Robert Bruininks

Kenneth Keller

Mark Yudolf

C. Peter Magrath

My President

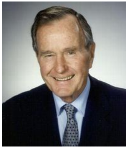
Malcolm Moos

More Presidents

Bill Clinton

Barack Obama

Jimmy Carter



G. H. W. Bush

G. W. Bush

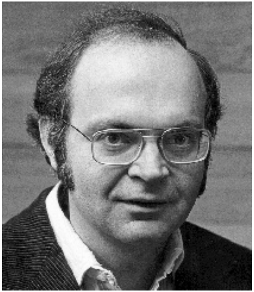
Gerald Ford

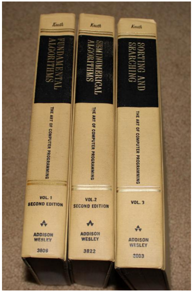
Ronald Reagan

My President

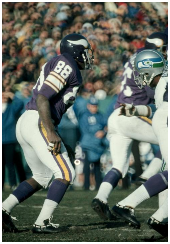
In the Beginning...

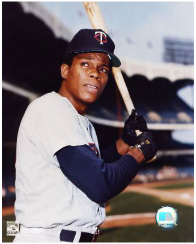
My Inspiration





In 1973...



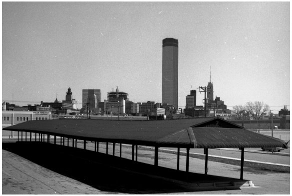


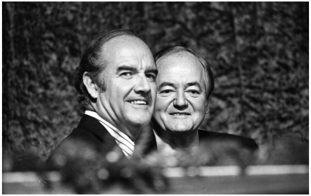
In 1973...

In 1973...

In 1973...

In 1973...



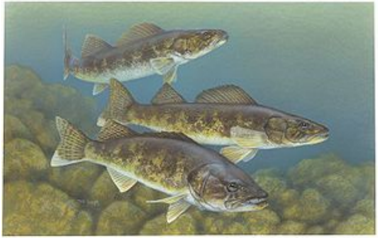


L'Étoile du Nord

State Flower (1902)

State Bird (1961)

State Tree (1953)



State Fish (1965)

L'Étoile du Nord

State Grain (1977)

State Butterfly (1998)

State Fungus (1984)

State Fruit (2006)

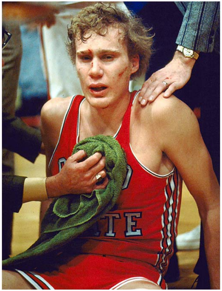
Minnesota Icons

Big Ten Sports

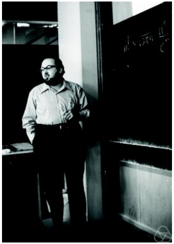
Big Ten Football

- Gopher football record since I've been here: 203-257
- Number of Gopher football coaches with winning records since I've been here: 0
- Number of years before I arrived that the Gophers were last in a major bowl game: 12

Big Ten Basketball



The Start of Combinatorics at Minnesota



Ed Bender's Wisdom

“The trick in mathematics is to find problems that are neither trivially impossible nor impossibly trivial.”

Pólya meets Schensted

George Pólya

Craige Schensted
"Ea"

Some Definitions

- G: a finite permutation group acting on $[n]$.
- Δ_μ: the orbits of μ-colorings of $[n]$ under the action of G.
- C_ρ: the permutations in S_n of type ρ.
- m, p and s: the usual symmetric function bases: monomial, power sum and Schur.
- $K_{\lambda,\mu}$: the Kostka number, the number of semistandard Young tableaux (SSYT) of type μ and shape λ.
- f_λ: the number of standard Young tableaux (SYT) of shape λ.
- χ^λ_ρ: the λ irreducible S_n character, at the conjugate class ρ.

A Calculation

$$\begin{aligned}
\sum_{\mu} |\Delta_\mu| m_\mu &= \frac{1}{|G|} \sum_{\rho} |G \cap C_\rho| p_\rho \\
&= \frac{1}{|G|} \sum_{\rho} |G \cap C_\rho| \sum_{\lambda} \chi^\lambda_\rho s_\lambda \\
&= \frac{1}{|G|} \sum_{\rho} |G \cap C_\rho| \sum_{\lambda} \sum_{\mu} \chi^\lambda_\rho K_{\lambda,\mu} m_\mu \\
&= \sum_{\mu} \sum_{\lambda} \left(\sum_{\rho} \frac{|G \cap C_\rho|}{|G|} \chi^\lambda_\rho \right) K_{\lambda,\mu} m_\mu \\
&= \sum_{\mu} \left(\sum_{\lambda} K_{\lambda,G} K_{\lambda,\mu} \right) m_\mu
\end{aligned}$$

Problems

It follows that

$$|\Delta_\mu| = \sum_\lambda K_{\lambda,\mu} K_{\lambda,G} \,.$$

$K_{\lambda,G}$ is the number of occurrences of the irreducible λ in the inducement of the trivial character of G up to S_n.
It is an integer.

Problem

Interpret $K_{\lambda,G}$ as a subset of standard Young tableaux of shape λ.

Problem

Give a Schensted-like proof that

$$|\Delta_\mu| = \sum_\lambda K_{\lambda,\mu} K_{\lambda,G} \,.$$

Notes

- If $G = S_\nu = S_{\nu_1} \times S_{\nu_2} \times \dots$, then $K_{\lambda,G} = K_{\lambda,\nu}$ and

$$|\Delta_\mu| = \sum_\lambda K_{\lambda,\mu} K_{\lambda,\nu}\,,$$

  follows from the Robinson-Schensted-Knuth correspondence.
- For any G, $K_{\lambda,G} \le f_\lambda$.

An Example

Write

$$h_G = \sum_\lambda K_{\lambda,G} s_\lambda \,.$$

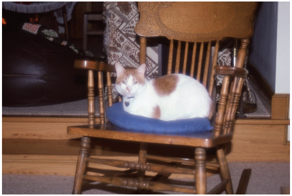
For cyclic groups Z_n, for $n = 2, 3, 4, 5, 6$:

$$h_{Z_2} = s_2$$
$$h_{Z_3} = s_3 + s_{1^3}$$
$$h_{Z_4} = s_4 + s_{2^2} + s_{2\,1^2}$$
$$h_{Z_5} = s_5 + s_{3\,2} + 2s_{3\,1^2} + s_{2^2\,1} + s_{1^5}$$
$$h_{Z_6} = s_6 + 2s_{4\,2} + 2s_{4\,1^2} + s_{3^2} + 2s_{3\,2\,1} + 2s_{3\,1^3} + 2s_{2^3} + s_{2^2\,1^2} + s_{2\,1^4}$$

Oberwolfach, 1980

Our Group Expands

We Have Somthing in Common



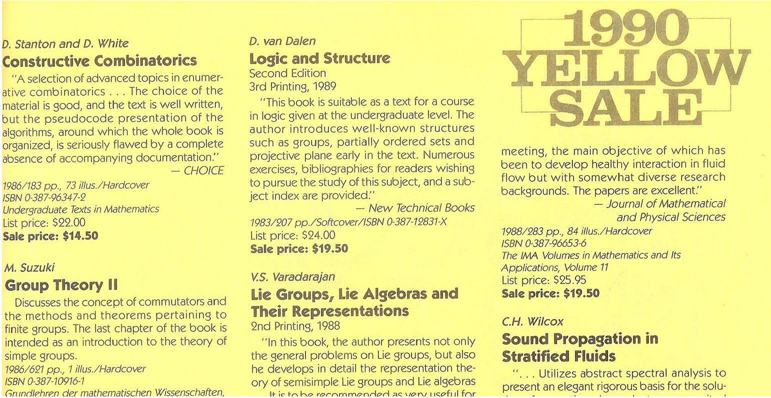
We Have Something in Common




We Have Something Else in Common

We Have Something Else in Common

We Write a Book



D. Stanton and D. White

Constructive Combinatorics

"A selection of advanced topics in enumerative combinatorics . . . The choice of the material is good, and the text is well written, but the pseudocode presentation of the algorithms, around which the whole book is organized, is seriously flawed by a complete absence of accompanying documentation."

— *CHOICE*

1986/183 pp., 73 illus./Hardcover
ISBN 0-387-96347-2
Undergraduate Texts in Mathematics
List price: $22.00
Sale price: $14.50

M. Suzuki

Group Theory II

Discusses the concept of commutators and the methods and theorems pertaining to finite groups. The last chapter of the book is intended as an introduction to the theory of simple groups.

1986/621 pp., 1 illus./Hardcover
ISBN 0-387-10916-1
Grundlehren der mathematischen Wissenschaften,

D. van Dalen

Logic and Structure
Second Edition
3rd Printing, 1989

"This book is suitable as a text for a course in logic given at the undergraduate level. The author introduces well-known structures such as groups, partially ordered sets and projective plane early in the text. Numerous exercises, bibliographies for readers wishing to pursue the study of this subject, and a subject index are provided."

— *New Technical Books*

1983/207 pp./Softcover/ISBN 0-387-12831-X
List price: $24.00
Sale price: $19.50

V.S. Varadarajan

Lie Groups, Lie Algebras and Their Representations
2nd Printing, 1988

"In this book, the author presents not only the general problems on Lie groups, but also he develops in detail the representation theory of semisimple Lie groups and Lie algebras

1990 YELLOW SALE

meeting, the main objective of which has been to develop healthy interaction in fluid flow but with somewhat diverse research backgrounds. The papers are excellent."

— *Journal of Mathematical and Physical Sciences*

1988/283 pp., 84 illus./Hardcover
ISBN 0-387-96653-6
The IMA Volumes in Mathematics and Its Applications, Volume 11
List price: $25.95
Sale price: $19.50

C.H. Wilcox

Sound Propagation in Stratified Fluids

". . . Utilizes abstract spectral analysis to present an elegant rigorous basis for the solu-

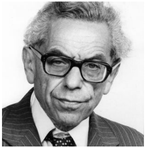
G. W. Peck

G.

W.

P

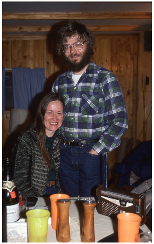
E



C

K

Background



Background

- W.- Williamson (1977): All natural matchings in the boolean algebra B_n are essentially the same and yield a symmetric chain decomposition (SCD).
- W. (1980): B_n reduced by a group action is rank unimodal.
- Stanley (1984): Unitary Peck posets reduced by a group action are Peck.
- Peck poset: graded, rank symmetric, rank unimodal, strongly Sperner.

The Problem

- The most famous example of Stanley's theorem: partitions inside a rectangle (B_n reduced by a wreath product of two symmetric groups). Rank generating function is the q-binomial coefficient.
- Existence of an SCD would immediately imply Peck.

Problem

For what groups G does B_n/G have an SCD? Can the SCD on B_n be modified to give an SCD on B_n/G?

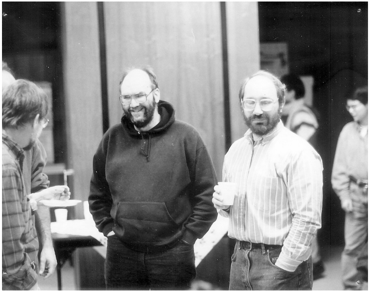
Recent Progress

- Problem dates back to Stanley (1980).
- Stated by Canfield and Mason (2006).
- Hersh and Schilling (2012): True for cyclic groups, using the natural boolean algebra matching.

IMA Year, 1987-1988

IMA Year, 1987-1988

New Arrivals

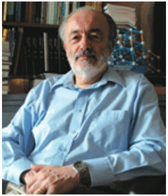


Minnesota Events

Invited Speakers:

- Noga Alon (Israel)
- R.J. Baxter (Melbourne, Australia)
- Francesco Brenti (Perugia, Italy)
- D. Jackson (Waterloo, Canada)
- Bernard Leclerc (Caen, France)
- Victor Reiner (USA)
- Michelle Wachs (Miami, USA)
- Günter Ziegler (Germany)

FPSAC '96



FPSAC '96

No one else is to blame for this problem...

The Cone of Log-Concavity

- C_N^k: cone generated by products (homogeneous of degree N) of Schur functions s_λ, where $l(\lambda) \leq k$.
- Determine the extreme vectors.
- $k = 1$: C_N^1 is the cone generated by h_λ, $\lambda \vdash N$, h_λ will be extreme.
- $k \geq N$: C_N^N is the cone generated by products of Schur functions $\implies$ s_λ will be extreme (Littlewood Richardson rule).
- $k = 2$: Jacobi-Trudi says cone is generated by products of the form
$$h_i h_j - h_{i+1} h_{j-1} \quad \text{and} \quad h_i \quad i \geq j \geq 1$$
- Call this last case *Cone of Log-Concavity*

Extreme Vector Examples

$$s_{3,1}s_2 = s_{3,2}s_1 + s_{1^2}s_4$$

For $N = 6$, there are 13 extreme vectors:

$$\begin{array}{lll} s_6 & s_4 s_{1^2} & s_3 s_{2,1} \\ s_{5,1} & s_{3,1} s_{1^2} & (s_{2,1})^2 \\ s_{4,2} & s_{2^2} s_2 & s_2 (s_{1^2})^2 \\ s_{3^2} & s_{2^2} s_{1^2} & (s_{1^2})^3 \\ s_{3,2} s_1 & & \end{array}$$

The Problem

Conjecture

A product $s_{\alpha_1} s_{\alpha_2} \dots$ is extreme in the cone of log-concavity if and only if no pair $\alpha_s = \lambda$ and $\alpha_t = \mu$ satisfies any one of the following conditions:

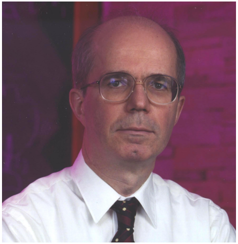
1. $\lambda = (\lambda_1 \geq \lambda_2 > 0)$, $\mu = (\mu_1 \geq \mu_2 > 0)$, *with*
$$\lambda_1 > \mu_1 \geq \lambda_2 > \mu_2\,;$$
2. $\lambda = (\lambda_1 > \lambda_2 > 0)$, $\mu = (\mu_1 > 0)$, *with*
$$\lambda_1 \geq \mu_1 \geq \lambda_2\,;$$
3. $\lambda = (\lambda_1 > 0)$, $\mu = (\mu_1 > 0)$.

Notes

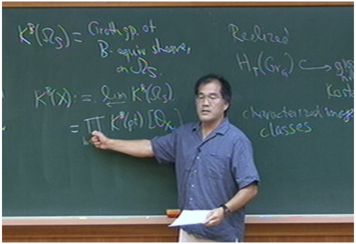
- If no such pair satisfies any of these conditions, we say the collection of Schur functions is *nested*.
- It is easy to show that if the collection is not nested, then the product is not extreme (see above example).
- It is also true (but not easy to prove) that if the set of partitions is nested and the collection of all parts is a distinct partition, then the product is extreme.
- Proof uses Littlewood-Richardson rule in a non-trivial way.
- Proof relies on Farkas' Lemma: $\mathbf{v}$ is extreme if and only if there is a hyperplane which separates $\mathbf{v}$ from all other generating vectors.

New Minnesota Icons

More Arrivals and a Departure



Sign-Balanced Posets



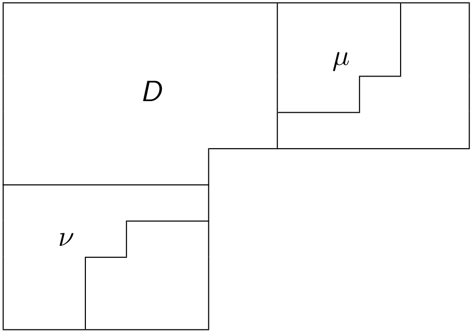
Background

- Ruskey (1993) asked: for which posets are the number of odd linear extensions and the number of even linear extensions equal (*sign-balanced*)?
- Most natural posets are sign-balanced (easy).
- For product of two chains $C_n \times C_m$ the answer was undecided.
- Ruskey(1993) proved sign-balanced when n and m even, conjectured sign-balanced when n and m odd, and conjectured not sign-balanced when $m \neq n \mod 2$ (and did not conjecture amount of imbalance)

Semi-Self-Complementary Shapes

- α and β rectangular Ferrers shapes with cellwise intersection E and cellwise union D.
- *Semi-self-complementary shapes (SSC)*: shape λ where
  1. λ contains D
  2. λ/D has two parts: μ, to the right of D; and ν, below D.
  3. μ and ν are complementary inside E
- *Twist (tw)* is $|\mu|$.

SSC Example



Shifted Shapes

- $Str(\alpha)$ is the strict shape associated with a rectangle α.
- Example: $\alpha = 6^3$, $Str(\alpha) = (8, 6, 4)$.
- g_λ is the number of SYT of shifted shape λ (strict).
- If μ and ν are two shapes, then $\mu \cup \nu$ is the partition whose parts are the multiset of the parts of μ and ν.
- Fact: $g_{Str(\alpha)} = f_\alpha$ (Combinatorial proofs: Worley (1984), Sagan (1987), Haiman (1989))

A Theorem

Theorem (W (2001))

If μ is a shape whose 2-quotient is a pair of rectangles, α and β, then the poset whose Hasse diagram is μ is sign-balanced if and only if $Str(\alpha) \cup Str(\beta)$ has a repeated part. Otherwise, the amount of imbalance is $\pm g_{Str(\alpha)\cup Str(\beta)}$.

Corollary (W (2001))

$C_n \times C_m$ is signed balanced if and only if $n = m \mod 2$. If $n \neq m \mod 2$, the amount of imbalance is $\pm g_\rho$ where $\rho = ((m+n-1)/2, (m+n-3)/2, \ldots, (|m-n|+1)/2)$.

The Problems

The theorem above follows from this theorem

Theorem (W (2001))

$$\sum_{\lambda}(-1)^{tw(\lambda)}f_{\lambda} = \begin{cases} \pm g_{Str(\alpha)\cup Str(\beta)}, & \text{if } Str(\alpha)\cup Str(\beta) \text{ distinct;} \\ 0, & \text{otherwise.} \end{cases}$$

where the sum is over SSC shapes.

Problem

Find a sign-reversing involution on standard Young tableaux of SSC shapes which proves this theorem.

Note: For $C_n \times C_m$, this involution exists (W).

Shifted Littlewood-Richardson Coefficients

$$f_\lambda = \sum_\rho c_{\rho,\lambda} g_\rho \,,$$

where $c_{\rho,\lambda}$ is the number of Littlewood-Richardson fillings of the shifted skew shape $(\lambda+\delta)/\rho$ of content δ, δ an appropriate triangular shape (Stembridge (1989)).

Shimozono's Refinement

Write:

$$\begin{aligned} \sum_{\lambda} (-1)^{tw(\lambda)} f_{\lambda} &= \sum_{\lambda} \sum_{\rho} (-1)^{tw(\lambda)} c_{\rho,\lambda} g_{\rho} \\ &= \sum_{\rho} \Big(\sum_{\lambda} (-1)^{tw(\lambda)} c_{\rho,\lambda} \Big) g_{\rho} \end{aligned}$$

where λ is SSC and ρ is strict.

Theorem (Shimozono (1999))

The inner sum above is 0 *unless* $\rho = Str(\alpha) \cup Str(\beta)$*, in which case it is* ± 1*.*

Proof uses Jing vertex operators and Schur Q symmetric functions.

Refined Problem

Problem

Give a sign-reversing involution which proves Shimozono's result.

Table of $c_{\rho,\lambda}$ for SSC shapes, $\alpha = 3^2$, $\beta = 2^2$ with $(-1)^{tw(\lambda)}$:

		ρ						
		7 3	7 2 1	6 4	6 3 1	5 4 1	5 3 2	4 3 2 1
	$[+]\, 5^2$	0	0	1	0	0	0	0
	$[-]\, 5\,4\,1$	1	0	2	1	1	0	0
	$[+]\, 5\,3\,2$	1	1	1	2	1	1	0
λ	$[+]\, 4^2\,1^2$	1	0	1	1	1	1	0
	$[-]\, 4\,3\,2\,1$	1	1	2	3	2	3	1
	$[+]\, 3^2\,2^2$	0	0	1	1	1	1	0

More Arrivals and Departures

Cyclic Sieving

Cyclic Sieving Phenomenon

- X a set; $X(q)$ a generating function, $X(1) = |X|$; C a cyclic action of order n on X.
- For $c \in C$, write $X(c)$ to mean $X(q)$ evaluated at the nth root of unity corresponding to c.
- $(X, X(q), C)$ is an instance of the *cyclic sieving phenomenon* (Reiner, Stanton, W, (2004)) if $X(c)$ is the number of $x \in X$ fixed by $c \in C$.
- Many instances of CSP discovered over the last several years (Sagan (2011)).

Promotion

Schützenberger (1963)

$$T = \begin{matrix} 1 & 2 & 6 \\ 3 & 4 & 7 \\ 5 & 8 & 9 \end{matrix} \qquad \begin{matrix} \color{red}{1} & \color{red}{2} & 6 \\ 3 & \color{red}{4} & \color{red}{7} \\ 5 & 8 & \color{red}{9} \end{matrix}$$

$$\begin{matrix} \color{red}{2} & \color{red}{4} & 6 \\ 3 & \color{red}{7} & \color{red}{9} \\ 5 & 8 & \end{matrix}$$

$$p(T) = \begin{matrix} 1 & 3 & 5 \\ 2 & 6 & 8 \\ 4 & 7 & 9 \end{matrix} \qquad \begin{matrix} 2 & 4 & 6 \\ 3 & 7 & 9 \\ 5 & 8 & \color{blue}{1} \end{matrix}$$

CSP on promotion

- X is standard tableaux of shape n^m.
- Promotion on X is a cyclic action C of order mn
- $f_\lambda(q)$ is the q-analog of the Frame-Robinson-Thrall hook formula for f_λ

Rhoades Result

Theorem (Rhoades (2010))
$(X, f_{n^m}(q), C)$ *is an instance of CSP.*

The proof uses Kazdan Lustig representation theory.

Is there a combinatorial proof?

A Combinatorial Framework, Part 1

- $f_\lambda(q)$ evaluated at a primitive kth root of unity is the number of k rim hook tableaux of shape λ.
- The number of k rim hook tableaux of shape λ is the number of k-tuples of SYT $R = (R_1, \ldots, R_k)$ (using one alphabet) whose shapes are the k-quotient of λ.
- For rectangle n^m, with $k|m$ or $k|n$, the k-quotient is a k-tuple of rectangles of the "same" size.
- The number of possible R which rectify to T $(r(R) = T)$ of shape λ (plactic product of the R_i) is a Littlewood-Richardson coefficient $c_{m,n,k,\lambda}$.
- Promotion p is a cyclic action of order mn/k on the quotient R.

An Example

For $m = 4,\ n = 6,\ k = 3.$

$$c_{4,6,3,(4,3,1)} = 2$$

$$R = (4 \quad 6, \quad 1 \quad 3, \quad \begin{matrix} 2 & 5 \\ 7 & 8 \end{matrix})$$

$$r(R) = \begin{matrix} 1 & 2 & 5 & 8 \\ 3 & 6 & 7 & \\ 4 & & & \end{matrix}$$

$$p(R) = (3 \quad 5, \quad 2 \quad 8, \quad \begin{matrix} 1 & 4 \\ 6 & 7 \end{matrix})$$

A Combinatorial Framework, Part 2

- A standard tableau T of rectangular shape m^n fixed by mn/k promotions is a *k-banded* standard tableau, having k bands of size mn/k.
- If T is k-banded and R is the corresponding k-quotient of T, then the first band $B_1(T)$ should be the rectification of R.
- The orbit of T (of size mn/k) should correspond to the orbit of R under promotion.
- Corresponding to the rectification of any band $B_i(T)$ to $B_1(T)$ is a Littlewood-Richardson word. But the Littlewood-Richardson coefficients are greater than the number of possible bandings. Some of the Littlewood-Richardson words do not correspond to banded tableaux.

Another Example

$m = 4$, $n = 6$, $k = 3$, two tableaux, where each band rectifies to $B_1(T)$.
For this T, $p^8(T) = T$:

$$T = \begin{array}{cccccc} \color{red}{1} & \color{red}{2} & \color{red}{5} & \color{red}{8} & \color{blue}{5} & \color{blue}{8} \\ \color{red}{3} & \color{red}{6} & \color{red}{7} & \color{blue}{2} & \color{blue}{7} & \color{green}{2} \\ \color{red}{4} & \color{blue}{1} & \color{blue}{3} & \color{green}{1} & \color{green}{3} & \color{green}{5} \\ \color{blue}{4} & \color{blue}{6} & \color{green}{4} & \color{green}{6} & \color{green}{7} & \color{green}{8} \end{array}$$

For this T, $p^8(T) \neq T$:

$$T = \begin{array}{cccccc} \color{red}{1} & \color{red}{2} & \color{red}{5} & \color{red}{8} & \color{blue}{2} & \color{blue}{5} \\ \color{red}{3} & \color{red}{6} & \color{red}{7} & \color{blue}{7} & \color{blue}{8} & \color{green}{2} \\ \color{red}{4} & \color{blue}{1} & \color{blue}{3} & \color{green}{1} & \color{green}{3} & \color{green}{5} \\ \color{blue}{4} & \color{blue}{6} & \color{green}{4} & \color{green}{6} & \color{green}{7} & \color{green}{8} \end{array}$$

The Table Definition

- For $m = 4$, $n = 6$ and $k = 3$
- First column gives possible shape λ for B_1
- Second column gives Littlewood-Richardson “flag” for possible 3-banded tableaux
- Third column gives $c_{m,n,k,\lambda}$ from Part 1
- Last column gives f_λ. Dotting the last two columns gives the number of such 3-banded tableaux (840)

The Table

λ	LR Flag	$c_{4,6,3,\lambda}$	f_λ
$6\,2$	$(1,1,1)$	1	20
$6\,1^2$	$(1,0,1)$	0	21
$5\,3$	$(1,1,1)$	1	28
$5\,2\,1$	$(1,2,1)$	2	64
$5\,1^3$	$(1,0,1)$	0	35
4^2	$(1,1,1)$	1	14
$4\,3\,1$	$(1,5,1)$	2	70
$4\,2^2$	$(1,3,1)$	3	56
$4\,2\,1^2$	$(1,1,1)$	1	90
$3^2\,2$	$(1,1,1)$	1	42
$3^2\,1^2$	$(1,1,1)$	1	56
$3\,2^2\,1$	$(1,2,1)$	2	70
2^4	$(1,1,1)$	1	14

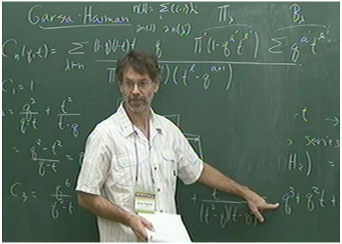
The Problem

Problem

Suppose $\alpha = n^m$ and k divides n or m. Prove Rhoades' result by finding a bijection between k-banded SYT of shape α and k-rim hook tableaux of shape α.

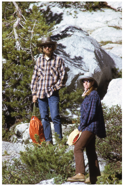
My Students

Here are three of the over 6000 undergraduates I have taught:



My Graduate Students

Thanks!






That's All Folks