Circular Planar Electrical Networks: Posets and Positivity

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Abstract

Following de Verdière-Gitler-Vertigan and Curtis-Ingerman-Morrow, we prove a host of new results on circular planar electrical networks. We first construct a poset EP_n of electrical networks with n boundary vertices, and prove that it is graded by number of edges of critical representatives. We then answer various enumerative questions related to EP_n, adapting methods of Callan and Stein-Everett. Finally, we study certain positivity phenomena of the response matrices arising from circular planar electrical networks. In doing so, we introduce electrical positroids, extending work of Postnikov, and discuss a naturally arising example of a Laurent phenomenon algebra, as studied by Lam-Pylyavskyy.

Contents

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1 Introduction

Circular planar electrical networks are a natural generalization of an idea from classical physics: that any electrical resistor network with two vertices connected to a battery behaves like one with a single resistor. When we embed a resistor network in a disk and allow arbitrarily many boundary vertices connected to batteries, the situation becomes more interesting. An inverse boundary problem for these electrical networks was studied in detail by de Verdière-Gitler-Vertigan [dVGV] and Curtis-Ingerman-Morrow [CIM]: given the *response matrix* of a network, that is, information about how the network responds to voltages applied at the boundary vertices, can the network be recovered?

In general, the answer is "no," though much can be said about the information that can be recovered. If, for example, the underlying graph of the electrical network is known and is *critical*, the conductances (equivalently, resistances) can be uniquely recovered [CIM, Theorem 2]. Moreover, any two networks which produce the same response matrix can be related by a certain class of combinatorial transformations, the *local equivalences* [dVGV, Théorème 4].

The goal of this paper is to study more closely the rich theory of circular planar electrical networks. Our first task is to define a poset EP_n of circular planar graphs, under the operations of contraction and deletion of edges. Using the important tool of *medial graphs* developed in [CIM] and [dVGV], we prove:

Theorem 1.1. *The poset EP_n is graded by number of edges of critical representatives.*

The poset EP_n also has an intricate topological structure. By [CIM, Theorem 4] and [dVGV, Théorème 3], the space of response matrices for circular planar electrical networks of order n decomposes as a disjoint union of open *cells*, each diffeomorphic to a product of copies of the positive real line. In light of this decomposition, we can describe EP_n as the poset of these cells under containment of closure. We give several conjectural properties of EP_n related to this topological structure.

We also embark on a study of the enumerative properties of EP_n. Medial graphs bear a strong resemblance to certain objects whose enumerative properties are known: stabilized-interval free (SIF) permutations, as studied by Callan [C], and irreducible linked diagrams, as studied by Stein-Everett [SE]. Exploiting this resemblance, we summarize and prove analogues of known results in the following theorem:

Theorem 1.2. *Put $X_n = |EP_n|$, the number of equivalence classes of electrical networks of order n. Then:*

(a) $X_1 = 1$ *and*

$$X_n = 2(n-1)X_{n-1} + \sum_{j=2}^{n-2}(j-1)X_jX_{n-j}.$$

(b) $[t^{n-1}]X(t)^n = n \cdot (2n-3)!!$, *where* $X(t)$ *is the generating function for the sequence* $\{X_i\}$.

(c) $X_n/(2n-1)!! \sim e^{-1/2}$.

$n \times n$ response matrices are characterized in [CIM, Theorem 4] as the symmetric matrices whose *circular minors* are non-negative, such that the sum of the entries in each row (and column) is zero. Furthermore, [CIM, Lemma 4.2] gives a combinatorial criterion, in terms of the underlying electrical network, for exactly which circular minors are strictly positive. However, until now, the combinatorial properties of these response matrices have remained largely unstudied, despite their inherently combinatorial descriptions.

A natural question that arises is: which sets of circular minors can be positive, while the others are zero? It is clear (for example, from the Grassmann-Plücker relations) that one cannot construct response matrices with arbitrary sets of positive circular minors. Postnikov [P] studied a similar question in the *totally nonnegative Grassmannian*, as follows: for $k \times n$ matrices A, with $k < n$ and all $k \times k$ minors nonnegative, which sets (in fact, matroids) of $k \times k$ minors can be the set of positive minors of A? These sets, called *positroids* by Knutson-Lam-Speyer [KLS], were found in [P] to index many interesting combinatorial objects. Two of these objects, plabic graphs and alternating strand diagrams, are highly similar to circular planar electrial networks and medial graphs, respectively, which we study in this paper. Our introduction of *electrical positroids* is therefore a natural extension of the theory of positroids. We give a novel axiomatization of electrical positroids, motivated by the Grassmann-Plücker relations, and prove the following:

Theorem 1.3. *A set S of circular pairs is the set of positive circular minors of a response matrix if and only if S is an electrical positroid.*

Another point of interest is that of *positivity tests* for response matrices. In [FZP], Fomin-Zelevinsky describe various positivity tests for *totally positive matrices*: given an $n \times n$ matrix, there exist sets of n^2 minors whose positivity implies the positivity of all minors. These sets of minors are described combinatorially by *double wiring diagrams.* Fomin-Zelevinsky later introduced *cluster algebras* in [FZ1], in part, to study similar positivity phenomena. In particular, their double wiring diagrams fit naturally within the realm of cluster algebras as manifestations of certain cluster algebra mutations.

In a similar way, we describe sets of $\binom{n}{2}$ minors of an $n \times n$ matrix M whose positivity implies the positivity of all circular minors, that is, that M is a response matrix for a top-rank (in EP_n) electrical network. However, these sets do not form clusters in a cluster algebra. Instead, they form clusters in a *Laurent phenomenon (LP) algebra*, a notion introduced by Lam-Pylyavskyy in [LP]. This observation leads to the last of our main theorems:

Theorem 1.4. *There exists an LP algebra $\mathcal{LM}_n$, isomorphic to the polynomial ring on $\binom{n}{2}$ generators, with an initial seed $\mathcal{D}_n$ of diametric circular minors. $\mathcal{D}_n$ is a positivity test for circular minors, and furthermore, all "Plücker clusters" in $\mathcal{LM}_n$, that is, clusters of circular minors, are positivity tests.*

In proving Theorem 1.4, we find that $\mathcal{LM}_n$ is, in a sense, "double-covered" by a cluster algebra $\mathcal{CM}_n$ that behaves very much like $\mathcal{LM}_n$ when we restrict to certain types of mutations. Further investigation of the clusters leads to an analogue of weak separation, as studied by Oh-Speyer-Postnikov [OSP] and Scott [S]. Conjecturally, the "Plücker clusters," of $\mathcal{LM}_n$ correspond exactly to the maximal pairwise weakly separated sets of circular pairs. Furthermore, we conjecture that these maximal pairwise weakly separated sets are related to each other by mutations corresponding to the Grasmann-Plücker relations. While we establish several weak forms of the conjecture, the general statement remains open.

The roadmap of the paper is as follows. In an attempt to keep the exposition of this paper as self-contained as possible, we carefully review terminology and known results in §2, where we also establish some basic properties of electrical networks. In §3, we define the poset EP_n, establishing the equivalence of the two descriptions given above, and prove Theorem 1.1 as Theorem 3.2.4. The study of enumerative properties of EP_n is undertaken in §4, where we prove the three parts of Theorem 1.2 as Theorems 4.1.6, 4.1.8, and 4.2.1. In the second half of the paper, we turn our attention to response matrices. In §5, we motivate and introduce electrical positroids, proving Theorem 1.3 as Theorem 5.1.7. Finally, in §6, using positivity tests as a springboard, we construct $\mathcal{LM}_n$ and prove Theorem 1.4 as Corollary 6.1.9, Lemma 6.2.6, and Theorem 6.2.17. We conclude by establishing weak forms of Conjecture 6.3.4, which relates the clusters of $\mathcal{LM}_n$ to positivity tests and our new analogue of weak separation.

2 Electrical Networks

We begin a systematic discussion of electrical networks by recalling various notions and results from [CIM]. We will also introduce some new terminology and conventions which will aid our exposition, in some cases deviating from [CIM].

2.1 Circular Planar Electrical Networks, up to equivalence

Definition 2.1.1. A **circular planar graph** Γ is a planar graph embedded in a disk D. Γ is allowed to have self-loops and multiple edges, and has at least one vertex on the boundary of D - such vertices are called **boundary vertices**. A **circular planar electrical network** is a circular planar graph Γ, together with a **conductance** $\gamma : E(\Gamma) \to \mathbb{R}_{>0}$.

To avoid cumbersome language, we will henceforth refer to these objects as **electrical networks.** We will also call the number of boundary vertices of an electrical network (or a circular planar graph) its **order**.

We can interpret this construction as an electrical network in the physical sense, with a resistor existing on each edge e with conductance $\gamma(e)$. Electrical networks satisfy **Ohm's Law** and **Kirchhoff's Laws**, classical physical phenomena which we neglect to explain in detail here. Given an electrical network (Γ, γ), suppose that we apply electrical potentials at each of the boundary vertices $V_1, \dots, V_n$, inducing currents through the network. Then, we get a map $f : \mathbb{R}^n \to \mathbb{R}^n$, where f sends the potentials $(p_1, \dots, p_n)$ applied at the vertices $V_1, \dots, V_n$ to the currents $(i_1, \dots, i_n)$ observed at $V_1, \dots, V_n$. We will take currents going out of the boundary to be negative and those going in to the boundary to be positive.

Remark 2.1.2. The convention for current direction above is the opposite of that used in [CIM], but we will prefer it for the ensuing elegance of the statement of Theorem 2.2.6a.

In fact, f is linear (see [CIM, §1]), and we have natural bases for the spaces of applied voltages and observed currents at the boundary vertices. Thus, we can make the following definition:

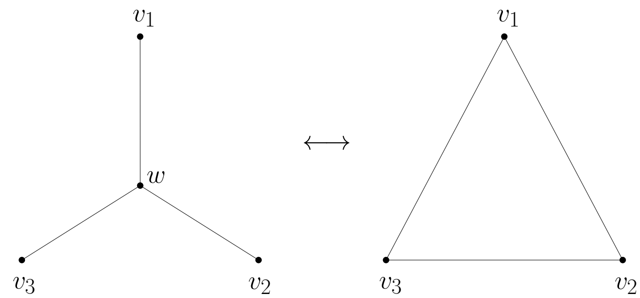

Figure 1: Y-Δ transformation.

Definition 2.1.3. Given an electrical network (Γ, γ), define the **response matrix** of the network to be the linear map f constructed above from applied voltages to observed currents, in terms of the natural bases indexed by the boundary vertices.

Definition 2.1.4. Two electrical networks $(\Gamma_1, \gamma_1), (\Gamma_2, \gamma_2)$ are **equivalent** if they have the same response matrix. In other words, the two networks cannot be distinguished only by applying voltages at the boundary vertices and observing the resulting currents. The equivalence relation is denoted by $\sim$.

We will study electrical networks up to equivalence. We have an important class of combinatorial transformations that may be applied to electrical networks, known as **local equivalences**, described below. These transformations may be seen to be equivalences by applications of Ohm's and Kirchhoff's Laws. Note that all of these local equivalences may be performed in reverse.

1. **Self-loop and spike removal.** Self-loops (cycles of length 1) and spikes (edges adjoined to non-boundary vertices of degree 1) of any conductances may always be removed.

2. **Replacement of edges in parallel.** Two edges e_1, e_2 between with common endpoints v, w may be replaced by a single edge of conductance $\gamma(e_1) + \gamma(e_2)$.

3. **Replacement of edges in series.** Two edges v_1w, wv_2 ($v_1 \neq v_2$) meeting at a vertex w of degree 2 may be replaced by a single edge v_1v_2 of conductance $((\gamma(v_1w)^{-1} + \gamma(wv_2)^{-1})^{-1}$.

4. **Y-Δ transformations.** (See Figure 1) Three edges v_1w, v_2w, v_3w meeting at a non-boundary vertex w of degree 3 may be replaced by three edges v_1v_2, v_2v_3, v_3v_1, of conductances

$$\frac{\gamma_1\gamma_2}{\gamma_1 + \gamma_2 + \gamma_3}, \frac{\gamma_2\gamma_3}{\gamma_1 + \gamma_2 + \gamma_3}, \frac{\gamma_3\gamma_1}{\gamma_1 + \gamma_2 + \gamma_3},$$

   where γ_i denotes the conductance $\gamma(v_iw)$.

In fact, local equivalences are sufficient to generate equivalence of electrical networks:

Theorem 2.1.5 ([dVGV, Théorème 4])**.** *Two electrical networks are equivalent if and only if they are related by a sequence of local equivalences.*

When dealing with electrical networks, we will sometimes avoid making any reference to the conductance map γ, and instead consider just the underlying circular planar graph Γ. In doing so, we will abuse terminology by calling circular planar graphs "electrical networks." In practice, we will only use the following notion of equivalence on circular planar graphs:

Definition 2.1.6. Let Γ_1, Γ_2 be circular planar graphs, each with the same number of boundary vertices. Then, Γ_1, Γ_2 are **equivalent** if there exist conductivities γ_1, γ_2 on Γ_1, Γ_2, respectively such that $(\Gamma_1, \gamma_1), (\Gamma_2, \gamma_2)$ are equivalent electrical networks. As with electrical networks, this equivalence is denoted $\sim$.

It is clear that we may still apply local equivalences with this notion of equivalence. Furthermore, we have an analogue of Theorem 2.1.5: two circular planar graphs are equivalent if and only if they can be related by local equivalences, where we "forget" the conductances.

2.2 Circular Pairs and Circular Minors

Circular pairs and circular minors are central to the characterization of response matrices. Accordingly, they will provide the foundation for our study of positivity in §5 and §6.

Definition 2.2.1. Let $P = \{p_1, p_2, \ldots, p_k\}$ and $Q = \{q_1, q_2, \ldots, q_k\}$ be disjoint ordered subsets of the boundary vertices of an electrical network (Γ, γ). We say that $(P; Q)$ is a **circular pair** if $p_1, \ldots, p_k, q_k, \ldots, q_1$ are in clockwise order around the circle. We will refer to k as the **size** of the circular pair.

Remark 2.2.2. We will take $(P; Q)$ to be the same circular pair as $(\widetilde{Q}; \widetilde{P})$, where $\widetilde{P}$ denotes the ordered set P with its elements reversed. Almost all of our definitions and statements are compatible with this convention; most notably, by Theorem 2.2.6a, because response matrices are positive, the circular minors $M(P; Q)$ and $M(\widetilde{Q}; \widetilde{P})$ are the same. Whenever there is a question as to the effect of choosing either $(P; Q)$ or $(\widetilde{Q}; \widetilde{P})$, we take extra care to point the possible ambiguity.

Definition 2.2.3. Let $(P; Q)$ and (Γ, γ) be as in Definition 2.2.1. We say that there is a **connection** from P to Q in Γ if there exists a collection of vertex-disjoint paths from p_i to q_i in Γ, and furthermore each path in the collection contains no boundary vertices other than its endpoints. We denote the set of circular pairs $(P; Q)$ for which P is connected to Q by $\pi(\Gamma)$.

Definition 2.2.4. Let $(P; Q)$ and (Γ, γ) be as in Definition 2.2.1, and let M be the response matrix. We define the **circular minor** associated to $(P; Q)$ to be the determinant of the $k \times k$ matrix $M(P; Q)$ with $M(P; Q)_{i,j} = M_{p_i, q_j}$.

Remark 2.2.5. We will sometimes refer to submatrices and their determinants both as minors, interchangeably. In all instances, it will be clear from context which we mean.

We are interested in circular minors and connections because of the following result from [CIM]:

Theorem 2.2.6. *Let M be an $n \times n$ matrix. Then:*

(a) M is the response matrix for an electrical network (Γ, γ) if and only if M is symmetric with row and column sums equal zero, and each of the circular minors $M(P; Q)$ is non-negative.

(b) If M is the response matrix for an electrical network (Γ, γ), the positive circular minors $M(P; Q)$ are exactly those for which there is a connection from P to Q.

Proof. (a) is immediate from [CIM, Theorem 4], which we will state as Theorem 2.3.6 later. (b) is [CIM, Theorem 4.2]. Note that, because we have declared current going into the circle to be negative, we do not have the extra factors of $(-1)^k$ as in [CIM]. □

We now define two operations on the circular planar graphs and electrical networks. Each operation decreases the total number of edges by one.

Definition 2.2.7. Let G be a circular planar graph, and let e be an edge with endpoints v, w. The **deletion** of e from G is exactly as named; the edge e is removed while leaving the rest of the vertices and edges of G unchanged. If v, w are not both boundary vertex of G, we may also perform a **contraction** of e, which identifies all points of e. If exactly one of v, w is a boundary vertex, then the image of e under the contraction is a boundary vertex. Note that edges connecting two boundary vertices cannot be contracted to either endpoint.

2.3 Critical Graphs

In this section, we introduce critical graphs, a particular class of circular planar graphs. The definition is at first somewhat mysterious, but their importance will quickly become clear.

Definition 2.3.1. Let G be a circular planar graph. G is said to be **critical** if, for any removal of an edge via deletion or contraction, there exists a circular pair $(P; Q)$ for which P is connected to Q through G before the edge removal, but not afterward.

Theorem 2.3.2 ([dVGV, Théorème 2])**.** *Every equivalence class of circular planar graphs has a critical representative.*

Theorem 2.3.3 ([CIM, Theorem 1])**.** *Suppose G_1, G_2 are critical. Then, G_1 and G_2 are Y-Δ equivalent (that is, related by a sequence of Y-Δ transformations) if and only if $\pi(G_1) = \pi(G_2)$.*

Proposition 2.3.4. *Let G_1, G_2 be arbitrary circular planar graphs. Then, $G_1 \sim G_2$ if and only if $\pi(G_1) = \pi(G_2)$.*

Proof. By Theorem 2.1.5, if $G_1 \sim G_2$, then G_1 and G_2 are related by a sequence of local equivalences. All local equivalences preserve $\pi(-)$; indeed, we have the claim for Y-Δ transformations by Theorem 2.3.3, and it is easy to check for all other local equivalences. It follows that $\pi(G_1) = \pi(G_2)$.

In the other direction, by Theorem 2.3.2, there exist critical graphs H_1, H_2 such that $G_1 \sim H_1$ and $G_2 \sim H_2$. By similar logic from the previous paragraph, we have $\pi(H_1) = \pi(G_1) = \pi(G_2) = \pi(H_2)$, and thus, by Theorem 2.3.3, $H_1 \sim H_2$. It follows that $G_1 \sim G_2$, so we are done. □

Definition 2.3.5. Fix a set B of n boundary vertices on a disk D. For any set of circular minors π, let $\Omega(\pi)$ denote the set of response matrices whose, with the set of positive minors being exactly those corresponding to the elements π. We will refer to the sets $\Omega(\pi)$ as **cells**, in light of the theorem that follows.

Theorem 2.3.6 ([CIM, Theorem 4])**.** *Suppose that G is critical and has N edges. Put $\pi = \pi(G)$. Then, the map $r_G : \mathbb{R}^N_{>0} \to \Omega(\pi)$, taking the conductances on the edges of G to the resulting response matrix, is a diffeomorphism.*

It follows that the space of response matrices for electrical networks of order n is the disjoint union of the cells $\Omega(\pi)$, some of which are empty. The non-empty cells $\Omega(\pi)$ are those which correspond to critical graphs G with $\pi(G) = \pi$. We will describe how these cells are attached to each other in Proposition 3.1.2.

Remark 2.3.7. Later, we will prefer (e.g. in Proposition 3.1.2) to index these cells by their underlying (equivalence classes of) circular planar graphs, referring to them as $\Omega(G)$. Thus, $\Omega(G)$ denotes the set response matrices for conductances on G.

Let us now characterize critical graphs in more conceptual ways. We quote a fourth characterization using medial graphs in Theorem 2.4.2.

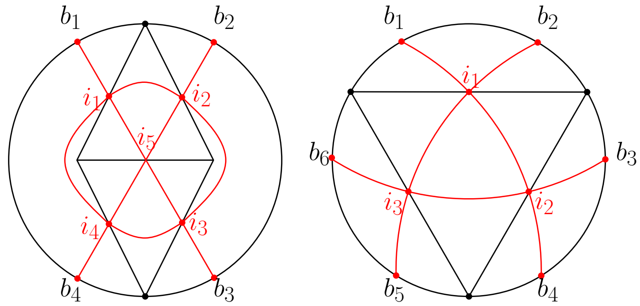

Figure 2: Medial Graphs

Theorem 2.3.8. *Let* (Γ, γ) *be an electrical network. The following are equivalent:*

(1) Γ *is critical.*

(2) Given the response matrix M *of* (Γ, γ), γ *can uniquely be recovered from* M *and* Γ.

(3) Γ *has the minimal number of edges among elements of its equivalence class.*

Proof. By [CIM, Lemma 13.2], (1) and (2) are equivalent. We now prove that (3) implies (1). Suppose for sake of contradiction Γ has the minimal number of edges among elements of its equivalence class, but is not critical. Then, there exists some edge that may be contracted or deleted to give Γ', such that $\pi(\Gamma) = \pi(\Gamma')$. Then, by Proposition 2.3.4, we have $\Gamma' \sim \Gamma$, contradicting the minimality of the number of edges of Γ.

Finally, to see that (1) implies (3), suppose for sake of contradiction that Γ is critical and equivalent to a graph Γ' with a strictly fewer edges. Γ' cannot be critical, or else Γ and Γ' would be Y-Δ equivalent and thus have an equal number of edges. However, if Γ' is not critical, we also have a contradiction by the previous paragraph. The result follows. □

2.4 Medial Graphs

One of our main tools in studying circular planar graphs (and thus, electrical networks) will be their medial graphs. In a sense, medial graphs are the dual object to circular planar graphs. See Figure 2 for examples.

Let G be a circular planar graph with n boundary vertices; color all vertices of G black, for convenience. Then, for each boundary vertex, add two red vertices to the boundary circle, one on either side, as well as a red vertex on each edge of G. We then construct the **medial graph** of G, denoted $\mathcal{M}(G)$, as follows.

Take the set of red vertices to be the vertex set of $\mathcal{M}(G)$. Two red non-boundary vertices in $\mathcal{M}(G)$ are connected by an edge if and only if their associated edges share a vertex and border the same face. Then, the red boundary vertices are each connected to exactly one other red vertex: if the red boundary vertex r lies clockwise from its associated black boundary vertex b, then r is connected to the red vertex associated to the first edge in clockwise order around b after the arc rb. Similarly, if r lies counterclockwise from b, we connect r to the red vertex associated to the first edge in counterclockwise order around b after the arc rb. Note that if no edges of G are incident at b, then the two red vertices associated to b are connected by an edge of $\mathcal{M}(G)$.

We will refer to the red vertices on the boundary circle as **medial boundary vertices**, as to distinguish them from the black **boundary vertices**, a term we will reserve for the boundary

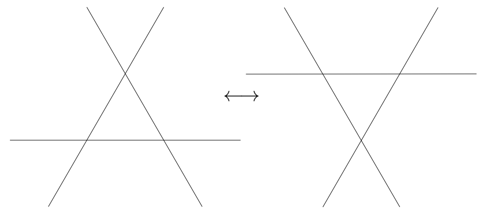

Figure 3: Motions

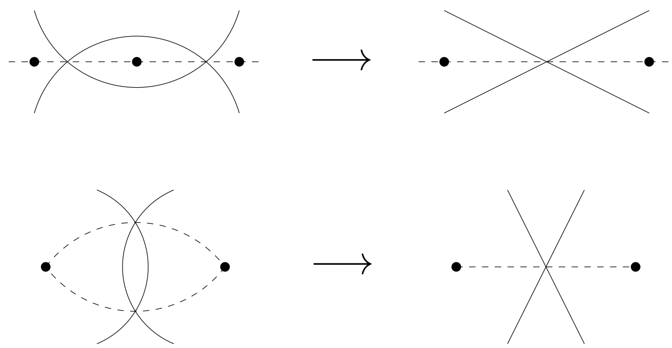

Figure 4: Resolution of Lenses

vertices of the original circular planar graph (electrical network). The **order** of a medial graph is the order of its underlying electrical network.

Note that the medial boundary vertices of $\mathcal{M}(G)$ have degree 1, and all other vertices have degree 4. Thus, we may form **geodesics** in $\mathcal{M}(G)$ in the following way. Starting at each medial boundary vertex, draw a path $e_1 e_2 \cdots e_n$ (labeled by its edges) so that if the edge e_i ends at the non-medial boundary vertex v, the edge e_{i+1} is taken to be the edge with endpoint v such that the edges e_i and e_{i+1} separate the other two edges incident at v. The geodesic ends when it reaches a second boundary vertex. The remaining geodesics are constructed in a similar way, but do not start and end at boundary vertices: instead, they must be finite cycles inside the circle.

For example, in Figure 2, we have three geodesics in the right hand diagram: b_1b_4, b_2b_5, and b_3b_6, where here we label the geodesics by their vertices. In the left hand diagram, we have the geodesics b_1b_3, b_2b_4, and $i_1i_2i_3i_4$.

Definition 2.4.1. Two geodesics are said to form a **lens** if they intersect at distinct p_1 and p_2, in such a way that they do not intersect between p_1 and p_2. A medial graph is said to be **lensless** if all geodesics connect two medial boundary vertices (that is, no geodesics are cycles), and no two geodesics form a lens.

The local equivalences of electrical networks may easily be translated into operations on their medial graphs. Most importantly, Y-Δ transformations become **motions**, as shown in 3, and replacing series or parallel edges with a single edge both correspond to **resolution** of lenses, as shown in Figure 4. Note, however, that a lens may only be resolved if no other geodesics pass through the lens. Defining two medial graphs to be **equivalent** if their underlying circular planar graphs are equivalent, we obtain an analogue of Theorem 2.1.5.

The power of medial graphs lies in the following theorem:

Theorem 2.4.2 ([CIM, Lemma 13.1])**.** *G is critical if and only if $\mathcal{M}(G)$ is lensless.*

In particular, if G is critical, the geodesics of $\mathcal{M}(G)$ consist only n "wires" connecting pairs of the $2n$ boundary medial vertices. Thus, any critical graph G gives a perfect matching of the medial boundary vertices. Furthermore, suppose $H \sim G$ is critical. By Theorem 2.3.3 and Proposition 2.3.4, G and H are related by Y-Δ transformations, so $\mathcal{M}(G)$ and $\mathcal{M}(H)$ are related by motions. In particular, $\mathcal{M}(G)$ and $\mathcal{M}(H)$ match the same pairs of boundary medial vertices, so we have a well-defined map from critical circular planar graph equivalence classes to matchings. In fact, this map is injective:

Proposition 2.4.3. *Suppose that the geodesics of two lensless medial graphs $\mathcal{M}(G), \mathcal{M}(H)$ match the same pairs of medial boundary vertices. Then, the medial $\mathcal{M}(G)$ and $\mathcal{M}(H)$ are related by motions, or equivalently, G and H are Y-Δ equivalent.*

Proof. Implicit in [CIM, Theorem 7.2]. □

Definition 2.4.4. Given the boundary vertices of a circular planar graph embedded in a disk D, take $2n$ medial boundary vertices as before. A **wiring diagram** is collection of n smooth curves (wires) embedded in D, each of which connects a pair of medial boundary vertices in such a way that each medial boundary vertex has exactly one incident wire. We require that wiring diagrams have no triple crossings or self-loops. As with electrical networks and medial graphs, the **order** of the wiring diagram is defined to be equal to n.

It is immediate from Proposition 2.4.3 that, given a set of boundary vertices, perfect matchings on the set of medial boundary vertices are in bijection with motion-equivalence classes of lensless wiring diagrams. Thus, we have an injection $G \mapsto \mathcal{M}(G)$ from critical graph equivalence classes to motion-equivalence classes of lensless wiring diagram, but this map is not surjective. We describe the image of this injection in the next definition:

Definition 2.4.5. Given boundary vertices $V_1, \ldots, V_n$ and a wiring diagram W on the same boundary circle, a **dividing line** for W is a line V_iV_j with $i \neq j$ such that there does not exist a wire connecting two points on opposite sides of V_iV_j. The wiring diagram is called **full** if it has no dividing lines.

It is obvious that fullness is preserved under motions. Now, suppose that we have a lensless full wiring diagram W; we now define a critical graph $\mathcal{E}(W)$. Let D be the disk in which our wiring diagram is embedded. The wires of W divide D in to faces, and it is well-known that these faces can be colored black and white such that neighboring faces have opposite colors.

The condition that W be full means that each face contains at most one boundary vertex. Furthermore, all boundary vertices are contained in faces of the same color; without loss of generality, assume that this color is black. Then place an additional vertex inside each black face which does not contain a boundary vertex. The boundary vertices, in addition to these added interior vertices, form the vertex set for $\mathcal{E}(W)$. Finally, two vertices of $\mathcal{E}(W)$ are connected by an edge if and only if their corresponding faces share a common point on their respective boundaries, which must be an intersection p of two wires of W. This edge is drawn as to pass through p. An example is shown in Figure 5.

It is straightforward to check that $\mathcal{M}$ and $\mathcal{E}$ are inverse maps. We have thus proven the following result:

Theorem 2.4.6. *The associations $G \mapsto \mathcal{M}(G)$ and $W \mapsto \mathcal{E}(W)$ are inverse bijections between equivalence classes of critical graphs and motion-equivalence classes of full lensless wiring diagrams.*

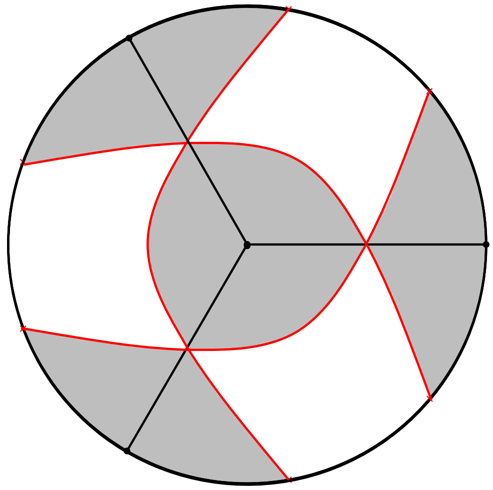

Figure 5: Recovering an electrical network from its (lensless) medial graph.

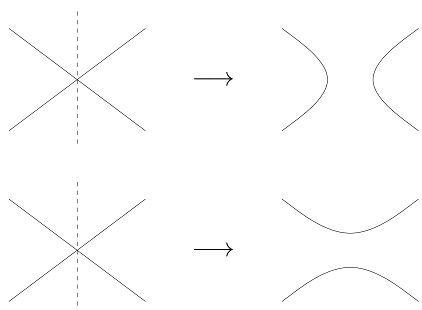

Figure 6: Breaking a crossing, in two ways.

Finally, let us discuss the analogues of contraction and deletion in medial graphs. Each operation corresponds to the **breaking** of a crossing, as shown in Figure 6. A crossing may be broken in two ways: breaking outward from the corresponding edge of the underlying electrical network corresponds to contraction, and breaking along the edge corresponds to deletion. In the same way that contraction or deletion of an edge in a critical graph is not guaranteed to yield a critical graph, breaking a crossing in lensless medial graphs does not necessarily yield a lensless medial graph.

Not all breakings of crossings are valid, as some crossings may be broken in a particular way to create a dividing line. In fact, it is straightforward to check that creating a dividing line by breaking a crossing corresponds to contracting a boundary edge, which we also do not allow. Thus, we allow all breakings of crossings as long as no dividing lines are created: such breakings are called **legal**.

3 The Electrical Poset EP_n

We now consider EP_n, the poset of circular planar graphs under contraction and deletion. We will find that, equivalently, EP_n is the poset of disjoint cells $\Omega(G)$ (see Remark 2.3.7) under containment in closure.

3.1 Construction

Before constructing EP_n, we need a lemma to guarantee that the order relation will be well-defined.

Lemma 3.1.1. *Let G be a circular planar graph, and suppose that H can be obtained from G by a sequence of contractions and deletions. Consider a circular planar graph G' with $G' \sim G$. Then, there exists a sequence of contractions and deletions starting from G' whose result is some $H' \sim H$.*

Proof. By induction, we may assume that H can be obtained from G by one contraction or one deletion. Furthermore, by Theorem 2.1.5, we may assume by induction that G and G' are related by a local equivalence. If this local equivalence is the deletion of a self-loop or boundary spike, the result is trivial. Next, suppose G' is be obtained from G by one Y-Δ transformation. We have several cases: in each, let the vertices of the Y (and Δ) to which the transformation is applied be A, B, C, and let the central vertex of the Y, which may be in G or G' be P. In each case, if the deleted or contracted edge of G is outside the Y or Δ, it is clear that the same edge-removal may be performed in G'.

- Suppose that a Y in G may be transformed to a Δ in G', and that H is obtained from G by contraction, without loss of generality, of AP. Then, deleting the edge BC from G' yields $H' \sim H$.
- Suppose that a Y in G may be transformed to a Δ in G', and that H is obtained from G by deletion, without loss of generality, of AP. Then, deleting AB and AC from G' yields $H' \sim H$.
- Suppose that a Δ in G may be transformed to a Y in G', and that H is obtained from G by deletion, without loss of generality, of AB. Then, contracting CP in G' yields $H' \sim H$.
- Suppose that a Δ in G may be transformed to a Y in G', and that H is obtained from G by deletion, without loss of generality, of AB. Then, contracting AP to A and AB to B in G' yields $H' \sim H$.

Next, consider the case in which we have parallel edges e, f connecting the vertices A, B in G, and that G' is obtained by removing e (analogous to replacing the parallel edges by a single edge). If, in G, we contract or delete an edge not connecting A and B to get H, we can perform the same operation in G' and then delete E to get $H' \sim H$. If, instead, we contract an edge between A and B to get H from G, we perform the same operation in G', and then delete e, which became a self-loop. Finally, if we delete an edge between A and B to get H, then we can delete the same edge in G' to get H, unless e is deleted from G, in which case we can take $H' = H$.

Now, suppose G' can be obtained from G by adding an edge e in parallel to an edge already in G. Then, if we contract or delete an edge f in G to get H, we can perform the same operation in G', then delete e, to get $H' \sim H$.

The case in which G' and G are related by contracting an edge in series with another edge follows from a similar argument. We have exhausted all local equivalences, completing the proof. □

For distinct equivalence classes $[G], [H]$, we may now define $[H] < [G]$ if, given any $G \in [G]$, there exists a sequence of contractions and deletions that may be applied to G to obtain an element of $[H]$. We thus have a (well-defined) **electrical poset of order** n, denoted EP_n, of equivalence classes of circular planar graphs or order n. If $H \in [H]$ and $G \in [G]$ with $[H] < [G]$, we will write $H < G$.

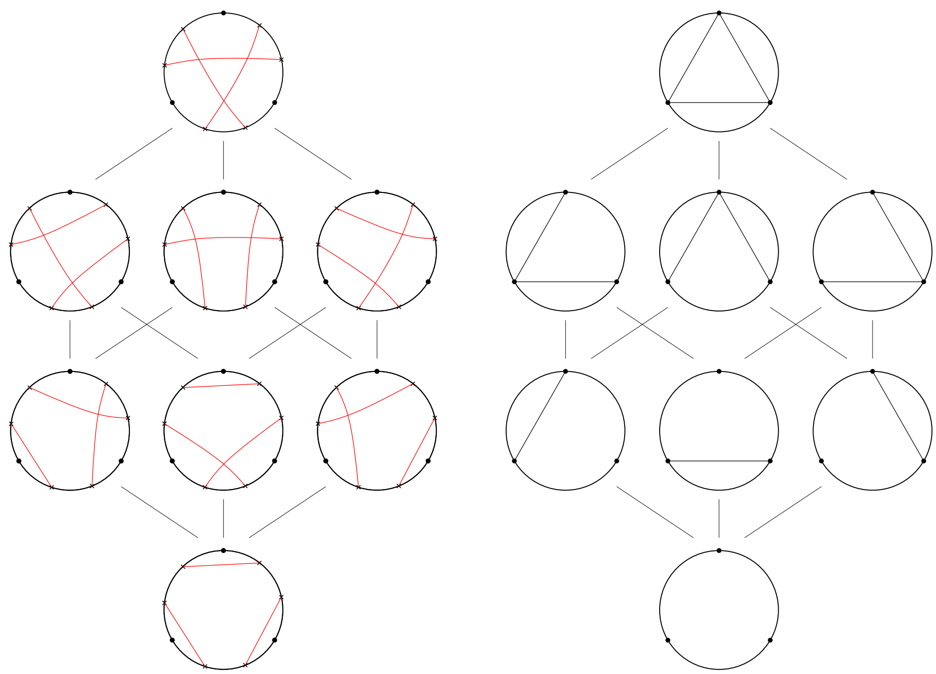

Figure 7: EP_3

Figure 7 shows EP_3, with elements represented as medial graphs (left) and electrical networks (right). Theorem 2.3.2 guarantees that the electrical networks may be taken to be critical. Note that EP_3 is isomorphic to the Boolean Lattice B_3, because all critical graphs of order 3 arise from taking edge-subsets of the top graph.

Let us now give an alternate description of the poset EP_n. Associated to each circular planar graph G, we have an open cell $\Omega(G)$ of response matrices for conductances on G, where $\Omega(G)$ is taken to be a subset of the space Ω_n of symmetric $n \times n$ matrices. It is clear that, if $G \sim G'$, we have, by definition, $\Omega(G) = \Omega(G')$.

Proposition 3.1.2. *Let G be a circular planar graph. Then,*

$$\overline{\Omega(G)} = \bigsqcup_{H \leq G} \Omega(H), \tag{3.1.3}$$

where $\overline{\Omega(G)}$ denotes the closure of $\Omega(G)$ in Ω_n, and the union is taken over equivalence classes of circular planar graphs $H \leq G$ in EP_n.

Because the $\Omega(G)$ are pairwise disjoint when we restrict ourselves to equivalence classes of circular planar graphs (a consequence of Theorems 2.2.6 and 2.3.3), we get:

Corollary 3.1.4. *$[H] \leq [G]$ in EP_n if and only if $\Omega(H) \subset \overline{\Omega(G)}$.*

Proof of Proposition 3.1.2. Without loss of generality, we may take G to be critical. Let N be the number of edges of G. By 2.3.6, the map $r_G : \mathbb{R}^N_{>0} \to \Omega(G) \subset \Omega_n$, sending a collection of conductances of the edges of G the resulting response matrix, is a diffeomorphism. We will describe a procedure for producing a response matrix for any electrical network whose underlying graph H is obtainable from G by a sequence of contractions and deletions (that is, $H \leq G$).

Given $\gamma \in \mathbb{R}^N_{>0}$, write $\gamma = (\gamma_1, \ldots, \gamma_N)$. Note that for each $i \in [1, N]$ and fixed conductances $\gamma_1, \ldots, \widehat{\gamma_i}, \ldots, \gamma_n$, the limit $\lim_{\gamma_i \to 0} r_G(\gamma)$ must exist; indeed, sending the conductance γ_i to zero is equivalent to deleting its associated edge. This fact is most easily seen by physical reasoning: an edge of zero conductance has no current flowing through it, and thus the network may as well not have this edge. Thus, $\lim_{\gamma_i \to 0} r_G(\gamma)$ is just $r_{G'}(\gamma_1, \ldots, \widehat{\gamma_i}, \ldots, \gamma_n)$, where G' is the result of deleting e from G. Similarly, we find that $\lim_{\gamma_i \to \infty} r_G(\gamma)$ is $r_{G''}(\gamma_1, \ldots, \hat{\gamma_i}, \ldots, \gamma_n)$, where G'' is the result of contracting e.

It follows easily, then, that for all H which can be obtained from G by a contraction or deletion, we have $\Omega(H) \subset \overline{\Omega(G)}$, because, by the previous paragraph, $\Omega(H) = \mathrm{Im}(r_H) \subset \overline{\Omega(G)}$. By induction, we have the same for all $H \leq G$.

It is left to check that any $M \in \overline{\Omega(G)}$ is in some cell $\Omega(H)$ with $H \leq G$. We have that M is a limit of response matrices $M_1, M_2, \ldots \in \Omega(G)$. The determinants of the circular minors of M are limits of determinants of the same minors of the M_i, and thus non-negative. It follows that M is the response matrix for some network H, that is, $M \in \Omega(H)$. We claim that $H \leq G$, which will finish the proof.

Consider the sequence $\{C_k\}$ defined by $C_k = r_G^{-1}(M_k)$, which is a sequence of conductances on G. For each edge $e \in G$, we get a sequence $\{C(e)_k\}$ of conductances of e in $\{C_k\}$. It is then a consequence of the continuity of r_G, r_G^{-1}, and the existence of the limits $\lim_{\gamma_i \to 0} r_G(\gamma), \lim_{\gamma_i \to \infty} r_G(\gamma)$, that the sequences $\{C(e)_k\}$ each converge to a finite nonnegative limit or otherwise go to $+\infty$.

Furthermore, we claim that for a boundary edge e (that is, one that connects two boundary vertices), $\{C(e)_k\}$ cannot tend to $+\infty$. Suppose, instead, that such is the case, that for some boundary edge $e = V_iV_j$, we have $C(e)_k \to \infty$. Then, note that imposing a positive voltage at V_i and and zero voltage at all other boundary vertices sends the current measurement at V_i to $-\infty$ as $C(e)_k \to \infty$. In particular, our sequence $M_1, M_2, \ldots$ cannot converge, so we have a contradiction.

To finish, it is clear, for example, using similar ideas to the proof of the first direction, that contracting the edges e for which $C(e)_k \to \infty$ (which can be done because such e cannot be boundary edges) and deleting those for which $C(e)_k \to 0$ yields H. The proof is complete. □

3.2 Gradedness

In this section, we prove our first main theorem, that EP_n is graded.

Proposition 3.2.1. *$[G]$ covers $[H]$ in EP_n if and only if, for a critical representative $G \in [G]$, an edge of G may be contracted or deleted to obtain a critical graph in $[H]$.*

Proof. First, suppose that G and H are critical graphs such that deleting or contracting an edge of G yields H. Then, if $[G] > [X] > [H]$ for some circular planar graph X, some sequence of at least two deletions or contractions of G yields $H' \sim H$. It is clear that H' has fewer edges than H, contradicting Theorem 2.3.8. It follows that $[G]$ covers $[H]$.

We now proceed to prove the opposite direction. Fix a critical graph G, and let e be an edge of G that can be deleted or contracted in such a way that the resulting graph H is not critical. By way of Lemmas 3.2.2 and 3.2.3, we will first construct $T \sim G$ with certain properties, then, from T, construct a graph G' such that $[G] > [G'] > [H]$. The desired result will then follow, because suppose $[G]$ covers $[H]$ and $G \in [G]$ is critical. Then, there exists an edge $e \in G$ which may be contracted or deleted to yield $H \in [H]$, and it will also be true that H is critical.

First, we translate to the language of medial graphs. When we break a crossing in the medial graph $\mathcal{M}(G)$, we may create lenses that must be resolved to produce a lensless medial graph. Suppose that our deletion or contraction of $e \in G$ corresponds to breaking the crossing between the geodesics ab and cd in $\mathcal{M}(G)$, where the points a, c, b, d appear in clockwise order on the boundary circle. Let $ab \cap cd = p$, and suppose that when the crossing at p is broken, the resulting geodesics are ad and cb.

For what follows, let $\mathcal{F} = \{f_1, \ldots, f_k\}$ denote the set of geodesics f_i in $\mathcal{M}(G)$ such that f_i intersects ab between a and p, and also intersects cd between d and p. We now construct T in two steps.

Lemma 3.2.2. *There exists a lensless medial graph K such that:*

- *K is equivalent to $\mathcal{M}(G)$,*
- *geodesics ab and cd still intersect at p, and breaking the crossing at p to give geodesics ad, bc yields a medial graph equivalent to $\mathcal{M}(H)$, and*
- *for $f_i, f_j \in \mathcal{F}$ which cross each other, the crossing $f_i \cap f_j$ lies outside the sector apd.*

Proof. The proof is similar to that of [CIM, Lemma 6.2]. Start with the medial graph $\mathcal{M}(G)$, and for each $f_i \in \mathcal{F}$, let $v_i = f_i \cap ab$. Also, for each $f_i \in \mathcal{F}$ which intersects another $f_j \in \mathcal{F}$ in the sector apd, let D_i be the closest point of intersection of some f_j along f_i to v_i in apd. Let D be the set of D_i.

If D is empty, there is nothing to check, so we assume that D is nonempty. Then, consider the subgraph of $\mathcal{M}(G')$ obtained by restricting to the geodesics in $\mathcal{F}$, along with ab and cd. In this subgraph, choose a point $D_i \in D$ such that the number r of regions within the configuration formed by f_i, f_j, and ap is a minimum, where f_j denotes the other geodesic passing through D_i.

We claim that $r = 1$: assume otherwise. Then, there exists a geodesic f_k intersecting f_j between v_j and D_i and intersecting ap between v_i and v_j, as, by definition, D_i is the first point of intersection on f_i after v_i. However, the area enclosed by f_k, f_j, and ap a number of regions strictly fewer than r. Hence, we could instead have chosen the point $D_j \in D$, with $D_j \neq D_i$, contradicting the minimality.

It follows that ap, f_i, and f_j form a triangle, and thus the crossing at D_i may be moved out of sector apd by a motion. Iterating this process, a finite number of motions may be applied in such so that no $f_i, f_j \in \mathcal{F}$ intersect in the sector apd. After applying these motions, we obtain a medial graph K equivalent to $\mathcal{M}(G')$ satisfying the first and third properties.

It is easy to see that K also satisfies the second property, as none of the motions involved use the crossing at p. Thus, if we translate the sequence of motions into Y-Δ transformations on circular planar graphs, starting with G, no Y-Δ transformation is applied involving the edge e corresponding to p. Thus, deleting or contracting e commutes with the Y-Δ transformations we have performed. □

It now suffices to consider the graph K. Let $f_1 \in \mathcal{F}$ be the geodesic intersecting ab at the point v_1 closest to p, and let $w_1 = f_1 \cap cd$.

Lemma 3.2.3. *There exists a lensless medial graph $K' \sim K$, such that:*

- *geodesics ab and cd intersect at p, as before, and breaking the crossing at p to give geodesics ad, bc yields a medial graph equivalent to $\mathcal{M}(H)$, and*
- *No other geodesic of K' enters the triangle with vertices v_1, p, w_1.*

Proof. We first consider the set $\mathcal{X}$ of geodesics that only intersect cd and f_1. With an argument similar to that of Lemma 3.2.2, we may first apply motions so that any intersection of two elements $\mathcal{X}$ occurs outside the triangle with vertices v_1, p, w_1. Then, we may apply motions at w_1 to move each of the geodesics in $\mathcal{X}$ outside of this triangle, so that they intersect f_1 in the sector bpd. After applying similar motions to the set of geodesics $\mathcal{Y}$ intersecting ab and f_1, we have K'. The fact that K' satisfies the first desired property follows from the same argument as that of Lemma 3.2.3. $\square$

We are now ready to finish the proof of Proposition 3.2.1. Let $T = \mathcal{E}(K')$ (see Theorem 2.4.6). Then, in T, because of the properties of K', contracting e to form the graph $H' \sim H$ forms a pair of parallel edges. Replacing the parallel edges with a single edge gives a circular planar graph H'', which is still equivalent to H. Suppose that e has endpoints B, C and the edges in parallel are formed with A. Then, we have the triangle ABC in T.

Write $S = \pi(T)$ (see Definition 2.2.3) and $S' = \pi(H')$. Because T is critical, $S' \neq S$, so fix $(P; Q) \in S - S'$. Then, it is straightforward to check that any connection $\mathcal{C}$ between P and Q must have used both B and C, but cannot have used the edge BC. Furthermore, $\mathcal{C}$ can use at most one of the edges AB, AC. Indeed, if both AB, AC are used, they appear in the same path γ, but replacing the two edges AB, AC with BC in γ gives a connection between P and Q, but we know that no such connection can use BC, a contradiction. Without loss of generality, suppose that $\mathcal{C}$ does not use AB. Then, deleting AB from T yields a graph G' with $(P; Q) \in G'$, hence G' is not equivalent to H. However, it is clear that deleting BC from G' yields $H'' \sim H$. It follows, then, that in the case in which e is contracted, we have G' such that $[G] > [G'] > [H]$, and hence $[G]$ does not cover $[H]$.

For the case in which we delete $e = ZC$ in T, the argument is similar. Deleting e in T yields a graph $H' \sim H$ with two edges AZ, ZB in series, which implies that T has a Y with vertices A, B, C, Z, where Z is the middle vertex. It is easy to see that Z is not a boundary vertex. Then, replacing AZ, ZB in H' with the edge AB yields a graph $H'' \sim H$. There exists a circular pair $(P; Q) \in \pi(T) - \pi(H)$, so we have a connection $\mathcal{C}$ between P and Q using the edge ZC. Then, $\mathcal{C}$ also must use exactly one of AZ and BZ: wthout loss of generality, assume it is AZ. Contracting BZ in T to yield the graph G' leaves $\mathcal{C}$ intact, and deleting ZC from G' gives $H'' \sim H$. As before, we thus have $[G] > [G'] > [H]$, so we are done. $\square$

Theorem 3.2.4. *EP_n is graded by number of edges of critical representatives.*

Proof. First, by Theorem 2.3.8, note that for any $[G] \in EP_n$, all critical representatives of $[G]$ have the same number of edges. Now, we need to show that if $[G]$ covers $[H]$, the number of edges in a critical representative of $[G]$ is one more than the same number for $[H]$. Let $G \in [G]$ be critical. By Proposition 3.2.1, an edge of G may be contracted or deleted to yield a critical representative $H \in [H]$, and it is clear that H has one fewer edge than G. $\square$

Definition 3.2.5. For all non-negative integers r, denote the set of elements of EP_n of rank r by $EP_{n,r}$.

Let us pause to point out connections between EP_n and two other posets, interpreting EP_n as the graded poset of lensless medial graphs with the covering relation arising from the legal breakings of crossings that preserve lenslessness.

First, EP_n bears a strong resemblance to the symmetric group S_n under the (strong) Bruhat order, as follows. Associated to each permutation $\sigma \in S_n$, there is a lensless wiring diagram, with n wires connecting two parallel lines ℓ_1, ℓ_2, both with marked points $1, 2, \ldots, n$. For each $i \in [n]$, there is a wire joining the point $i \in \ell_1$ to $\sigma(i) \in \ell_2$. Then, the covering relation in S_n is exactly that of EP_n, except for the fact that each crossing can be broken in exactly one legal way.

Also, consider the poset W_n of equivalence classes of lensless wiring diagrams, not necessarily full. Here, the equivalence relation here is generated by motions and resolution of lenses. The order relation arises from breaking of crossings, in a similar way to EP_n, but we are no longer concerned

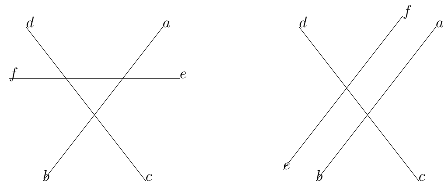

Figure 8: Possible starting configurations for two breakings using six medial boundary vertices.

about the creation of dividing lines. W_n can be proven to be graded in a similar way to the proof of Theorem 3.2.4, and it is furthermore not difficult to check that EP_n is in fact an interval in W_n.

3.3 Toward Eulerianness

In this section, we conjecture several important properties of the poset EP_n. We prove that all closed intervals of length 2 have four elements, which reduces the conjectured properties to the existence of an L-labeling on EP_n.

Lemma 3.3.1. *Suppose $x \in EP_{n,r-1}, z \in EP_{n,r+1}$ with $x < z$. Then, there exist exactly two $y \in EP_{n,r}$ with $x < y < z$.*

Proof. Take x and z to be (equivalence classes of) lensless medial graphs. By Theorem 3.2.4, x may be obtained from z by a sequence of two legal resolutions of crossings. Suppose that x contains the intersecting wires (labeled by their endpoints) ab and cd, whose intersection is broken (legally, that is, without creating dividing lines) by instead taking wires ac, bd. There are two cases for the next covering relation, from which x results: either one of ac, bd is involved, or a crossing between two new wires is broken.

In the first case, suppose that a crossing between bd and ef is broken to give wires bf, de. Up to equivalence under motions, we have one of the two configurations in Figure 3.3.2, constituting subcases A and B. We need to show that, in both cases, there is exactly one other sequence of two legal breakings of crossings, starting from z, that gives x.

In subcase A, there are, at first glance, two possible other ways to get from the set of wires $\{ab, cd, ef\}$ to the set $\{ac, de, bf\}$: the first is through $\{ab, cf, de\}$ and the second is through $\{ae, cd, bf\}$. However, note that the latter case produces a lens, regardless of how the wires are initially positioned to cross each other. Furthermore, assuming the legality of the sequence of breakings $\{ab, cd, ef\} \to \{ac, bd, ef\} \to \{ac, bf, de\}$, it is straightforward to check that $\{ab, cd, ef\} \to \{ab, cf, de\} \to \{ae, cd, bf\}$ is also a legal sequence of breakings. In subcase B, it is clear that the only other way to get from z to x is through $\{ab, cf, de\}$, and indeed, it is again not difficult to check that we get legal resolutions here.

Now, suppose instead that we have the legal sequence of resolutions

$$\{ab, cd, ef, gh\} \to \{ac, bd, ef, gh\} \to \{ac, bd, eg, fh\} \tag{3.3.2}$$

The only other possible way to get from z to x is through $\{ab, cd, eg, fh\}$. Here, there are a number of cases to check in order to verify legality of the sequence of two breakings involved. The essentially different starting configurations are enumerated in Figure 9. In each, one may check that

$$\{ab, cd, ef, gh\} \to \{ab, cd, eg, fh\} \to \{ac, bd, eg, fh\} \tag{3.3.3}$$

is a sequence of legal breakings of which does not create lenses, which will be a consequence of the fact that the same is true of (3.3.2). The details are omitted. □

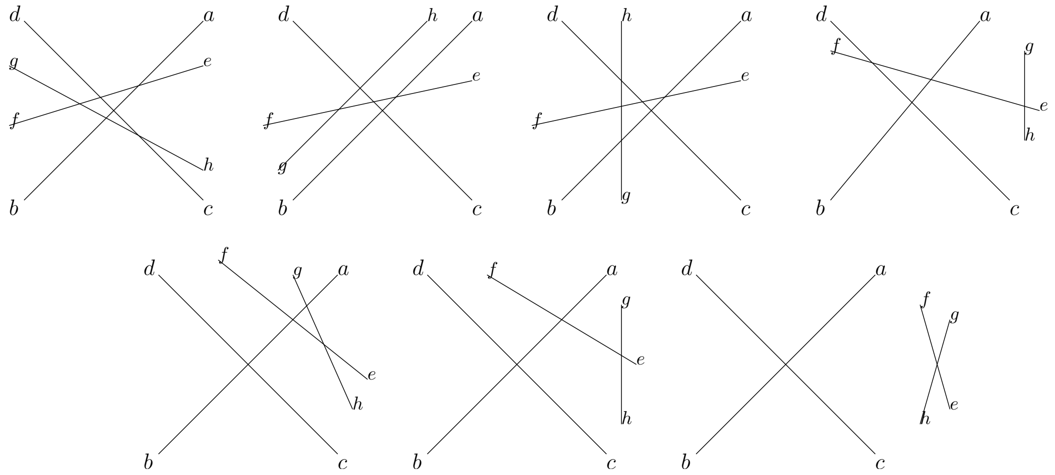

Figure 9: Possible starting configurations for two breakings using eight medial boundary vertices.

Conjecture 3.3.4. *EP_n is lexicographically shellable, and hence Cohen-Macaulay, spherical, and Eulerian.*

We refer the reader to [BW] for definitions. Indeed, if we have an L-labelling for EP_n, it would follow that the order complex $\Delta(EP_n)$ is shellable and thus Cohen-Macaulay (see [BW, Theorem 3.4, Theorem 5.4(C)]). By Lemma 3.3.1, [B, Proposition 4.7.22] would apply, and we would conclude that EP_n is spherical and hence Eulerian.

EP_n has been verified to be Eulerian for $n \leq 7$, and the homology of $EP_n - \{\widehat{0}, \widehat{1}\}$ agrees with that of a sphere of the correct dimension, $\binom{n}{2} - 2$, for $n \leq 4$. On the other hand, no L-labeling of EP_n is known for $n \geq 4$.

4 Enumerative Properties

We now investigate the enumerative properties of EP_n, defined in §3. In the sections that follow, all wiring diagrams are assumed to be lensless, and are considered up to motion-equivalence.

4.1 Total size $X_n = |EP_n|$

In this section, we extend the work of [C] to prove the first two enumerative results concerning $|EP_n|$, the number of equivalence classes of critical graphs (equivalently, full wiring diagrams) of order n. There is a strong analogy between stabilized-interval free (SIF) permuations, as described in [C], and our medial graphs, as follows. A permutation σ may be represented as a 2-regular graph Σ embedded in a disk with n boundary vertices. Then, σ is SIF if and only if there are no dividing lines, where here a dividing line is a line ℓ between two boundary vertices such that no edge of Σ connects vertices on opposite sides of ℓ.

To begin, we define two operations on wiring diagrams in order to build large wiring diagrams out of small, and vice versa. In both definitions, fix a lensless (but not necessarily full) wiring diagram M of order n, with boundary vertices labeled $V_1, V_2, \ldots, V_n$.

Definition 4.1.1. Let $w = XY$ be a wire of M. Construct the **crossed expansion of M at w**, denoted $M^w_{+,c}$ as follows: add a boundary vertex V_{n+1} to M, with associated medial boundary vertices A, B, so that the medial boundary vertices A, B, X, Y appear in order around the circle.

Then, delete w from M and replace it with the crossing wires AX, BY to form $M^w_{+,c}$. Similarly, define the **uncrossed expansion of M at w**, denoted $M^w_{+,u}$, to be the lensless wiring digram obtained by deleting w and replacing it with the non-crossing wires AY, BX.

Definition 4.1.2. Let V_i be a boundary vertex with associated medial boundary vertices A, B, such that we have the wires $AX, BY \in M$, and $X \neq B, Y \neq A$. Define the **refinement of M at V_i**, denoted M^i_- to be the lensless wiring diagram of order $n-1$ obtained by deleting the wires AX, BY as well as the vertices A, B, V_i, and adding the wire XY.

Each construction is well-defined up to equivalence under motions by Theorem 2.4.3. It is clear that expanding M, then refining the result at the appropriate vertex, recovers M. Similarly, refining M, then expanding the result after appropriately relabeling the vertices, recovers M if the correct choice of crossed or uncrossed is made.

Lemma 4.1.3. *Let M be a full wiring diagram, with boundary vertices $V_1, V_2, \ldots, V_n$. Then:*

(a) $M^w_{+,c}$ is full for all wires $w \in M$.

(b) Either $M^w_{+,u}$ is full, or otherwise $M^w_{+,u}$ has exactly one dividing line, which must have V_{n+1} as one of its endpoints.

Proof. First, suppose for sake of contradiction that $M^w_{+,c}$ has a dividing line ℓ. If ℓ is of the form V_iV_{n+1}, then ℓ must exit the sector formed by the two crossed wires coming out of the medial boundary vertices associated to V_{n+1}. If this is the case, we get an intersection between $M_{+,c}$ and a wire, a contradiction. If instead, $\ell = V_iV_j$ with $i, j \neq n+1$, then ℓ is a dividing line in M, also a contradiction. We thus have (a). Similarly, we find that any dividing line of $M^w_{+,u}$ must have V_{n+1} as an endpoint. However, if $V_iV_{n+1}, V_{i'}V_{n+1}$ are dividing lines, then $V_iV_{i'}$ is as well, a contradiction, so we have (b). ☐

Lemma 4.1.4. *Let M be a full wiring diagram, with boundary vertices $V_1, V_2, \ldots, V_n$. Furthermore, suppose M^n_- exists and is not full. Then, M^n_- has a unique dividing line V_iV_j with $1 \le i < j \le n-1$ and $j - i$ maximal.*

Proof. By assumption, M^n_- has a dividing line, so suppose for sake of contradiction that $\ell_1 = V_{i_1}V_{j_1}, \ell_2 = V_{i_2}V_{j_2}$ are both dividing lines of M' with $d = j_1 - i_1 = j_2 - i_2$ maximal. Without loss of generality, assume $i_1 < i_2$ (and $i_1 < j_1, i_2 < j_2$). If $j_1 \ge i_2$, then $V_{i_1}V_{j_2}$ is also a dividing line with $j_2 - i_1 > d$, a contradiction. On the other hand, if $j_1 < i_2$, at least one of ℓ_1, ℓ_2 is a dividing line for M, again a contradiction. ☐

If M, i, j are as above, we now define two wiring diagrams M_1 and M_2; see Figure 10 for an example. First, let M_1 be the result of restricting M to the wires associated to the vertices V_k, for $k \in [i, j] \cup \{n\}$. Note that M_1 is a wiring diagram of order $j - i + 1$ with boundary vertices $V_i, V_{i+1}, \ldots, V_j$ (and not V_n). Then, let M_2 be the wiring diagram of order $n - (j - i + 1)$ obtained by restricting M to the wires associated to the vertices V_k, for $k \notin [i, j] \cup \{n\}$.

Lemma 4.1.5. *M_1 and M_2, as above, are full.*

Proof. It is not difficult to check that any dividing line of M_1 must also be a dividing line of M, a contradiction. A dividing line $V_{i'}V_{j'}$ of M_2 must also be a dividing line of M^n_-, but then $j' - i' > j - i$, contradicting the maximality from Lemma 4.1.4. ☐

We are now ready to prove the main theorem of this section.

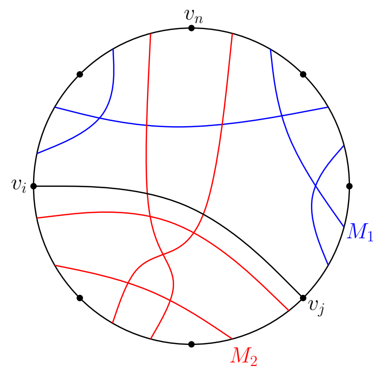

Figure 10: M_1 and M_2, from M.

Theorem 4.1.6. *Put $X_n = |EP_n|$, which here we take to be the number of full wiring diagrams of order n. Then, $X_1 = 1$, and for $n \geq 2$,*

$$X_n = 2(n-1)X_{n-1} + \sum_{k=2}^{n-2}(k-1)X_kX_{n-k}.$$

Proof. $X_1 = 1$ is obvious. For $n > 1$, we would like to count the number of full wiring diagrams M of order n, whose boundary vertices are labeled $V_1, V_2, \ldots, V_n$, in clockwise order, with medial boundary vertices A_i and B_i at each vertex, so that the order of points on the circle is A_i, V_i, B_i in clockwise order. If A_nB_n is a wire, constructing the rest of M amounts to constructing a full wiring diagram of order $n-1$, so there are X_{n-1} such full wiring diagrams in this case.

Otherwise, consider the refinement M^n_-. All M for which M^n_- is full can be obtained by expanding at one of the $n-1$ wires of a full wiring diagram M' of order $n-1$. By Lemma 4.1.3, the expanded wiring diagram is full unless it has exactly one dividing line V_kV_n, and furthermore it is easy to see that any such graphs is an expansion of a full wiring diagram of order $n-1$.

There are $2(n-1)$ ways to expand M', and each expansion gives a different wiring diagram of order n, for $2(n-1)X_{n-1}$ total expanded wiring diagrams. However, by the previous paragraph, the number of these which are not full is $\sum_{k=1}^{n-1} X_kX_{n-k}$, as imposing a unique dividing line V_kV_n forces us to construct two full wiring diagrams on either side, of orders $k, n-k$ respectively. Thus, we have $2(n-1)X_{n-1} - \sum_{k=1}^{n-1} X_kX_{n-k}$ full wiring diagrams of order n such that refining at V_n gives another full wiring diagram.

It is left to count those M such that contracting at V_n leaves a non-full wiring diagram M'. By Lemma 4.1.5, such an M gies us a pair of full wiring diagrams of orders $i+j+1, n-(i+j+1)$, where V_iV_j is as in Lemma 4.1.4. Conversely, given a pair of boundary vertices $V_i, V_j \neq V_n$ of M and full wiring diagrams of orders $j-i+1, n-(j-i+1)$, we may reverse the construction $M \mapsto (M_1, M_2)$ to get a wiring diagram of order n: furthermore, it is not difficult to check that this wiring diagram is full.

It follows that the number of such M is

$$\sum_{1 \leq i < j \leq n-1} X_{j-i+1}X_{n-(j-i+1)} = \sum_{k=1}^{n-2} kX_kX_{n-k}.$$

Summing our three cases together, we find

$$X_n = X_{n-1} + 2(n-1)X_{n-1} - \sum_{k=1}^{n-1} X_kX_{n-k} + \sum_{k=1}^{n-2} kX_kX_{n-k}$$

$$= 2(n-1)X_{n-1} + \sum_{k=2}^{n-2}(k-1)X_kX_{n-k},$$

using the fact that $X_1 = 1$. The theorem is proven. □

Remark 4.1.7. The sequence $\{X_n\}$ is found in the Online Encyclopedia of Integer Sequences, see [OEIS]. The original motivation for the sequence is unknown.

We also have an analogue of the other main result of [C].

Theorem 4.1.8. *Let $X(t) = \sum_{n=0}^{\infty} X_n t^n$ be the generating function for the sequence $\{X_n\}$, where we take $X(0) = 0$. Then, we have $[t^{n-1}]X(t)^n = n \cdot (2n-3)!!$.*

Proof. Consider n boundary vertices on a circle, labeled $V_1, V_2, \ldots, V_n$ in clockwise order. Then, label $2n$ medial boundary vertices $W_1, W_2, \ldots, W_{2n}$ in clockwise order so that W_{2n-1} and W_{2n} lie between V_n and V_1 on the circle. Note that $n \cdot (2n-3)!!$ counts the number of wiring diagrams so that the wire with endpoint W_{2n} has second endpoint W_z, for some z odd. Call such wiring diagrams $2n$**-odd**. We need a bijection between $2n$-odd wiring diagrams and lists of n full wiring diagrams with sum of orders equal to $n-1$.

From here, the rest of the proof is nearly identical to the analogous result given on [C, p. 3], so we give only a sketch. We will refer the reader often to [C] for more details.

Let W be a $2n$-odd wiring diagram, with boundary vertices and medial boundary vertices labeled as above. For $i = 1, 2, \ldots, n$, let p_i denote the pair of medial boundary vertices $\{W_{2i-1}, W_{2i}\}$. Consider the set of dividing lines of W. We first partition the p_i in to minimal consecutive blocks $I = \{p_k, p_{k+1}, \ldots, p_\ell\}$, where indices are not taken modulo n, such that no wire has one endpoint in some $p_i \in I$ and the other in some $p_j \notin I$. Let π denote this partition, with blocks $\pi_1, \pi_2, \ldots, \pi_d$. We order the blocks in such a way that if $p_i \in \pi_a$, and $p_j \in \pi_b$, then, if $i < j$, we have $a < b$. Note that, in particular, $p_n \in \pi_d$.

Now, for each block π_a, write $|\pi_a| = x_a$. For $a < d$, π_a may be further partitioned in to a non-crossing partition of total size s, according to the dividing lines in the corresponding subgraph of W. Each such partition corresponds to a Dyck path $\mathcal{P}_a$ of length $2x_a$, by a bijection described in [C], and it is not difficult to check that, because π_a was constructed to be a minimal connected component, $\mathcal{P}_a$ only touches the x-axis at its endpoints.

On π_d, we first perform the following operation similar to refinement, as in Definition 4.1.2. Let the second endpoints of the wires w_α, w_β coming from W_{2n-1}, W_{2n}, respectively, be W_α, W_β, respectively. Then, delete the wires w_α, w_β, and replace them with a single wire between W_α, W_β. If, however, W_{2n-1}, W_{2n} are connected by a single wire, simply delete this wire. In either case, the resulting block π'_d now has order $x_d - 1$, and it, too, may be further partitioned in to a non-crossing partition, corresponding to a Dyck path $\mathcal{P}_d$ of length $2x_d - 2$. Unlike $\mathcal{P}_a$, with $a < d$, $\mathcal{P}_d$ may touch the x-axis more than twice.

We now cut the Dyck paths $\mathcal{P}_a$ in a similar way to that of [C]. For $a < d$, we cut $\mathcal{P}_a$ in the following way: remove the last upstep u, thus breaking $\mathcal{P}_a$ in to a path P_a, followed by an upstep u, and then followed by a descent D_a. As for $\mathcal{P}_d$, recall that, due to the bijection between non-crossing partitions and Dyck paths, the upsteps in $\mathcal{P}_d$ correspond to to the elements $p_i \in \pi_d$. Let u_0 denote the upstep corresponding to the p_i containing W_z, the second endpoint of the wire with endpoint W_{2n}. Then, break $\mathcal{P}_d$ in to the paths R, S, where R is the part of $\mathcal{P}_d$ appearing before u_0, and S consists of alls steps after those of R. In the case that $z = 2k-1$, note that $x_d = 1$ and thus $\mathcal{P}_d$ is empty. In this case, R, S are also taken to be empty.

Finally form the concatenated path $D_1uD_2u\cdots D_{k-1}uSRP_1P_2\cdots P_{k-1}$, as in [C]. There are $n-1$ upsteps in this path, which begins and ends on the x-axis. In between these $n-1$ upsteps are

n (possibly empty) descents, which, using the bijection between non-crossing partitions and Dyck paths, correspond to full sub-wiring diagrams of the p_a. Therefore, we get the desired list of n full wiring diagrams of total order $n-1$, and the process is reversible by an argument similar to that of [C]. The details are left to the reader.

□

4.2 Asymptotic Behavior of $X_n = |EP_n|$

In this section, we adapt methods from [SE] to prove:

Theorem 4.2.1. *We have*

$$\lim_{n\to\infty} \frac{X_n}{(2n-1)!!} = \frac{1}{\sqrt{e}}.$$

In other words, the density of full wiring diagrams in the set of all wiring diagrams is $e^{-1/2}$.

Lemma 4.2.2. *For $n \geq 6$, $(2n-1)X_{n-1} < X_n < 2nX_{n-1}$.*

Proof. We proceed by strong induction on n: the inequality is easily verified for $n = 6, 7, 8$ using Theorem 4.1.6. Furthermore, note that $X_n < 2nX_{n-1}$ for $n = 2, 3, 4, 5$ as well. Now, assume $n \geq 9$.

By Lemma 4.1.6, it is enough to show

$$X_{n-1} < \sum_{j=2}^{n-2} (j-1)X_j X_{n-j} < 2X_{n-1}. \tag{4.2.3}$$

We first show the left hand side of (4.2.3). Now, we have

$$\begin{aligned}\sum_{j=2}^{n-2} (j-1)X_j X_{n-j} &> X_2 X_{n-2} + (n-4)X_{n-3}X_3 + (n-3)X_{n-2}X_2 \\ &= 2(n-2)X_{n-2} + 8(n-4)X_{n-3} \\ &> \left(\frac{n-2}{n-1} + \frac{2(n-4)}{(n-2)(n-1)}\right) X_{n-1} \\ &> X_{n-1},\end{aligned}$$

where we have applied the inductive hypothesis.

It remains to prove the right hand side of (4.2.3). First, suppose that n is odd, with $n = 2k-1, k \geq 4$. Let $Q_i = X_i/X_{i-1}$ for each i; we know that $Q_i > 2i-1$ for all $i \geq 5$. Then, we have

$$\begin{aligned}\sum_{j=2}^{n-2} (j-1)X_j X_{n-j} &= (2k-3) \sum_{j=2}^{k-1} X_j X_{2k-1-j} \\ &= (2k-3)X_{n-1} \sum_{j=2}^{k-1} \frac{X_j}{Q_{2k-2}Q_{2k-3}\cdots Q_{2k-j}} \\ &< (2k-3)X_{n-1} \sum_{j=2}^{k-1} \frac{X_j}{(4k-5)(4k-7)\cdots(4k-2j-1)}\end{aligned}$$

However, we claim that the terms in the sum are strictly decreasing. This amounts to the inequality $(4k-2j-1)X_{j-1} > X_j$ for $3 \leq j \leq k-1$, which follows by the inductive hypothesis as $4k-2j-1 > 2j$. Thus,

$$\begin{aligned}
&(2k-3)X_{n-1} \sum_{j=2}^{k-1} \frac{X_j}{(4k-5)(4k-7)\cdots(4k-2j-1)} \\
&< (2k-3)X_{n-1} \left(\frac{X_2}{4k-5} + \frac{(k-3)X_3}{(4k-5)(4k-7)} \right) \\
&= X_{n-1} \left(\frac{4k-6}{4k-5} + \frac{(4k-12)(4k-6)}{(4k-5)(4k-7)} \right) \\
&< 2X_{n-1},
\end{aligned}$$

where we substitute $X_2 = 2, X_3 = 8$. The case in which n is even may be handled similarly, and the induction is complete. □

Corollary 4.2.4. *There exists a limit*

$$C = \lim_{n\to\infty} \frac{X_n}{(2n-1)!!},$$

and furthermore, $C > 0$.

Proof. The sequence $X_n/(2n-1)!!$ is bounded above by 1 and is eventually strictly increasing by Lemma 4.2.2, so the limit C exists. Furthermore, $C > 0$ because $X_n/(2n-1)!!$ is eventually increasing. □

To prove Theorem 4.2.1, we will estimate the number of non-full wiring graphs of order n. Let D_n denote the number of wiring diagrams formed in the following way: for $1 \leq j \leq n-2$, choose j pairs of adjacent boundary vertices, and for each pair, connect the two medial boundary vertices between them. Then, with the remaining $2n-2j$ vertices, form a full wiring diagram of order $n-j$, which in particular has no dividing lines whose endpoints are adjacent boundary vertices. It is clear that all such diagrams are non-full.

For completeness, we will also include in our count the wiring diagram where all pairs of adjacent boundary vertices give dividing lines, but because we are interested in the asymptotic behavior of D_n, this addition will be of no consequence. It is easily seen that

$$D_n = 1 + \sum_{j=1}^{n-2} \binom{n}{j} X_{n-j}.$$

Now, let E_n be the number of non-full wiring diagrams not constructed above. Consider the following construction: choose an ordered pair of distinct, non-adjacent boundary vertices on our boundary circle. Then, on each side of the directed segment, construct any wiring diagram. This construction yields

$$Y_n = n \sum_{j=2}^{n-2} (2n-2j-1)!!(2j-1)!!$$

total (not necessarily distinct) wiring diagrams, which clearly overcounts E_n.

We now state two lemmas:

Lemma 4.2.5. $D_n/X_n \sim \sqrt{e}-1$.

Lemma 4.2.6. $Y_n/X_n \sim 0$.

From here, we will be able to establish the desired asymptotic.

Proof of Theorem 4.2.1. X_n, D_n, and E_n together count the total number of wiring diagrams, which is equal to $(2n-1)!!$. Thus,

$$\frac{(2n-1)!!}{X_n} = \frac{X_n + D_n + E_n}{X_n} \sim 1 + (\sqrt{e}-1) + 0 = e^{1/2},$$

assuming Lemmas 4.2.5 and 4.2.6 (we have $Y_n/X_n \sim 0$, so $E_n/X_n \sim 0$ as well), so the desired conclusion is immediate from taking the reciprocal. □

Thus, it remains to prove Lemmas 4.2.5 and 4.2.6, which we defer to Appendix A.

To conclude this section, we propose the following generalization of Theorems 4.1.8 and 4.2.1:

Conjecture 4.2.7. *Let λ be a positive integer. Consider the sequence $\{X_{n,\lambda}\}$ defined by $X_{1,\lambda} = 1$, and*

$$X_n = \lambda(n-1)X_{n-1,\lambda} + \sum_{k=2}^{n-2}(j-1)X_{j,\lambda}X_{n-k,\lambda}.$$

Then, let $X_\lambda(t)$ be the generating function for the sequence $\{X_{n,\lambda}\}$. Then,

$$[t^{n-1}]X_\lambda(t)^n = n \cdot (\lambda n - (\lambda - 1))\underbrace{!!\cdots!}_{\lambda}$$

and

$$\lim_{n\to\infty} \frac{X_{\lambda,n}}{(\lambda n - (\lambda-1))\underbrace{!!\cdots!}_{\lambda}} = \frac{1}{\sqrt[n]{e}},$$

where $(\lambda n - (\lambda-1))\underbrace{!!\cdots!}_{\lambda} = (\lambda n - (\lambda - 1))(\lambda(n-1) - (\lambda-1)) \cdots (\lambda+1) \cdot 1.$

A proof exhibiting and exploiting a combinatorial interpretation for the sequence $\{X_{n,\lambda}\}$ would be most desirable, as we have done for $\lambda = 2$. However, no such interpretation is known for $\lambda > 2$. The case $\lambda = 1$ is handled in [C] and [ST, §3], though the latter does not use the interpretation of $X_{n,1}$ as SIF permutations of $[n]$ to obtain the asymptotic.

Interestingly, if we define $X_{n,-1}$ analogously, we get $X_{n,-1} = (-1)^{n+1}C_n$, where C_n denotes the n-th Catalan number, see [OEIS].

4.3 Rank sizes $|EP_{n,r}|$

Proposition 4.3.1. *For non-negative $c \le n-2$, we have $|EP_{n,\binom{n}{2}-c}| = \binom{n-1+c}{c}$. Furthermore, $|EP_{n,\binom{n}{2}-(n-1)}| = \binom{2n-2}{n-1} - n$.*

Proof. For convenience, put $N = \binom{n}{2}$. We claim that for $c \le n-2$, any wiring diagram of order n with $N-c$ crossings is necessarily full. Suppose instead that we have a dividing line, dividing our circle in to two wiring diagrams of orders with $j, n-j$. Then, there are at most

$$\binom{j}{2} + \binom{n-j}{2} \le \binom{n-1}{2} = N - (n-1)$$

crossings, so if $c \leq n-2$ we cannot have a dividing line.

Thus, for $c \leq n-2$, it suffices to compute the number of circular wiring diagrams with $N-c$ crossings. By [R, (1)], this number is the coefficient of the q^{N-c} term of the polynomial

$$T_n(q) = (1-q)^{-n} \sum_{j=0}^{n} (-1)^j \left[\binom{2n}{n-j} - \binom{2n}{n-j-1} \right] q^{\binom{j+1}{2}}, \tag{4.3.2}$$

which, as noted in [R, p. 218], is $\binom{n+c-1}{n-1}$ for $c \leq n-1$. This immediately gives the desired result for $c \leq n-2$.

For $c = n-1$, we have, by the above, $\binom{2n-2}{n-1}$ wiring diagrams with $N-c$ crossings; we need to count the number of such wiring diagrams that contain a dividing line. However, note that if our dividing line separates the circle in to wiring diagrams of orders $j, n-j$ for $1 < j \leq n/2$, there are at most $\binom{n-2}{2}+1$ crossings (using a similar argument to that in the first paragraph), which is strictly less than $N-(n-1)$, so we must have $j=1$.

Furthermore, by the first paragraph, if $j=1$, we need exactly $\binom{n-1}{2}$ crossings. Thus, a non-full wiring diagram with $N-(n-1)$ crossings must connect two adjacent medial boundary vertices between two boundary vertices, and connect all of the other medial boundary vertices in such the unique way such that we have the maximal possible number of crossings between the $n-1$ wires. There are clearly n such non-full wiring diagrams, giving $|EP_{n,N-(n-1)}| = \binom{2n-2}{n-1} - n$, as desired. □

Proposition 4.3.1 gives an exact formula for $|EP_{n,r}|$ for r large, but no general formula is known for general r. For fixed r and n sufficiently large, one will only have finitely many cases to enumerate for possible configurations of an electrical network, but the casework becomes cumbersome quickly. However, the Möbius Inversion Formula gives us an expression for the generating function for the number of full wiring diagrams of order n, counted by number of crossings.

Let NC_n be the (graded) poset of non-crossing partitions on n, ordered by refinement. By [BS, Proposition 2.3], we have $\mu(\widehat{0}, \widehat{1}) = (-1)^{n-1} C_{n-1}$ in NC_n, and furthermore, for any $\pi \in NC_n$, the interval $(\widehat{0}, \pi)$ is isomorphic to a product of the partition lattices NC_k, where k ranges over the block sizes of π. Given $\pi \in NC_n$, π may be represented as a set of dividing lines in a disk D with boundary vertices $V_1, V_2, \ldots, V_n$ as follows: draw the dividing line V_iV_j if i, j are in the same block of π. Furthermore, the set of dividing lines for a wiring diagram yields a non-crossing partition $[n]$ in the same way.

Let k_π denote the number of blocks in π. It is clear that drawing these dividing lines of π breaks D in to $n+1-k_\pi$ regions in which wires can be drawn (see Figure 11 for an example). Let $a_{\pi,1}, \ldots, a_{\pi,n+1-k\pi}$ denote the numbers of boundary vertices drawn in these regions. Finally, let $X_n(q)$ be the rank-generating function for EP_n, that is, the polynomial in q such that the coefficient of q^r is $|EP_{n,r}|$. Then, by Möbius Inversion, we get:

Proposition 4.3.3.

$$X_n(q) = \sum_{\pi \in NC_n} \left((-1)^{n-k_\pi} \prod_{i=1}^{k_\pi} C_{\pi_i - 1} \prod_{j=1}^{n+1-k_\pi} T_{a_j}(q) \right), \tag{4.3.4}$$

where $k_\pi, a_{\pi,j}$ are as before, π_i denotes the number of elements in the i-th block of π, and the polynomial $T_m(q)$ is as in (4.3.2).

To conclude this section, we cannot resist making the following conjecture:

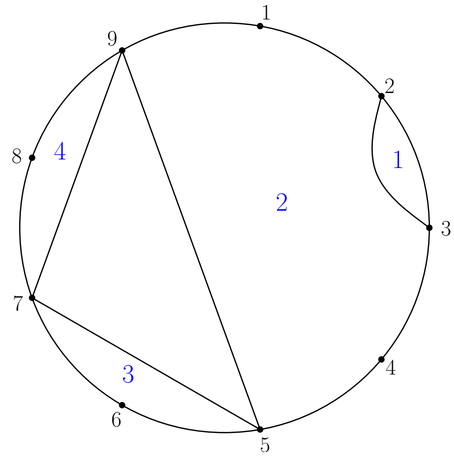

Figure 11: [1][23][4][579][6][8] breaks the disk in to four wiring regions.

Conjecture 4.3.5. *EP_n is rank-unimodal, for $n \geq 4$.*

In support of Conjecture 4.3.5, let us list the rank sizes of EP_n below, for small values of n.

Rank Sizes	
n	Rank Size
1	1
2	1, 1
3	1, 3, 3, 1
4	1, 6, 14, 16, 10, 4, 1
5	1, 10, 40, 85, 110, 97, 65, 35, 15, 5, 1,
6	1, 15, 90, 295, 609, 873, 948, 840, 636, 421, 246, 126, 56, 21, 6, 1
7	1, 21, 175, 805, 2366, 4872, 7567, 9459, 10031, 9359, 7861, 6027, 4249, 2765, 1661, 917, 462, 210, 84, 28, 7, 1
8	1, 28, 308, 1876, 7350, 20272, 42090, 69620, 96334, 115980, 125044, 123176, 112380, 95836, 76868, 58220, 41734, 28344, 18236, 11096, 6364, 3424, 1716, 792, 330, 120, 36, 8, 1

5 Electrical Positroids

For the rest of the paper, we shift our focus to the combinatorial properties of response matrices. By Theorem 2.2.6, $n \times n$ response matrices are characterized in the following way: a square matrix M is the response matrix for an electrical network (Γ, γ) if and only if M is symmetric, its row and column sums are zero, and its circular minors $M(P;Q)$ are non-negative. Furthermore, $M(P;Q)$ is positive if and only if there is a connection from P to Q in Γ. The sets S of circular pairs for which there exists a response matrix M with $M(P;Q)$ is positive if and only if $(P;Q) \in S$, then, are thus our next objects of study.

The case of the totally nonnegative Grassmannian was studied in [P]: for $k \times n$ (with $k < n$) matrices with non-negative maximal minors, the possible sets of positive maximal minors are called *positroids*, and are a special class of matroids. Our objects will be called *electrical positroids*, which we first construct axiomatically, then prove are exactly those sets S of positive circular minors in response matrices.

5.1 Grassmann-Plücker Relations and Electrical Positroid Axioms

Here, we present the axioms for electrical positroids, which arise naturally from the Grassmann-Plücker Relations.

Definition 5.1.1. Let M be a fixed matrix, whose rows and columns are indexed by some sets I, J. We write $\Delta^{i_1 i_2 \cdots i_m, j_1 j_2 \cdots j_n}$ for the determinant of the matrix M' formed by deleting the rows corresponding to $i_1, i_2, \ldots, i_m \in I$ and $j_1, j_2, \ldots, j_n \in J$, provided M' is square.

While the meaning $\Delta^{i_1 i_2 \cdots i_m, j_1 j_2 \cdots j_n}$ depends on the underlying sets I, J, these sets will always be implicit.

Proposition 5.1.2. *We have the following two* ***Grassmann-Plücker*** *relations.*

(a) Let M be an $n \times n$ matrix, with a, b elements of its row set and c, d elements of its column set. Furthermore, suppose that the row a appears above row b and column c appears to the left of column d. Then,

$$\Delta^{a,c}\Delta^{b,d} = \Delta^{a,d}\Delta^{b,c} + \Delta^{ab,cd}\Delta^{\emptyset,\emptyset}. \tag{5.1.3}$$

(b) Let M be an $(n+1) \times n$ matrix, with a, b, c elements of its row set (appearing in this order, from top to bottom), and let d an element of its column set. Then,

$$\Delta^{b,\emptyset}\Delta^{ac,d} = \Delta^{a,\emptyset}\Delta^{bc,d} + \Delta^{c,\emptyset}\Delta^{ab,d}. \tag{5.1.4}$$

While the Grasmann-Plücker relations are purely algebraic in formulation, they encode combinatorial information concerning the connections of circular pairs in a circular planar graph Γ. As a simple example, consider four boundary vertices a, b, d, c in clockwise order of an electrical network (Γ, γ), and let $\pi = \pi(\Gamma)$. If M is the response matrix of (Γ, γ), then $M' = M(\{a, b\}, \{c, d\})$ is the circular minor associated to the circular pair $(a, b; c, d)$; thus, M' has non-negative determinant. Furthermore, the entries of M' are 1×1 circular minors of M, so they, too, must be non-negative.

Now, suppose that the left hand side of (5.1.3) is positive, that is, $\Delta^{a,c}\Delta^{b,d} > 0$. Equivalently, there are connections between b and d and between a and c in Γ. Then, at least one of the two terms on the right hand side must be strictly positive; combinatorially, this means that either there are connections between b and c and between a and d, or there is a connection between $\{a, b\}$ and $\{c, d\}$. One can derive similar combinatorial rules by assuming one of the terms on the right hand side is positive, and deducing that the left hand side must be positive as well.

The first six of the electrical positroid axioms given in Definition 5.1.6 summarize all of the information that can be extracted in this way from the Grassmann-Plücker relations.

Definition 5.1.5. If $a \in P$, write $P - a$ for the ordered set formed by removing a from P.

Definition 5.1.6. A set S of circular pairs a **electrical positroid** if it satisfies the following eight axioms:

1. For ordered sets $P = \{a_1, a_2, \ldots, a_N\}$ and $Q = \{b_1, b_2, \ldots, b_N\}$, with $a_1, \ldots, a_N, b_N, \ldots, b_1$ in clockwise order (that is, $(P; Q)$ is a circular pair), consider any $a = a_i$, $b = a_j$, $c = b_k$, $d = b_\ell$ with $i < j$ and $k < \ell$. Then:

   (a) If $(P - a; Q - c), (P - b; Q - d) \in S$, then either $(P - a; Q - d), (P - b; Q - c) \in S$ or $(P - a - b; Q - c - d), (P; Q) \in S$.

(b) If $(P-a;Q-d),(P-b;Q-c) \in S$, then $(P-a;Q-c),(P-b;Q-d) \in S$.

(c) If $(P-a-b;Q-c-d),(P;Q) \in S$, then $(P-a;Q-c),(P-b;Q-d) \in S$.

2. For $P = \{a_1, a_2, \ldots, a_{N+1}\}$ and $Q = \{b_1, b_2, \ldots, b_N\}$, with $a_1, a_2, \ldots, a_{N+1}, b_N, \ldots, b_1$ in clockwise order, consider any $a = a_i$, $b = a_j$, $c = a_k$, $d = b_\ell$ with $i < j < k$. Then:

   (a) If $(P-b;Q),(P-a-c;Q-d) \in S$, then either $(P-a;Q),(P-b-c;Q-d) \in S$ or $(P-c;Q),(P-a-b;Q-d) \in S$.

   (b) If $(P-a;Q),(P-b-c;Q-d) \in S$, then $(P-b;Q),(P-a-c;Q-d) \in S$.

   (c) If $(P-c;Q),(P-a-c;Q-d) \in S$, then $(P-b;Q),(P-a-c;Q-d) \in S$.

Finally:

3. (**Subset axiom**) For $P = \{a_1, a_2, \ldots, a_n\}$ and $Q = \{b_1, b_2, \ldots, b_n\}$ with $(P;Q)$ a circular pair, if $(P;Q) \in S$, then $(P-a_i;Q-b_i) \in S$.

4. $(\emptyset;\emptyset) \in S$.

Theorem 5.1.7. *A set S of circular pairs is an electrical positroid if and only if there exists a response matrix whose positive circular minors are exactly those corresponding to S.*

Given a response matrix M, it is straightforward to check that the set S of circular pairs corresponding to the positive circular minors of M satisfies the first six axioms, by Proposition 5.1.2. S also satisfies the Subset Axiom, by Theorem 2.2.6. Finally, adopting the convention that the empty determinant is equal to 1, we have the last axiom. To prove Theorem 5.1.7, we thus need to show that any electrical positroid S may be realized as the set of positive circular minors of a response matrix, or equivalently the set of connections in a circular planar graph.

5.2 Proof of Theorem 5.1.7

We now prove Theorem 5.1.7. First, recall the important convention that $(P;Q) = (\widetilde{Q};\widetilde{P})$. We leave it to the reader to check, whenever appropriate, that all of the definitions and statements we make in this section are compatible with this convention.

Fix a boundary circle with n boundary vertices, which we label $1, 2, \ldots, n$ in clockwise order. In this section, all labels are considered modulo n. We have shown, via the Grassmann-Plücker Relations, that the set of circular pairs corresponding to the positive circular minors of a response matrix is an electrical positroid. We now prove that, for all electrical positroids S, there exists a critical graph G for which $\pi(G) = S$, which will establish Theorem 5.1.7. The idea of the argument is as follows.

Assume, for sake of contradiction, that there exists some electrical positroid S for which there does not exist such a critical graph G with $\pi(G) = S$. Then, let S_0 have maximal size among all such electrical positroids. Note that S_0 does not contain all circular pairs $(P;Q)$, because otherwise $S_0 = \pi(G_{\max})$, where $G_{\max}$ denotes a critical representative of the top-rank element of EP_n.

We will then add circular pairs to S_0 according to the boundary edge and boundary spike properties (cf. [CIM, §4]), discussed below, to form an electrical positroid S_1. By the maximality of S_0, $S_1 = \pi(G_1)$ for some critical graph G_1. We will then delete a boundary edge or contract a boundary spike in G_1 to obtain a graph G_0, and show that $\pi(G_0) = S_0$.

We begin by defining two properties of circular pairs, the $(i, i+1)$-boundary edge property and the i-boundary spike property. Let us first adopt a notational convention.

Definition 5.2.1. Given a circular pair $(P;Q)$, let $(P+x;Q+y)$ denote the unique circular pair (if it exists) with $P+x = P \cup \{x\}$ and $Q+y = Q \cup \{y\}$ as sets. In the ordered sets $P+x, Q+y$, x, y are inserted in the appropriate positions so that $(P+x;Q+y)$ is indeed a circular pair.

Given arbitrary P, Q, x, y, $(P+x;Q+y)$ may not be a circular pair. However, whenever we make reference to a pair of this form without commenting on its existence, we assert implicitly that it is, in fact, a circular pair.

Definition 5.2.2. A set S of circular pairs is said to have the $(i, i+1)$**-BEP (boundary edge property)** if, for all circular pairs $(P;Q) \in S$, if $(P+i;Q+(i+1))$ is a circular pair, then $(P+i;Q+(i+1)) \in S$.

Remark 5.2.3. According to Remark 2.2.2, if S has the $(i, i+1)$-BEP, then if $(P;Q) \in S$ and $(P+(i+1);Q+i)$ is a circular pair, then $(P+(i+1);Q+i) \in S$.

Definition 5.2.4. A set S of circular pairs is said to have the i**-BSP (boundary spike property)** if, for any circular pairs $(P;Q) \in S$ and x, y such that $(P+x;Q+i), (P+i;Q+y) \in S$, we have $(P+x;Q+y) \in S$.

Lemma 5.2.5. *Recall the definitions of boundary edges and boundary spikes from [CIM, §4]. Let G be a circular planar graph, and write $S = \pi(G)$.*

(a) There exists $H \sim G$ with a boundary edge $(i, i+1)$ if and only if S has the $(i, i+1)$-BEP.

(b) There exists $H \sim G$ has a boundary spike at i if and only if S has the i-BSP.

Proof. We prove (a); the proof of (b) is similar. Without loss of generality, we may assume that $i = 1$. It easy to check that if G has a boundary edge, then S must have the corresponding BEP. Conversely, suppose that S has the $(1,2)$-BEP. Then, let G' be the graph obtained by adding an edge $(1,2)$ in G such that the added edge does not cut through any faces of G. Clearly, $\pi(G') - S$ consists only of circular pairs $(P;Q)$ such that $(P+1;Q+2) \in S$ or $(P+2;Q+1) \in S$. However, S contains all such circular pairs, so in fact $\pi(G') = \pi(G)$. Then, by Theorem 2.3.3, $G' \sim G$.

It is left to check that $G \sim H$, for some circular planar graph H with the boundary edge $(1,2)$. Let a, b be the two medial boundary vertices between 1 and 2 in $\mathcal{M}(G)$. Note that adding the edge $(1,2)$ to G corresponds to introducing an additional crossing in $\mathcal{M}(G)$ between the (distinct) wires with endpoints a and b. Introducing this new crossing yields an equivalent medial graph, so it must have created it a lens. From here, it is easily seen, after applying [CIM, Lemma 6.3], that motions may be applied in $\mathcal{M}(G')$ so that this lens corresponds to parallel edges between the boundary vertices 1 and 2 in some $H \sim G' \sim G$. The desired conclusion follows. □

Lemma 5.2.6. *If S has all n BEPs and all n BSPs, then S contains all circular pairs.*

Proof. We proceed by induction on the size of $(P;Q)$ that $(P;Q) \in S$ for all circular pairs $(P;Q)$. First, suppose that $|P| = 1$. First, $(i;i+1) \in S$ for all i, because it has all BEPs and $(\emptyset;\emptyset) \in S$. Then, because S has the i-BSP, and $(i-1;i), (i;i+1) \in S$ we obtain $(i-1;i+1) \in S$. Continuing in this way gives that S contains all circular pairs $(P;Q)$ with $|P| = 1$.

Now, suppose that S contains all circular pairs of size $k-1$. Let $(a_1, \dots a_k; b_1, \dots b_k)$ be a circular pair of size k. By assumption, $(a_2, \dots a_k; b_2, \dots b_k) \in S$. Because S has all BEPs, $(b_1 + 1, a_2, \dots a_k; b_1, b_2, \dots b_k) \in S$ and $(b_1 + 2, a_2 \dots a_k; b_1 + 1, b_2, \dots b_k) \in S$, so by the (b_1+1)-BSP, $(b_1 + 2, a_2, \dots a_k; b_1, b_2, \dots b_k) \in S$. Continuing in this way gives $(a_1, \dots a_k; b_1, \dots b_k) \in S$, so we have the desired claim. □

In particular, Lemma 5.2.6 tells us that there exists an i such that S_0, as defined in the beginning of this section, either does not have the $(i, i+1)$-BEP for some i, or does not have the i-BSP for some i. We first assume that S_0 does not have all BEPs; without loss of generality, suppose that S_0 does not have the $(n, 1)$-BEP.

We will now add circular pairs to S_0 to obtain an electrical positroid S_1 that does have the $(n, 1)$-BEP. Specifically, we add to S_0 every circular pair $(P+1; Q+n)$, where $(P; Q) \in S_0$ has $1 < a_1 < b_1 < n$ (here $P = \{a_1, \ldots, a_k\}, Q = \{b_1, \ldots, B_k\}$), to obtain S_1. According to Remark 2.2.2, this construction also puts any $(P+n; Q+1) \in S_1$, where $(P; Q) \in S_0$ and $1 < b_k < a_k < n$.

Lemma 5.2.7. *S' is an electrical positroid, and has the $(n, 1)$-BEP.*

Proof. The proof is straightforward, so it is omitted. □

By assumption, S_0 is the maximal electrical positroid for which any circular planar graph G has $\pi(G) \neq S_0$. Thus, there exists a graph G_1 be a graph such that $\pi(G_1) = S_1$, and G_1 may be taken to have a boundary edge $(n, 1)$ by Lemma 5.2.5. Then, let G_0 be the result of deleting the boundary edge $(n, 1)$. To obtain a contradiction, it is enough to prove that $S_0 = \pi(G_0)$.

We now present a series of technical lemmas.

Definition 5.2.8. Consider a circular pair $(P; Q) \in S_0$ for which $1, n \notin P \cup Q$. We will assume, for the rest of this section, that $(P+1; Q+n)$ is a circular pair. $(P; Q)$ is said to be is **incomplete** if $(P+1; Q+n) \notin S_0$, and **complete** if $(P+1; Q+n) \in S_0$.

Lemma 5.2.9. *Let $(P; Q) = (a_1, \ldots, a_k; b_1, \ldots, b_k) \in S_0$ be an incomplete circular pair, such that $(P+1; Q+n)$ is a circular pair (and is not in S_0). Furthermore, assume that $(P; Q)$ is **minimal**, that is, $(P - a_k; Q - b_k)$ is complete. Then, for all $0 \leq i \leq k-1$, $(a_i; b_{i+1}), (a_{i+1}; b_i) \in S_0$.*

Proof. Immediate from Axiom 1a of Definition 5.1.6. □

Lemma 5.2.10. *Let $(a, b, c; d, e, f)$ be a circular pair. Then, if $(a; d), (a; f), (b; e), (c; d), (c; f) \in S_0$, then $(a; d), (b; e), (b; f), (c; e) \in S_0$.*

Proof. Immediate from Axiom 1b. □

Lemma 5.2.11. *If $(a_1, \ldots a_n; b_1, \ldots b_n) \in S_0$, $(a_{n+1}; b_{n+1}) \in S_0$, and $a_n, a_{n+1}, b_{n+1}, b_n$ appear in clockwise order, then $(a_1, \ldots, a_{n-1}, a_{n+1}; b_1, \ldots, b_{n-1}, b_{n+1}) \in S_0$.*

Proof. If $(a_n; b_{n+1}) \in S$ and $(a_{n+1}; b_n) \in S$, the claim follows from Axiom 2b, Axiom 2c and induction on n. Otherwise, it follows from Axiom 1a and induction on n. □

Lemma 5.2.12. *Let $(P; Q) = (a_1, \ldots, a_k; b_1, \ldots, b_k) \in S_0$ be a complete circular pair. Then, $(P - a_i; Q - b_i)$ is complete for all $i = 1, 2, \ldots, k$.*

Proof. Applying Axiom 2c with a_1, a_i, a_k, b_1 to $(P; Q - b_k)$ gives $(P - a_i; Q - b_k) \in S$. Then another application of Axiom 2c, to $(Q; P - a_i)$ with b_1, b_i, b_k, a_i gives the desired result. □

Lemma 5.2.13. *Let $(P, a, b, c, Q; R, d, e, f, T)$ be a circular pair, where P, Q, R, T are sequences of boundary vertices. Suppose that*

$$(a; d), (a; e), (b; d), (b; e), (b; f), (c; e), (c; f) \in S,$$
$$(P, a, b; R, d, e) \in S, \text{ and}$$
$$(P, a, c, Q; R, d, f, T) \in S$$

Then, $(P, a, b, Q; R, d, e, T) \in S$.

Proof. First, write $P = P' \cup \{p\}, R = R' \cup \{r\}\}$, where p and r are the last elements of the ordered sets P, R, respectively. Then, if $(P', b; R, f) \in S$, an application of Axiom 2b on $(f, e, p, P'; b, r, R')$ with f, e, r, p yields $(P, b; R, f) \in S$. Similarly, we find by induction that $(b; f) \in S \Rightarrow (P, b; R, f) \in S$. Then, we have $(P, a, b; R, d, f) \in S$ by Axiom 2b applied to $(f, e, d, R; b, a, P)$ with f, e, d, a. Similarly, write $Q = \{q\} \cup Q', T = \{t\} \cup T'$, where q, t are the first elements of Q, T, respectively. By Axiom 2b applied to $(P, a, b, c, q, Q'; R, d, f, t, T')$ with b, c, q, t, we see that $(P, a, b, Q; R, d, f, T) \in S$. The lemma then follows from Axiom 2c applied to $(T, f, e, d, R; Q, b, a, P)$ with T, f, e, Q. □

Lemma 5.2.14. *Let P, Q, R, T be sequences of indices, and let $(1, P, a, b, c, Q; n, R, d, e, f, T)$ be a circular pair. Suppose*

$$\begin{aligned}(a; d), (a; e), (b; d), (b; e), (b; f), (c; e), (c; f) \in S,\\ (1, P, a, b, Q; n, R, d, e, T) \in S, \text{ and}\\ (P, a, c, Q; R, d, f, T) \in S.\end{aligned}$$

Then $(1, P, a, c, Q; n, R, d, f, T) \in S$.

Proof. With the same notation as in the previous lemma, $(a, c, Q'; d, e, T') \in S \Rightarrow (a, c, Q; d, e, T) \in S$ by Axiom 2c on $(T', t, f, e, d; Q', q, c, a)$ with t, f, e, q. Then, an inductive argument shows that $(a, c, Q; d, e, T) \in S$. A similar argument shows that $(P, a, c, Q; R, d, e, T) \in S$. Then, Axiom 2c applied to $(1, P, a, b, c, Q; n, R, d, e, T)$ with $1, b, c, n$ implies that $(1, P, a, c, Q; n, R, d, e, T) \in S$ and applying Axiom 2c again to $(1, P, a, c, Q; n, R, d, e, f, T)$ with $n, e, f, 1$ yields the desired result. □

Lemma 5.2.15. *Consider a circular pair $(P; Q) = (a_1, \ldots, a_k; b_1, \ldots, b_k\}$, and let $(P+a; Q+b)$ be an incomplete circular pair with $a_k < a < b < b_k$ in clockwise order. Then, any electrical positroid Z satisfying $S_0 \cup \{(P + 1; Q + n)\} \subset Z \subset S_1$ contains $(P + a + 1; Q + b + n)$.*

Proof. It is easy to see that any element of $Z \backslash S_0$ must be of the form $(P'+1; Q'+n)$, for some P', Q'. By Axiom 1a, $(P+1; Q+n) \in Z$ and $(P+a; Q+b) \in Z$ implies that either $(P+a+1; Q+b+n) \in Z$, or $(P + a + 1; Q + b + n) \notin Z$ and $(P + 1; Q + b), (P + a, Q + 2) \in Z$. We are done in the former case, so assume for sake of contradiction that we have the latter. $(P + 1; Q + b), (P + a, Q + 2)$ are not of the form $(P' + 1; Q' + n)$, so cannot lie in $Z \setminus S$; thus, $(P + 1; Q + b), (P + a, Q + 2) \in S$. Finally, Axiom 1b yields us $(P + 1; Q + n) \in S$, a contradiction, so we are done. □

Definition 5.2.16. Two pairs of indices (i, j) and (i', j') are said to **cross** if $i < i' < j' < j$ and $(i; j'), (i'; j) \in S$.

Definition 5.2.17. For ease of notation, denote the sequence of indices $a_k, \ldots, a_\ell$ by $A_{k,\ell}$.

We now algorithmically construct a set $\mathcal{P}$ of circular pairs, which we will call **primary** circular pairs. We will use this notion to eventually prove Lemma 5.2.22, a key ingredient in our proof of the main theorem. The construction is as follows: begin by placing $(1; n) \in \mathcal{P}$. Then, for each $(P; Q) = (A_{1,i-1}; B_{1,i-1}) \in \mathcal{P}$, if we also have $(P; Q) \in S_0$, perform the following operation.

- Let a be the first index appearing clockwise from a_{i-1} such that there exists c with (a, c) crossing (a_{i-1}, b_{i-1}), and also $(A_{2,i-1}, a; B_{2,i-1}, c) \in S$. If a does not exist, stop. Otherwise, with a fixed, take c to be the first index appearing counterclockwise from b_{i-1} satisfying these properties.
- If a exists, add $(A_{1,i-1}, a; B_{1,i-1}, c)$ to $\mathcal{P}$, and remove $(P; Q) = (A_{1,i-1}; B_{1,i-1})$.

- Similarly, let b to be the largest index counterclockwise from b_{i-1} such that there exists d with (d, b) crossing (a_{i-1}, b_{i-1}) and $(a_2, \dots d; b_2, \dots b_i) \in S$. If b does not exist, stop. Otherwise, with b fixed, take d to be the first index clockwise from a_{i-1} with these properties.

- If $a \neq d$ and $b \neq c$ (note that if $a = d$, then $b = c$), then add $(A_{1,i-1}, d; B_{1,i-1}, b)$ to $\mathcal{P}$. Note that $c \leq d$ or else, by 1a, c could originally have been set to d.

It is easily seen that at any time, the algorithm may be performed on the elements of $\mathcal{P}$ in any order, and that it will eventually terminate, when the operation described above results in no change in $\mathcal{P}$ for all $(P; Q) \in \mathcal{P}$.

Definition 5.2.18. For a circular pair $(P; Q) = (p_1, \dots, p_k; q_1, \dots, q_k)$, define $E(P; Q) = \{\{p_i, q_i\} \mid i \in \{1, \dots, k\}\}$. We will take $E(P; Q)$ to be an ordered set and abusively refer to its elements as **connections**.

Lemma 5.2.19. *For any incomplete circular pair $(P; Q)$, there exists a circular pair $(P'; Q') \in \mathcal{P}$ such that any electrical positroid Z satisfying $S_0 \cup \{(P'; Q')\} \subset Z \subset S_1$ contains $(P + 1; Q + n)$.*

Proof. By Lemma 5.2.15, we may assume that $(P; Q)$ is a minimal incomplete circular pair. Let $(P+1; Q+n) = (1, a_1, \dots, a_k; n, b_1, \dots, b_k) = (1, A_{1,k}; n, B_{1,k})$ (see Definition 5.2.17). Consider the primary circular pairs whose first i connections are the same as those of $(P; Q)$. By the construction of $\mathcal{P}$, there are at most two such primary circular pairs, which we denote by

$$(P; Q)_1 = (A_{1,i}, C_{i+1,m}; B_{1,i}, D_{i+1,m})$$
$$(P; Q)_2 = (A_{1,i}, E_{i+1,m'}; b_{1,i}, f_{i+1,m'}).$$

By Lemma 5.2.9 and the construction of $\mathcal{P}$, we have that $c_{i+1} \leq a_{i+1}$ and $d_{i+1} \geq b_{i+1}$ or $e_{i+1} \leq a_{i+1}$ and $f_{i+1} \geq b_{i+1}$ (or else we would have been able to set $d_{i+1} = b_{i+1}$ or $e_{i+1} = a_{i+1}$). Furthermore, exactly one of these pairs of inequalities holds. Let us assume that the former holds, as the latter case is identical, and in this case, call $(P; Q)_1$ the primary circular pair associated to $(P; Q)$. If, on the other hand, $(P; Q)_1$ (as above) is the only primary circular pair sharing its i connections with $(P; Q)$, then $c_{i+1} \leq a_{i+1}$ and $d_{i+1} \geq b_{i+1}$, and we still refer to $(P; Q)_1$ as the primary circular pair associated to $(P; Q)$.

We now prove the lemma by retrograde induction on i, where here i is such that the first i connections of $(P; Q)$ are shared with some primary circular pair. If $i = k$, we are done by Lemma 5.2.15, and if the first i connections of $(P; Q)$ are exactly the primary circular pair in question, we are done by the Subset Axiom. Otherwise, we first need $(A; B) = (A_{1,i}, c_{i+1}, A_{i+2,k}; B_{1,i}, d_{i+1}, B_{i+2,k}) \in S_0$, which follows from Lemmas 5.2.10 and 5.2.13, where the conditions of these lemmas are satisfied as a result of Lemma 5.2.9. It is easy to see that the primary circular pair associated to $(A; B)$ is the same as that for $(P; Q)$. It follows, then, by the inductive hypothesis, that $(1, A_{1,i}, c_{i+1}, A_{i+2,k}; B_{1,i}, d_{i+1}, B_{i+2,k}, n) \in Z$. When $a_{i+1} \neq c_{i+1}$ and $b_{i+1} \neq d_{i+1}$, Lemma 5.2.14 yields the desired result, and if one of $a_{i+1} = c_{i+1}$ or $b_{i+1} = d_{i+1}$, we are done by a similar argument. □

Lemma 5.2.20. *There is exactly one circular pair in $\mathcal{P}$ that does not lie in S_0, which we call the S_0-**primary circular pair**.*

Proof. By Lemma 5.2.19, $\mathcal{P} \setminus S_0$ has at least one element, because S_0 does not have the $(n, 1)$-BEP. Assume, for sake of contradiction, that $\mathcal{P} \setminus S_0$ has two elements, of the form

$$(A_{1,i-1}, c, P; B_{1,i-1}, d, Q)$$

$$(A_{1,i-1}, e, P'; B_{1,i-1}, f, Q').$$

Because $(A_{1,i-1}, c, P; B_{1,i-1}, d, Q) \notin S_0$ and $(A_{1,i-1}, P; B_{1,i-1}, Q), (A_{2,i-1}, c, P; B_{2,i-1}, d, Q) \in S_0$, we must have $(A_{2,i-1}, c, P; B_{1,i-1}, Q) \in S_0$, by Axiom 1a. Thus, $(A_{2,i-1}, c; B_{1,i-1}) \in S_0$, by the Subset Axiom. By the same argument applied to e, f, we must have $(A_{1,i-1}; B_{2,i-1}, f) \in S_0$, so Axiom 1b gives $(A_{2,i-1}, c; B_{2,i-1}, f) \in S_0$. However, because $f > d$, we have a contradiction of the definition of d. Thus, $|\mathcal{P} \setminus S_0| = 1$. □

Lemma 5.2.21. *For any incomplete circular pair $(P; Q)$, any electrical positroid Z satisfying $S_0 \cup \{(P + 1; Q + n)\} \subset Z \subset S_1$ contains the S_0-primary circular pair.*

Proof. Proceed by retrograde induction on i, where i is such that the first i connections of $(P; Q)$ are the same as those of some primary circular pair. By the Subset Axiom, we can assume that $(P; Q)$ is minimal. The base case is immediate from the Subset Axiom, so suppose that $i < k$. Let $(P; Q) = (A_{1,k}; B_{1,k})$. Then, we need to show that, if $(P+1; Q+n) \in Z$, then $(1, A_{1,i}, c_{i+1}, A_{i+2,k}; n, B_{1,k}) \in Z$.

First, suppose that both $c_{i+1} < a_{i+1}$ and $b_{i+1} < d_{i+1}$. Then, the desired claim is exactly Lemma 5.2.13, as long as $i+1 < m$. Assume, then, that $i+1 = m$. First, an application of Lemma 5.2.13 yields $(A_{1,i}, c_{i+1}, A_{i+2,k}; B_{1,i}, d_{i+1}, B_{i+2,k}) \in S_0$, which implies $(A_{1,i}, c_{i+1}, A_{i+2,k}, B_{1,k}) \in S_0$ and $(A_{1,k}; B_{1,i}, d_{i+1}, B_{i+2,k}) \in S_0$. Furthermore, if $(A_{i,i}, c_{i+1}, A_{i+1,k}; n, B_{1,k}) \in S_0$, then Axiom 2b yields $(1, A_{1,k}; n, B_{1,k}) \in S_0$, a contradiction.

Similarly, we have $(1, A_{1,i}, c_{i+1}, A_{i+2,k}; B_{1,i}, d_{i+1}, B_{i+1,k}) \notin S_0$. As a result, applying Axiom 2a to $(1, A_{1,i}, c_{i+1}, A_{i+1,k}; n, B_{1,k})$ with $1, c_{i+1}, a_{i+1}, n$ gives $(1, A_{1,i}, c_{i+1}, A_{i+2,k}; n, B_{1,k}) \in S_0$. One more application of Axiom 2a to $(1, A_{1,i}, c_{i+1}, A_{i+2,k}; n, B_{1,i}, d_{i+1}, B_{i+1}, k)$ with $n, d_{i+1}, b_{i+1}, 1$ yields the desired result.

If one of the indices $a_{i+1} = c_{i+1}$ or $b_{i+1} = d_{i+1}$, then we are also done by a similar argument. □

Corollary 5.2.22. *For any two incomplete circular pairs $(P; Q)$ and $(P'; Q')$, any electrical positroid Z satisfying $S \cup \{(P + 1; Q + n)\} \subset Z \subset S_1$ must also contain $(P' + 1; Q' + n)$.*

Proof. By Lemma 5.2.21, Z must contain the S_0-primary circular pair. The claim then follows by Lemma 5.2.19. □

By the above results, if we start with our set S_0 and some incomplete circular pair $(P; Q) \in S_0$, "completing" $(P; Q)$ by adding $(P + 1; Q + n)$ to S_0 will require that we have completed every incomplete pair. We now finish the proof of Theorem 5.1.7, in the boundary edge case.

Let $T_0 \subset S_0$ denote the subset of circular pairs in S_0 without the connection $(1, n)$, and define T_1, T_0' similarly for S_1, S_1', respectively. By construction, it is easily seen that $T_0 = T_1 = T_0'$. While T_0 may not necessarily be an electrical positroid, we have:

Lemma 5.2.23. *There exists an electrical positroid T with $T_0 \subset T \subset S_0 \cap S_0'$.*

Proof. We give an algorithm to construct such an electrical positroid T. We begin by setting $T = T_0$; note that T satisfies the last two electrical positroid axioms, but may not satisfy the first six. Each of the first six axioms are of the form $\mathcal{A}, \mathcal{B} \in T \Rightarrow \mathcal{C}, \mathcal{D} \in T$, or otherwise $\mathcal{A}, \mathcal{B} \in T \Rightarrow \mathcal{C}, \mathcal{D} \in T$ or $\mathcal{E}, \mathcal{F} \in T$. At each step of the algorithm, if T is an electrical positroid, we stop, and if not, we pick an electrical positroid axiom α (among the first six) not satisfied by $\mathcal{A}, \mathcal{B} \in T$. We then show that we can add elements of $S_0 \cap S_0'$ to T so that α is satisfied by $\mathcal{A}, \mathcal{B}$, and so that T also still satisfies the Subset Axiom.

It is clear that adding circular pairs to T in this way is possible when α is one of Axioms 1b, 1c, 2b, and 2c: we take the add circular pairs $\mathcal{C}, \mathcal{D}$, as above, as well as all of the circular pairs formed by subsets of their respective connections. We will show that this operation is also possible when α is one of Axioms 1a and 2a. From here, it will be clear that the algorithm must terminate, because we can only add finitely many elements to T. Therefore, we will eventually find T with the desired properties.

In each of the cases below, the circular pairs added to T are always assumed to be added along with each of their subsets, that is, the circular pairs formed by subsets of their connections. In this way, the Subset Axiom is satisfied by T at all steps in the algorithm.

We first consider Axiom 1a, which we assume to fail in T when applied to $(P-a;Q-c),(P-b;Q-d)\in T$. If $(P-a;Q-c),(P-b;Q-d)\in S_0$, either $(P-a;Q-d),(P-b;Q-c)\in S_0$ or $(P-a-b;Q-c-d),(P;Q)\in S_0$. It is easy to see that $1\in P$ and $n\in Q$ (or vice versa, but we can swap P and Q and reverse their orders), or else Axiom 1a already would have been satisfied by $(P-a;Q-c),(P-b;Q-d)\in T$. We proceed by casework:

- $(a,c)=(1,n)$. Then, because Axiom 1a fails, we have $(P-b;Q-c)\notin T_0,S_0,S_0'$. Thus, we may add $(P-a-b;Q-c-d),(P;Q)$ to T, and these lie in $S\cap S''$.

- $a=1,c\neq n$. We have $(P-a;Q-c),(P-b;Q-d)\in T$. First, suppose that $(P-a;Q-d)\notin T$. Because $(P-a;Q-d)$ does not contain the connection $(1,n)$, we have $(P-a;Q-d)\notin S_0,S_0'$. Then, $(P-a-b;Q-c-d),(P;Q)\in S_0,S_0'$, so we may add may $(P;Q)$ to T, so that Axiom 1a is satisfied with $(P-a;Q-c),(P-b;Q-d)\in T$ (note that $(P-a-b;Q-c-d)$ is already in T).

  Now, suppose instead that $(P-a;Q-d)\in T$. If $(P-b-1;Q-c-n)\notin T$, then $(P-b;Q-c)\notin S_0,S_0'$ by the Subset Axiom. Then, $(P-a-b;Q-c-d),(P;Q)\in S_0,S_0'$, and so we may add $(P;Q)$ to T to satisfy Axiom 1a. Now, assume that $(P-b-1;Q-c-n)\in T$. For any electrical positroid $\overline{S}$, Axiom 2b applied to $(P-b;Q)$ and $d,c,n,1$ gives that $(P-b;Q-d)\in\overline{S}$ and $(P-b-1;Q-c-n)\in\overline{S}$ implies $(P-b;Q-c)\in\overline{S}$ and $(P-b-1;Q-n-d)\in\overline{S}$. By the discussion above, we have $(P-b;Q-d),(P-b-1;Q-c-n)\in T,S_0,S_0'$, and so we may add $(P-b;Q-c)$ to T. The case in which $a\neq 1,c=n$ is identical.

- The case $a\neq 1,c\neq n$ may be handled using similar logic; the details are left to the reader.

Finally, consider Axiom 2a, which we assume to fail for $(P-b;Q),(P-a-c;Q-d)\in T$. As before, we may assume $1\in P,n\in Q$.

- $(a,d)=(1,n)$, or $(d,a)=(n,1)$. Similar to the first case above.

- $a=1,d\neq n$. Then, we have $(P-b;Q),(P-a-c;Q-d)\in T,S_0,S_0'$. As in the second case for Axiom 1a, we may assume that we have $(P-a;Q)\in T,S_0,S_0'$ and $(P-a-b;Q-d)\in T,S_0,S_0'$, or else both S_0 and S_0' would contain exactly one of $(P-a;Q),(P-b-d;Q-d)$ and $(P-c;Q),(P-a-b;Q-d)$. Moreover, we may assume that we have $(P-1-b-c;Q-n-d)\in T,S_0,S_0'$ by similar logic. Because $(P-b;Q)\in T,S_0,S_0'$ and $(P-1-b-c;Q-n-d)\in T,S_0,S_0'$, we may apply Axiom 1c to find that $(P-b-c;Q-d)\in S_0,S_0'$. Thus, we can add $(P-b-c;Q-d)$ to T, so that we still have $T\subset S_0\cap S_0'$. The case $a=n,d\neq 1$ is identical.

- The cases $a\neq 1,d=n$ and $a\neq 1,d\neq n$ may be handled using similar logic; we again omit the details.

Thus, in all cases, our algorithm is well-defined, and we are done. □

Proof of Theorem 5.1.7. By Lemma 5.2.22, we must in fact have $S_0 = T = S_0'$, provided that neither S_0 nor S_0' is equal to S_1, which is true by construction (recall that G_0' is critical). The proof is complete, in the boundary edge case.

It is left to consider the case in which S_0 has the $(i, i+1)$-BEP, for each i, but fails to have the i-BSP, for some i. Without loss of generality, suppose that S_0 does not have the 1-BSP. We now form S_1 as the union of S_0 and the set of all circular pairs $(P + x; Q + y)$ such that $(P + x; Q + 1), (P + 1, Q + y) \in S_0$, where $(P; Q)$ is a circular pair with $1, x \notin P, 1, y \notin Q$.

In Appendix B, we form a circular planar graph G_1 such that $\pi(G_1) = S_1$ and G_1 has a boundary spike at 1. Then, contracting this boundary spike to obtain the graph G_0, we find that $\pi(G_0) = S_0$. Therefore, with the additional results of Appendix B, the theorem is proven. □

6 The LP Algebra $\mathcal{LM}_n$

We now study the LP Algebra $\mathcal{LM}_n$. Our starting point will be *positivity tests*; a particular positivity test will form the initial seed in $\mathcal{LM}_n$. We then proceed to investigae the algebraic and combinatorial properties of clusters in $\mathcal{LM}_n$.

6.1 Positivity Tests

Let M be a symmetric $n \times n$ matrix with row and column sums equal to zero. In this section, we describe tests for deciding if M is the response matrix for an electrical network in the top rank of EP_n. Equivalently, we describe tests for deciding if all of the circular minors of M are positive. These tests are similar to certain tests for total positivity described in [FZP]. Throughout the remainder of this section, all indices around the circle are considered modulo n, and we will refer to circular pairs and their corresponding minors interchangeably.

Definition 6.1.1. A set S of circular pairs is a **positivity test** if, for all matrices M whose minors corresponding to S are positive, every circular minor of M is positive (equivalently, M is the response matrix for a top-rank electrical network).

We begin by describing a positivity test of size $\binom{n}{2}$. Fix n vertices on a boundary circle, labeled $1, 2, \ldots, n$ in clockwise order.

Definition 6.1.2. For two points $a, b \in [n]$, let $d(a, b)$ denote the number of boundary vertices on the arc formed by starting at a and moving clockwise to b, inclusive.

Definition 6.1.3. A circular pair $(P; Q) = (p_1, \cdots, p_k; q_1, \cdots, q_k)$ is called **solid** if both sequences $p_1, \ldots, p_k$ and $q_1, \ldots, q_k$ appear consecutively in clockwise order around the circle. Write $d_1 = d_1(P; Q) = d(p_k, q_k)$, and $d_2 = d_2(P; Q) = d(q_1, p_1)$. We will call a solid circular pair $(P; Q)$ **picked** if one of the following conditions holds:

- $d_1 \leq d_2$ and $1 \leq p_1 \leq \frac{n}{2}$, or
- $d_1 \geq d_2$ and $1 \leq q_k \leq \frac{n}{2}$

Definition 6.1.4. Let M be a fixed symmetric $n \times n$ matrix. Define the set of **diametric pairs** $\mathcal{D}_n$ to be the set of solid circular pairs $(P; Q)$ such that either $|d_1 - d_2| \leq 1$ or $|d_1 - d_2| = 2$ and $(P; Q)$ is picked. We will refer to the elements of $\mathcal{D}_n$ as circular pairs and minors interchangeably.

It is easily checked that $|\mathcal{D}_n| = \binom{n}{2}$.

Remark 6.1.5. For a solid circular pair $(P;Q)$, we have that $|d_1 - d_2| \equiv n \pmod 2$, so $\mathcal{D}_n$ consists of the solid circular pairs with $|d_1 - d_2| = 1$ when n is odd, and the solid circular pairs with either $|d_1 - d_2| = 0$, or $|d_1 - d_2| = 2$ and $(P;Q)$ is picked when n is even.

Recall (see Remark 2.2.2) that the circular pairs $(P;Q)$ and $(\widetilde{Q};\widetilde{P})$ will be regarded as the same. Note, for example, that $(P;Q) \in \mathcal{D}_n$ if and only if $(\widetilde{Q};\widetilde{P}) \in \mathcal{D}_n$, so the definition of $\mathcal{D}_n$ is compatible with this convention.

Proposition 6.1.6. *If M is taken to be an $n \times n$ symmetric matrix of indeterminates, any circular minor is a positive rational expression in the determinants of the elements of $\mathcal{D}_n$.*

Proof. We will make use of the Grassmann-Plücker relations, (5.1.3) and (5.1.4), so we repeat them here:

For $(a,b;c,d)$ a circular pair,

$$\Delta^{a,c}\Delta^{b,d} = \Delta^{a,d}\Delta^{b,c} + \Delta^{ab,cd}\Delta^{\emptyset,\emptyset} \tag{5.1.3}$$

and for a,b,c,d in clockwise order,

$$\Delta^{b,\emptyset}\Delta^{ac,d} = \Delta^{a,\emptyset}\Delta^{bc,d} + \Delta^{c,\emptyset}\Delta^{ab,d} \tag{5.1.4}$$

We will first show, by induction on $|d_1 - d_2|$, that any solid circular pair is a positive rational expression in the elements of $\mathcal{D}_n$. There is nothing to check when $|d_1 - d_2|$ is equal to 0 (when n is even) or 1 (when n is odd). If $(P;Q) = (p_1,\ldots,p_k;q_1,\ldots,q_k)$ is a solid circular pair such that $|d_1 - d_2| = 2$ (hence, n is even) and $(P;Q)$ is not picked, then either $(p_1-1,p_1,\ldots,p_k;q_1+1,q_1,\ldots,q_k)$ or $(p_1,\ldots,p_k,p_k+1;q_1,\ldots,q_k,q_k-1)$ is a solid circular pair with $|d_1 - d_2| = 2$, and it must be picked. Assume, without loss of generality, that it is the former, and let $p_0 = p_1 - 1$, $q_0 = q_1 + 1$. Letting $\Delta = (p_0,\ldots,p_k;q_0,\ldots,q_k)$, we have, by (5.1.3), that:

$$\Delta^{p_0,q_0} = \frac{\Delta^{p_0,q_k}\Delta^{p_k,q_0} + \Delta^{p_0p_k,q_0q_k}\Delta^{\emptyset,\emptyset}}{\Delta^{p_k,q_k}}. \tag{6.1.7}$$

Because $(p_1,\ldots,p_k;q_1,\ldots,q_k) = \Delta^{p_0,q_0}$ is not picked, Δ^{p_k,q_k} corresponds to a picked circular pair with $|d_1 - d_2| = 2$, and $\Delta^{p_0,q_k}, \Delta^{p_k,q_0}, \Delta^{p_0p_k,q_0q_k}$, and $\Delta^{\emptyset,\emptyset}$ all have $|d_1 - d_2| = 0$, so we have that $(P;Q)$ is a rational expression of elements of $\mathcal{D}_n$.

Now, for $m \geq 3$, assume that all solid pairs with $|d_1 - d_2| < m$ are positive rational expression in the elements of $\mathcal{D}_n$, and consider a solid pair $(P;Q)$ with $|d_1 - d_2| = m$. Then, either $(p_1 - 1, p_1,\ldots,p_k;q_1+1,q_1,\ldots,q_k)$ or $(p_1,\ldots,p_k,p_k+1;q_1,\ldots,q_k,q_k-1)$ is a solid circular pair with $|d_1 - d_2| < m$. Assume, without loss of generality, that the former is the case. Then, we may again set $p_0 = p_1 - 1$, $q_0 = q_1 + 1$, and $\Delta = (p_0,\ldots,p_k;q_0,\ldots,q_k)$, and (6.1.7) still holds. Each term on the right hand side corresponds to a solid pair with a smaller value of $|d_1 - d_2|$, so $(P;Q)$ is a positive rational expression in the elements of $\mathcal{D}_n$, by the inductive hypothesis.

We now show that any circular pair is a positive rational expression in the elements of $\mathcal{D}_n$. For a sequence $P = p_1,\ldots,p_k$ of points ordered clockwise around the circle, let $c_P \in \{1,\ldots,k\}$ be the largest index such that $p_1,\ldots,p_{c_P}$ are consecutive. If $c_P < k$, then define $d_3(P) = d(p_{c_P}, p_{c_P+1})$ and $d_4(P) = d(p_1,p_k)$. If, on the other hand, $c_P = k$, define $d_3(P) = d_4(P) = 0$. Similarly, for a sequence $Q = q_1,\ldots,q_k$ of points ordered counterclockwise around the circle, let $c_Q \in \{1,\ldots,k\}$ be the smallest index such that $q_k,\ldots,q_{c_Q}$ are consecutive. If $c_P > 1$, then define $d_3(Q) = d(p_{c_Q+1}, p_{c_Q})$ and $d_4(Q) = d(q_k,q_1)$. If, on the other hand, $c_Q = 1$, define $d_3(P) = d_4(P) = 0$.

For a circular pair $(P;Q)$, define $\Phi((P;Q)) = d_3(P) + d_4(P) + d_3(Q) + d_4(Q)$, and note that $\Phi((P;Q)) = \Phi((\widetilde{Q};\widetilde{P}))$, so Φ is well-defined when we impose the convention $(P;Q) = (\widetilde{Q};\widetilde{P})$. We finish the proof by showing that any circular pair is a positive rational expression in the elements of $\mathcal{D}_n$ by induction on Φ.

If $\Phi((P;Q)) = 0$, then $(P;Q)$ is solid and hence a rational expression in the elements of $\mathcal{D}_n$. Now, assume that for any $m > 0$, every circular pair $(P';Q')$ with $\Phi((P';Q')) < m$ is a rational expression in the elements of $\mathcal{D}_n$, and consider a circular pair $(P;Q)$ with $\Phi((P;Q)) = m$. Assume, without loss of generality, that $c_P \neq k$, and let $\ell = p_{c_P} + 1$. Applying (5.1.3) to $\Delta = (p_1, \ldots, p_{c_P}, \ell, p_{c_P+1}, \ldots, p_k; q_1, \ldots, q_k)$, we get:

$$\Delta^{\ell,\emptyset} = \frac{\Delta^{p_1,\emptyset}\Delta^{\ell p_k,q_k} + \Delta^{p_k,\emptyset}\Delta^{p_1\ell,q_k}}{\Delta^{p_1p_k,q_k}}. \tag{6.1.8}$$

Each term on the right hand side is easily seen to have a smaller value of Φ than m, so we are done by induction. □

Corollary 6.1.9. *$\mathcal{D}_n$ is a positivity test.*

6.2 $\mathcal{CM}_n$ and $\mathcal{LM}_n$

The positive rational expressions from the previous section are reminiscent of a cluster algebra structure (see [FP, §3] for definitions). In fact, (5.1.3) and (5.1.4) are exactly the exchange relations for the local moves in double wiring diagrams [FZP, Figure 9]. Due to parity issues similar to those encountered when attempting to associate a cluster algebra to a non-orientable surface in [DP], the structure of positivity tests is slightly different from a cluster algebra. We present the structure in two different ways: first, as a Laurent phenomenon (LP) algebra $\mathcal{LM}_n$ (see [LP, §2,3] for definitions), and secondly as a cluster algebra $\mathcal{CM}_n$ similar to the double cover cluster algebra in [DP]. $\mathcal{LM}_n$, we will find, is isomorphic to the polynomial ring on $\binom{n}{2}$ variables, but more importantly encodes the information of the positivity of the circular minors of a fixed $n \times n$ matrix.

We begin by describing an undirected graph U_n that encodes the desired mutation relations among our initial seed. The vertex set of U_n will be $V_n = \mathcal{D}_n \cup \{(\emptyset;\emptyset)\}$.

Definition 6.2.1. A solid circular pair $(p_1, \ldots, p_k; q_1, \ldots, q_k)$ is called **maximal** if $2k + 2 > n$ or $2k + 2 = n$ and $d_1 = d_2$. A solid circular pair $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ is called **limiting** if $|d_1 - d_2| = 2$, $(P;Q)$ is picked, and $1 = p_1$ or $1 = q_k$.

Let us now describe the edges of U_n: see Figure 12 for an example. For each $(P;Q) \in V_n$ that is not maximal, limiting, or empty (that is, equal to $(\emptyset;\emptyset)$), there is a unique way to substitute values in 5.1.3 such that $(P;Q)$ appears on the left hand side, and all four terms on the right hand side are in V_n. We draw edges from $(P;Q)$ to these four vertices in U_n. Finally, if $(P;Q), (R;S) \in V_n$ are limiting, we draw an edge between them if their sizes differ by 1. The edges drawn in these two cases constitute all edges of V_n.

For any maximal circular pair $(P;Q)$, it can be proven that there exists a symmetric matrix A such that A is positive on any circular pair except $A(P;Q) \leq 0$. In fact, if $(P;Q)$ is maximal and has $|d_1 - d_2| \leq 1$, then the set of all circular pairs other than $(P;Q)$ is an electrical positroid. Hence, in our quivers, we will take the vertices corresponding to the maximal circular pairs and $(\emptyset;\emptyset)$ to be frozen.

U_n can then be embedded in the plane in a natural way with the circular pairs of size k lying on the circle of radius k centered at $(\emptyset;\emptyset)$, and all edges except those between vertices corresponding to limiting circular pairs either along those circles or radially outward from $(\emptyset;\emptyset)$.

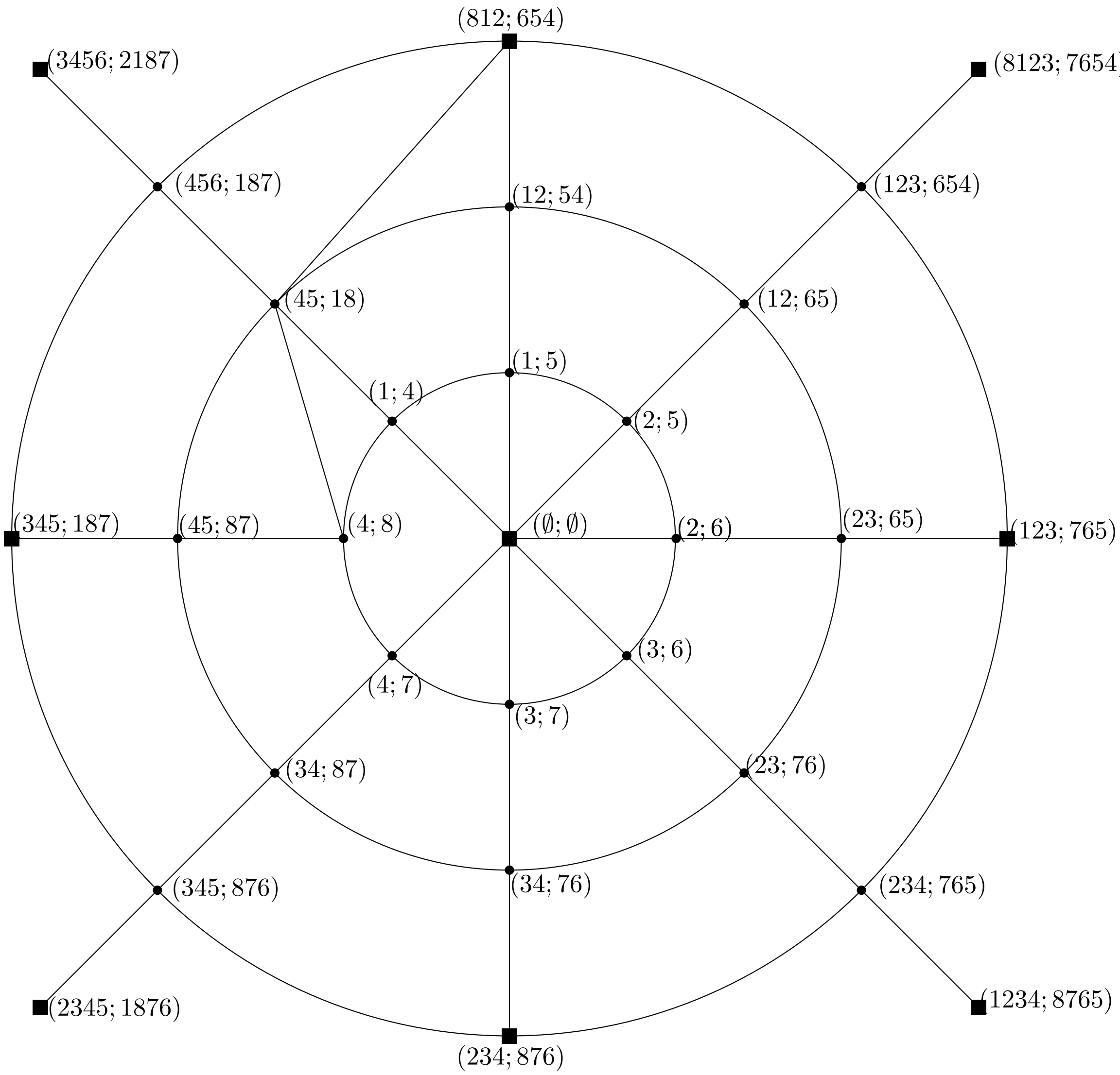

Figure 12: The graph U_8 depicting the desired exhange relations among $\mathcal{D}_8$. Vertices marked as squares correspond to frozen variables. $(4;8)$, $(45;18)$ and $(812;654)$ are the limiting circular pairs.

If we could orient the edges of U_n such that they alternate between in- and out-edges at each non-frozen vertex, then the resulting quiver would give a cluster algebra such that mutations at vertices whose associated cluster variables are neither frozen not limiting correspond to the relation Grassmann-Plücker relation (5.1.3). Furthermore, these mutation relations among the vertices of V_n constitute all of the Grassmann-Plücker relations for which five of the six terms on the right hand side are elements in V_n, and the term which is not in V_n is on the left hand side of the relation (5.1.3) or (5.1.4). However, for $n \geq 5$, such an orientation of the edges of U_n is impossible, because the dual graph of U_n contains odd cycles. We thus define:

Definition 6.2.2. Let $\mathcal{LM}_n$ be the LP algebra constructed as follows: the initial seed $\mathcal{S}_n$ has cluster variables equal to the minors in V_n, with the maximal pairs and $(\emptyset;\emptyset)$ frozen, and, for any other $(P;Q) \in V_n$, the exchange polynomial $F_{(P;Q)}$ is the same as what is obtained from a quiver with underlying graph U_n, such that the edges around the vertex associated to $(P;Q)$ in U_n alternate between in- and out-edges.

For example, in $\mathcal{LM}_8$, the exchange polynomial associated to the cluster variable $x_{(12;54)}$ is $x_{(45;18)}x_{(12;65)}+x_{(1;5)}x_{(812;654)}$. We need the additional technical condition that $F_{(P;Q)}$ is irreducible as a polynomial in the cluster variables V_n, but the irreducibility is clear.

We next define a cluster algebra $\mathcal{CM}_n$ which is a double cover of positivity tests, in the following sense: we begin consider an $n \times n$ matrix M', which we no longer assume to be symmetric. We write **non-symmetric circular pairs** in the row and column sets of M' as $(P;Q)'$, so that $(P;Q)'$ and $(\widetilde{Q};\widetilde{P})'$ now represent different circular pairs. We will say that two expressions A, B in the entries of M' **correspond** if swapping the rows and columns for each entry in A gives B, and we will write $B = c(A)$. For instance, $(P;Q)' = c((Q;P)')$.

The set of cluster variables V_n' in our initial seed will consist of pairs $(P;Q)'$ such that $(P;Q) \in V_n$. Note that $|V_n'| = 2\binom{n}{2}+1$, as V_n' contains $(P;Q)'$ and $(\widetilde{Q};\widetilde{P})'$ for each $(P;Q) \in \mathcal{D}_n$, and finally $(\emptyset;\emptyset)$. $(P;Q)'$ will be frozen in V_n' if $(P;Q)$ was frozen in V_n.

We construct the undirected graph U_n' with vertex set V_n' by adding edges in the same way that U_n was constructed. The only difference in our description is that if $(P;Q),(R;S) \in V_n$ are limiting, then they will be adjacent only if their sizes differ by 1 and $P \cap R \neq \emptyset$. See Figure 13 for an example.

Unlike in U_n, the edges of U_n' can be oriented such that they are alternating around each non-frozen vertex. Let $\mathcal{Q}_n$ be the quiver from either orientation. Then, let $\mathcal{CM}_n$ be the cluster algebra with initial quiver $\mathcal{Q}_n$.

Breaking the symmetry of M' removed the parity problems from U_n, so that we could define a cluster algebra, but we are still interested in using U_n' to study M when M is symmetric. Toward this goal, we can restrict ourselves so that whenever we mutate at a cluster variable v, we then mutate at $c(v)$ immediately afterward. Call this restriction the **symmetry restriction**.

Lemma 6.2.3. *After the mutation sequence $\mu_{x_1}, \mu_{c(x_1)}, \mu_{x_2}, \mu_{c(x_2)}, \ldots, \mu_{x_r}, \mu_{c(x_r)}$ from the initial seed in $\mathcal{CM}_n$, the number of edges from x to y in the quiver is equal to the number of edges from $c(y)$ to $c(x)$ for each x, y in the final quiver.*

Proof. We proceed by induction on r; for $r = 0$, we have the claim by construction. Now, suppose that we have performed the mutations $\mu_{x_1}, \mu_{c(x_1)}, \mu_{x_2}, \mu_{c(x_2)}, \ldots, \mu_{x_{r-1}}, \mu_{c(x_{r-1})}$ and currently have the desired symmetry property. By the inductive hypothesis, x_r and $c(x_r)$ are not adjacent, or else we would have had edges between them in both directions, which would have been removed after mutations. Thus, no edges incident to $c(x_r)$ are created or removed upon mutating at x_r. Hence, mutating at $c(x_r)$ afterward makes the symmetric changes to the graph, as desired. $\square$

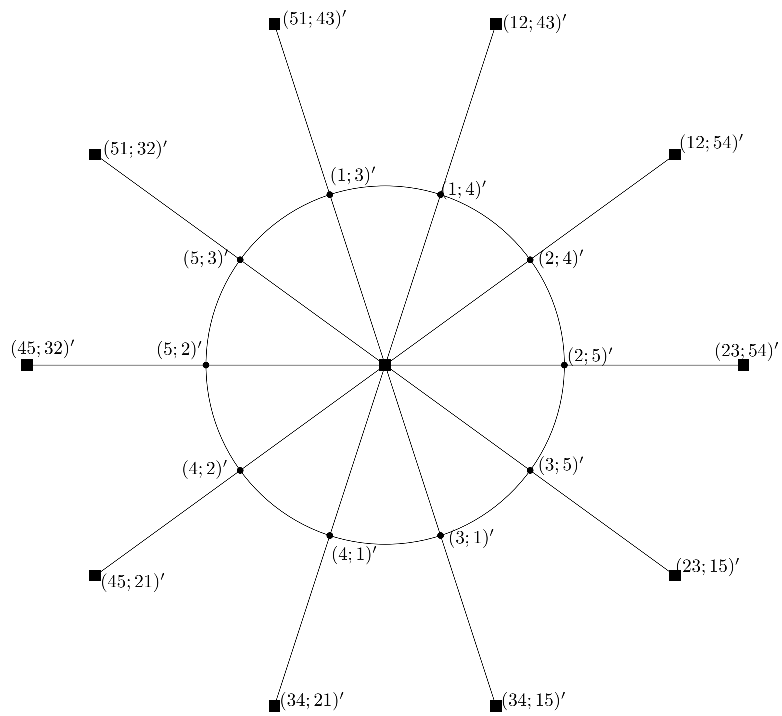

Figure 13: The graph U'_5. In the quiver $\mathcal{Q}_n$, the edges alternate directions around a non-frozen vertex.

Definition 6.2.4. Let $\mathbb{C}[M]$ and $\mathbb{C}[M']$ denote the polynomial rings in the off-diagonal entries of M and M' respectively; recall that M is symmetric, so $M_{ij} = M_{ji}$. Then, we can define the **symmetrizing homomorphism** $C : \mathbb{C}[M'] \to \mathbb{C}[M]$ by its action on the off-diagonal entries of M':

$$C(M'_{ij}) = C(M'_{ji}) = M_{ij}.$$

If S is a set of polynomials in $\mathbb{C}[M']$, then write $C(S) = \{C(s) \mid s \in S\}$.

Lemma 6.2.5. *Let L'_1 be the cluster of $\mathcal{CM}_n$ that results from starting at the initial cluster and performing the sequence of mutations $\mu_{x_1}, \mu_{c(x_1)}, \mu_{x_2}, \mu_{c(x_2)}, \ldots, \mu_{x_r}, \mu_{c(x_r)}$. Let L_2 be the cluster of $\mathcal{LM}_n$ that results from starting at the initial cluster and performing the sequence of mutations $\mu_{x_1}, \mu_{x_2}, \ldots, \mu_{x_r}$. Then, $C(L'_1) = L_2$.*

Proof. Using Lemma 6.2.3, the proof is similar to [LP, Proposition 4.4]. □

In light of Lemma 6.2.5, we may understand the clusters in $\mathcal{LM}_n$ by forming "double-cover" clusters in $\mathcal{CM}_n$. A sequence μ of mutations in $\mathcal{LM}_n$ corresponds to a sequence μ' of twice as many mutations in $\mathcal{CM}_n$, where we impose the symmetry restriction, and the cluster variables in $\mathcal{LM}_n$ after applying μ are the symmetrizations of those in $\mathcal{CM}_n$ after applying μ'.

Lemma 6.2.6. *Any cluster S of $\mathcal{LM}_n$ consisting entirely of circular pairs is a positivity test.*

Proof. In $\mathcal{CM}_n$, the exchange polynomial has only positive coefficients, so each variable in any cluster is a rational function with positive coefficients in the variables of any other cluster. In particular, each non-symmetric circular pair in $V_n - (\emptyset; \emptyset)'$ is a rational function with positive coefficients in the variables of any cluster reachable under the symmetry restriction. Hence, by

Lemma 6.2.5, each circular pair in $\mathcal{D}_n$ can be written as a rational function with positive coefficients of the variables in S. The desired result follows easily. □

As with double wiring diagrams for totally positive matrices [FZP], and plabic graphs for the totally nonnegative Grassmannian [P], we now restricting ourselves to certain types of mutations in $\mathcal{LM}_n$. A natural choice is mutations with exchange relations of the form 5.1.3 or 5.1.4. These mutations keep us within clusters consisting entirely of circular minors, the "Plücker clusters."

We begin by restricting ourselves only to mutations with exchange relations of the form 5.1.3. Because the initial seed $\mathcal{S}_n$ consists only of solid circular pairs, we will only be able to mutate to other clusters consisting entirely of solid circular pairs. Our goal is to chracterize these clusters. We will be able to write down such a characterization using Corollary 6.2.16 and Lemma 6.2.5, and give a more elegant description of the clusters in Proposition 6.3.6.

Definition 6.2.7. Let $(P;Q)' = (p_1,\dots,p_k;q_1,\dots,q_k)'$ be a non-symmetric, non-empty circular pair. Define the statistics $D(P;Q)'$, $T(P;Q)'$, and $k(P;Q)'$ by:

$$D(P;Q)' = d_1(P;Q)' - d_2(P;Q)' = d(p_k,q_k) - d(q_1,p_1)$$

$$T(P;Q)' = \begin{cases} \frac{p_1+q_1}{2} \pmod{n} & \text{if } p_1 < q_1 \\ \frac{p_1+q_1+n}{2} \pmod{n} & \text{if } p_1 > q_1 \end{cases}$$

$$k(P;Q)' = |P|, \text{ that is, the size of } (P;Q)'$$

Remark 6.2.8. A non-symmetric solid circular pair $(P;Q)'$ is uniquely determined by the triple $(D(P;Q)', T(P;Q)', k(P;Q)')$. A necessary condition for a triple (D,T,k) to correspond to a non-symmetric solid circular pair is that $|D|+2k \le n$. When the terms are non-symmetric solid circular pairs, (5.1.3) can be written using these triples as:

$$(D-2,T,k)(D+2,T,k) = (D,T-1/2,k)(D,T+1/2,k) + (D,T,k+1)(D,T,k-1). \quad (6.2.9)$$

Definition 6.2.10. We call two non-symmetric solid circular pairs corresponding to the triples (D_1,T_1,k_1) and (D_2,T_2,k_2) **adjacent** if $T_1 = T_2$ and $|k_1 - k_2| = 1$, or $k_1 = k_2$ and $T_1 - T_2 \equiv \pm 1/2 \pmod{n}$. We call $(P;Q)'$ and $(R;S)'$ **diagonally adjacent** if there are two non-symmetric solid circular pairs $(A;B)',(C;D)'$ which are both adjacent to both $(P;Q)'$ and $(R;S)'$. We call $(A;B)',(C;D)'$ the **connection** of $(P;Q)',(R;S)'$.

Note that, in the initial quiver $\mathcal{Q}_n$, adjacent and diagonally adjacent circular pairs correspond to vertices which are adjacent in particular ways. Specifically, adjacent circular pairs correspond to vertices which are adjacent on the same concentric circle, or along the same radial spoke of U'_n. Diagonally adjacent circular pairs correspond to those which are adjacent via all other edges, the "diagonal" edges. We can now classify clusters of $\mathcal{CM}_n$ which can be reached only using the mutations with exchange relation (5.1.3).

Definition 6.2.11. We call a set S of $2\binom{n}{2}+1$ non-symmetric solid circular pairs a **solid cluster** if it has the following properties:

- $(\emptyset;\emptyset)' \in S$,
- for each integer $1 \le k \le \frac{n}{2}$, and each $T \in \{0.5, 1, 1.5, 2, \dots, n\}$, unless $k = \frac{n}{2}$ and T is an integer, there is a D such that the non-symmetric solid circular pair corresponding to (D,T,k) is in S, and

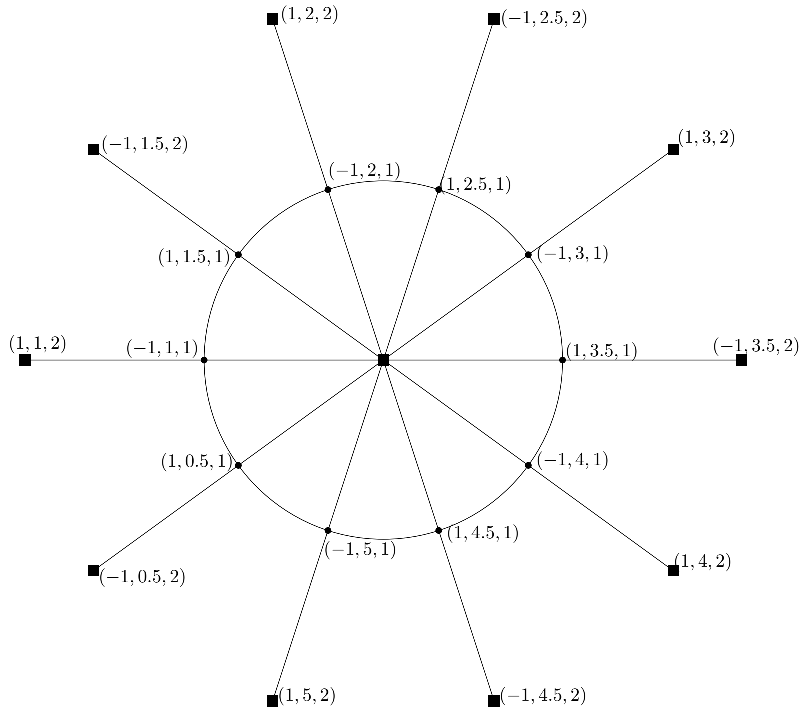

Figure 14: The graph U_5' with non-symmetric solid circular pairs labeled by triples (D, T, k). In $\mathcal{Q}_n$, the edges alternate directions around each non-frozen vertex. Compare to Figure 13.

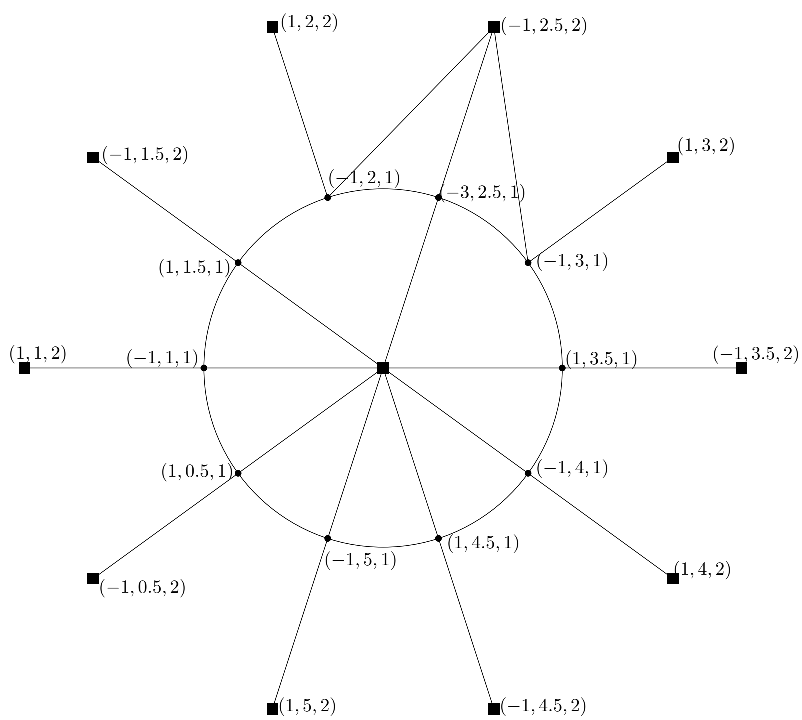

Figure 15: The graph U_5' after a mutation at $(1, 2.5, 1)$, with non-symmetric solid circular pairs labeled by triples (D, T, k). In the quiver, the edges alternate directions around each non-frozen vertex.

- if $(P;Q)', (R;S)' \in S$ and $(P;Q)'$ is adjacent to $(R;S)'$, then $|D(P;Q)' - D(R;S)'| = 2$.

Remark 6.2.12. There is a natural embedding of a solid cluster S in the plane, similar to our embedding of U'_n. We place $(\emptyset;\emptyset)'$ at any point, and then pairs of size k on the circle of radius k centered at that point. Moreover, we place adjacent pairs of the same size consecutively around each circle, and adjacent pairs of different sizes collinear with $(\emptyset;\emptyset)'$.

Definition 6.2.13. For a solid cluster S of $\mathcal{CM}_n$ and associated quiver $\mathcal{B}$, we call (S, B) a **solid seed** if it has the following properties:

- vertices corresponding to maximal non-symmetric solid circular pairs are frozen,
- there is an edge between any pair of adjacent vertices that are not both frozen,
- there is an edge between diagonally adjacent vertices $(P;Q)', (R;S)'$ if their connection $(A;B)', (F;G)'$ satisfies $|D(A;B)' - D(F;G)'| = 4$,
- there is an edge from a size 1 vertex $(P;Q)'$ to $(\emptyset;\emptyset)'$ if it would make the degree of $(P;Q)'$ even,
- all edges of $\mathcal{B}$ are in drawn in one of the four ways described above, and
- all edges are oriented so that, in the embedding described in Remark 6.2.12, edges alternate between in- and out-edges around any non-frozen vertex.

If, furthermore, $s \in S$ if and only if $C(s) \in S$, or equivalently, the non-symmetric solid circular pair corresponding to (D, T, k) is in S if and only if that corresponding to $(-D, T, k)$ is, then we call (S, b) a **symmetric solid seed**.

See, for example, Figure 15.

Remark 6.2.14. In a solid seed $(S, \mathcal{B})$, a variable $(P;Q)' \in S$ has an exchange polynomial of the form (5.1.3) whenever its corresponding vertex in $\mathcal{B}$, and has edges to the vertices corresponding to its four adjacent variables in $\mathcal{B}$, and no other vertices.

Lemma 6.2.15. *In $\mathcal{CM}_n$, from the initial seed with cluster V'_n and quiver $\mathcal{Q}'_n$, mutations of the form (5.1.3) may be applied to obtain the seed $(W'_n, \mathcal{R}'_n)$ if and only if $(W'_n, \mathcal{R}'_n)$ is a solid seed. Here, we do not impose the symmetry restriction.*

Proof. First, assume that $(W'_n, \mathcal{R}'_n)$ via mutations of the form (5.1.3). First, it is easy to check that $(V'_n, \mathcal{Q}'_n)$ is a solid seed. Then, our mutations do indeed turn non-symmetric solid circular pairs into other non-symmetric solid circular pairs. Furthermore, when we perform a mutation of the form (6.2.9) at the vertex v, the values of T and k do not change, and the value of D changes from being either 2 more than the values of D at the vertices adjacent to v to being 2 less, or vice versa. Hence, the resulting seed is also solid, so, by induction, $(W'_n, \mathcal{R}'_n)$ is solid.

Conversely, assume $(W'_n, \mathcal{R}'_n)$ is solid. We begin by noting that, by Remark 6.2.14, whenever the four terms on the right hand side of (6.2.9) and one term on the left hand side are in our cluster, then we can perform the corresponding mutation.

Now, define $(I'_n, \mathcal{QI}'_n)$ to be the unique symmetric solid seed such that, for each $(P;Q)' \in I'_n$, $D(P;Q)' \in \{-2, -1, 0, 1, 2\}$. For any solid seed $(W'_n, \mathcal{R}'_n)$, we give a mutation sequence $\mu_{(W'_n, \mathcal{R}'_n)}$ using only mutations of the form (6.2.9) that transforms $(W'_n, \mathcal{R}'_n)$ into (I'_n, QI'_n). Hence, we will be able to get from the seed $(V'_n, \mathcal{Q}'_n)$ to $(W'_n, \mathcal{R}'_n)$ by performing $\mu_{(V'_n, \mathcal{Q}'_n)}$, followed by $\mu_{(W'_n, \mathcal{R}'_n)}$ in reverse order.

It is left to construct the desired mutation sequence. We define $\mu_{(W'_n,\mathcal{R}'_n)}$ as follows: while the current seed is not $(I'_n, \mathcal{QI}'_n)$, choose a vertex v of the quiver, with associated cluster variable $(P;Q)'$, for which the value of $|D(P;Q)'|$ is maximized. We must have $|D(P;Q)'| > 2$, and by maximality, for each vertex $(R;S)'$ adjacent to $(P;Q)'$, we must have $|D(R;S)'| = |D(P;Q)'| - 2$. Hence, we can mutate at $(P;Q)$ to reduce $|D(P;Q)'|$ by at least 2. This process may be iterated to decrease the sum, over all cluster variables $(P;Q)'$ in our seed, of the $|D(P;Q)'|$, until we reach the seed $(I'_n, \mathcal{QI}'_n)$. The proof is complete. □

Corollary 6.2.16. *In $\mathcal{CM}_n$, from our initial seed with cluster V'_n and quiver $\mathcal{Q}'_n$, we can apply symmetric pairs of mutations (that is, mutations with the symmetry restriction) of the form (5.1.3) to obtain the seed $(W'_n, \mathcal{R}'_n)$ if and only if $(W'_n, \mathcal{R}'_n)$ is a symmetric solid seed.*

Proof. The proof is almost identical to that of Lemma 6.2.15. It suffices to note that in a symmetric solid seed, $(P;Q)'$ has a maximal value of $|D|$ if and only if $c(P;Q)'$ does, so the mutation sequence $\mu_{(W'_n,\mathcal{R}'_n)}$ can be selected to obey the symmetry restriction. □

Using Corollary 6.2.16 and Lemma 6.2.5, we could also prove a similar result for the LP algebra $\mathcal{LM}_n$. However, we will wait to do so until Lemma 6.3.6 where we will describe it more elegantly using the notion of weak separation.

We have now have the required machinery to prove our main theorem of this section.

Theorem 6.2.17. *Fix a symmetric $n \times n$ matrix M of distinct indeterminates. Then, the $\mathbb{C}$-Laurent phenomenon algebra $\mathcal{LM}_n$ is isomorphic to the polynomial ring (over $\mathbb{C}$) on the $\binom{n}{2}$ non-diagonal entries of M.*

Proof. We may directly apply the analogue of [FP, Proposition 3.6] for LP algebras (for which the proof is identical) to $\mathcal{LM}_n$ as defined in Definition 6.2.2. It is well-known that minors of a matrix of indeterminates are irreducible, so we immediately have that all of our seed variables are pairwise coprime. We also need to check that each initial seed variable is is coprime to the variable obtained by mutating its associated vertex in $\mathcal{Q}_n$, and that this new variable is in the polynomial ring generated by the non-diagonal entries of M. For non-limiting (and non-frozen) minors, this is clear, because each such mutation replaces a minor with another minor via (5.1.3). For limiting minors, this is not the case, and we defer the proof to Appendix C.

It remains to check, then that each of the $n(n-1)$ non-diagonal entries of M' appear as cluster variables in some cluster of $\mathcal{CM}_n$. However, because 1×1 minors are solid, the result is immediate from Lemma 6.2.16 and Lemma 6.2.5. □

Because all 1×1 minors are solid, mutations of the form (5.1.3) were sufficient to establish Theorem 6.2.17. Once we allow mutations of the form (5.1.4), the clusters become more difficult to describe. However, let us propose the following conjecture, which has been established computationally for $n \leq 6$.

Conjecture 6.2.18. *Every cluster of $\mathcal{LM}_n$ consisting entirely of circular pairs can be reached from the initial cluster using only mutations of the form (5.1.3) or (5.1.4).*

6.3 Weak Separation

We next introduce an analogue of weakly separated sets from [LZ] for circular pairs. We recall the definition used in [S], [OSP], which is more natural in this case[1]:

[1] Our definition varies slightly from that in the literature in the case where the two sets do not have the same size.

Definition 6.3.1. Two sets $A, B \subset [n]$ are **weakly separated** if there are no $a, a' \in A \setminus B$ and $b, b' \in B \setminus A$ such that $a < b < a' < b'$ or $b < a < b' < a'$.

Our analogue is as follows:

Definition 6.3.2. Two circular pairs $(P; Q)$ and $(R; S)$ are **weakly separated** if $P \cup R$ is weakly separated from $Q \cup S$, and $P \cup S$ is weakly separated from $Q \cup R$.

Remark 6.3.3. Note that $(P; Q)$ is weakly separated from itself and from $(\widetilde{Q}; \widetilde{P})$. Furthermore, $(P; Q)$ is weakly separated from $(R; S)$ if and only if $(\widetilde{Q}; \widetilde{P})$ is, so under the convention $(P; Q) = (\widetilde{Q}; \widetilde{P})$, weak separation is well-defined.

Conjecture 6.3.4. *Let C be a set of circular minors, for an $n \times n$ generic response matrix.*

P*: C is a minimal positivity test.*

S*: C is a maximal set of pairwise weakly separated circular pairs.*

C*: C is a cluster of $\mathcal{LM}_n$.*

Conjecture 6.3.4 has been computationally verified for $n \leq 6$. We now prove various weak forms of this conjecture. First, for all clusters C of $\mathcal{LM}_n$ that are reachable from the initial seed via Grasmann-Plücker Relations (cf. Conjecture 6.2.18), the elements of C are pairwise weakly separated:

Proposition 6.3.5. *If C is a set of pairwise weakly separated circular pairs such that, for some substitution of values into (5.1.3) or (5.1.4), all the terms on the right hand side, and one term $(P; Q)$ on the left hand side, are in C, then the remaining term $(R; S)$ on the left hand side is weakly separated from all of $C - (P; Q)$.*

Proof. Let a, b, c, d be as in (5.1.3) or (5.1.4). It is clear that $(R; S)$ can only be non-weakly separated from an element of $C - (P; Q)$, if a, b, c, d are boundary vertices forcing the non-weak separation.. However, this is easily seen to be impossible. □

When restricting ourselves to clusters of solid minors, the analogue of Corollary 6.2.16 for $\mathcal{LM}_n$ matches exactly with a weak form of the equivalence S $\Leftrightarrow$ C in Conjecture 6.3.4.

Proposition 6.3.6. *A set C of solid circular pairs can be reached (as a cluster) from the initial cluster $\mathcal{S}_n$ in $\mathcal{LM}_n$ using only mutations of the form (5.1.3) if and only if C is a set of $\binom{n}{2}$ pairwise weakly separated solid circular pairs.*

Proof. The elements of the initial cluster $\mathcal{S}_n$ in $\mathcal{LM}_n$, which consists of the diametric pairs $\mathcal{D}_n$, are easily seen to be pairwise weakly separated. Then, by Proposition 6.3.5, any cluster we can reach from $\mathcal{S}_n$ using only mutations of the form (5.1.3) must also be pairwise weakly separated.

Conversely, consider any set C of $\binom{n}{2}$ pairwise weakly separated solid circular pairs. Let $C' = \{(P; Q)' \mid (P; Q) \in C\}$, and notice that $|C'| = 2\binom{n}{2}$. By Corollary 6.2.16 and Lemma 6.2.5, it is enough to prove that that $C' \cup \{(\emptyset; \emptyset)\}$ is a solid cluster (see Definition 6.2.11) in $\mathcal{CM}_n$. From here it will follow by definition that C' is a symmetric solid seed, meaning C can be reached from $\mathcal{S}_n$ in $\mathcal{LM}_n$ using only mutations of the form (5.1.3), as desired.

Before proceeding, it is straightforward to check that circular pairs $(P; Q) = (p_1, \ldots, p_a; q_1, \ldots, q_a)$ and $(R; S) = (r_1, \ldots, r_b; s_1, \ldots, s_b)$ are weakly separated if and only if the following four intersections are non-empty:

$$\{p_1, q_1\} \cap (R \cup S), \ \{p_a, q_a\} \cap (R \cup S), \ \{r_1, s_1\} \cap (P \cup Q), \ \{r_b, s_b\} \cap (P \cup Q).$$

We now prove that $C' \cup \{(\emptyset;\emptyset)\}$ is a solid cluster. First, notice that if non-symmetric circular pairs $(P;Q)' = (p_1,\dots,p_a;q_1,\dots,q_a)'$ and $(R;S)' = (r_1,\dots,r_b;s_1,\dots,s_b)'$ are such that $k(P;Q)' = k(R;S)'$ and $T(P;Q)' = T(R;S)'$, but $D(P;Q)' \neq D(R;S)'$, then $(P;Q)$ and $(R;Q)$ are not weakly separated. Hence, at most one of $(P;Q)'$ and $(R;S)'$ is in C'. As there are exactly $2\binom{n}{2}$ choices of T and k that give valid non-symmetric solid circular pairs, there must be one element of C' corresponding to each choice of (T,k).

Second, consider any adjacent $(P;Q)'$ and $(R;S)'$ in C'. Without loss of generality, one of

- $k(P;Q)' = k(R;S)'$ and $T(P;Q)' = T(R;S)' + \frac{1}{2}$,
- $T(P;Q)' = T(R;S)'$ and $k(P;Q)' = k(R;S)' + 1$.

holds. In either case, because $(P;Q)'$ and $(R;S)'$ are weakly separated, we can see that $|D(P;Q)' - D(R;S)'| = 2$. It follows that $C' \cup \{(\emptyset;\emptyset)\}$ is a solid cluster, so we are done. □

We now relate C and P. Recall that, by Lemma 6.2.6, any if C satisfies C, then C is a positivity test. Furthermore, $|C| = \binom{n}{2}$. We can prove, similarly to [LZ, Theorem 1.2], that:

Proposition 6.3.7. *If C satisfies S, then $|C| \leq \binom{n}{2}$.*

In fact, we can prove a slightly stronger result by interpreting a circular pair as a set of edges.

Definition 6.3.8. For a circular pair $(P;Q) = (p_1,\dots,p_k;q_1,\dots,q_k)$, define $E(P;Q) = \{\{p_i,q_i\} \mid i \in \{1,\dots,k\}\}$ (cf. Definition 5.2.18). Similarly, for a set $D \subset \{\{i,j\} \mid 1 \leq i < j \leq n\}$ of edges such that no two edges in D cross, let $P(E)$ be the circular pair for which $E(P(D)) = D$.

Proposition 6.3.9. *If C is a set of pairwise weakly separated circular pairs with*

$$E = \bigcup_{(P;Q)\in C} E(P;Q),$$

then $|C| \leq |E|$.

Proof. Procced by induction on $|E|$. The case $|E| = 0$ is trivial, so assume the result is true for $|E| < m$. Suppose that we have C, E with $|E| = m$, and assume for sake of contradiction that $|C| > m$. Choose some $\{a,b\} \in E$ such that, for any other $\{c,d\} \in E$, c and d do not both lie on the arc drawn from a to b in the clockwise direction (this arc is taken to include both a and b). Now, letting $E' = E \setminus \{\{a,b\}\}$, define the projection map $J : 2^E \to 2^{E'}$ by:

$$J(D) = \begin{cases} D \setminus \{\{a,b\}\} & \text{if } \{a,b\} \in D, \\ D & \text{otherwise.} \end{cases}$$

We may define J for circular pairs analogously: $J(P;Q) = V(J(E(P;Q)))$, and let $C' = \{J(P;Q) \mid (P;Q) \in C\}$. Let us now prove two lemmas.

Lemma 6.3.10. *The elements of C' are pairwise weakly separated.*

Proof. Assume, for sake of contradiction, that we have $(P;Q),(R;S) \in C$, such that $J(P;Q)$ and $J(R;S)$ are not weakly separated. If $\{a,b\} \in E(P;Q), E(R;S)$ or $\{a,b\} \notin E(P;Q), E(R;S)$, the claim is clear. Thus, we may assume without loss of generality, that $a \in P$, $b \in Q$, and $\{a,b\} \notin E(R;S)$. Because $J(P;Q)$ and $J(R;S)$ are not weakly separated, suppose, without loss of generality, that $w, y \in R \cup (P \setminus \{a\})$ and $x, z \in S \cup (Q \setminus \{b\})$ such that w, x, y, z are in clockwise order, and furthermore $w, y \notin S \cup (Q \setminus \{b\})$ and $x, z \notin R \cup (P \setminus \{a\})$. Note that, if $a, b \notin \{w, x, y, z\}$, then w, x, y, z would also show that $(P; Q)$ is not weakly separated from $(R; S)$.

Assume that $a = w$; the other cases are similar. First, suppose that $b \neq x$. Then, we must have $w \in R$, and we obtain a similar contradiction to before. On the other hand, if $b = x$, then $a = w \in R$ and $b = x \in S$. But, since $\{a, b\} \notin E(R; S)$, a and b are in distinct non-intersecting edges in $E(R; S)$. Because these edges are also in E, we have a contradiction of the construction of $\{a, b\}$. The lemma follows. □

Lemma 6.3.11. *There is at most one $(P; Q) \in C$ with $\{a, b\} \in E(P; Q)$ and $J(P; Q) \in C$.*

Proof. Assume for sake of contradiction that we have distinct circular pairs $(P; Q), (R; S) \in C$, with $\{a, b\} \in E(P; Q), E(R; S)$, and $J(P; Q), J(R; S) \in C$. Without loss of generality, assume that $a \in P, R$ and $b \in Q, S$. By the fact that $J(P; Q), J(R; S) \in C$ and the construction of $\{a, b\}$, there exist two points u, v on the clockwise arc from b to a not containing its endpoints, which are both in exactly one of P, Q, R, S.

First, if $u \in Q$ and $v \in P$, then the points a, b, u, v force $J(P; Q)$ and $(R; S)$ not to be weakly separated. However, $J(P; Q), (R; S) \in C$, so we have a contradiction.

If $u \in Q$ and $v \in R$, then we get a similar contradiction if $d(b, v) < d(b, u)$, so we have that a, b, u, v are in clockwise order. Because $|Q| = |P|$ and $|R| = |S|$, there must be an x such that $x \in P \cup S$ but $x \notin R \cup Q$.

We have four cases for the position of x, relative to a, b, u, v. If a, x, b are in clockwise order, then we get a contradiction of our construction of $\{a, b\}$. If b, x, u are in clockwise order, then a, b, x, u either contradicts that $(P; Q)$ is a circular pair or that $(P; Q)$ is weakly separated from $J(R; S)$. The case in which v, x, a are in clockwise order is similar. Finally, if u, x, v are in clockwise order, then either a, b, u, x or a, b, x, v contradicts that either $(P; Q)$ is weakly separated from $J(R; S)$ or that $J(P; Q)$ is weakly separated from $(R; S)$.

The other cases follow similarly. □

We can now finish the proof of Proposition 6.3.9. By Lemma 6.3.10, the elements of C' are pairwise weakly separated, and we also have $E' = \bigcup_{(P;Q) \in C'} E(P; Q)$. Thus, by the inductive hypothesis, $|C'| \leq |E'| = |E| - 1$. However, it is easy to see from Lemma 6.3.11 that $|C'| \geq |C| - 1$, so the induction is complete. □

Now, Proposition 6.3.7 follows easily by taking $E = \{\{i, j\} \mid 1 \leq i < j \leq n\}$ in Proposition 6.3.9. Proposition 6.3.9 also has another natural corollary:

Corollary 6.3.12. *For any set S of pairwise weakly separated circular pairs, there is an injective map $e : S \to \{\{i, j\} \mid 1 \leq i < j \leq n\}$ such that $e(P; Q) \in E(P; Q)$ for each $(P; Q) \in S$.*

Proof. Proposition 6.3.9 gives exactly the condition required to apply Hall's marriage theorem. □

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A Proofs of Lemmas 4.2.5 and 4.2.6

Recall the definitions of D_n, E_n, X_n from §4.2. We will prove Lemmas 4.2.5 and 4.2.6, that $D_n/X_n \sim \sqrt{e}-1$ and $E_n/X_n \sim 0$, respectively.

Proof of Lemma 4.2.5. We may as well consider $D_n - 1 = \sum_{j=1}^{n-2} \binom{n}{j} X_{n-j}$. Using the notation $Q_i = X_i/X_{i-1}$, as in the proof of Lemma 4.2.2, we have

$$\begin{aligned}
\frac{\sum_{j=1}^{n-2} \binom{n}{j} X_{n-j}}{X_n} &= \sum_{j=1}^{n-2} \frac{1}{j!} \cdot \frac{n(n-1)\cdots(n-j+1)}{Q_nQ_{n-1}\cdots Q_{n-j+1}} \\
&= \sum_{j=1}^{n-2} \frac{1}{2^j j!} \cdot \frac{2n(2n-2)\cdots(2n-2j+2)}{Q_nQ_{n-1}\cdots Q_{n-j+1}} \\
&= \sum_{j=1}^{n-2} \frac{1}{2^j j!} + \sum_{j-1}^{n-2} \frac{1}{2^j j!} \left(\frac{2n(2n-2)\cdots(2n-2j+2)}{Q_nQ_{n-1}\cdots Q_{n-j+1}} - 1 \right).
\end{aligned}$$

As $n \to \infty$, first summand above converges to $\sqrt{e}-1$, so it is left to check that the second summand converges to zero.

Note that, by Lemma 4.2.2,

$$\frac{2n(2n-2)\cdots(2n-2j+1)}{Q_nQ_{n-1}\cdots Q_{n-j+2}} > 1.$$

Now,

$$\begin{aligned}
0 &< \sum_{j=1}^{n-2} \frac{1}{2^j j!} \left(\frac{2n(2n-2)\cdots(2n-2j+2)}{Q_nQ_{n-1}\cdots Q_{n-j+1}} - 1 \right) \\
&< \sum_{j=1}^{n-5} \frac{1}{2^j j!} \left(\frac{2n(2n-2)\cdots(2n-2j+2)}{Q_nQ_{n-1}\cdots Q_{n-j+1}} - 1 \right) \\
&\quad + Kn \left[\frac{1}{2^{n-4}(n-4)!} + \frac{1}{2^{n-3}(n-3)!} + \frac{1}{2^{n-2}(n-2)!} \right],
\end{aligned}$$

for some positive constant K, because

$$\begin{aligned}
&\frac{2n(2n-2)\cdots(2n-2j+2)}{Q_nQ_{n-1}\cdots Q_{n-j+1}} \\
&< 2n \cdot \frac{2n-2}{Q_n} \cdot \frac{2n-4}{Q_{n-1}} \cdots \frac{2n-2j+2}{Q_{n-j+2}} \cdot \frac{1}{Q_{n-j+1}} \\
&< Kn,
\end{aligned}$$

as by Lemma 4.2.2, all but a fixed number of the fractions are less than 1, and those which are not are constant. It is then easy to see that the term

$$Kn \left[\frac{1}{2^{n-4}(n-4)!} + \frac{1}{2^{n-3}(n-3)!} + \frac{1}{2^{n-2}(n-2)!} \right]$$

goes to zero as $n \to \infty$. Now, applying Lemma 4.2.2 again (noting that the indices are all at least 6),

$$\sum_{j=1}^{n-5} \frac{1}{2^j j!} \left(\frac{2n(2n-2)\cdots(2n-2j+2)}{Q_nQ_{n-1}\cdots Q_{n-j+1}} - 1 \right)$$

$$
\begin{aligned}
&< \sum_{j=1}^{n-5} \frac{1}{2^j j!}\left(\frac{2n(2n-2)\cdots(2n-2j+2)}{(2n-1)(2n-3)\cdots(2n-2j+1)}-1\right)\\
&< \sum_{j=1}^{n-5} \frac{1}{2^j j!}\cdot\left(\frac{2n}{2n-2j+1}-1\right)\\
&< \sum_{j=1}^{n-5} \frac{1}{2^j j!}\cdot\frac{2j-1}{2n-2j+1}\\
&< \sum_{j=1}^{n-5} \frac{1}{2^{j-1}(j-1)!}\cdot\frac{1}{2n-2j+1}.
\end{aligned}
$$

It is enough to show that the above sum goes to zero as $n\to\infty$. To do this, we split it in to two sums:

$$
\begin{aligned}
&\sum_{j=1}^{n-5} \frac{1}{2^{j-1}(j-1)!}\cdot\frac{1}{2n-2j+1}\\
&= \sum_{1\le j<n/2} \frac{1}{2^{j-1}(j-1)!}\cdot\frac{1}{2n-2j+1} + \sum_{n/2\le j\le n-5} \frac{1}{2^{j-1}(j-1)!}\cdot\frac{1}{2n-2j+1}\\
&< \sum_{1\le j<n/2} \frac{1}{2^{j-1}(j-1)!}\cdot\frac{1}{n} + \sum_{n/2\le j\le n-5} \frac{1}{2^{j-1}(j-1)!}\\
&< \frac{\sqrt{e}}{n} + \sum_{n/2\le j\le n-5} \frac{1}{2^{j-1}(j-1)!}.
\end{aligned}
$$

The first summand clearly tends to zero as $n\to\infty$. The rest of the sum must tend to zero as well, as it is the tail of a convergent sum, so the proof is complete. □

Proof of Lemma 4.2.6. First, note that by Corollary 4.2.4, X_i is within a (positive) constant factor of $(2i-1)!!$ for each i. Thus, to prove that $E_n/X_n \sim 0$, we may as well prove that

$$
n\sum_{j=2}^{n-2} \frac{(2j-1)!!(2n-2j-1)!!}{(2n-1)!!} \sim 0.
$$

It is straightforward to check that the largest terms of the sum are when $j=2,n-2$, and these terms are of inverse quadratic order. Thus,

$$
n\sum_{j=2}^{n-2} \frac{(2j-1)!!(2n-2j-1)!!}{(2n-1)!!} < nO(n^{-2}) = O(n^{-1}),
$$

and the conclusion follows. □

B Proof of Theorem 5.1.7 in the BSP case

We now finish the proof of Theorem 5.1.7. Recall that S_0 is an electrical positroid for which no circular planar graph G has $\pi(G)=S_0$, and that S_0 is chosen to be maximal among electrical positroids with this property. By assumption, S_0 has the $(i,i+1)$-BEP for each i, and does not have the 1-BSP. Furthermore, recall the construction of S_1 from the end of §5.2. Then, we have:

Lemma B.1. *S_1 is an electrical positroid, and has the 1-BSP.*

Proof. Straightforward. ☐

By assumption, $S_1 = \pi(G_1)$, for some circular planar graph G_1, which has a boundary spike at 1. Let G_0 be graph obtained after contracting the boundary spike in G_1. We will prove that $\pi(G_0) = S_0$, which will yield the desired contradiction. The proof is similar to the case handled in §5.2.

Recall the notation from Definition 5.2.17, where we let $A_{k,\ell}$ denote the sequence $a_k, \dots, a_\ell$.

Definition B.2. A circular pair $(P;Q) = (A_{1,n}; B_{1,n})$ is said to be **incomplete** if $(P;Q) \notin S$ but $(P;Q') = (A_{1,n}; 1, B_{2,n}) \in S$ and $(P';Q) = (1, A_{1,n}; B_{1,n}) \in S$. If, on the other hand, $(P;Q) \in S$ in addition to $(P;Q')$ and $(P';Q)$, $(P;Q)$ is said to be **complete**.

We also define the set $\mathcal{P}$ of **primary** circular pairs as in §5.2, where we take circular pairs of the form $(P;Q) = (A_{1,n}; B_{1,n})$ with the property that $(A_{1,n}; 1, B_{2,n}), (1, A_{2,n}, B_{2,n}) \in S$. It is easy to see that the analogue of Lemma 5.2.9 holds when $(P;Q)$ is incomplete. Then, because S_0 has all BEPs, the primary circular pairs $(A_{1,n}; B_{1,n})$ will all have $a_1 = 2, b_1 = n$. We now prove a series of lemmas, mirroring those in 5.2.

Lemma B.3. *For an incomplete circular pair $(P;Q) = (A_{1,n}, B_{1,n})$, any electrical positroid Z satisfying $S_0 \cup \{(P;Q)\} \subset Z \subset S_1$ con $(P+a; Q+b)$ with $a > a_{|P|}, b < b_{|P|}$ when $(P+a;Q+b)$ is incomplete.*

Proof. First by Axiom 1a in Z, $(P;Q) \in Z$ and $(P+a-a_1; Q+b-b_1) \in Z$ implies that we either have our claim, or we have $(P+a-a_1;Q) \in Z$ and $(P;Q+b-b_1) \in Z$. We first apply Axiom 2a to $1, a_1, a; b_1$ on the circular pair $(P+a+1, Q+b)$. We have $(P+1+a-a_1; Q+b) \in S$ by definition, and we also have $(P;Q+b-b_1) \in Z$, so this implies that we either have our claim, or we have $(P+1;Q+b) \in Z$. The latter then implies that $(P;Q+b-b_1) \in S$. A similar argument for Axiom 2a on $a_1; 1, b_1, b$ gives either our claim or that $(P+a-a_1;Q) \in S$. Then, Axiom 2a implies that $(P;Q) \in S$, a contradiction. Thus, we have our claim. ☐

Lemma B.4. *For any incomplete circular pair $(P;Q)$, there exists a circular pair $(P';Q') \in \mathcal{P}$ such that any electrical positroid Z satisfying $S_0 \cup \{(P;Q)\} \subset Z \subset S_1$ contains $(P;Q)$.*

Proof. Proceed by induction on i, where i is such that the first i connections (see Definition 5.2.18) of $(P;Q)$ are the same as those of a primary circular pair. The base case may be handled similarly as in the proof of Lemma 5.2.19. Now, let $(R;T) = (A_{1,n}; B_{1,n})$ be the primary circular pair such that if $(P;Q) = (C_{1,m}; D_{1,m})$, then $a_i \le c_i, b_i \le d_i$ for all i; $(R;T)$ exists by an identical argument as in the proof of Lemma 5.2.19. Call $(R;T)$ the primary circular pair associated to $(P;Q)$.

Recalling that $a_1 = 2, b_1 = n$, we first need to show that $(2, A_{2,i+1}, C_{i+2,m}; 1, B_{2,i+1}, D_{i+2,m}) \in S$. The same result replacing $(2,1)$ with $(1,n)$ would follow from an identical argument. By the definition of the $(1,2)$-BEP, we need to show that $(A_{2,i+1}, C_{i+2,m}; B_{2,i+1}, D_{i+2,m}) \in S$. In the case that $i > 2$, we do so by applying Lemmas 5.2.9 and Lemma 5.2.13. In the case that $i = 0$, we may apply the Subset Axiom. Finally, in the case that $i = 1$, we may apply Lemma 5.2.11.

We now claim that, if $(A;B) = (A_{1,i+1}, C_{i+2,m}; B_{1,i+1}, D_{i+2,m}) \in Z$, then $(P';Q') \in Z$. The lemma will then follow, because $(A;B)$ and $(P;Q)$ are easily seen to have the same primary associated circular pair. If $i > 0$, the proof of the claim is identical to that of 5.2.19, so assume that $i = 0$. In this case, we have $(a_1, C_{2,m}; b_1, D_{2,m}), (C_{1,m}; 1, D_{2,m}) \in Z$. If $a_1 \ne c_1$, Axiom 1b implies that we have $(C_{1,m}; b_1, D_{2,m}) \in Z$. Then, as $(1, C_{2,m}; D_{1,m}) \in Z$, we are done by Axiom 1b. ☐

Lemma B.5. *There is exactly one circular pair in $\mathcal{P}$ that does not lie in S_0, which we call the S_0-**primary circular pair.***

Proof. The argument is the same to that of Lemma 5.2.20. □

Lemma B.6. *For any incomplete circular pair $(P;Q)$, any electrical positroid Z satisfying $S_0 \cup \{(P;Q)\} \subset Z \subset S_1$ contains the S_0-primary circular pair.*

Proof. Proceed by retrograde induction on i, where i is such that the first i connections of $(P;Q)$ are the same as those of the S_0-primary circular pair. If $i > 0$, we can argue exactly as in the proof of Lemma 5.2.21. Thus, assume that $i = 0$.

Let $(R;T) = (A_{1,n}; B_{1,n})$ be the S_0-primary circular pair. By Lemma B.5, $(P;Q) = (C_{1,m}; D_{1,m})$ has the property that $a_i \leq c_i, b_i \leq d_i$ for all i. Also, by how the construction of S_1, for any circular pair (C,D), $(C+1;D+n) \in S_0 \Leftrightarrow (C+1;D+n) \in S_1$ and $(C+2;D+1) \in S_0 \Leftrightarrow (C+2;D+1) \in S_1$. Therefore, it follows that $(P+1;Q+n), (P+2;Q+1) \notin Z$. Then, by Axiom 1a, $(2, C_{2,m}; D_{1,m} \in Z$, and another application of Axiom 1a yields $(2, C_{2,m}; n, D_{2,m}) \in Z$, completing the proof. □

Lemma B.7. *For any two incomplete circular pairs $(P;Q)$ and $(P';Q')$ any electrical positroid Z containing S and contained in S' with $(P;Q)$ must contain $(P';Q')$.*

Proof. By Lemma B.4, Z contains the primary circular pair. The claim then follows by Lemma B.6. □

Lemma B.8. *Let $T = S_0 \cap S_0'$. Then, T is an electrical positroid.*

Proof. The proof follows the same outline as that of Lemma 5.2.23; here, we verify that T satisfies each electrical positroid axiom. By construction, S_0 and S_0' only differ in the circular pairs $(P;Q)$ for which $(P - a_1 + 1; Q), (P; Q - b_1 + 1) \in S_0 \cap S_0'$. In particular, $1 \neq P, Q$. T is easily seen to satisfy the electrical positroid axioms other than 1a and 2a.

We first consider Axiom 1a: suppose that $(P-a;Q-c), (P-b;Q-d) \in T$; we show that either $(P-a;Q-d), (P-b;Q-c) \in T$ or $(P;Q), (P-a-b;Q-c-d) \in T$. We have the following cases:

- $a, c \neq 1$. Suppose that $(P-a;Q-d) \in S_0'$. Then, either $(P-a;Q-d) \in S_0$ or $(P-a;Q-b_1-d+1) \in S_0$. Axiom 1b applied to $(P;Q-d+1)$ with $a, b, 1, b_1$ gives $(P-a;Q-d) \in S_0$. Similarly, the roles of S_0' and S_0 may be swapped, and we may apply the same argument with $(P-b;Q-c)$. Thus, either S_0 and S_0' both contain $(P-a;Q-d)$ and $(P-b;Q-c)$ or both do not, in which case they both contain $(P;Q)$ and $(P-a-b;Q-c-d)$.

- $a = 1, b \neq a_2$. Suppose that $(P-a;Q-d) \in S_0'$ and $(P-b;Q-c) \in S_0'$. Then, $(P-b;Q-c) \in S_0$. If $(P-a;Q-d) \in S_0$, we are done. Otherwise, if $(P-a;Q-d) \notin S_0$, as S_0 is an electrical positroid, we find $(P;Q) \in S_0$ and $(P-a-b;Q-c-d) \in S_0$. Then, we must have $(P;Q) \in S_0'$, in which case we are done, or either $(P-1-b;Q-c-d) \in S_0'$ or $(P-a_2-b;Q-c-d) \in S_0'$. In the latter case, Axiom 2b applied to $(p;Q-d)$ with $1, a_2, b, c$ gives that $(P-1-b;Q-c-d) \in S_0'$. Thus, S_0 and S_0' contain $(P-a-b;Q-c-d)$ and $(P;Q)$, so we are done in this case as well.

- The cases $a = 1, b = a_2, c \neq b_1$ and $a = 1, b = a_2, c = b_1$ are handled by similar logic; the details are omitted. The case $c = 1$ is symmetric with $a = 1$.

We now consider Axiom 2a: suppose that $(P-b;Q), (P-a-c;Q-d) \in T$; we show that either $(P-a;Q), (P-b-c;Q-d) \in S$ or $(P-c;Q), (P-a-b;Q-d) \in T$. We have the following cases:

- $a, d \neq 1, d \neq b_1$. If $(P-a;Q) \in S_0'$, then either $(P-a;Q) \in S_0$ or $(P-a;Q-b_1+1) \in S_0$. Then, an application of Axiom 1b to $(P;Q+1)$ with $a, b, 1, b_1$ yields that $(P-a;Q) \in S_0$. The same argument holds if $(P-a;Q) \in S_0$ to show that $(P-a;Q) \in S_0'$ does as well. Now, suppose $(P-b-c;Q-d) \in S'0$. Then, either $(P-b-c;Q-d) \in S_0$, which case we are done, or $(P-a_1-b-c+1;Q-d) \in S_0$. Then, the Subset Axiom, $(P-a_1-b-c;Q-b_1-d) \in S0$. Applying Axiom 1c to $(P-b;Q)$ with (a_1, c, b_1, d) then yields that $(P-b-c;Q-d) \in S_0$, as desired. The same argument holds if we swap the roles of S_0 and S_0'.
- $a \neq 1, d = b_1$. If $(P-a;Q) \in S_0'$, by the same argument as with the case $a, d \neq 1, d \neq b_1$, we have $(P-a;Q) \in S_0$. If $(P-b-c;Q-d) \in S_0'$ as well, then either $(P-b-c;Q-d) \in S_0$, in which case we are done, or $(P+1-a_1-b-c;Q-d) \in S_0$. In the latter case, because S_0 is an electrical positroid, $(P-c;Q) \in S_0$. Then, applying Axiom 2b to $(P+1-c;Q)$ with $(1, a_1, b, d)$ gives that $(P-b-c;Q-d) \in S_0$, so we are done.
- The cases $a = 1$ and $d = 1$ are handled with similar logic; the details are omitted.

We have exhausted all cases, so the proof of the lemma is complete. □

Now, by Lemma B.7, we have $S_0 = S_0'$, so the proof of Theorem 5.1.7 is complete.

C Mutation at limiting minors in $\mathcal{LM}_n$

Recall that, to complete the proof of Theorem 6.2.17, we need an additional technical result, which we state and prove here.

Proposition C.1. *From the initial cluster of $\mathcal{LM}_n$, mutating at a non-frozen limiting solid circular pair $(P;Q)$ gives a new cluster variable which is a polynomial in the entries of M, and relatively prime to $(P;Q)$.*

Proof. Consider the limiting solid circular pair $(P;Q)$ of size k. Fix the ground set

$$(I;J) = \left(\frac{n}{2}, \frac{n}{2}+1, \ldots, \frac{n}{2}+k; 2, 1, n, n-1, \ldots, n-k+1\right),$$

so that Δ denotes the determinant of the submatrix of M with rows indexed by I and columns indexed by J. Furthermore, let

$$b = \frac{n}{2}+k-1,\ c = \frac{n}{2}+k,\ d = 2,\ e = 1,\ f = n-k+2,\ g = n-k+1.$$

Then, the cluster variable associated to the vertex $(P;Q)$ is $\Delta^{c,dg}$, and its corresponding exchange polynomial in the initial seed of $\mathcal{LM}_n$ is

$$\Delta^{\emptyset,d} \cdot \Delta^{c,fg} \cdot \Delta^{bc,deg} + \Delta^{\emptyset,g} \cdot \Delta^{c,de} \cdot \Delta^{bc,dfg}.$$

The new cluster variable from mutating at $(P;Q)$ is

$$\frac{\Delta^{\emptyset,d} \cdot \Delta^{c,fg} \cdot \Delta^{bc,deg} + \Delta^{\emptyset,g} \cdot \Delta^{c,de} \cdot \Delta^{bc,dfg}}{\Delta^{c,dg}} = \Delta^{b,de} \cdot \Delta^{c,fg} - \Delta^{b,fg} \cdot \Delta^{c,de},$$

where the last equality may be checked directly. We wish to show that $\Delta^{c,fg} - \Delta^{b,fg} \cdot \Delta^{c,de}$ is relatively prime to $\Delta^{c,dg}$, which is irreducible, so it is enough to check that $\Delta^{c,dg}$ does not divide $\Pi = \Delta^{b,de} \cdot \Delta^{c,fg} - \Delta^{b,fg} \cdot \Delta^{c,de}$.

Let us outline the argument. If it is the case that $\Delta^{c,dg}$ divides Π, then each term in the expansion of Π must be divisible by a monomial in the expansion of $\Delta^{c,dg}$. However, we claim that this cannot be true. Indeed, any monomial in the expansion of $\Delta^{c,dg}$ contains exactly one factor of a variable x_{bz} in the row of M corresponding to b, and the column of M corresponding to $z \neq 2, n-k+1$. However, it is easily checked that there are terms of Π, after expanding and collecting like terms, containing variables X_{bz} with $z = 2$, so the claim is established. □

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