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TORIC PARTIAL ORDERS

MIKE DEVELIN, MATTHEW MACAULEY, AND VICTOR REINER

Abstract. We define toric partial orders, corresponding to regions of graphic toric hyperplane arrangements, just as ordinary partial orders correspond to regions of graphic hyperplane arrangements. Combinatorially, toric posets correspond to finite posets under the equivalence relation generated by converting minimal elements into maximal elements, or sources into sinks. We derive toric analogues for several features of ordinary partial orders, such as chains, antichains, transitivity, Hasse diagrams, linear extensions, and total orders.

1. Introduction

We define finite *toric partial orders* or *toric posets*, which are cyclic analogues of partial orders, but differ from an established notion of *partial cyclic orders* already in the literature; see Remark 1.12 below. Toric posets can be defined in combinatorial geometric ways that are analogous to partial orders or posets:

- Posets on a finite set V correspond to open polyhedral cones that arise as chambers in *graphic hyperplane arrangements* in $\mathbb{R}^V$; toric posets correspond to chambers occurring within *graphic toric hyperplane arrangements* in the quotient space $\mathbb{R}^V/\mathbb{Z}^V$.
- Posets correspond to *transitive closures* of acyclic orientations of graphs; toric posets correspond to a notion of *toric transitive closures* of acyclic orientations.
- Both transitive closure and toric transitive closure will turn out to be *convex closures*, so that there is a notion of *toric Hasse diagram* for a toric poset, like the Hasse diagram of a poset.

We next make this more precise, indicating where the main results will be proven.

1.1. **Posets geometrically.** We first recall (e.g. from Stanley [32], Greene [15], Greene and Zaslavsky [16, §7], Postnikov, Reiner and Williams [27, §§3.3-3.4]) geometric features of posets, specifically their relations to graphic hyperplane arrangements and acyclic orientations, emphasizing notions with toric counterparts.

Let V be a finite set of cardinality $|V| = n$; often we will choose $V = [n] := \{1, 2, \ldots, n\}$. One can think of a *partially ordered set* or *poset* P on V as a binary

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relation $i <_P j$ which is

- *irreflexive*: $i \not<_P i$,
- *antisymmetric*: $i <_P j$ implies $j \not<_P i$, and
- *transitive*: $i <_P j$ and $j <_P k$ implies $i <_P k$.

However, one can also identify P with a certain *open polyhedral cone* in $\mathbb{R}^V$

$$c = c(P) := \{x \in \mathbb{R}^V : x_i < x_j \text{ if } i <_P j\}. \tag{1.1}$$

Note that the cone c determines the poset $P = P(c)$ as follows: $i <_P j$ if and only if $x_i < x_j$ for all x in c.

Each such cone c also arises as a connected component in the complement of at least one *graphic hyperplane arrangement* for a graph G, and often arises in several such arrangements, as explained below. Given a simple graph $G = (V, E)$, the *graphic arrangement* $\mathcal{A}(G)$ is the union of all hyperplanes in $\mathbb{R}^V$ of the form $x_i = x_j$ where $\{i, j\}$ is in E. Each point $x = (x_1, \ldots, x_n)$ in the *complement* $\mathbb{R}^V - \mathcal{A}(G)$ determines an *acyclic orientation* $\omega(x)$ of the edge set E: for an edge $\{i, j\}$ in E, since $x_i \neq x_j$, either

- $x_i < x_j$ and $\omega(x)$ directs $i \to j$, or
- $x_j < x_i$ and $\omega(x)$ directs $j \to i$.

It is easily seen that the fibers of this map $\alpha_G : x \longmapsto \omega(x)$ are the connected components of the complement $\mathbb{R}^V - \mathcal{A}(G)$, which are open polyhedral cones called *chambers*. Thus the map α_G induces a bijection between the set $\mathrm{Acyc}(G)$ of all acyclic orientations ω of G and the set $\mathrm{Cham}\,\mathcal{A}(G)$ of chambers c of $\mathcal{A}(G)$:

$$\begin{tikzcd} \mathbb{R}^V - \mathcal{A}(G) \arrow[rr, "\alpha_G"] \arrow[dr] & & \mathrm{Acyc}(G) \\ & \mathrm{Cham}\,\mathcal{A}(G) \arrow[ur, dotted] & \end{tikzcd} \tag{1.2}$$

These two sets are well known [16, Theorem 7.1], [32] to have cardinality

$$|\,\mathrm{Acyc}(G)| = |\,\mathrm{Cham}\,\mathcal{A}(G)| = T_G(2, 0)$$

where $T_G(x, y)$ is the *Tutte polynomial* of G [33].

Posets are also determined by their extensions to *total orders* $w_1 < \cdots < w_n$, which are indexed by permutations $w = (w_1, \ldots, w_n)$ of V. The total orders index the chambers

$$c_w := \{x \in \mathbb{R}^V : x_{w_1} < x_{w_2} < \cdots < x_{w_n}\}$$

in the complement of the *complete graphic arrangement* $\mathcal{A}(K_V)$, also known as the *reflection arrangement of type* A_{n-1} or *braid arrangement*. Given a poset P, its set $\mathcal{L}(P)$ of all *linear extensions* or *extensions to a total order* has the property that

$$\overline{c(P)} = \bigcup_{w \in \mathcal{L}(P)} \overline{c}_w$$

where $\overline{(\cdot)}$ denotes topological closure. Thus when one *fixes* the graph G, chambers c (or posets $P(c)$) arising as $\alpha_G^{-1}(\omega)$ for various ω in $\mathrm{Acyc}(G)$ are determined by their sets $\mathcal{L}(P(c))$ of linear extensions.

The same poset P or chamber $c = c(P)$ generally arises in many graphic arrangements $\mathcal{A}(G)$, as one varies the graph G, leading to ambiguity in its labeling by a pair (G, ω) with ω in $\mathrm{Acyc}(G)$. Nevertheless, this ambiguity is well controlled, in that there are two canonical choices $(\bar{G}(P), \bar{\omega}(P))$ and $(\hat{G}^{\mathrm{Hasse}}(P), \omega^{\mathrm{Hasse}}(P))$ with the following properties:

- A graph G has $c(P)$ occurring in $\operatorname{Cham}\mathcal{A}(G)$ if and only if $\hat{G}^{\mathrm{Hasse}}(P) \subseteq G \subseteq \bar{G}(P)$ where $\subseteq$ is inclusion of edge sets. In this case, $\alpha_G(c(P)) = \omega$ where ω is the restriction $\bar{\omega}(P)|_G$.
- The map which sends $(G, \omega) \longmapsto (\bar{G}(P), \bar{\omega}(P))$ is *transitive closure*. It adds into G all edges $\{i, j\}$ which lie on some *chain* (= *totally ordered subset*) C of P, and directs $i \to j$ if $i <_C j$. Alternatively phrased, transitive closure adds the directed edge $i \to j$ to (G, ω) whenever there is a directed path from i to j in (G, ω).

The existence of a unique *inclusion-minimal* choice $(\hat{G}^{\mathrm{Hasse}}(P), \omega^{\mathrm{Hasse}}(P))$, called the *Hasse diagram* for P, follows from this well-known fact [11, 12]: the *transitive closure* $A \longmapsto \bar{A}$ on the acyclic subsets A of all possible oriented edges $\overleftrightarrow{K}_V = \{(i, j) \in V \times V : i \neq j\}$, is a *convex closure*, meaning that

$$\text{for } a \neq b \text{ with } a, b \notin \bar{A} \text{ and } a \in \overline{A \cup \{b\}}, \text{ one has } b \notin \overline{A \cup \{a\}}. \tag{1.3}$$

1.2. **Toric posets.** We do not initially define a toric poset P on the finite set V via some binary (or ternary) relation. Rather we define it in terms of chambers in a *toric graphic arrangement* $\mathcal{A}_{\mathrm{tor}}(G) = \pi(\mathcal{A}(G))$, the image of the graphic arrangement $\mathcal{A}(G)$ under the quotient map $\mathbb{R}^V \xrightarrow{\pi} \mathbb{R}^V/\mathbb{Z}^V$. These are important examples of *unimodular toric arrangements* discussed by Novik, Postnikov and Sturmfels in [26, §§4-5]. Various aspects of the geometry, topology and combinatorics of toric arrangements generally may be found in the work of D'Adderio and Moci [6], d'Antonio and Delucchi [7], De Concini and Procesi [8], Ehrenborg, Readdy and Slone [13], Macmeikan [19, 20], Moci [22], and others.

Definition 1.1. A connected component c of the complement $\mathbb{R}^V/\mathbb{Z}^V - \mathcal{A}_{\mathrm{tor}}(G)$ is called a *toric chamber* for G; denote by $\operatorname{Cham}\mathcal{A}_{\mathrm{tor}}(G)$ the set of all toric chambers of $\mathcal{A}_{\mathrm{tor}}(G)$.

A *toric poset* P is a set c that arises as a toric chamber for at least one graph G. We will write $P = P(c)$ and $c = c(P)$, depending upon the context.

Example 1.2. When $n = 2$, so $V = \{1, 2\}$, there are only two simple graphs $G = (V, E)$, a graph G_0 with no edges and the complete graph K_2 with a single edge $\{1, 2\}$. There is only one poset over G_0, because $\mathbb{R}^V - \mathcal{A}(G) = \mathbb{R}^V$. There are two posets over $G = K_2$ because the arrangement $\mathcal{A}(G)$ has two chambers – the regions above and below the line $x_1 = x_2$, respectively.

The toric poset case has a few subtle differences. For both G_0 and K_2, the torus $\mathbb{R}^2/\mathbb{Z}^2$ remains connected after removing the arrangement $\mathcal{A}_{\mathrm{tor}}(G)$, and hence they each have only one toric chamber; call these chambers $c_0(= \mathbb{R}^2/\mathbb{Z}^2)$ for the graph G_0, and $c(= \mathbb{R}^2/\mathbb{Z}^2 - \{x_1 = x_2\})$ for the graph K_2. They represent two different toric posets $P(c_0)$ and $P(c)$, even though their topological closures $\bar{c} = \bar{c}_0 (= c_0) = \mathbb{R}^2/\mathbb{Z}^2$ are the same.

A point x in $\mathbb{R}^V/\mathbb{Z}^V$ does not have uniquely defined coordinates $(x_1, \ldots, x_n)$. However, it is well defined to speak of the *fractional part* x_i mod 1, that is, the unique representative of the class of x_i in $\mathbb{R}/\mathbb{Z}$ that lies in $[0, 1)$. Therefore a point x in $\mathbb{R}^V/\mathbb{Z}^V - \mathcal{A}_{\mathrm{tor}}(G)$, still induces an acyclic orientation $\omega(x)$ of G, as follows: for each edge $\{i,j\}$ in E, since $x_i \neq x_j \bmod \mathbb{Z}$, either

- $x_i \bmod 1 < x_j \bmod 1$, and $\omega(x)$ directs $i \to j$, or
- $x_j \bmod 1 < x_i \bmod 1$, and $\omega(x)$ directs $j \to i$.

Denote this map $x \mapsto \omega(x)$ by $\mathbb{R}^V/\mathbb{Z}^V - \mathcal{A}_{\mathrm{tor}}(G) \xrightarrow{\bar{\alpha}_G} \mathrm{Acyc}(G)$. Unfortunately, two points lying in the same toric chamber c in $\mathrm{Cham}_{\mathrm{tor}}\,\mathcal{A}_{\mathrm{tor}}(G)$ need not map to the same acyclic orientation under $\bar{\alpha}_G$. This ambiguity leads one naturally to the following equivalence relation on acyclic orientations.

Definition 1.3. When two acyclic orientations ω and ω' of G differ only by converting one source vertex of ω into a sink of ω', say that they differ by a *flip*. The transitive closure of the flip operation generates an equivalence relation on $\mathrm{Acyc}(G)$, called *toric equivalence*, that we denote by $\equiv$.

A thorough investigation of this source-to-sink flip operation and equivalence relation was undertaken by Pretzel in [28], and studied earlier by Mosesjan [24]. Its connection with the theory of tilings and height functions was studied by Propp [29] who introduced an important distributive lattice structure on each equivalence class. It has also appeared at other times in various contexts[1] in the literature [4, 14, 18, 31]. Its relation to geometry of toric chambers $c = c(P)$ or toric posets $P = P(c)$ is our first main result, proven in §2.

Theorem 1.4. *The map* $\bar{\alpha}_G$ *induces a bijection between* $\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G)$ *and* $\mathrm{Acyc}(G)/\!\equiv$ *as follows:*

$$\begin{tikzcd}
\mathbb{R}^V/\mathbb{Z}^V - \mathcal{A}_{\mathrm{tor}}(G) \arrow[r, "\bar{\alpha}_G"] \arrow[d] & \mathrm{Acyc}(G) \arrow[d] \\
\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G) \arrow[r, dotted, "\bar{\alpha}_G"'] & \mathrm{Acyc}(G)/\!\equiv
\end{tikzcd} \tag{1.4}$$

In other words, two points x, x' in $\mathbb{R}^V/\mathbb{Z}^V - \mathcal{A}_{\mathrm{tor}}(G)$ have $\bar{\alpha}_G(x) \equiv \bar{\alpha}_G(x')$ if and only if x, x' lie in the same toric chamber c in $\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G)$.

The two sets $\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G)$ and $\mathrm{Acyc}(G)/\!\equiv$ appearing in the theorem are known to have cardinality

$$|\,\mathrm{Acyc}(G)/\!\equiv\,| = |\,\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G)| = T_G(1,0)$$

where $T_G(x,y)$ is the Tutte polynomial of G; see [17] and [26, Theorem 4.1]. A more general statement was proven by Moci [22, Cor. 5.16] who introduced his *arithmetic Tutte polynomial* $M_{\mathcal{A}}(x,y)$ for *any* toric arrangement $\mathcal{A}$. Its evaluation $M_{\mathcal{A}}(1,0)$ again counts the chambers of $\mathcal{A}$, and for unimodular arrangements one has $M_{\mathcal{A}}(x,y) = T_{\mathcal{A}}(x,y)$.

Example 1.5. A *tree* G on n vertices has Tutte polynomial $T_G(x,y) = x^{n-1}$. It will have $T_G(2,0) = 2^{n-1}$ acyclic orientations ω and induced partial orders, but only $T_G(1,0) = 1$ toric chamber or toric partial order: any two acyclic orientations of a tree are equivalent by a sequence of source-to-sink moves.

[1] Pretzel called the source-to-sink flip *pushing down maximal vertices*; in [18], it was called a *click*. In the category of representations of a quiver, it is related to Bernšteĭn, Gel′fand and Ponomarev's *reflection functors* [1].

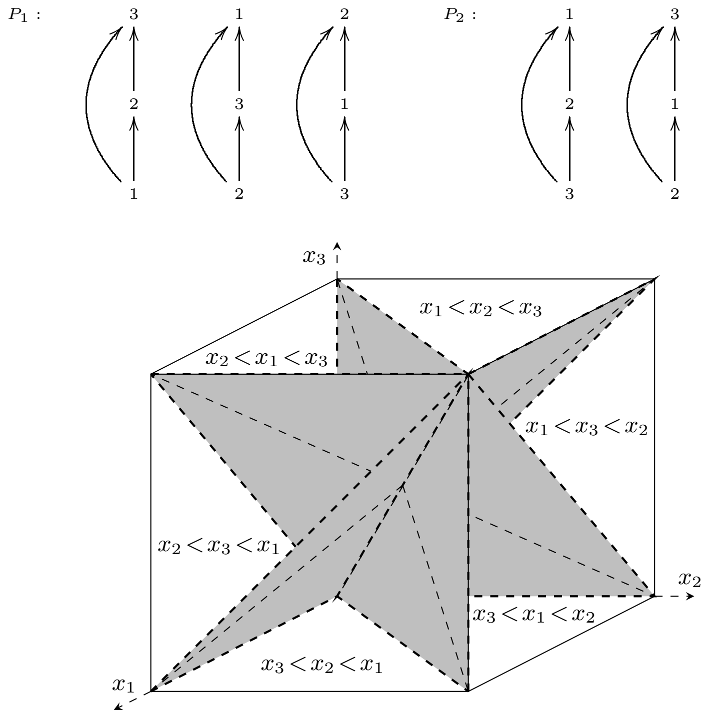
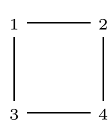


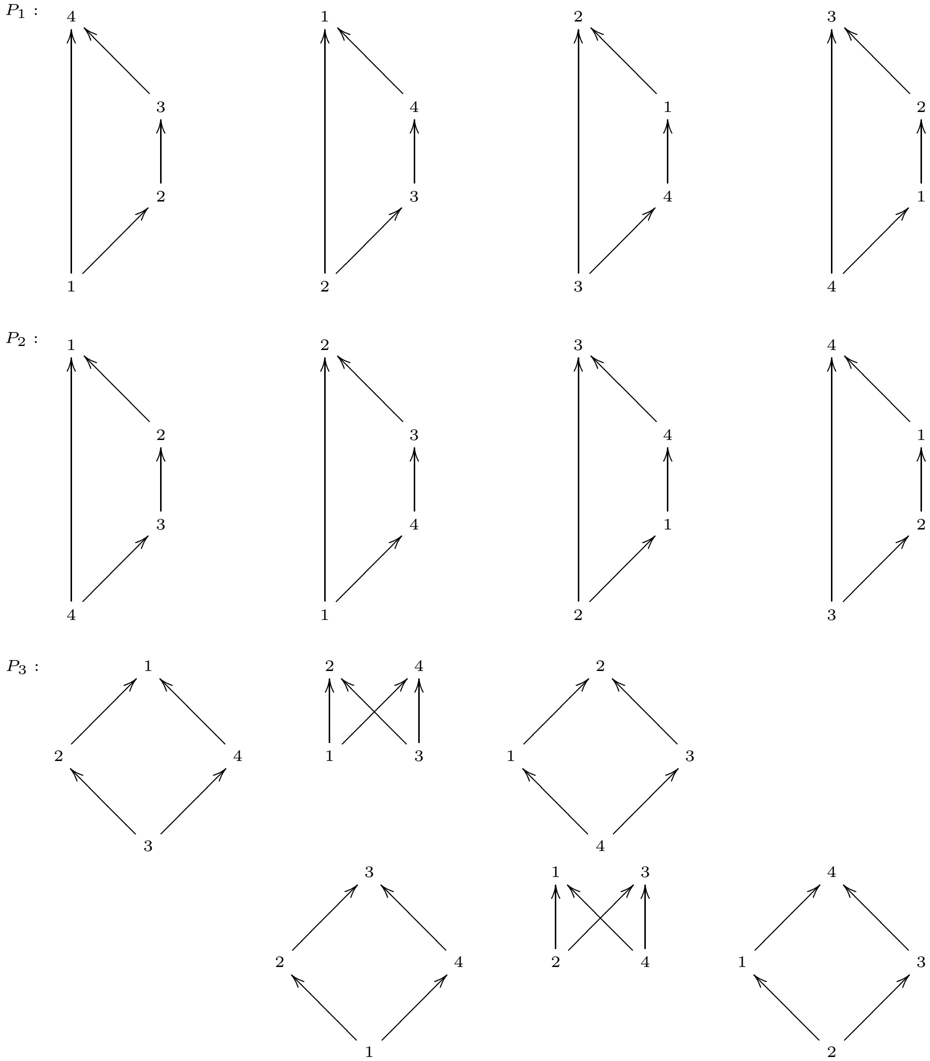
FIGURE 1. The six acyclic orientations of $G = K_3$ fall into two toric equivalence classes. The corresponding toric graphic arrangement has two toric chambers.

Example 1.6. Let $G = (V, E)$ be the complete graph over $V = \{1, 2, 3\}$. The six acyclic orientations of $G = K_3$ fall into two distinct toric equivalence classes, which are shown at the top in Figure 1. The 3-torus can be viewed as the unit cube $[0, 1]^3$ with opposite faces identified. It is easy to see that under the quotient map $\mathbb{R}^3 \to \mathbb{R}^3/\mathbb{Z}^3$, the six chambers of $\mathcal{A}(G)$ become two distinct toric chambers, $c_1 = c(P_1)$ and $c_2 = c(P_2)$, as shown at the bottom in Figure 1. Since removing an edge from G yields a tree, removing any toric hyperplane from $\mathcal{A}(G)$ merges the two toric chambers into one.

Example 1.7. Consider $V = \{1, 2, 3, 4\}$ and the graph $G = (V, E)$ depicted here:



It has Tutte polynomial $T_G(x, y) = x^3 + x^2 + x + y$, and hence has $T_G(2, 0) = 2^3 + 2^2 + 2 + 0 = 14$ acyclic orientations ω. These ω fall into $T_G(1, 0) = 1^3 + 1^2 + 1 + 0 = 3$ different toric equivalence classes $[\omega]$, having cardinalities $4, 4, 6$, respectively, corresponding to three different toric posets P_i or chambers c_i in $\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G)$:



Toric total orders (see §5) are indexed by the $(n-1)!$ cyclic equivalence classes of permutations

$$[w] := [(w_1, w_2, \ldots, w_n)] = \left\{ \begin{array}{l} (w_1, w_2, \ldots, w_{n-1}, w_n), \\ (w_2, \ldots, w_{n-1}, w_n, w_1), \\ \qquad\vdots \\ (w_n, w_1, w_2, \ldots, w_{n-1}) \end{array} \right\} \tag{1.5}$$

and correspond to the toric chambers $c_{[w]}$ in the complement of the *toric complete graphic arrangement* $\mathcal{A}_{\mathrm{tor}}(K_V)$. For a particular toric poset $P = P(c)$, one says that $[w]$ is a *toric total extension* of P if $c_{[w]} \subseteq c$. Denote by $\mathcal{L}_{\mathrm{tor}}(P)$ the set of all such toric total extensions $[w]$ of P. Although it is possible (see Example 5.3 below) for two different toric posets P to have the same set $\mathcal{L}_{\mathrm{tor}}(P)$, the following assertion (combining Proposition 3.2 and Corollary 5.2 below) still holds.

Proposition 1.8. *When one fixes the graph G, the toric chamber c (or its poset $P = P(c)$) for which $\bar{\alpha}_G(c) = [\omega]$ is completely determined by its topological closure $\overline{c}$. Furthermore one has $\overline{c} = \bigcup_{w \in \mathcal{L}_{\mathrm{tor}}(P)} \overline{c}_{[w]}$, so that this closure depends only on the set of toric total extensions $\mathcal{L}_{\mathrm{tor}}(P)$.*

Example 1.9. The graph G from Example 1.7 and its three toric posets P_1, P_2, P_3 partition the $(4-1)! = 6$ different toric total orders on $V = \{1,2,3,4\}$ into their sets of toric total extensions $\mathcal{L}_{\mathrm{tor}}(P_i)$ as follows:

$$\begin{aligned}
\mathcal{L}_{\mathrm{tor}}(P_1) &= \{[(1,2,3,4)]\},\\
\mathcal{L}_{\mathrm{tor}}(P_2) &= \{[(1,4,3,2)]\},\\
\mathcal{L}_{\mathrm{tor}}(P_3) &= \{[(1,2,4,3)], [(1,3,2,4)], [(1,3,4,2)], [(1,4,2,3)]\}.
\end{aligned}$$

As with posets, the same toric poset $P = P(c)$ arises as a chamber c in *many* toric graphic arrangements $\mathcal{A}_{\mathrm{tor}}(G)$. However, as with posets, this ambiguity is well controlled, in that there are two canonical choices of equivalence classes $(\bar{G}^{\mathrm{tor}}(P), [\bar{\omega}^{\mathrm{tor}}(P)])$ and $(\hat{G}^{\mathrm{torHasse}}(P), [\omega^{\mathrm{torHasse}}(P)])$ with the following properties:

- A graph G has $c(P)$ occurring in $\operatorname{Cham} \mathcal{A}_{\mathrm{tor}}(G)$ if and only if

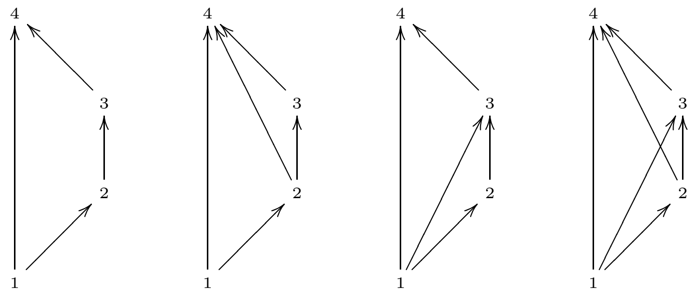
$$\hat{G}^{\mathrm{torHasse}}(P) \subseteq G \subseteq \bar{G}^{\mathrm{tor}}(P)$$

  where $\subseteq$ is inclusion of edges. In this case, if $\bar{\alpha}_G(c(P)) = [\omega]$, then ω can be taken to be the restriction to G of a particular orientation in the class $[\bar{\omega}^{\mathrm{tor}}(P)]$.
- The map which sends $(G, \omega) \longmapsto (\bar{G}^{\mathrm{tor}}, \bar{\omega}^{\mathrm{tor}})$ may be described by what will be called (in §7) *toric transitive closure*: one adds into G all edges $\{i, j\}$ which lie on some *toric chain* C in P. Here a toric chain (see §6) is a subset $C \subset V$ which is totally ordered in *every* poset associated with an orientation in the class $[\omega]$. An added edge will be directed $i \to j$ if one passes through first i and then j along some *toric directed path* $(i_1, \ldots, i_m)$ in (G, ω), as defined in §4 below; see Definition 4.1. Alternatively phrased, toric transitive closure adds the directed edge $i \to j$ to (G, ω) whenever there is a toric directed path in (G, ω) containing a directed subpath from i to j.

The existence of the unique *inclusion-minimal* choice $(\hat{G}^{\mathrm{torHasse}}(P), [\omega^{\mathrm{torHasse}}(P)])$, which we will call the *toric Hasse diagram* of P, follows from our second main result, proven in §8.

Theorem 1.10. *Considered as a closure operation $A \longmapsto \bar{A}^{\mathrm{tor}}$ on acyclic subsets A of the set of all possible oriented edges $\overleftrightarrow{K}_V = \{(i,j) \in V \times V : i \neq j\}$, toric transitive closure is a convex closure, that is, it satisfies* (1.3) *above.*

Example 1.11. The toric poset $P_1 = P(c_1)$ from Example 1.7 appears as a chamber c_1 in $\operatorname{Cham} \mathcal{A}_{\mathrm{tor}}(G_i)$ for exactly four graphs G_1, G_2, G_3, G_4, each shown below with an orientation ω_i such that $\bar{\alpha}_{G_i}(c_1) = [\omega_i]$:



For any of these four pairs (G_i, ω_i) with $i = 1, 2, 3, 4$, one has that the leftmost pair is its Hasse diagram $(\hat{G}_i^{\text{torHasse}}, \omega_i^{\text{torHasse}})$, and the rightmost pair is its toric transitive closure $(\bar{G}_i^{\text{tor}}, \bar{\omega}_i^{\text{tor}})$.

In contrast, the toric poset $P_3 = P(c_3)$ from Example 1.7 has no toric directed paths with more than 2 elements, so it has no toric chains of size greater than 2, and therefore no new edges are added in its toric transitive closure. Thus, $(\hat{G}_1^{\text{torHasse}}, \omega^{\text{torHasse}}) = (\bar{G}_1^{\text{tor}}, \bar{\omega}^{\text{tor}})$ for any of the six orientations ω of G_1 such that $\bar{\alpha}_{G_1}(c_3) = [\omega]$.

We close this Introduction with two remarks, one on terminology, the other giving further motivation.

Remark 1.12. Aside from the connection to toric hyperplane arrangements, we have chosen the name "toric partial order", as opposed to the arguably more natural term "cyclic partial order", because the latter is easily confused with *partial cyclic orders*, the following pre-existing concept in the literature, going back at least as far as Megiddo [23].

Definition 1.13. A *partial cyclic order* on V is a ternary relation $T \subseteq V \times V \times V$, that is,

- *antisymmetric*: if $(i, j, k) \in T$, then $(k, j, i) \notin T$;
- *transitive*: if $(i, j, k) \in T$ and $(i, k, \ell) \in T$, then $(i, j, \ell) \in T$;
- *cyclic*: if $(i, j, k) \in T$, then $(j, k, i) \in T$.

Definition 1.14. When a partial cyclic order on V is *complete* in the sense that for every triple $\{i, j, k\} \subseteq V$ of distinct elements, T contains some permutation of (i, j, k), then T is called a *total cyclic order*. A total cyclic order on V is easily seen to be the same as a toric total order: specify a cyclic equivalence class $[w]$ as in (1.5), and then check that $[w]$ is determined by knowing its restrictions $[w|_{\{i,j,k\}}]$ for all triples $\{i, j, k\}$.

Partial cyclic orders have been widely studied, and have some interesting features not shared by ordinary partial orders. For example, every partial order can be extended to a total order, but not every partial cyclic order can be extended to a total cyclic order; an example of this on 13 vertices is given in [23].

Remark 1.15. We mention a further analogy between posets and toric posets, related to Coxeter groups, that was one of our motivations for formalizing this concept.

Recall [2] that a *Coxeter system* (W, S) is a group W with generating set $S = \{s_1, \ldots, s_n\}$ having presentation $W = \langle S : (s_i s_j)^{m_{i,j}} = e \rangle$ for some $m_{i,j}$ in $\{1, 2, 3, \ldots\} \cup \{\infty\}$, where $m_{i,i} = 1$ for all i and $m_{i,j} \geq 2$ for $i \neq j$. Associated to (W, S) is the *Coxeter graph* on vertex set S with an edge $\{s_i, s_j\}$ labeled by $m_{i,j}$ whenever $m_{i,j} > 2$, so that s_i, s_j do not commute; ignoring the edge labels, we will call this the unlabeled Coxeter graph. A *Coxeter element* for (W, S) is an element of the form $s_{w_1} s_{w_2} \cdots s_{w_n}$ for some choice of a total order w on S.

Theorem 1.16. *Fix a Coxeter system (W, S) with unlabeled Coxeter graph G, and consider the map sending an acyclic orientation ω in $\mathrm{Acyc}(G)$ having poset $P = \alpha_G(\omega)$ to the Coxeter element $s_{w_1} s_{w_2} \cdots s_{w_n}$ for any choice of a linear extension w in $\mathcal{L}(P)$.*

(i) *This map is well defined, and induces a bijection (see* [2, §V.6] *and* [3]*)*

$$\mathrm{Acyc}(G) \longleftrightarrow \{ \textit{ Coxeter elements for } (W, S) \}.$$

(ii) *It also induces a well-defined map on the toric equivalence classes $[\omega]$ to the* ***W-conjugacy classes*** *of all Coxeter elements, and gives a bijection (see* [14, 17, 18, 30] *and* [26, Remark 5.5]*)*

$$\mathrm{Acyc}(G)/\equiv \quad \longleftrightarrow \quad \{W\textit{-conjugacy classes of Coxeter elements for } (W, S)\}.$$

We believe toric partial orders will play a key role in resolving more questions about W-conjugacy classes.

2. Toric arrangements and proof of Theorem 1.4

Theorem 1.4. *The map $\bar{\alpha}_G$ induces a bijection between $\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G)$ and $\mathrm{Acyc}(G)/\equiv$ as follows:*

$$\begin{tikzcd}
\mathbb{R}^V/\mathbb{Z}^V - \mathcal{A}_{\mathrm{tor}}(G) \arrow[r, "\bar{\alpha}_G"] \arrow[d] & \mathrm{Acyc}(G) \arrow[d] \\
\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G) \arrow[r, dotted, "\bar{\alpha}_G"'] & \mathrm{Acyc}(G)/\equiv
\end{tikzcd}$$

In other words, two points x, x' in $\mathbb{R}^V/\mathbb{Z}^V - \mathcal{A}_{\mathrm{tor}}(G)$ have $\bar{\alpha}_G(x) \equiv \bar{\alpha}_G(x')$ if and only if x, x' lie in the same toric chamber c in $\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G)$.

Before embarking on the proof, we introduce one further geometric object intimately connected with

- the graphic arrangement $\mathcal{A}(G) = \bigcup_{\{i,j\} \in E} \{x \in \mathbb{R}^V : x_i = x_j\} \subset \mathbb{R}^V$, and
- the toric graphic arrangement $\mathcal{A}_{\mathrm{tor}}(G) = \pi(\mathcal{A}(G))$, its image under $\mathbb{R}^V \xrightarrow{\pi} \mathbb{R}^V/\mathbb{Z}^V$.

Definition 2.1. Define the *affine graphic arrangement* in $\mathbb{R}^V$ by

$$\mathcal{A}_{\mathrm{aff}}(G) := \pi^{-1}(\mathcal{A}_{\mathrm{tor}}(G)) = \pi^{-1}(\pi(\mathcal{A}(G))) = \bigcup_{\substack{\{i,j\} \in E \\ k \in \mathbb{Z}}} \{x \in \mathbb{R}^V : x_i = x_j + k\}. \tag{2.1}$$

Call the connected components $\hat{c}$ of the complement $\mathbb{R}^V - \mathcal{A}_{\mathrm{aff}}(G)$ *affine chambers*, and denote the set of all such chambers $\mathrm{Cham}\,\mathcal{A}_{\mathrm{aff}}(G)$.

The reason for introducing $\mathcal{A}_{\text{aff}}(G)$ and $\operatorname{Cham}\mathcal{A}_{\text{aff}}(G)$ is the following immediate consequence of the path-lifting property for $\mathbb{R}^V \xrightarrow{\pi} \mathbb{R}^V/\mathbb{Z}^V$ as a (universal) covering map (see e.g. [25, Chap. 13]), along with the definition (2.1) of $\mathcal{A}_{\text{aff}}(G)$ as the full inverse image under π of $\mathcal{A}_{\text{tor}}(G)$.

Proposition 2.2. *Two points x, y in $\mathbb{R}^V/\mathbb{Z}^V - \mathcal{A}_{\text{tor}}(G)$ lie in the same chamber c in $\operatorname{Cham}\mathcal{A}_{\text{tor}}(G)$ if and only if they have two lifts $\hat{x}, \hat{y}$ lying in the same affine chamber $\hat{c}$ in $\operatorname{Cham}\mathcal{A}_{\text{aff}}(G)$.*

The point will be that, since affine chambers $\hat{c}$ are (open) convex polyhedral regions in $\mathbb{R}^V$, it is sometimes easier to argue about lifted points $\hat{x}$ rather than x itself.

Our proof of Theorem 1.4 proceeds by showing the map

$$\mathbb{R}^V/\mathbb{Z}^V - \mathcal{A}_{\text{tor}}(G) \xrightarrow{\bar{\alpha}_G} \operatorname{Acyc}(G)$$

descends to

- a well-defined map $\operatorname{Cham}\mathcal{A}_{\text{tor}}(G) \xrightarrow{\bar{\alpha}_G} \operatorname{Acyc}(G)/\equiv$,
- which is surjective,
- and injective.

2.1. **Well-definition.** We must show that when x, y lie in the same toric chamber c in $\operatorname{Cham}\mathcal{A}_{\text{tor}}(G)$, then $\bar{\alpha}_G(x) \equiv \bar{\alpha}_G(y)$. As in Proposition 2.2, pick lifts $\hat{x}, \hat{y}$ in $\mathbb{R}^V$ and a path $\hat{\gamma}$ between them in some affine chamber $\hat{c}$. Because these chambers are open, one can assume without loss of generality that $\hat{\gamma}$ takes steps in coordinate directions only, and therefore that $\hat{x}, \hat{y}$ differ in only a single coordinate: say $\hat{x}_i \neq \hat{y}_i$, but $\hat{x}_j = \hat{y}_j$ for all $j \neq i$. Furthermore, as $\bar{\alpha}_G(x)$ changes only when a coordinate of $\hat{x}$ passes through an integer, without loss of generality, one may assume

$$\hat{x}_i \bmod 1 = 1 - \varepsilon,$$
$$\hat{y}_i \bmod 1 = \varepsilon$$

for some arbitrarily small $\varepsilon > 0$. Since the points on $\hat{\gamma}$ all avoid $\mathcal{A}_{\text{aff}}(G)$, and the i^{th} coordinate will pass through 0 at some point on the path $\hat{\gamma}$, each of the coordinates $\hat{x}_j (= \hat{y}_j)$ for indices j with $\{i, j\}$ in E must have $0 < \hat{x}_j \bmod 1 < 1$. Hence one can choose ε small enough that all j for which $\{i, j\}$ in E satisfy

$$(\hat{y}_i \bmod 1 =)\, \varepsilon < \hat{x}_j \bmod 1 < 1 - \varepsilon\, (= \hat{x}_i \bmod 1).$$

One finds that $\bar{\alpha}_G(\hat{x})$ and $\bar{\alpha}_G(\hat{y})$ differ by changing i from sink to a source, so $\bar{\alpha}_G(\hat{x}) \equiv \bar{\alpha}_G(\hat{y})$, as desired.

2.2. **Surjectivity.** It suffices to check that the map $\mathbb{R}^V/\mathbb{Z}^V - \mathcal{A}_{\text{tor}}(G) \xrightarrow{\bar{\alpha}_G} \operatorname{Acyc}(G)$ is surjective. Given an acyclic orientation ω of G, pick any linear extension $w_1 < \cdots < w_n$ of its associated partial order $\alpha_G^{-1}(\omega)$ on V. Then choose real numbers $0 < x_{w_1} < \cdots < x_{w_n} < 1$, so that

$$x = (x_1, \ldots, x_n) = (x_1 \bmod 1, \ldots, x_n \bmod 1)$$

and hence $\bar{\alpha}_G(x) = \omega$.

2.3. **Injectivity.** The key to injectivity is the following lemma.

Lemma 2.3. *Suppose x lies in a toric chamber c in* $\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G)$, *and* $\bar{\alpha}_G(x) = \omega$. *Then for any $\omega' \equiv \omega$, there exists some x' in the same toric chamber c having* $\bar{\alpha}_G(x') = \omega'$.

Proof. It suffices to check this when ω' is obtained from ω by changing a source vertex i in ω to a sink in ω'. Since $\bar{\alpha}_G(x) = \omega$, one must have for each j with $\{i, j\}$ in E that

$$(0 \leq) x_i \bmod 1 < x_j \bmod 1 (< 1).$$

Lift x to $\hat{x} = (x_1 \bmod 1, \ldots, x_n \bmod 1)$, and choose ε small enough so that each j with $\{i, j\}$ in E has $x_j \bmod 1 < 1 - \varepsilon$. Define $\hat{y}$ to have all the same coordinates as $\hat{x}$ except for $\hat{y}_i = -\varepsilon$, so that $\hat{y}_i \bmod 1 = 1 - \varepsilon$, and hence $y := \pi(\hat{y})$ has $\bar{\alpha}_G(y) = \omega'$ by construction. Note that the straight-line path $\hat{\gamma}$ from $\hat{x}$ to $\hat{y}$ changes only the i^{th} coordinate, decreasing it from $\hat{x}_i$ to $\hat{y}_i = -\varepsilon$, and hence never crosses any of the affine hyperplanes in $\mathcal{A}_{\mathrm{aff}}(G)$. Therefore $\hat{x}, \hat{y}$ lie in the same affine chamber, and x, y lie in the same toric chamber c. ☐

Now suppose that points x, x' in two toric chambers c, c' have $\bar{\alpha}_G(x) \equiv \bar{\alpha}_G(x')$, and we must show that $c = c'$. By Lemma 2.3, without loss of generality one has $\bar{\alpha}_G(x) = \omega = \bar{\alpha}_G(x')$. Thus one can lift x, x' to $\hat{x}, \hat{x}'$ having $\hat{x}_i, \hat{x}'_i$ in $[0, 1)$ for all i, and hence $\alpha_G(\hat{x}) = \omega = \alpha_G(\hat{x}')$. For each edge $\{i, j\}$ in E, say directed $i \to j$ in ω, one has both

$$0 \leq \hat{x}_i < \hat{x}_j < 1,$$
$$0 \leq \hat{x}'_i < \hat{x}'_j < 1.$$

Thus every point $\hat{y}$ on the straight-line path $\hat{\gamma}$ between $\hat{x}$ and $\hat{x}'$ also satisfies $0 \leq \hat{y}_i < \hat{y}_j < 1$, avoiding all affine hyperplanes in $\mathcal{A}_{\mathrm{aff}}(G)$. Thus $\hat{x}, \hat{x}'$ lie in the same affine chamber $\hat{c}$, so that x, x' lie in the same toric chamber, as desired. This completes the proof of injectivity, and hence the proof of Theorem 1.4. ☐

One corollary to Theorem 1.4 is a (slightly) more concrete description of a toric chamber c.

Corollary 2.4. *For a graph $G = (V, E)$ and toric chamber c in* $\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G)$ *with* $\bar{\alpha}_G(c) = [\omega]$, *one has*

$$c = \bigcup_{\omega' \in [\omega]} \bar{\alpha}_G^{-1}(\omega') = \bigcup_{\omega' \in [\omega]} \{x \in \mathbb{R}^V/\mathbb{Z}^V : x_i \bmod 1 < x_j \bmod 1 \textit{ if } \omega' \textit{ directs } i \to j\}.$$

3. Toric extensions

Recall that for two (ordinary) posets P, P' on a set V, one says that P' *is an extension of* P when $i <_P j$ implies $i <_{P'} j$. It is easily seen how to reformulate this geometrically: P' is an extension of P if and only if one has an inclusion of their open polyhedral cones $c(P') \subseteq c(P)$, as defined in (1.1). This motivates the following definition.

Definition 3.1. Given two toric posets P, P' say that P' *is a toric extension of* P if one has an inclusion of their open chambers $c(P') \subseteq c(P)$ within $\mathbb{R}^V/\mathbb{Z}^V$.

An obvious situation where this can occur is when one has $G = (V, E)$ and $G' = (V, E')$ two graphs on the same vertex set V, with G an *edge-subgraph* of G' in the sense that $E \subseteq E'$,

Proposition 3.2. *Fix $G = (V, E)$ a simple graph.*

(i) Toric chambers in $\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G)$ are determined by their topological closures: for any pair of chambers c_1, c_2 in $\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G)$, if $\bar{c}_1 = \bar{c}_2$, then $c_1 = c_2$.

(ii) If G is an edge-subgraph of G', then $\bar{c} = \bigcup_{c'} \bar{c}'$, where the union runs over all toric chambers c' in $\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G')$ for which $P(c')$ is a toric extension of $P(c)$.

Proof. For (i), first note that any toric chamber c in $\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G)$ has boundary $\bar{c} - c$ contained in $\mathcal{A}_{\mathrm{tor}}(G)$. Now assume two toric chambers c_1, c_2 in $\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G)$ have $\bar{c}_1 = \bar{c}_2$, and we wish to show $c_1 = c_2$. Any point x of c_1 has $x \in c_1 \subseteq \bar{c}_1 = \bar{c}_2$. However, x cannot lie in $\mathcal{A}_{\mathrm{tor}}(G)$ since c_1 is disjoint from $\mathcal{A}_{\mathrm{tor}}(G)$, so x does not lie in $\bar{c}_2 - c_2 \subset \mathcal{A}_{\mathrm{tor}}(G)$ by our first observation. Hence x lies in c_2. But then c_1, c_2 are connected components of $\mathbb{R}^V/\mathbb{Z}^V - \mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G)$, sharing the point x, so $c_1 = c_2$.

For (ii), we first argue that

$$\bar{c} = \pi\left(\overline{\pi^{-1}(c)}\right) \tag{3.1}$$

using the fact that the covering map $\mathbb{R}^V \xrightarrow{\pi} \mathbb{R}^V/\mathbb{Z}^V$ is locally a homeomorphism. For any point x in $\mathbb{R}^V/\mathbb{Z}^V$ there is an open neighborhood U which lifts to an open neighborhood $\hat{U}$, mapping homeomorphically under π to U. Hence x is the limit of a sequence $\{x_i\}_{i=1}^{\infty}$ of points in c if and only if its lift $\hat{x} = \pi|_{\hat{U}}^{-1}(x)$ is a limit of the sequence of points $\{\pi|_{\hat{U}}^{-1}(x_i)\}_{i=1}^{\infty}$ in $\pi^{-1}(c)$. This shows (3.1).

Since a toric chamber c has $\pi^{-1}(c)$ given by a union of affine chambers $\hat{c}$ in $\mathrm{Cham}\,\mathcal{A}_{\mathrm{aff}}(G)$, in light of (3.1), it suffices to show that any affine chamber $\hat{c}$ in $\mathrm{Cham}\,\mathcal{A}_{\mathrm{aff}}(G)$ has closure $\bar{\hat{c}}$ given by the union of the closures $\overline{\hat{c}'}$ taken over all affine chambers $\hat{c}'$ in $\mathrm{Cham}\,\mathcal{A}_{\mathrm{aff}}(G')$ that satisfy $\hat{c}' \subseteq \hat{c}$. However, this is clear since $\hat{c}$ is a polyhedron bounded by hyperplanes taken from $\mathcal{A}_{\mathrm{aff}}(G)$, while $\mathcal{A}_{\mathrm{aff}}(G')$ simply refines this decomposition with more hyperplanes. $\square$

4. Toric directed paths

A particular special case of Proposition 3.2 is worth noting: every graph $G = (V, E)$ is an edge-subgraph of the *complete graph* K_V. As noted in the Introduction, acyclic orientations ω of K_V correspond to total orders $w_1 < \cdots < w_n$, indexed by permutations $w = (w_1, \ldots, w_n)$ of $V = [n] := \{1, 2, \ldots, n\}$. It is easy to characterize the toric equivalence relation $\equiv$ on these total orders, and hence the toric chambers $\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(K_V)$, in terms of cyclic shifts of these linear orders. However, it is worthwhile to define this concept a bit more generally– it turns out to be crucial in §6.

Definition 4.1. Given a simple graph $G = (V, E)$ and an acyclic orientation ω of G, say that a sequence $(i_1, i_2, \ldots, i_m)$ of elements of V forms a *toric directed path* *in* ω if (G, ω) contains all of these edges:

(4.1)
$$\begin{tikzcd}
i_m & \\
 & i_{m-1} \arrow[ul] \\
 & \vdots \arrow[u] \\
 & i_2 \arrow[u] \\
i_1 \arrow[uuuu] \arrow[ur] &
\end{tikzcd}$$

In particular, for small values of m, a toric directed path in ω

- of size $m = 2$ is a directed edge (i_1, i_2),
- of size $m = 1$ is a degenerate path (i_1) for any i_1 in V, and
- of size $m = 0$ is the empty subset $\varnothing \subset V$.

Proposition 4.2. *An acyclic orientation* ω *of* G *contains a toric directed path* $(i_1, i_2, \ldots, i_m)$ *if and only if every acyclic orientation* ω' *in its toric equivalence class contains a (unique) toric directed path*

$$(i_\ell, i_{\ell+1}, \ldots, i_m, i_1, i_2, \ldots, i_{\ell-1})$$

which is one of its cyclic shifts, that is, it lies in the cyclic equivalence class $[(i_1, \ldots, i_m)]$.

Proof. A toric directed path $(i_1, i_2, \ldots, i_m)$ has only one source, namely i_1, and only one sink, namely i_m. The assertion follows by checking that the effect of a source-to-sink flip at i_1 (resp. i_m) is a cyclic shift to the toric directed path $(i_2, \ldots, i_m, i_1)$ (resp. $(i_m, i_1, i_2, \ldots, i_{m-1})$). □

Remark 4.3. We point out a reformulation of the sink-to-source or toric equivalence relation $\equiv$ on $\mathrm{Acyc}(G)$, due to Pretzel [28], leading to a reformulation of toric directed paths, useful in §10 on toric antichains.

Given a simple graph $G = (V, E)$, say that a cyclic equivalence class $I = [(i_1, \ldots, i_m)]$ of ordered vertices is a *directed cycle* of G if $m \geq 3$ and G contains all of the (undirected) edges $\{i_j, i_{j+1}\}_{j=1,2,\ldots,m}$, with subscripts taken modulo m. Given such a directed cycle I define *Coleman's* ν*-function* [5]

$$\mathrm{Acyc}(G) \xrightarrow{\nu_I} \mathbb{Z}$$

where $\nu_I(\omega)$ for an acyclic orientation ω of G is defined to be the number of edges $\{i_j, i_{j+1}\}$ in I which ω orients $i_j \to i_{j+1}$ minus the number of edges $\{i_j, i_{j+1}\}$ which ω orients $i_{j+1} \to i_j$. It is easy to see that ν_I is preserved by flips, and thus extends in a well-defined manner to toric equivalence classes $[\omega]$. In fact, Pretzel [28] showed that this is a complete invariant for toric equivalence:

Proposition 4.4. *Fixing the graph* $G = (V, E)$*, two acyclic orientations* ω, ω' *in* $\mathrm{Acyc}(G)$ *have* $\omega \equiv \omega'$ *if and only if* $\nu_I(\omega) = \nu_I(\omega')$ *for every directed cycle* I *of* G.

Toric directed paths then have an obvious characterization in terms of their ν_I function.

Corollary 4.5. *Given a directed cycle $I = [(i_1, \ldots, i_m)]$ in G, an acyclic orientation ω in* $\operatorname{Acyc}(G)$ *contains a toric directed path lying in the cyclic equivalence class I if and only if $|\nu_I(\omega)| = m - 2$.*

5. Toric total orders

An important special case of toric directed paths occurs when one considers acyclic orientations of the complete graph K_V. Acyclic orientations of K_V correspond to permutations $w = (w_1, \ldots, w_n)$ of V (or *total orders*), and always form toric directed paths in w. Hence their toric equivalence classes are the equivalence classes $[w]$ of permutations up to cyclic shifts, or *toric total orders*. This concept coincides with the pre-existing concept of *total cyclic order* from Definition 1.14, even though toric *partial* orders are not the same as *partial* cyclic orders. Therefore, we can use these terms interchangeably.

By Theorem 1.4, these toric total orders $[w]$ index the chambers $c_{[w]}$ in $\operatorname{Cham} \mathcal{A}_{\text{tor}}(K_V)$. By Corollary 2.4, one has this more concrete description of such chambers:

$$c_{[w]} = \bigcup_{i=1}^{n} \{x \in \mathbb{R}^V/\mathbb{Z}^V : x_{w_i} \bmod 1 < \cdots < x_{w_n} \bmod 1 < x_{w_1} \bmod 1 < \cdots < x_{w_{i-1}} \bmod 1\}. \tag{5.1}$$

Definition 5.1. Given a toric poset $P = P(c)$ on V, say that a toric total order $[w]$ on V is a *toric total extension* of P if the toric chamber $c_{[w]}$ of $\operatorname{Cham} \mathcal{A}_{\text{tor}}(K_V)$ is contained in c. Denote by $\mathcal{L}_{\text{tor}}(P)$ the set of all such toric total extensions $[w]$ of P.

The following corollary is then a special case of Proposition 3.2.

Corollary 5.2. *Fix a simple graph $G = (V, E)$. Then any toric chamber/poset $c = c(P)$ in* $\operatorname{Cham} \mathcal{A}_{\text{tor}}(G)$ *has topological closure*

$$\bar{c} = \bigcup_{[w] \in \mathcal{L}_{\text{tor}}(P)} \bar{c}_{[w]}$$

and is completely determined by its set $\mathcal{L}_{\text{tor}}(P)$ of toric total extensions: if c_1, c_2 in $\operatorname{Cham} \mathcal{A}_{\text{tor}}(G)$ *have $\mathcal{L}_{\text{tor}}(P(c_1)) = \mathcal{L}_{\text{tor}}(P(c_2))$, then $c_1 = c_2$.*

Example 5.3. Corollary 5.2 fails when one does *not* fix the graph G. For example, when $V = \{1, 2, 3\}$, all seven of the *non-complete* graphs $G \neq K_V = K_3$ share the property that $\operatorname{Cham} \mathcal{A}_{\text{tor}}(G)$ has only one chamber $c = c(P)$ with $\mathcal{L}_{\text{tor}}(P) = \{[(1, 2, 3)], [(1, 3, 2)]\}$, whose closure $\bar{c}$ is the entire torus $\mathbb{R}^3/\mathbb{Z}^3$. However, the unique toric chambers c for these 7 graphs are all different, when considered as *open* subsets of $\mathbb{R}^3/\mathbb{Z}^3$, and therefore each represents a *different* toric poset $P = P(c)$.

On the other hand, the complete graph $K_V = K_3$ has two different toric equivalence classes of acyclic orientations, representing two different chambers within the same toric arrangement $\mathcal{A}_{\text{tor}}(K_3)$, and two different toric posets: $P(c_{[(1,2,3)]})$ and $P(c_{[(1,3,2)]})$.

6. Toric chains

We introduce the toric analogue of a chain (= totally ordered subset) in a poset, and explicate its relation to the toric directed paths from Definition 4.1 and the toric total extensions from Definition 5.1 (or equivalently, total cyclic extensions).

As motivation, note that in an ordinary poset $P(c)$, a chain $C = \{i_1, \ldots, i_m\} \subseteq V$ has the following geometric description: there is a total ordering $(i_1, \ldots, i_m)$ of C such that every point x in the open polyhedral cone $c = c(P)$ has $x_{i_1} < x_{i_2} < \cdots < x_{i_m}$.

Definition 6.1. Fix a toric poset $P = P(c)$ on a finite set V. Call a subset $C = \{i_1, \ldots, i_m\} \subseteq V$ a *toric chain* in P if there exists a cyclic equivalence class $[(i_1, \ldots, i_m)]$ of linear orderings of C with the following property: for every x in the open toric chamber $c = c(P)$ there exists some $(j_1, \ldots, j_m)$ in $[(i_1, \ldots, i_m)]$ for which

$$x_{j_1} \bmod 1 < x_{j_2} \bmod 1 < \cdots < x_{j_m} \bmod 1. \tag{6.1}$$

In this situation, we will say that $P|_C = [(i_1, \ldots, i_m)]$.

Remark 6.2. Note that

- singleton sets $\{i\}$ and the empty subset $\varnothing \subset V$ are always toric chains in P,
- subsets of toric chains are toric chains, and
- a pair $\{i, j\}$ is a toric chain in $P = P(c)$ if and only if every point x in the open toric chamber c has $x_i \bmod 1 \neq x_j \bmod 1$; in particular, this will be true whenever c appears as a toric chamber in $\mathrm{Cham}\,\mathcal{A}_{\mathrm{tor}}(G)$ for a graph G having $\{i, j\}$ as an edge of G.

Though the definition of toric chain does not refer to a particular graph G, there are several convenient characterizations that involve a graph. In the following proposition, we list five equivalent conditions. The exception when $|C| \neq 2$ is needed because the last condition is vacuously true whenever $|C| = 2$; in this case, only the first four are equivalent.

Proposition 6.3. *Fix a toric poset $P = P(c)$ on a finite set V, and $C = \{i_1, \ldots, i_m\} \subseteq V$. The first four of the following five conditions are equivalent, and when $m = |C| \neq 2$, they are also equivalent to the fifth:*

(a) C is a toric chain in P, with $P|_C = [(i_1, \ldots, i_m)]$.

(b) For every graph $G = (V, E)$ and acyclic orientation ω of G having $\bar{\alpha}_G(c) = [\omega]$, the subset C is a chain in the poset $P(G, \omega)$, ordered in some cyclic shift of the order $(i_i, \ldots, i_m)$.

(c) For every graph $G = (V, E)$ and acyclic orientation ω of G having $\bar{\alpha}_G(c) = [\omega]$, the subset C occurs as a subsequence of a toric directed path in ω, in some cyclic shift of the order $(i_i, \ldots, i_m)$.

(d) There exists a graph $G = (V, E)$ and acyclic orientation ω of G having $\bar{\alpha}_G(c) = [\omega]$ such that C occurs as a subsequence of a toric directed path in ω, in some cyclic shift of the order $(i_1, \ldots, i_m)$.

(e) Every total cyclic extension $[w]$ in $\mathcal{L}_{\mathrm{tor}}(P(c))$ has the same restriction $[w|_C] = [(i_1, \ldots, i_m)]$.

The following easy and well-known lemma will be used in the proof.

Lemma 6.4. *When two elements i, j are incomparable in a finite poset Q on V, one can choose a linear extension $w = (w_1, \ldots, w_n)$ in $\mathcal{L}(Q)$ that has i, j appearing consecutively, say $(w_s, w_{s+1}) = (i, j)$.*

Proof. Begin w with any linear extension $w_1, w_2, \ldots, w_{s-1}$ for the order ideal $Q_{<i} \cup Q_{<j}$, followed by $w_s = i, w_{s+1} = j$, and finish with any linear extension $w_{s+2}, w_{s+3}, \ldots, w_n$ for $Q - (Q_{\leq i} \cup Q_{\leq j})$. $\square$

Proof of Proposition 6.3. Note that if $|C| \leq 1$, all five conditions (a)-(e) are vacuously true, so without loss of generality $|C| \geq 2$. We will first show (a) implies (b) implies (c) implies (d) implies (e). Then we will show that (e) implies (a) when $|C| \geq 3$, and (d) implies (a) when $|C| = 2$.

(a) implies (b). Assume that C is a toric chain of P, with $P|_C = [(i_1, \ldots, i_m)]$, and take a graph G and orientation ω such that $\bar{\alpha}_G(c) = [\omega]$.

We first show by contradiction that C must be totally ordered in $Q := P(G, \omega)$. Assume not, and say i, j in C are incomparable in Q. By Lemma 6.4 there is a linear extension $w = (w_1, \ldots, w_n)$ in $\mathcal{L}(Q)$ having i, j appear consecutively, say $(w_s, w_{s+1}) = (i, j)$. Choose x in $\mathbb{R}^n$ with $0 \leq x_{w_1} < \cdots < x_{w_n} < 1$ and let x' be obtained by x by exchanging x_i, x_j, that is, $x'_i = x_j$ and $x'_j = x_i$. Since $x = x \bmod 1$ and $x' = x' \bmod 1$, one has $\bar{\alpha}_G(x) = \omega = \bar{\alpha}_G(x')$, and hence x, x' lie in $c = c(P)$. The condition (6.1) on x, x' implies that $[w|_C] = [w'|_C]$ should give the same cyclic order on C, which forces $m = 2$ and $C = \{i, j\}$. However, the average $x'' = \frac{x+x'}{2}$ gives a third point in c having $x''_i \bmod 1 = x''_i = x''_j = x''_j \bmod 1$, contradicting (6.1).

Once one knows that C is totally ordered in Q, consideration of (6.1) for the point x chosen as above implies that $w|_C$ lies in $[(i_1, \ldots, i_m)]$, and hence the same is true of $Q|_C$.

(b) implies (c). Assume for the toric poset $P = P(c)$, every graph G and orientation ω with $\bar{\alpha}_G(c) = [\omega]$ has C totally ordered in $P(G, \omega)$ by a cyclic shift $(j_1, \ldots, j_m)$ in $[(i_1, \ldots, i_m)]$. We will show that C actually occurs in this order as a subsequence of some toric directed path in ω.

By Proposition 4.2, one is free to alter ω within the class $[\omega]$. So choose ω within $[\omega]$ among all those for which $P(G, \omega)$ on V totally orders C as $j_1 < \cdots < j_m$, but minimizing the cardinality $|Z|$ where

$$Z := \{z \in V : z \text{ there is a directed } \omega \text{ path from } j_m \text{ to } z\}.$$

Note that Z is non-empty, since it contains j_m. We claim that minimality forces $|Z| = 1$, that is, $Z = \{j_m\}$. To argue the claim by contradiction, assume $Z \neq \{j_m\}$. Then one can find an ω-sink $z \neq j_m$ in Z, as V is finite, and ω is acyclic. Perform a sink-to-source flip at z to create a new orientation ω' in $[\omega]$. Then ω' still has $P(G, \omega')$ totally ordering C as $j_1 < \cdots < j_m$, but its set Z' has $|Z'| < |Z|$ because $Z' \subset Z - \{z\}$.

Now $Z = \{j_m\}$ means that j_m is an ω-sink. Create ω' by flipping j_m from sink to source. Since j_1 is supposed to be comparable with j_m in $P(G, \omega')$, one must have $j_m <_{P(G,\omega')} j_1$, that is, there is an ω'-path of the form $j_m \to k \to \cdots \to j_1$; possibly $k = j_1$ here. But this means that prior to the sink-to-source flip of j_m, one had a toric directed ω-path $k \to \cdots \to j_1 \to j_2 \to \cdots \to j_m$ that contained C, as desired.

(c) implies (d). Trivial.

(d) implies (e). Assume the graph G has $\bar{\alpha}_G(c) = [\omega]$ and C occurs in the order $(i_1, \ldots, i_m)$ as a subsequence of a toric directed path in ω. We must show that every total cyclic extension $[w]$ of $P = P(c)$ has restriction $[w|_C] = [(i_1, \ldots, i_m)]$.

By Definition 5.1, one has $c_{[w]} \subseteq c$. By (5.1), one can pick a point x in $c_{[w]}$, so that

$$x_{w_1} \bmod 1 < \cdots < x_{w_n} \bmod 1.$$

Since x also lies in c, one has $\bar{\alpha}_G(x) = \omega' \equiv \omega$. Proposition 4.2 implies that ω' contains as a toric directed path some cyclic shift $(j_1, \ldots, j_m)$ of $(i_1, \ldots, i_m)$. Hence

$$x_{j_1} \bmod 1 < \cdots < x_{j_m} \bmod 1,$$

which forces $w|_C = (j_1, \ldots, j_m)$, as desired.

(e) implies (a) when $|C| \geq 3$. Assume that every total cyclic extension $[w]$ of $P = P(c)$ has $w|_C$ lying in the same cyclic equivalence class $[(i_1, \ldots, i_m)]$. We want to show that every point x in c satisfies (6.1). Recall from Corollary 2.4 that there is at least one graph G and toric equivalence class $[\omega]$ containing $\bar{\alpha}_G(x)$, that is, $\bar{\alpha}_G(c) = [\omega]$. It suffices to show that the partial order $Q := P(G, \omega)$ on V induced by any orientation ω in this toric equivalence class has restriction $Q|_C$ to the subset C giving a total order $(j_1, \ldots, j_m)$, and this total order lies in $[(i_1, \ldots, i_m)]$.

Suppose that $Q|_C$ is *not* a total order; say elements i, j in C are incomparable in Q. By Lemma 6.4, one can then choose linear extensions w, w' in $\mathcal{L}(Q)$ that both have i, j consecutive, and differ only in swapping i, j, say $(w_s, w_{s+1}) = (i, j)$ and $(w'_s, w'_{s+1}) = (j, i)$. Pick points x, x' that satisfy

$$0 \leq x_{w_1} < \cdots < x_{w_n} < 1,$$
$$0 \leq x'_{w'_1} < \cdots < x'_{w'_n} < 1.$$

Since $x = x \bmod 1, x' = x' \bmod 1$, one finds that x, x' lie in $c_{[w]}, c_{[w']}$, respectively. Also one has $\bar{\alpha}_G(x) = \omega = \bar{\alpha}_G(x')$ so that both x, x' lie in c. Hence $c_{[w]}, c_{[w']} \subseteq c$, that is, both $[w], [w']$ are total cyclic extensions in $\mathcal{L}_{\mathrm{tor}}(P(c))$. However, since $|C| \geq 3$, there exists some third element k in $C - \{i, j\}$, and $[w], [w']$ differ in their cyclic ordering of $\{i, j, k\}$. This contradicts assumption (e), so $Q|_C$ is a total order.

Once one knows that $Q|_C$ is a total order $j_1 < \cdots < j_m$, the above argument shows that $(j_1, \ldots, j_m)$ lies in the cyclic equivalence class $[w|_C]$ for every w in $\mathcal{L}_{\mathrm{tor}}(P)$, which is $[(i_1, \ldots, i_m)]$ by assumption.

(d) implies (a) when $|C| = 2$. Suppose $\bar{\alpha}_G(c) = [\omega]$ and C occurs as a subsequence of a toric directed path in ω, with $i_1 < i_2$. By Proposition 4.2, if $\omega' \equiv \omega$, then C occurs in a toric directed path in ω'. This means that for any x with $\bar{\alpha}_G(x) = \omega'$, we have $x_{i_1} \bmod 1 \neq x_{i_2} \bmod 1$, and so either $x_{i_1} \bmod 1 < x_{i_2} \bmod 1$ or $x_{i_2} \bmod 1 < x_{i_1} \bmod 1$ must hold for every x in c. Thus C is a toric chain of $P(c)$. $\square$

7. Toric transitivity

We next clarify the edges that are "forced" in a toric partial order, an analogue of transitivity that we refer to as *toric transitivity*.

Theorem 7.1. *Fix a toric poset $P = P(c)$ on V, and assume that $G = (V, E)$ has c appearing as a toric chamber in* $\operatorname{Cham} \mathcal{A}_{\mathrm{tor}}(G)$, *say $\bar{\alpha}_G(c) = [\omega]$. Then for any non-edge pair $\{i, j\} \notin E$, either*

- *(i) i, j lie on a toric chain in P, in which case c is also a toric chamber for $G^+ = (V, E \cup \{i, j\})$, and there is a unique extension ω^+ of ω such that $\bar{\alpha}_{G^+}(c) = \omega^+$, or*
- *(ii) i, j lie on no common toric chain in P, and then the hyperplane $x_i = x_j \bmod 1$ intersects the open toric chamber c.*

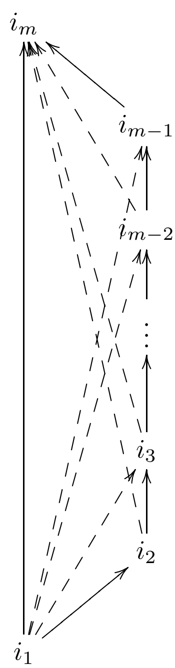
Proof. Assertion (i) follows from Proposition 6.3: when i, j lie on a toric chain C in P, assertion (b) of that proposition says that they lie on a toric directed path in ω for every representative of the class $[\omega]$, and hence the inequality $x_i \bmod 1 < x_j \bmod 1$ (or its reverse inequality) is already implied by the other inequalities defining the points of $\bar{\alpha}_G^{-1}(\omega)$ that come from the edges of G induced by C.

For assertion (ii), note that whenever there exist no points x of the open toric chamber c having $x_i \bmod 1 = x_j \bmod 1$, then every x in c has either $x_i \bmod 1 < x_j \bmod 1$ or $x_j \bmod 1 < x_i \bmod 1$. This shows that $\{i, j\}$ is itself a toric chain in $P = P(c)$; see Remark 6.2. ☐

This suggests the following definition.

Definition 7.2. Given a graph $G = (V, E)$ and ω in $\operatorname{Acyc}(G)$, the *toric transitive closure* of the pair (G, ω) is the pair $(\bar{G}^{\mathrm{tor}}, \bar{\omega}^{\mathrm{tor}})$ defined as follows. The edges of $\bar{G}^{\mathrm{tor}}$ are obtained by adding to the edges of G all pairs $\{i, j\}$ that are a subset of some toric directed path in ω; see the dotted edges in (7.1) below. The acyclic orientation $\bar{\omega}^{\mathrm{tor}}$ orients the edge $i \to j$ if the toric directed path contains a path from i to j, rather than from j to i.

(7.1)



Corollary 7.3. *The toric transitive closure of (G,ω) depends only upon the toric poset $P = P(c)$ which satisfies $\bar{\alpha}_G(c) = [\omega]$, in the following sense: given two graphs $G_i = (V, E_i)$ for $i = 1, 2$, and ω_i in* $\mathrm{Acyc}(G_i)$ *with $\bar{\alpha}_{G_i}(c) = [\omega_i]$, then*

(i) $\bar{G}_1^{\mathrm{tor}} = \bar{G}_2^{\mathrm{tor}}$, *and*
(ii) $\bar{\omega}_1^{\mathrm{tor}} \equiv \bar{\omega}_2^{\mathrm{tor}}$.

Proof. Assertion (i) follows from the fact that $\{i,j\}$ appears as an edge in $\bar{G}^{\mathrm{tor}}$ if and only if it is a subset of some toric chain of P, and adding $\{i,j\}$ does not affect the toric poset $P = P(c)$, according to Theorem 7.1(i). For assertion (ii), note that iterating Theorem 7.1(i) gives

$$\bar{\alpha}^{-1}_{\bar{G}^{\mathrm{tor}}}(\bar{\omega}_1^{\mathrm{tor}}) = \bar{\alpha}^{-1}_{G_1}(\omega_1) = c = \bar{\alpha}^{-1}_{G_2}(\omega_2) = \bar{\alpha}^{-1}_{\bar{G}^{\mathrm{tor}}}(\bar{\omega}_2^{\mathrm{tor}}).$$

Assertion (ii) then follows from Theorem 1.4. ☐

Remark 7.4. Note that the toric transitive closure of $\bar{A}^{\mathrm{tor}}$ is always a subset of the ordinary transitive closure $\bar{A}$, since any toric directed path that contains (i,j) as a subsequence also contains an ordinary directed path from i to j. In particular, if A is acyclic, then $\bar{A}^{\mathrm{tor}}$ is acyclic.

8. Proof of Theorem 1.10

Here we wish to regard a pair (G,ω) of a simple graph $G = (V,E)$ and acyclic orientation ω in $\mathrm{Acyc}(G)$ as a subset $A \subset \overleftrightarrow{K}_V$ of the set of all possible directed edges $\overleftrightarrow{K}_V = \{(i,j) \in V \times V : i \neq j\}$. Then the toric transitive closure operation $(G,\omega) \longmapsto (\bar{G}^{\mathrm{tor}}, \bar{\omega}^{\mathrm{tor}})$ from Definition 7.2 may be regarded as a *closure operator* on $\overleftrightarrow{K}_V$, that is, a map $A \longmapsto \bar{A}^{\mathrm{tor}}$ from $2^{\overleftrightarrow{K}_V}$ to itself, satisfying

- $A \subseteq \bar{A}^{\mathrm{tor}}$,
- $A \subseteq B$ implies $\bar{A}^{\mathrm{tor}} \subseteq \bar{B}^{\mathrm{tor}}$, and
- $\bar{\bar{A}}^{\mathrm{tor}} = \bar{A}^{\mathrm{tor}}$.

Recall the statement of Theorem 1.10:

Theorem 1.10. *Let A be an acyclic subset of $\overleftrightarrow{K}_V$. The toric transitive closure operation $A \longmapsto \bar{A}^{\mathrm{tor}}$ is a convex closure, that is,*

$$\text{for } a \neq b \text{ with } a, b \notin \bar{A}^{\mathrm{tor}} \text{ and } a \in \overline{A \cup \{b\}}^{\mathrm{tor}}, \text{ one has } b \notin \overline{A \cup \{a\}}^{\mathrm{tor}}.$$

For the purposes of the proof, introduce one further bit of terminology.

Definition 8.1. For ω in $\mathrm{Acyc}(G)$ and a toric directed path $C = (i_1, \ldots, i_m)$ in ω of size $m \geq 3$, as in (4.1), call (i_1, i_m) the *long edge* of C, and call the other edges $(i_1,i_2), (i_2,i_3), \ldots, (i_{m-1}, i_m)$ the *short edges* of C.

Proof of Theorem 1.10. Proceed by contradiction: suppose $(i,j) \neq (k,\ell)$ are *not* in $\bar{A}^{\mathrm{tor}}$, but both

- (k,ℓ) lies in $\overline{A \cup (i,j)}^{\mathrm{tor}}$, say because (i,j) creates a toric directed path C also containing (k,ℓ), which was not already present in $\bar{A}^{\mathrm{tor}}$, and
- (i,j) lies in $\overline{A \cup (k,\ell)}^{\mathrm{tor}}$, say because (k,ℓ) creates a toric directed path D also containing (i,j), which was not already present in $\bar{A}^{\mathrm{tor}}$.

Introduce the (ordinary) partial order Q on V which is the (ordinary) transitive closure of $\bar{A}^{\mathrm{tor}} \cup \{(i,j),(k,\ell)\}$. We use this to argue a contradiction in various cases.

Case 1. *Either* (i, j) *is the long edge of* C, *or* (k, ℓ) *is the long edge of* D. By relabeling, assume without loss of generality that (i, j) is the long edge of C. Then in Q, one has

$$i \leq k < \ell \leq j \tag{8.1}$$

with at least one of the two weak inequalities being strict.

Subcase 1*a*. (k, ℓ) *is also the long edge of* D. Then in Q one also has $k \leq i < j \leq \ell$, which with (8.1) gives

$$k \leq i \leq k < \ell \leq j \leq \ell$$

forcing the contradiction $(i, j) = (k, \ell)$.

Subcase 1*b*. (k, ℓ) *is a short edge of* D. Then since C has (i, j) as its long edge and gives a toric directed path containing (k, ℓ) (while $\bar{A}^{\mathrm{tor}}$ had no such path), C must contain a directed path from k to ℓ with at least two steps. Combining this with $D - \{(k, \ell)\}$ gives a toric directed path in $\bar{A}^{\mathrm{tor}}$ that contains (i, j); a contradiction.

Case 2. *Both* $(i, j), (k, \ell)$ *are short edges of* C, D, *respectively.* In this case, $\bar{A}^{\mathrm{tor}}$ cannot contain a path from i to j, else replacing (i, j) in C with this path would give the contradiction that (i, j) is in $\bar{A}^{\mathrm{tor}}$. Similarly, $\bar{A}^{\mathrm{tor}}$ cannot contain a path from k to ℓ. Also note that, since C (or D) is a directed path containing all four of $\{i, j, k, \ell\}$, the four of them are totally ordered in Q. We now argue in subcases based on how Q totally orders $\{i, j, k, \ell\}$.

Subcase 2*a*. *Either* Q *has* $i < j \leq k < \ell$ *or* $k < \ell \leq i < j$. In this case, adding (i, j) to $\bar{A}^{\mathrm{tor}}$ cannot help to create a directed path from k to ℓ, contradicting the existence of C.

Subcase 2*b*. *Either* Q *has* $i \leq k < \ell \leq j$, *with at least one of the weak inequalities strict, or* $k \leq i < j \leq \ell$, *with at least one of the weak inequalities strict.*. Assume without loss of generality, by relabeling, that one is in the first case $i \leq k < \ell \leq j$. But then adding (i, j) to $\bar{A}^{\mathrm{tor}}$ again cannot help to create a directed path from k to ℓ, contradicting the existence of C.

Subcase 2*c*. *Either* Q *has* $i \leq k \leq j \leq \ell$, *with at least two consecutive strict inequalities, or* $k \leq i \leq \ell \leq j$, *with at least two consecutive strict inequalities.* Assume without loss of generality, by relabeling, that one is in the first case $i \leq k \leq j \leq \ell$. But then the consecutive strict inequalities either imply the existence within $\bar{A}^{\mathrm{tor}}$ of a directed path from i to j, or one from k to ℓ; a contradiction.

□

9. Toric Hasse diagrams

For convex closures $A \longmapsto \bar{A}$, it is well known that for any subset A, its *extreme points*

$$\mathrm{ex}(A) := \{a \in A : a \notin \overline{A - \{a\}}\}$$

gives the unique set which is minimal under inclusion among all subsets having the same closure as A; see [10]. For ordinary transitive closure of an acyclic orientation (G, ω) as a subset of $\overleftrightarrow{K_V}$, its extreme points are exactly the subset of directed edges (i, j) in the usual *Hasse diagram* for its associated partial order P. This suggests the following definition.

Definition 9.1. Given a graph $G = (V, E)$ and ω in $\mathrm{Acyc}(G)$, corresponding to a subset A of $\overleftrightarrow{K_V}$, its *toric Hasse diagram* is the pair $(\hat{G}^{\mathrm{torHasse}}, \omega^{\mathrm{torHasse}})$ corresponding to its subset of extreme points $\mathrm{ex}(A)$ with respect to the toric transitive closure operation $A \longmapsto \bar{A}^{\mathrm{tor}}$. The toric Hasse diagram of a toric poset P is (G).

Definition 7.2 allows one to rephrase this as follows:

- $\hat{G}^{\mathrm{torHasse}}$ is obtained from G by removing all *chord edges* $\{i_j, i_k\}$ with $|j-k| \geq 2$ from all toric directed paths $C = \{i_1, \ldots, i_m\}$ in ω that have $m = |C| \geq 4$, and
- $\omega^{\mathrm{torHasse}}$ is the restriction $\omega|_{\hat{G}^{\mathrm{torHasse}}}$.

One then has the following analogue of Corollary 7.3.

Corollary 9.2. *The toric Hasse diagram of (G, ω) depends only on the toric poset $P = P(c)$ having $\bar{\alpha}_G(c) = [\omega]$, in the following sense: given two graphs $G_i = (V, E_i)$ for $i = 1, 2$, and ω_i in $\mathrm{Acyc}(G_i)$ with $\bar{\alpha}_{G_i}(c) = [\omega_i]$, then*

(i) $\hat{G}_1^{\mathrm{torHasse}} = \hat{G}_2^{\mathrm{torHasse}}$, *and*
(ii) $\omega_1^{\mathrm{torHasse}} \equiv \omega_2^{\mathrm{torHasse}}$.

Proof. Same as the proof of Corollary 7.3. The key point is that the toric directed paths $C = \{i_1, \ldots, i_m\}$ in ω are the toric chains in P, and when $|C| \geq 4$, removing chords from C still keeps it a toric chain. $\square$

10. Toric antichains

Since chains in posets have a good toric analogue, one might ask if the same is true for antichains. Recall that an *antichain* of an ordinary poset P on V is a subset $A = \{i_1, \ldots, i_m\} \subseteq V$ characterized

- *combinatorially* by the condition that no pair $\{i, j\} \subset A$ with $i \neq j$ are comparable, that is, they lie on no chain of P, or
- *geometrically* by the equivalent condition that the $(|V|-m+1)$-dimensional linear subspace $\{x \in \mathbb{R}^V : x_{i_1} = x_{i_2} = \cdots = x_{i_m}\}$ intersects the open polyhedral cone/chamber $c(P)$ in $\mathbb{R}^V$.

In the toric situation, these two conditions lead to different notions of toric antichains.

Definition 10.1. Given a toric poset $P = P(c)$ on the finite set V, say that $A = \{i_1, \ldots, i_m\} \subseteq V$ is a

- *combinatorial toric antichain* of P if no $\{i, j\} \subset A$ with $i \neq j$ lie on a common toric chain of P,
- *geometric toric antichain* if the subspace $\{x \in \mathbb{R}^V/\mathbb{Z}^V : x_{i_1} = x_{i_2} = \cdots = x_{i_m}\}$ intersects the open toric chamber $c = c(P)$.

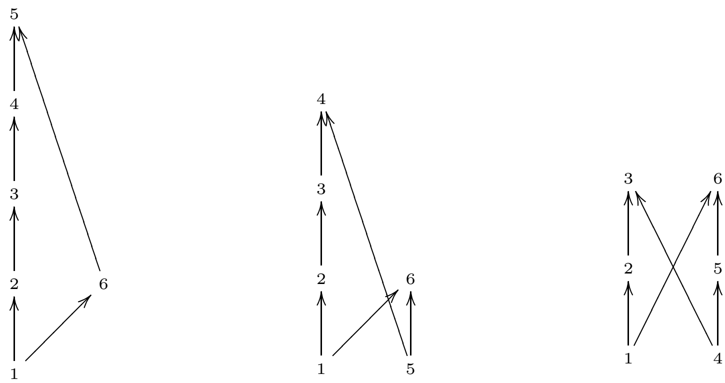
By analogy to the notion of the width of a poset, which is the size of its largest antichain, define the *geometric (resp. combinatorial) toric width* of a toric poset to be the size of the largest geometric (resp. combinatorial) toric antichain.

Given a toric poset $P = P(c)$ and a graph $G = (V, E)$ with $\bar{\alpha}_G(c) = [\omega]$, the definition and Corollary 2.4 imply that $A \subseteq V$ is a geometric toric antichain of P if and only if A is an antichain of $P(G, \omega')$ for some $\omega' \equiv \omega$. The following proposition should also be clear.

Proposition 10.2. *In a toric poset P, every geometric toric antichain is a combinatorial toric antichain. Thus its geometric toric width is bounded above by its combinatorial toric width.*

The next example shows that the inequality between these two notions of toric width can be strict.

Example 10.3. Consider the toric poset $P = P(c)$ whose toric Hasse diagram is the circular graph $G = C_6$ and for which $\bar{\alpha}_G(c)$ contains the following representatives ω_1, ω_2 and ω_3 of $\mathrm{Acyc}(G)$:



All three of these orientations satisfy $\nu_I(\omega_i) = 2$ for the directed cycle $I = [(1,2,3,4,5,6)]$ of G, where ν_I is Coleman's ν-function from Remark 4.3. Moreover, Proposition 4.4 says that $\nu_I(\omega) = 2$ must hold for any other ω in $[\omega_i]$. It is easy to check that for any such ω, the directed graph (G, ω) must be isomorphic to either (G, ω_1), (G, ω_2), or (G, ω_3).

Consequently, P has no toric chains except for those of cardinality $0, 1, 2$, that is, the empty set $\varnothing$, the six singletons and the six edge pairs in G. From this one can easily check that the combinatorial toric antichains of P are the empty set $\varnothing$, the six singletons, the pairs $\{i, j\}$ which do not form edges of G, and the two triples $\{1, 3, 5\}, \{2, 4, 6\}$. In particular, P has combinatorial toric width 3.

However, we claim neither of these triples $\{1, 3, 5\}, \{2, 4, 6\}$ can be a geometric toric antichain, so that the geometric toric width of P is 2. To argue that $\{1, 3, 5\}$ is not a geometric toric antichain, consider three paths of length 2 in G between the elements of $\{1, 3, 5\}$, that is, the paths

$$1 - 2 - 3,$$
$$3 - 4 - 5,$$
$$5 - 6 - 1.$$

The only way one could avoid having an ω-directed path between two elements of $\{1, 3, 5\}$ would be if ω orients both edges in each of the three paths listed above in opposite directions. But this would lead to $\nu_I(\omega) = 0$ which is impossible for ω in $[\omega_i]$. The argument for $\{2, 4, 6\}$ is similar.

Despite the difference in the two notions of toric width, one might still hope that one of the notions gives a toric analogue for one or both of these two classic results on chains and antichains in ordinary posets.

Theorem 10.4. *For any (ordinary) finite poset P, one has:*

(i) Dilworth's Theorem [9]*:*

$$\max\{|A| : A \text{ an antichain in } P\} = \min\{\ell : V = \bigcup_{i=1}^{\ell} C_i, \text{ with } C_i \text{ chains in } P\}.$$

(ii) Mirsky's Theorem [21]*:*

$$\max\{|C| : C \text{ a chain in } P\} = \min\{\ell : V = \bigcup_{i=1}^{\ell} A_i, \text{ with } A_i \text{ antichains in } P\}.$$

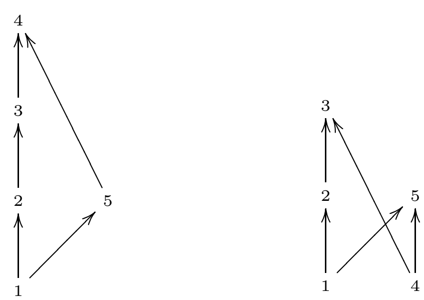
One at least has the following inequalities, coming from the easy observation that a toric chain and toric antichain (whether combinatorial or geometric) can intersect in at most one element.

Proposition 10.5. *For a toric poset P, both versions (geometric or combinatorial) of a toric antichain lead to the following inequalities holding:*

$$\begin{aligned}
&\max\{|A| : A \text{ a toric antichain in } P\} \\
&\qquad \leq \min\{\ell : V = \bigcup_{i=1}^{\ell} C_i, \text{ with } C_i \text{ toric chains in } P\}, \\
&\max\{|C| : C \text{ a toric chain in } P\} \\
&\qquad \leq \min\{\ell : V = \bigcup_{i=1}^{\ell} A_i, \text{ with } A_i \text{ toric antichains in } P\}.
\end{aligned}$$

However, the following example shows that both inequalities in Proposition 10.5 can be strict: neither of our two notions of toric antichain leads to a version of Dilworth's Theorem, nor of Mirsky's Theorem.

Example 10.6. Consider the toric poset $P = P(c)$ whose toric Hasse diagram is the circular graph $G = C_5$ and for which $\bar{\alpha}_G(c)$ contains the following representatives ω_1 and ω_2 of $\mathrm{Acyc}(G)$:



Both orientations above satisfy $\nu_I(\omega_i) = 1$ for the directed cycle $I = [(1, 2, 3, 4, 5)]$ of G. Proposition 4.4 says that $\nu_I(\omega) = 1$ must hold for any other ω in $[\omega_i]$, and so for such an ω, the directed graph (G, ω) must be isomorphic to either (G, ω_1) or (G, ω_2).

Consequently, P has no toric chains except for those of cardinality $0, 1, 2$, that is, the empty set $\varnothing$, the five singletons and the five edge pairs in G. In particular, the maximum size of a toric chain is 2. From this one can also easily check that the combinatorial toric antichains of P are the empty set $\varnothing$, the five singletons, and the five pairs $\{i, j\}$ which do not form edges of G. In fact, all of these are also geometric toric antichains, so in this example the two notions coincide, and for either one the toric width is 2.

However, as $|V| = 5$, there is no partition of V into two toric chains (the analogue of Dilworth's Theorem fails), nor into two toric antichains (the analogue of Mirsky's Theorem fails).

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Department of Audience Research, Facebook Inc., 1601 Willow Road, Menlo Park, California 94025

E-mail address: `develin@post.harvard.edu`

Department of Mathematical Sciences, Clemson University, Clemson, South Carolina 29634-0975

E-mail address: `macaule@clemson.edu`

Department of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455

E-mail address: `reiner@math.umn.edu`