Parametrizations of k-Nonnegative Matrices

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Abstract

Totally nonnegative (positive) matrices are matrices whose minors are all nonnegative (positive). We generalize the notion of total nonnegativity, as follows. A k-nonnegative (resp. k-positive) matrix has all minors of size k or less nonnegative (resp. positive). We give a generating set for the semigroup of k-nonnegative matrices, as well as relations for certain special cases, i.e. the $k = n - 1$ and $k = n - 2$ unitriangular cases. In the above two cases, we find that the set of k-nonnegative matrices can be partitioned into cells, analogous to the Bruhat cells of totally nonnegative matrices, based on their factorizations into generators. We will show that these cells, like the Bruhat cells, are homeomorphic to open balls, and we prove some results about the topological structure of the closure of these cells, and in fact, in the latter case, the cells form a Bruhat-like CW complex. We also give a family of minimal k-positivity tests which form sub-cluster algebras of the total positivity test cluster algebra. We describe ways to jump between these tests, and give an alternate description of some tests as double wiring diagrams.

1 Introduction

A totally nonnegative (respectively totally positive) matrix is a matrix whose minors are all nonnegative (respectively positive). Total positivity and nonnegativity are well-studied phenomena and arise in areas such as planar networks, combinatorics, dynamics, statistics and probability. The study of total positivity and total nonnegativity admit many varied applications, some of which are explored in "Totally Nonnegative Matrices" by Fallat and Johnson [5].

In this report, we generalize the notion of total nonnegativity and positivity as follows. A *k-nonnegative* (resp. *k-positive*) matrix is a matrix where all minors of order k or less are nonnegative (resp. positive). Because of our goal to produce results for k-nonnegative

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and k-positive matrices generalizing those that already exist for the semigroups of totally nonnegative and totally positive matrices, we will consider matrices in this report to be in the semigroup of square invertible matrices with either the k-nonnegativity or the k-positivity condition.

Following the lead of Fomin and Zelevinsky in [11], we consider the following two questions:

(1) How can k-nonnegative matrices be parametrized?

(2) What are positivity tests for k-positive matrices?

These questions are interesting, since answers to these questions would allow efficient generation and testing of matrices with this property. To answer these questions, at least to some extent, we provide factorizations and relations for certain k, describe Bruhat-like cells for k-nonnegativity and give a cluster algebra framework for finding k-positivity tests.

In Section 2 we detail our most general results on factorizations of k-nonegative matrices, after describing some relevant background. We describe partial factorizations of k-nonnegative matrices and partly characterize the locations of zero minors in the matrix.

In Section 3 we explore two special cases: $(n-1)$-nonnegative $n \times n$ matrices and $(n-2)$-nonnegative unitriangular $n \times n$ matrices, giving a specific generating set for the semigroup of k-nonnegative $n \times n$ real matrices as well as a set of relations. These generators have minimality properties, and our list of relations is complete enough to allow us to define analogues of Bruhat cells in these cases. These analogues share many properties of the standard cells of totally nonegative matrices. Section 3 concludes by detailing some progress towards understanding the topology of the cells of k-nonnegative matrices.

In Section 4, we describe the k-positivity cluster algebras. These are sub-cluster algebras of the well-known total positivity cluster algebra. We show a method of deriving k-positivity tests from a family of these cluster algebras as well as a convenient indexing of this family in terms of Young diagrams. A representative double wiring diagram is also given for each member of the family, which can be mutated according to a subset of the typical rules to get other k-positivity tests.

2 Preliminaries

2.1 Background

We begin by establishing some notation that will be used throughout the paper. For any matrix X, $X_{I,J}$ refers to the submatrix of X indexed by a subset of its rows I and a subset of its columns J, and $|X_{I,J}|$ will refer to the minor, i.e. the determinant of this submatrix. We say a minor $|X_{I,J}|$ is of *order* k if $|I| = |J| = k$. We also use $[n]$ to refer to the set $\{1, \dots, n\}$. Thus, $X_{[m],[m]}$ is the submatrix formed by taking the first m rows and columns.

The set of all totally nonnegative (resp. totally positive) matrices with real entries is closed under multiplication and, thus, forms a semigroup. This can be seen from the following identity:

Theorem 2.1 (Cauchy-Binet Identity)**.** *Let X be an $n \times m$ matrix and let Y be a $m \times n$ matrix with $n \leq m$. Then, we have*

$$\det(XY) = \sum_{J \subseteq [m], |J| = n} |X_{[n],J}||Y_{J,[n]}|$$

In particular, a consequence of the Cauchy-Binet Identity is that minors corresponding to submatrices of XY of a given order can be related to the minors corresponding to submatrices of X and Y of the same order. If we restrict our attention to invertible square matrices, then it follows that the set of invertible totally nonnegative (resp. totally positive) matrices also forms a semigroup. Similarly, the set of all upper unitriangular and the set of all lower unitriangular totally nonnegative matrices are both semigroups. Knowing this, two natural questions arise:

(1) How do we parametrize the set of all totally nonnegative invertible matrices?

(2) How do we test a given matrix in $\mathrm{GL}_n(\mathbb{R})$ for total positivity?

We first summarize the known answers to these two questions and then, in the remainder of the paper, discuss the answers to these questions in the context of k-nonnegative and k-positive invertible matrices. Most of the following summary can be found in [11]. We start by discussing the relationship between a generic totally nonnegative invertible matrix and totally nonnegative unitriangular matrices.

Theorem 2.2 (LDU Factorization)**.** *Let X be an invertible $n \times n$ totally nonnegative matrix. Then, one can write*

$$X = LDU$$

where L is a lower unitriangular matrix, D is a diagonal matrix, U is an upper unitriangular matrix, and L, D, U are totally nonnegative.

In order to answer (1), we would first like to find the generators of the semigroup of totally nonnegative matrices. A *Chevalley generator* is a particular type of matrix which differs from the identity matrix in precisely one entry. We use $e_i(a)$ to denote matrices that only differ by an $a > 0$ placed in the $(i, i+1)$-st entry and $f_i(a)$ to denote matrices that differ by an $a > 0$ placed in the $(i+1, i)$-st entry. More generally, *elementary Jacobi matrices* differ from the identity in exactly one entry either on, directly above, or directly below the main diagonal. Thus, an elementary Jacobi matrix is a Chevalley generator or a diagonal matrix which differs from the identity in one entry on the main diagonal. The following result is from Loewner and Whitney:

Theorem 2.3 (Whitney's Theorem; Theorem 2.2.2 of [5])**.** *Any invertible totally nonnegative matrix is a product of elementary Jacobi matrices with nonnegative matrix entries.*

In fact, using Theorem 2.2, we can say more about the factorization given in Theorem 2.3 using the following.

Corollary 2.4. *An upper unitriangular matrix X can be factored into Chevalley generators $e_i(a)$ with nonnegative parameters $a \geq 0$. Similarly, a lower unitriangular matrix X can be factored into Chevalley generators $f_i(a)$ with nonnegative parameters $a \geq 0$.*

Next, in order to discuss parametrizations of totally nonnegative matrices, we first need to discuss the relations between these generators. Let $h_k(a)$ be the elementary Jacobi matrix which differs from an identity matrix at the single entry (k, k) where it takes on the value a.

For the semigroup of unitriangular totally nonnegative matrices, the relations are relatively simple:

Theorem 2.5 (Section 2.2 of [10])**.** *These following identities hold:*

- $e_i(a)e_i(b) = e_i(a + b)$
- $e_i(a)e_{i+1}(b)e_i(c) = e_{i+1}\left(\frac{bc}{a+c}\right) e_i(a + c)e_{i+1}\left(\frac{ab}{a+c}\right)$
- $e_i(a)e_j(b) = e_j(b)e_i(a)$ *if* $|i - j| > 1$

They also hold if we replace $e_i(\cdot)$ with $f_i(\cdot)$.

More relations are necessary for a full set of relations.

Theorem 2.6 (4.17 and Theorem 1.9 of [10])**.** *In addition to the relations above, the following identities hold:*

- $h_{k+1}(a)e_k(b) = e_k(b/a)h_{k+1}(a)$
- $h_k(a)e_k(b) = e_k(ab)h_k(a)$
- $h_k(a)e_j(b) = e_j(b)h_k(a)$ *if* $k \neq j, j + 1$
- $h_{k+1}(a)f_k(b) = f_k(ab)h_{k+1}(a)$
- $h_k(a)f_k(b) = f_k(b/a)h_k(a)$
- $h_k(a)f_j(b) = f_j(b)h_k(a)$ *if* $k \neq j, j + 1$
- $e_i(a)f_j(b) = f_j(b)e_i(a)$ *if* $i \neq j$
- $e_i(a)f_i(b) = f_i(b/(1 + ab))h_i(1 + ab)h_{i+1}(1/(1 + ab))e_i(a/(1 + ab))$
- $h_i(a)h_i(b) = h_i(ab)$
- $h_i(a)h_j(b) = h_j(b)h_i(a)$

The relations obeyed by the e_i's are the same as the braid relations between adjacent transpositions in the Coxeter presentation of the symmetric group. The strong Bruhat order of the symmetric group determined by these relations is deeply connected to parametrizations of totally nonnegative matrices and totally nonnegative unitriangular matrices. More information about this can be found in Section 3 - Factorizations.

Next, we list known results about positivity tests for $\mathrm{GL}_n(\mathbb{R})$ matrices along with the necessary definitions. Most of these can be found in "Totally Positive Matrices" [17].

Lemma 2.7 (Fekete [7])**.** *Assume X is an $n \times m$ matrix with $n \geq m$ such that all minors of order $m-1$ with columns $[m-1]$ are positive and all minors of order m with consecutive rows are positive. Then all minors of X of order m are positive.*

A minor $|X_{I,J}|$ is called *solid* if both I and J consist of several consecutive indices. $|X_{I,J}|$ is called *initial* if it is solid and $\{1\} \in I \cup J$. $|X_{I,J}|$ is called *column-solid* if J consists of several consecutive indices and *row-solid* if I consists of several consecutive indices.

Theorem 2.8 (Theorem 2.2 of [17])**.** *Assume all solid minors of X are positive. Then X is totally positive.*

In fact, checking a smaller set of these minors will suffice.

Theorem 2.9 (Theorem 2.3 of [17])**.** *Assume all solid minors of X with rows $[k]$ and also all solid minors of X with columns $[k]$ are positive for $k = 1, 2, \ldots$. Then X is totally positive.*

A minor $X_{I,J}$ is called a leading principal minor of order k if $I = J = [k]$. Although these minors do not give a positivity test for $\mathrm{GL}_n(\mathbb{R})$ matrices, they satisfy another strong condition.

Theorem 2.10 (Lemma 15 of [11])**.** *The leading principal minors $X_{[k],[k]}$ of an invertible totally nonnegative matrix X are positive for $k = 1, \ldots, n$.*

Recently, k-nonnegative and k-positive matrices have been the subject of study in several papers by Fallat, Johnson, and others [4] [13] [14] [19] [6]. Rather than investigate the conditions under which the standard multiplication of two k-nonnegative (resp. k-positive) matrices is k-nonnegative (resp. k-positive), as we do in this work, these previous papers have studied the preservation of k-nonnegativity (k-positivity) under the Hadamard, or entrywise, products of matrices. [4] studies the behavior of n-positivity and 2-positivity under Hadamard powers. [6] shows that Hankel matrices, square matrices which are symmetric and constant across the anti-diagonals, are fairly well-behaved under Hadamard multiplication. [13] shows that a partial order on the group of permutations S_n given by 2-positive $n \times n$ matrices is equivalent to the Bruhat partial order and that a given positive matrix is 2-positive if and only if it satisfies certain inequalites related to the Bruhat order.

Another topic of interest is the "k-positive completion problem". A partial matrix is one in which some entries are specified and some are unspecified. A partial k-positive matrix is a partial matrix whose fully specified minors of order $\leq k$ are positive. A completion of a partial matrix is a particular choice of values for the unspecified entries. The matrix completion problem asks which partial k-positive matrices have a k-positive completion. Using the results of [13], [14] finds a complete solution to the 2-positive completion problem. The case of completing partial matrices with just one unspecified entry are solved for greater k in [19].

While it is well-known that the k-nonnegative matrices form a semigroup, this paper's work is, to the authors' knowledge, the first attempt to fully characterize the generators of this semigroup and find an analagous Bruhat cell decomposition as in the case of totally nonnegative matrices [10].

2.2 Equivalent Conditions and Elementary Generalizations

A natural question that arises when discussing k-nonnegative matrices (or, more generally, when discussing any condition on a matrix's minors) is whether we need to check all minors (usually an intractable computation), or just some subset of minors. For example, a well-known result, from [11], is that only column-solid minors are necessary to determine total nonnegativity.

The following three statements from [5] provide satisfactory answers to this question. While we independently proved these results, our proofs differ insignificantly from the above source, and so are not presented here.

We will generally assume all matrices are pulled from $GL_n(\mathbb{R})$; that is, they are invertible and square. Note that we sometimes abbreviate totally nonnegative as TNN and k-nonnegative as kNN. However, the following three statements hold true for matrices in the space of all $m \times n$ matrices as well as matrices in $GL_n(\mathbb{R})$.

Theorem 2.11 ([5] 3.1.6)**.** *If all solid minors of X of order at most k are positive, then X is k-positive.*

Theorem 2.12 ([6] 2.5)**.** *k-positive matrices are dense in the class of k-nonnegative matrices.*

Notice that this holds in the invertible case because invertible matrices are an open subspace in the space of all matrices.

Theorem 2.13 ([6] 2.3)**.** *If all initial minors of X of order at most $k-1$ are positive and all solid order k minors of X are positive, then X is k-positive.*

We will reword the above theorem in a way that will prove useful in Section 4.

Definition. The *k-initial minor matrix M* of a matrix X is defined as follows:

$$M_{ij} = \left|X_{[i-\ell+1,i],[j-\ell+1,j]}\right| \qquad \text{where } \ell = \min(k, i, j)$$

In other words, the value at position (i, j) is the value of the solid minor of largest order not exceeding k, such that (i, j) is the lower right corner of the corresponding submatrix.

For example, the n-initial minor matrix (also referred to as just the initial minors matrix) contains all of the initial minors of X, and a 1-initial minor matrix contains all of the entries of X.

Notice that the k-initial minor matrix gives us exactly the minors for the above necessary condition of k-positivity.

Corollary 2.14. *Let X be a matrix. Then X is k-positive if and only if the k-initial minor matrix has all positive entries.*

Remark 2.15. By a slight modification of the proof of Lemma 7 of [11], any choice of positive k-initial minors uniquely determines a matrix. This gives us an explicit bijection between k-positive matrices and 1-positive matrices, and therefore we have bijections between k-positive matrices and j-positive matrices.

Now, we present the following k-nonnegativity test. We have not found this in the literature, but it follows from a known proof technique, presented here.

Theorem 2.16. *An invertible matrix X is k-nonnegative if all column-solid (or alternatively, row-solid) minors of X of order k or less are nonnegative.*

Proof. Let $Q_n(q) = (q^{(i-j)^2})_{i,j=1}^n$ for $q \in (0,1)$. This matrix has the two nice properties that it is totally positive and $\lim_{q\to 0^+} Q_n(q) = I_n$. Let $X_q = Q_n(q)X$. Apply Cauchy-Binet on an order $r \leq k$ column-solid minor:

$$|(X_q)_{I,J}| = \sum_{\substack{S\subset[n]\\ |S|=r}} |Q_n(q)_{I,S}|\,|X_{S,J}|$$

This must be positive, since the column-solid minors of X are nonnegative and X is invertible. By 2.11 X_q must be k-positive. Taking limit $q \to 0^+$, we conclude that X is k-nonnegative. To get the analogous statement for row-solid minors, we can use $X_q = XQ_n(q)$. □

In propositions that follow, we will also use the fact, proved in [5], that the following matrix maps preserve k-nonnegativity.

Proposition 2.17 (Theorem 1.4.1 of [5])**.** *The linear maps listed below preserve k-nonnegativity and k-positivity:*

- $A \mapsto A^T$
- $A \mapsto \tilde{T}A\tilde{T}$ *for $\tilde{T}$ the permutation matrix of* $w_0 = (n\ n{-}1 \cdots 2\ 1)$ *(that is, $\tilde{T}\vec{e}_i = \vec{e}_{n-i+1}$).*

Notice that much of the information in Section 7.2 of [5] applies to k-nonnegative matrices, as long as the submatrices considered are less than k (notably, Theorem 7.1.9 and Theorem 7.2.8). This manifests in a generalization of Corollary 7.2.10, which we proved independently. However, since the proof is equivalent to that in [5], we just give the theorem statement:

Definition. Consider an $n\times n$ matrix M and a $c\times d$ solid submatrix indexed by $I = [i, i{+}c{-}1]$ and $J = [j, j+d-1]$. Then the *shadows* of $M_{I,J}$ are $M_{[1,i+c-1],[j,n]}$ and $M_{[i,n],[1,j+d-1]}$.

Theorem 2.18. *Suppose that M is an invertible k-nonnegative matrix and $M_{I,J}$ is a rank-deficient solid submatrix of order $c \leq k$. Then one of the shadows has rank equal to that of $M_{I,J}$ (in particular, the one to the side of the diagonal of M on which more entries of $M_{I,J}$ lie).*

This tells us where we can find rank-deficient matrices.

Proposition 2.19. *Let M be an invertible matrix. Consider a $c \times d$ solid submatrix A of M, with rank r less than the maximal rank $s = \min(c,d)$, with the bound property. Then none of the kth superdiagonals and subdiagonals of M intersect with A maximally, for $k < s - r$.*

Proof. If any of the above conditions fail to hold, then we have that row operations can bring an element on the diagonal to zero, along with all of the entries to the NW or the SE. This implies that M is singular. □

We can actually use this to prove a generalization of Theorem 3.1.10 from [5]. Call a minor a *corner minor* if it is indexed either by $[1, k]$ and $[n-k+1, n]$ or $[n-k+1, n]$ and $[1, k]$ for some k.

Corollary 2.20. *Suppose X is a $k+1$-nonnegative $n \times n$ matrix. Then X is k-positive if and only if all corner minors of order $\leq k$ are positive.*

Further, X is $k+1$-positive if and only if all corner minors of order $\leq k$ and all solid minors of order $k+1$ are positive.

Proof. The forward implication is obvious. Prove the contrapositive of the backwards implication: suppose X is not k-positive. Then 2.11 gives us that some solid minor of order at most k is non-positive. By 2.18 the corresponding submatrix has the bound property, so a corner minor of the corresponding order must take value zero.

The final statement is from 2.11. □

The description of generators for TNN matrices given by the Loewner-Whitney theorem has no obvious generalization for kNN matrices. However, these generators arise from the context of an LDU decomposition. We consider the algorithm for decomposition at an elementary level; this approach provides a generalization of the proof that gives us a statement about how close we can get to completion of an LDU decomposition.

First, it will be useful to explicitly state the following technical lemma. It immediately follows from Cauchy-Binet.

Lemma 2.21. *Suppose X' is a matrix X, transformed through a Chevalley operation. Then minors of X' are equal to minors of X except in the following cases:*

1. *If $X' = Xe_k(a)$ (adding a copies of column k to column $k+1$) then $\left|X'_{I,J}\right| = |X_{I,J}| + a\left|X_{I,J\setminus k+1\cup k}\right|$ when J contains $k+1$ but not k;*
2. *If $X' = e_k(a)X$ (adding a copies of row $k+1$ to row k) then $\left|X'_{I,J}\right| = |X_{I,J}| + a\left|X_{I\setminus k\cup k+1,J}\right|$ when I contains k but not $k+1$.*

For f_k, swap $k+1$ and k in the above statements.

Now, we give a definition that will prove helpful in describing the LDU decomposition process.

Definition. Call a k-nonnegative matrix M k-irreducible if $M = RS$ in the semigroup of invertible kNN matrices implies $R, S \notin \{f_i(a), e_i(a) \mid a > 0\}$.

Theorem 2.22. *Every k-nonnegative matrix X can be factored into a product of finitely many Chevalley generators and a k-irreducible matrix.*

Proof. Suppose X is not k-irreducible. Then there is some inverse Chevalley operation we can perform to X, maintaining nonnegativity. Without loss of generality suppose $e_i(a)^{-1}X$ is k-nonnegative for some $i \in [n]$ and $a \in \mathbb{R}_{>0}$ (corresponding to removing a copies of row $i+1$

to row i). We claim it is possible to choose $b > 0$ so that $e_i(b+\delta)^{-1}X$ is not k-nonnegative for any $\delta > 0$.

We want to determine when $e_i(x)^{-1}X$ is k-nonnegative in terms of x. It suffices to consider row-solid order $\leq k$ minors containing row i and not row $i+1$. These determinants are linear functions d_γ in x of the form $A - xB$ for some minors A, B of X. Thus, $d^{-1}([0,\infty))$ is closed for any d and the intersection $\cap_\gamma d_\gamma^{-1}([0,\infty))$ is also closed and compact; there must be some d with an upper bound, since otherwise we would break invertibility of X. We know this set is nonempty because a is in it. Thus, there is a maximal b in the intersection and applying an inverse Chevalley with any greater value will make the quotient matrix not k-nonnegative. It is also clear that this maximal b is of the form A/B (i.e. the minimum such A/B). Thus, $e_i(b+\delta)^{-1}X$ is not k-nonnegative for any $\delta > 0$.

So, in this way, we factor out a Chevalley generator, leaving a matrix with one more zero minor of order at most k. We can iterate this process, which must stop eventually because the number of minors of size at most k is finite. The resulting matrix must be k-irreducible. □

Note that while the above states that these k-irreducible matrices act "nicely", these will not give our desired minimal set of generators. In fact, since Chevalley generators are not commutative or normal (in the sense that multiplying a matrix on the left by a Chevalley is not equivalent to multiplying by a Chevalley on the right), we get cases where k-irreducible matrices can be factored as Xe_iY, where X, Y are k-irreducible. Such a case is seen in the $k = n-1$ section.

Now, we describe the extent to which we can factor Chevalley matrices from a generic kNN matrix.

Theorem 2.23. *If a matrix A is k-nonnegative, we can express it as a product of Chevalley matrices (specifically, only e_is) and a single kNN matrix where the ij-th entry is zero when $|j-i| > n-k$.*

That is, if a matrix is k-irreducible, the ij-th entry is zero when $|j-i| > n-k$.

Proof. We use the following lemma:

Lemma 2.24. *Let A be a kNN matrix. Then for a_{ij} such that the following hold, either $a_{ij} = 0$ or $e_i(-a_{ij}/a_{i+1\,j})A$ is kNN.*

(1) $i < j$;

(2) $a_{xy} = 0$ for $x \leq i$ and $y \geq j$, not including a_{ij} itself;

(3) $i < k$.

So we can reduce our matrix to one where a_{ij} is zero by factoring out a Chevalley matrix.

Proof. First, notice that our row operation is well-defined, since $a_{i+1\,j} = 0 \implies a_{ij} = 0$ from 2.18 and (1). Further, notice that from 2.16 and 2.21 the only minors we need to worry about are those row-solid minors containing row i but not row $i+1$. From (2), this means that any I, J to check for nonnegativity has $I = [h, i]$ for some $h > i - x$ and J having no indices greater than or equal to j. Let I, J define such a minor. Then using 2.21,

$$\begin{aligned}
|e_i(-a_{ij}/a_{i+1\,j})A_{I,J}| &= |A_{I,J}| - \frac{a_{ij}}{a_{i+1\,j}} \left|A_{I\setminus i \cup i+1, J}\right| \\
&= \frac{1}{a_{i+1\,j}} \left(a_{i+1\,j} |A_{I,J}| - a_{ij} \left|A_{I\setminus i \cup i+1, J}\right|\right) \\
&= \frac{1}{a_{i+1\,j}} |A_{I \cup i+1, J \cup j}|
\end{aligned}$$

And because the minor is of order one greater than the order of the original minor, when we only have minors of order less than k, the resulting matrix must be k-nonnegative. This is true by (3). □

We can consider iterating this factorization, using the criteria to find another entry to eliminate. The top-right corner satisfies the criterion for the lemma, and for a matrix where that entry is zero, the entry directly below satisfies the criterion, and so on. We can eliminate $k-1$ entries in the last column, one by one top-down, then $k-2$ entries in the second-to-last, and continue until we all entries desired to zero. Take the transpose of everything in the above argument to get the zeros in the bottom-left corner. □

Note that if we set $k = n$, we get the Loewner-Whitney theorem, so we have simply generalized an elementary proof for this.

We can actually say slightly more about the locations of 0 entires in k-irreducible matrices. The following results from a simple application of Lemma 2.24.

Lemma 2.25. *Let M be a k-nonnegative k-irreducible matrix for $k > 2$. Then*

$$m_{ij} = 0 \implies \begin{cases} m_{i-1,j-1} = 0 & \text{if } i \le k \text{ or } j \le k \\ m_{i+1,j+1} = 0 & \text{if } i > n-k \text{ or } j > n-k \end{cases}$$

The other question about irreducibility is describing the minors that prevent Chevalley matrices from dividing k-nonnegative matrices. For the following small case we can do so as follows:

Lemma 2.26. *If a matrix M is 2-irreducible and invertible, then there is a solid zero minor of order two in every pair of consecutive rows and every pair of consecutive columns.*

Proof. Suppose not. Then there exist two columns $i, i+1$ of M in which there are no solid zero minors of order 2. However, consider the column operation $Q = Me_{i-1}(-\epsilon)$; because M is 2-irreducible, by Theorem 2.16, there must be a column-solid minor that prevents Q from being 2NN for any $\epsilon > 0$. That is, by 2.21, we either have a column-solid zero minor in columns i and $i+1$ or a zero in the ith column.

If we have a zero in the ith column, Theorem 2.18 tells us that we either have a solid zero minor in the desired columns, or that the entry immediately to the left of the 0 in the ith column is also 0, in which case there must be another column-solid zero minor in columns i, $i+1$.

If we have a column-solid zero minor (say in rows a and b), then we must have a solid zero minor, as we will show. Since the minor is zero, row b is a multiple of row a (for instance, $\vec{a} = c\vec{b}$). Thus,

$$\begin{vmatrix} m_{a,i} & m_{a,i+1} \\ m_{a+1,i} & m_{a+1,i+1} \end{vmatrix} = -c \begin{vmatrix} m_{a+1,i} & m_{a+1,i+1} \\ m_{b,i} & m_{b,i+1} \end{vmatrix}$$

If $c = 0$ then we obviously have a solid zero minor. Otherwise, since the above determinants must both be nonnegative, they both must be zero, so we have a solid zero minor. □

We believe the same is true in general:

Conjecture 2.27. *If a matrix M is k-irreducible and invertible for $k < n$, then for every k-size interval of rows and columns, there is a solid zero minor.*

It is obviously true for $k = 1$, and the above lemma proves the statement for $k = 2$. From the classification of $(n-1)$-irreducible matrices done later, the conjecture will follow for this case. However, attempts to prove this conjecture using Pl ucker relations and other determinantal identities have not yet been successful.

This conjecture along with Theorem 2.23, give us the following interesting statement.

Corollary 2.28. *If a matrix M is k-irreducible and invertible for $k < n$, then the k-initial minor matrix has a zero in each row and column.*

3 Factorizations

In this section, we will describe the problem of finding generators for the semigroup of k-nonnegative matrices. First, we discuss obvious or trivial cases. Then, we give the generators for k-nonnegative matrices that are tridiagonal and pentadiagonal unitriangular. These generators will lead to a complete, parametrized set of generators for $(n-1)$-nonnegative matrices and $(n-2)$-nonnegative unitriangular matrices. The relations for these matrices are also given, and will be the basis for which we construct our Bruhat cell analogues, whose description will conclude this section.

First, notice that $(n-1)$-nonnegative unitriangular matrices are TNN, so this particular case is uninteresting. Second, consider the case of $k = 1$. When we restrict to the subsemigroup of unitriangular 1-nonnegative matrices, generating sets are known: for example, it is easy to see that we can generate the semigroup with $e_{ij}(a) = I + \delta_{ij}(a)$, for $i < j$ and $a > 0$. If we let $a \in \mathbb{R}$, the forms above generate the group of unitriangular matrices in a well-studied way ([16] §5). Restricting a to be positive preserves closure of the relations, so the complete list of generators and relations in this case are known.

One might expect that the niceness of the $k = 1$ unitriangular case to extend to the general $k = 1$ case. However, this does not seem to be the case. Notice that most of the results from

Section 2 we discuss apply only very weakly to the $k = 1$ case. Generally, the smaller the k, the less TNN structure the semigroup seems to have. Thus, in some sense this is the "worst" case. We have few results for this case, but we present some smaller items of note.

First, notice that *row-operation* generators, $e_{ij}(a) = I + \delta_{ij}(a)$ for $i \neq j$ and $a > 0$, are in this semigroup.

Thus, instead of considering 1-irreducible matrices, we can consider the following definition:

Definition. Consider a matrix M. Let $R_i = \{k \mid M_{ik} \neq 0\}$ and $C_i = \{k \mid M_{ki} \neq 0\}$ (so the indices representing nonzero elements in that row or column). Then M is *op-irreducible* if for all i, j, $R_i \subset R_j$ implies $i = j$ and $C_i \subset C_j$ implies $i = j$. In other words, no two R_i are comparable with each other, and no two C_i are comparable with each other.

Equivalently, a matrix M is op-irreducible when $M = RS$ in the semigroup of 1-nonnegative invertible $n \times n$ matrices implies that neither R nor S is a row-operation generator.

This definition has an analogous theorem to 2.22, with a similar proof.

Theorem 3.1. *If M is not op-irreducible, it can be expressed as a product of row-operation generators and a single op-irreducible matrix.*

Proof. Without loss of generality M has two rows m_i and m_j such that $\{k \mid m_{ik} \neq 0\} \subseteq \{k \mid m_{jk} \neq 0\}$. Let α be the largest ratio between any m_{ia} and m_{ja} (where a is an index where the two rows are both nonzero). Then the row operation sending m_j to $m_j - m_i/\alpha$ results in a matrix with one more zero than before.

If the resulting matrix is not op-irreducible, we can continue with this algorithm. We add one zero each time, so this algorithm eventually terminates. What we are left with must be op-irreducible. □

Remark 3.2. Non-diagonal op-irreducible matrices are not TNN. Further, adding all op-irreducible matrices to the generators of TNN matrices gives a generating set for 1-nonnegative matrices.

Because the definition of an op-irreducible matrix is not dependent on the values in the nonzero entries (meaning replacing changing the values in nonzero entries does not affect op-irreducibility as long as they are being changed to other nonzero values), a natural question becomes what shapes (what patterns of nonzero entries and zero entries) of a matrix are op-irreducible. This question is hard.

Remark 3.3. The number of shapes of op-irreducible $n \times n$ matrices up to permutation of rows and columns form a sequence. The first six elements of this sequence are $2, 1, 2, 5, 20, 296$. If we specify that there must be an invertible matrix with that shape, the only difference becomes that the first element of the sequence is 1.

Example 3.4. For $n = 4$, the shapes are as follows (the asterisks mark nonzero entries):

$$\begin{bmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{bmatrix} \begin{bmatrix} * & & & \\ & & * & * \\ & * & & * \\ & * & * & \end{bmatrix} \begin{bmatrix} * & * & & \\ * & & * & \\ & * & & * \\ & & * & * \end{bmatrix} \begin{bmatrix} & * & * & * \\ * & * & & \\ * & & * & \\ * & & & * \end{bmatrix}$$

This example gives the false impression that an induction argument may be able to enumerate all possible op-irreducible shapes. This is mostly a coincidence of small numbers; for larger cases, there are no obvious patterns among op-irreducible shapes. See Appendix C for op-irreducible 5×5 shapes that are not obviously derived from original cases.

Remark 3.5. Once we have the shapes of op-irreducible matrices, we immediately get parameter families for our generators, when we add Chevalley matrices, diagonal matrices, and permutation matrices. These can be given for n up to five: for $n = 2$ we need to add nothing more, for $n = 3$ [3] has a generating set, and for $n = 4, 5$ we have presented the op-irreducible matrices sufficient to add. However, even for cases larger than three, we suspect that this is far from minimal; we may not even need an op-irreducible matrix of every shape in a minimal generating set.

Finally, since the problem is tied to numerous other topics like Sperner families and matrices with fixed row and column sums, we can rewrite the problem of counting op-irreducible matrix shapes in a number of ways. We give one here. A *clutter* is a hypergraph where no edge properly contains another.

Remark 3.6. The number of shapes of op-irreducible matrices up to permutations is equal to the number of clutters with n vertices and n edges whose duals are also clutters (up to isomorphism).

3.1 Tridiagonal and Pentadiagonal Unitriangular Matrices

2.23 suggests that generators of larger k may be straightforward to find, since more elements of k-irreducible matrices are determined. This turns out to be true, but the techniques apply to general kNN matrices where only three bands of the matrix are nonzero. The two cases that allow for invertibility are *tridiagonal* matrices, where only the diagonal, subdiagonal, and superdiagonal are nonzero, and *pentadiagonal unitriangular matrices*, where only the diagonal, superdiagonal, and super-superdiagonal are nonzero.

We are able to give generators for k-nonnegative tridiagonal and pentadiagonal unitriangular matrices in general. The two cases are fairly similar.

Lemma 3.7. *For M a k-nonnegative tridiagonal matrix, we can write M as a product of Chevalley generators, diagonal matrices, and matrices of the form*

$$\begin{bmatrix} I_p & & \\ & H_q & \\ & & I_{n-p-q} \end{bmatrix}$$

Where H_q is a $q \times q$ invertible tridiagonal k-nonnegative, k-irreducible matrix with ones on the subdiagonal, no zeros on the superdiagonal, and $q > k$.

Proof. Induct on n. Clearly we have the statement for $n = 1$.

Now, consider a matrix M. By factoring out diagonal matrices we can assume that all of the entries in the subdiagonal are one. Note that via 2.19, no diagonal entries can be zero.

If there are no zeros off the diagonal then M is of the desired form. Suppose $m_{i,i+1} = 0$. Define $K_{i,j}(x)$ to be a zero matrix except for the bottom-left corner, whose value is x. Then as block diagonals we have

$$M = \begin{bmatrix} M_{[i],[i]} & K_{i,n-i}(m_{i,i+1}) \\ & M_{[i+1,n],[i+1,n]} \end{bmatrix} = \begin{bmatrix} M_{[i],[i]} & \\ & I_{n-i} \end{bmatrix} e_i\left(\frac{m_{i,i+1}}{m_{i,i}m_{i+1,i+1}}\right) \begin{bmatrix} I_i & \\ & M_{[i+1,n],[i+1,n]} \end{bmatrix}$$

It is easy to see that the matrices resulting from the factorization are k-nonnegative; the minors of the factors can easily be related to minors of M. Further, these matrices are obviously invertible.

Thus, we are left with one generator and two subcases, which we can further decompose by the inductive hypothesis if there are further zeros on the off-diagonals. The analogous case where $m_{i+1,i} = 0$ is given by the transpose of the above, and if both values $m_{i+1,i} = m_{i,i+1} = 0$, then we get the same formula as above, but without the Chevalley matrix in the middle.

If we end up with factors with tridiagonal blocks of size less than k, then k-nonnegativity implies TNN in these cases, and said factors can be decomposed into Chevalley generators by 2.3. We can further decompose our resulting blocks by factoring out Chevalley generators; if we ever lose our desired form, as shown above, we can factor into subcases and proceed. Thus, we can ensure that the subcases are k-irreducible as well. □

Now, we only need to worry about classifying H_q matrices as described in the above theorem. We consider an $n \times n$ matrix J of such a form. Because J must be a tridiagonal matrix with ones on the subdiagonal, there are only $2n - 1$ entries that are unknown. Let $a_i = j_{i,i}$ and $b_i = j_{i,i+1}$.

We first observe that a minor in a tridiagonal matrix (where non-diagonal entries are units) can be expressed in terms of a continued fraction. We will notate continued fractions in the following way:

$$[a_0; a_1, \dots, a_m; b_1, \dots, b_m] := a_0 - \cfrac{b_1}{a_1 - \frac{b_2}{a_2 - \cdots}}$$

This is different from the standard notation, which adds recursively rather than subtracts.

From observation we can see the following:

Lemma 3.8. *Let $C_i(j) = \left|M_{[i,i+j-1],[i,i+j-1]}\right|$. Then the following recursive relation is satisfied:*

$$C_i(0) = 1,\ C_i(1) = a_i,\ C_i(r) = a_{i+r-1}C_i(r-1) - b_{i+r-2}C_i(r-2)$$

This may be familiar as the recurrence defining the *generalized continuant*. We can give another relation by re-writing the above:

Lemma 3.9. *$C_i(r) = C_i(r-1)[a_{i+r-1}; \dots a_i; b_{i+r-2}, \dots, b_i]$ when $C_i(k) \neq 0$ for $k < r$.*

Proof. It is obviously true for the base cases of the recurrence. Rewrite the equation as follows:

$$\begin{aligned}\frac{C_i(r)}{C_i(r-1)} &= a_{i+r-1} - b_{i+r-2}\frac{C_i(r-2)}{C_i(r-1)} \\ &= a_{i+r-1} - \frac{b_{i+r-2}}{[a_{i+r-2}; a_{i+r-3}, \dots, a_i; b_{i+r-3}, \dots, b_i]} \\ &= [a_{i+r-1}; \dots a_i; b_{i+r-2}, \dots, b_i]\end{aligned}$$

□

Finally, to relate this to nonnegativity tests, we use the following lemma:

Lemma 3.10. *Column-solid minors of tridiagonal matrices evaluate to one of the following:*

- *Products of entries on the same diagonal*
- *A solid minor multiplied by a product of entries on the same diagonal.*

Since we have reduced our problems to tridiagonal matrices with 1s on the subdiagonal and nonzero entries on the superdiagonal, we can consider factoring Chevalley generators out of

Theorem 3.11. *Let J be an invertible tridiagonal matrix with 1s on the subdiagonal and nonzero entries on the superdiagonal. Then J is k-nonnegative and k-irreducible if and only if the following hold:*

$$\begin{aligned} a_i, b_i &> 0 \\ [a_{k+1}; \dots a_2; b_k, \dots, b_2] &= 0 \\ [a_{n-1}; \dots a_{n-k}; b_{n-2}, \dots, b_{n-k}] &= 0 \\ [a_x; \dots a_{x-k+1}; b_{x-1}, \dots, b_{x-k+1}] &\geq 0 \qquad \text{if } x \geq k \text{ and not } n-1, k+1 \\ [a_x; \dots a_1; b_{x-1}, \dots, b_1] &> 0 \qquad \text{if } x < k \end{aligned}$$

Proof. First, notice that M is kNN if and only if column-solid minors of order at most k are nonnegative by 2.16. From 3.10, this is equivalent to all minors of the form $C_i(j)$ being nonnegative for $j \leq k$, and all of the b_is being nonnegative (that is, positive, since we know they are nonzero by assumption). For $j < k$, we cannot have $C_i(j) = 0$: it breaks invertibility in the $j = 1$ case, and we get that $C_i(j) \neq 0$ by induction; if not, then $C_i(j+1)$ is negative. Thus, we can use 3.9, and say that $C_i(j)$ are all nonnegative precisely when the base case, a_i, are positive, as well as all of the corresponding continued fractions. Among these continued fractions, notice that if $[a_k; \dots a_i; b_{k-1}, \dots, b_i] > 0$, then so are the continued fractions achieved by truncating at any $j \leq k$.

This gives us a necessary and sufficient condition for k-nonnegativity:

$$\begin{aligned} b_i &> 0 \\ [a_x; \dots a_{x-k+1}; b_{x-1}, \dots, b_{x-k+1}] &\geq 0 \qquad \text{if } x \geq k \\ [a_x; \dots a_1; b_{x-1}, \dots, b_1] &> 0 \qquad \text{if } x < k \end{aligned}$$

Second, to guarantee k-irreducibility, notice that the only Chevalley generators we need to worry about factoring out are e_{n-1} and f_1 (from the left), and e_1 and f_{n-1} (from the right). If we cannot factor any of these from J, then the order k principal minors $C_2(k)$ and $C_{n-k}(k)$ are zero (if not, then we break the form being correct or we break k-nonnegativity). This gives us the criteria in the statement. □

Remark 3.12. Invertibility is an issue. The matrix:

$$\begin{bmatrix} 1 & 1 & & & \\ 1 & 1 & 1 & & \\ & 1 & 1 & 1 & \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \end{bmatrix}$$

satisfies all of the criterion for $n = 5$ and $k = 2$, but is not invertible. It is not obvious what more restrictions need to be added to give us invertibility.

Theorem 3.13. *The above criterion can be simplified into a $(2n-3)$-parameter family. Thus, the subset of invertible matrices in the family, along with Chevalley generators and diagonal matrices, generates all tridiagonal invertible kNN matrices.*

Proof. The criterion only gives lower bounds for our a_i and b_i, so we can turn them into parametrizations very easily.

Let our parameters be α_i for $i \in [1, n] \setminus \{k+1, n-1\}$, and β_i for $i \in [1, n-1]$. Let $b_i = \beta_i$. We specify $\beta_i \in \mathbb{R}_{>0}$.

Let $a_i = \alpha_i + [a_{x-1}, \dots, a_1; b_{x-1}, \dots, b_1]$, for $i < k$. We specify that $\alpha_i \in \mathbb{R}_{>0}$. Let $a_i = \alpha_i + [a_{x-1}, \dots, a_{x-k+1}; b_{x-1}, \dots, b_{x-k+1}]$, for $i \geq k$ and where α_i is defined. We specify that $\alpha_i \in \mathbb{R}_{\geq 0}$.

Finally, we cannot choose a_{k+1} and a_{n-1}, only solve using the above equations. Notice that because each inequality "tells us" a lower bound for each parameter, this simple family gives us precisely what we want. Showing this is just a matter of computation. □

Remark 3.14. We can actually reduce this to an $(n-3)$-parameter family, just by scaling the superdiagonal to ones via diagonal matrices. However, this proves more natural in the cases we have seen, so we present it in this manner.

Remark 3.15. If we imagine the $C_i(j)$ as a triangle indexed by i and j, then the k-initial minor matrix's diagonal entries are equal to $C_1(i)$ when $i \leq k$ and $C_{i-k+1}(k)$ when $i \geq k$. Further, the superdiagonal of the minor matrix gives us the value of the b_is. Thus, if we know our matrix is from our parameter family, the minor matrix determines the element from the family.

We can also give a minimality condition with these generators.

Theorem 3.16. *Let M be a matrix of the above form. Then if $RS = M$ in the semigroup of invertible k-nonnegative $n \times n$ matrices, one of R or S is a diagonal matrix.*

That is, any generating set of the semigroup must include all elements from the parameter family, up to scaling by diagonal matrices.

Proof. Suppose we have $RS = M$. From 2.19 we know that R and S have nonzero diagonals. Thus, we know that $r_{i,i+2}$, $s_{i,i+2}$ and their transpose analogues are all 0 from the formula for matrix multiplication. Further, we know that one of $r_{i,i+1}$ and $s_{i+1,i+2}$ are 0, and one of $r_{i,i+1}$ and $s_{i,i+1}$ is positive. Together, these show that R and S can only be as described above. □

And now a similar analysis for pentadiagonal unitriangular matrices. These are very similar, since a pentadiagonal unitriangular matrix is a tridiagonal matrix with ones on the subdiagonal, with an additional row and column added.

Lemma 3.17. *For M a k-nonnegative invertible pentadiagonal unitriangular matrix, we can write M as a product of Chevalleys, diagonal matrices, and matrices of the form*

$$\begin{bmatrix} I_p & & \\ & H_q & \\ & & I_{n-p-q} \end{bmatrix}$$

Where H_q is a $q \times q$ invertible pentadiagonal unitriangular k-nonnegative, k-irreducible matrix with all entries nonzero that can be nonzero, and $q > k$.

We notate similarly as before: entries on the superdiagonal are a_is, and entries on the super-superdiagonal are b_is.

Theorem 3.18. *Let S be a pentadiagonal unitriangular matrix with all entries nonzero that can be nonzero. Then S is k-nonnegative and k-irreducible if and only if the following hold:*

$$\begin{aligned} a_i, b_i &> 0 \\ [a_x; \dots a_{x-k+1}; b_{x-1}, \dots, b_{x-k+1}] &= 0 \qquad \text{if } x \in \{k, k+1, n-1, n-2\} \\ [a_x; \dots a_{x-k+1}; b_{x-1}, \dots, b_{x-k+1}] &\geq 0 \qquad \text{if } x \geq k \text{ and not listed above} \\ [a_x; \dots a_1; b_{x-1}, \dots, b_1] &> 0 \qquad \text{if } x < k \end{aligned}$$

The proof is very similar; the only difference is that more minors get set to zero from k-irreducibility, two more than for the tridiagonal case. Notice that we do not get the same issue with invertibility as before.

Theorem 3.19. *The above criterion can be simplified into a $(2n-7)$-parameter family. Thus, the family, along with Chevalley generators, generates all pentadiagonal unitriangular kNN matrices.*

Theorem 3.20. *Let M be a matrix of the above form. Then if $RS = M$ in the semigroup of invertible k-nonnegative $n \times n$ matrices, one of R or S is the identity.*

That is, any generating set of the semigroup must include all elements from the parameter family, up to scaling by diagonal matrices.

3.2 $k = n - 1$

Notice that in the case $k = n - 1$, by 2.23, all k-nonnegative matrices can be decomposed into Chevalley generators and tridiagonal matrices. Thus, the tridiagonal generators given above are precisely those that we need to add to get a full generating set. As it turns out, characterizing the generators in this case are fairly easy, and we have a parameter family that always results in invertible matrices. The $(2n - 3)$-parameter family of generators is as follows:

$$K(\vec{a}, \vec{b}) = \begin{bmatrix} a_1 & a_1 b_1 & & & & \\ 1 & a_2 + b_1 & a_2 b_2 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & a_{n-2} + b_{n-3} & a_{n-2} b_{n-2} & \\ & & & 1 & b_{n-2} & b_{n-1} Y \\ & & & & 1 & b_{n-1} X \end{bmatrix}$$

where $a_1, \ldots, a_{n-2}, b_1, \ldots, b_{n-1}$ are positive numbers and

$$Y = \prod_{i=1}^{n-2} b_i$$
$$X = \left| K_{[2,n-2],[2,n-2]} \right| = \sum_{k=1}^{n-2} \left(\prod_{\ell=2}^{k} b_{\ell-1} \prod_{\ell=k+1}^{n-2} a_\ell \right)$$

The values of the solid minors are simple:

Lemma 3.21. *The solid minors of $K(\vec{a}, \vec{b})$ are as follows. First, the minors on the subdiagonal and superdiagonal:*

$$\left| K(\vec{a}, \vec{b})_{[i,j],[i+1,j+1]} \right| = \prod_{k=i}^{j} a_k b_k$$
$$\left| K(\vec{a}, \vec{b})_{[i,n-1],[i+1,n]} \right| = b_1 \cdots b_{n-1} \prod_{k=i}^{n-1} a_k b_k$$
$$\left| K(\vec{a}, \vec{b})_{[i+1,j+1],[i,j]} \right| = 1$$

Then, the principal minors:

$$\begin{aligned}
\left|K(\vec{a},\vec{b})_{[i,j],[i,j]}\right| &= \sum_{k=i-1}^{j}\left(\prod_{\ell=i}^{k} b_{\ell-1}\prod_{\ell=k+1}^{j} a_\ell\right) && i,j<n,\ \textit{where } a_{n-1},b_0=0\\
\left|K(\vec{a},\vec{b})_{[i,n],[i,n]}\right| &= \left(\prod_{k=i}^{n} b_{k-1}\prod_{k=i}^{n-2} a_k\right)\left|K(\vec{a},\vec{b})_{[2,i-1],[2,i-1]}\right| && i>2\\
&= \left(\prod_{k=i}^{n} b_{k-1}\prod_{k=i}^{n-2} a_k\right)\sum_{k=1}^{i-1}\left(\prod_{\ell=2}^{k} b_{\ell-1}\prod_{\ell=k+1}^{i-1} a_\ell\right)\\
\left|K(\vec{a},\vec{b})_{[2,n],[2,n]}\right| &= 0\\
\left|K(\vec{a},\vec{b})\right| &= -a_1\cdots a_{n-2}b_1\cdots b_{n-1}
\end{aligned}$$

All other minors are trivially zero.

We now turn our interest to relations involving generators of the form $K(\vec{a},\vec{b})$. It can be seen by direct computation that the following relations hold:

$$e_i(x)K(\vec{a},\vec{b}) = K(\vec{A},\vec{B})e_{i+1}(x'),\ \text{ where } 1\le i\le n-2 \tag{3.2.1}$$

$$e_{n-1}(x)K(\vec{a},\vec{b}) = K(\vec{A},\vec{B})f_{n-1}(x')h_{n-1}(c) \tag{3.2.2}$$

$$f_{i+1}(x)K(\vec{a},\vec{b}) = K(\vec{A},\vec{B})h_{i+1}(1/w)f_i(x)h_i(w),\ \text{ where } 1\le i\le n-2 \tag{3.2.3}$$

$$f_1(x)K(\vec{a},\vec{b}) = K(\vec{A},\vec{B})e_1(x')h_1(c) \tag{3.2.4}$$

$$h_i(x)K(\vec{a},\vec{b}) = K(\vec{A},\vec{B})h_{i-1}(x),\ \text{ where } 2\le i\le n \tag{3.2.5}$$

$$h_1(x)K(\vec{a},\vec{b}) = K(\vec{A},\vec{B}) \tag{3.2.6}$$

$$K(\vec{a},\vec{b})h_n(x) = K(\vec{A},\vec{B}) \tag{3.2.7}$$

The values of the parameters are relatively uninteresting, so they are located in Appendix B. However, they have one important attribute: the expressions for new parameters are always subtraction-free rational expressions of the old parameters. Thus, similarly to the relations between Chevalley generators, they reflect equality of sets of matrices with the same factorization.

Finally, one relation is missing. We are unaware of the technical details, but the characterization of Bruhat cells in 3.30 shows that such a relation exists, and to foreshadow future intent, that ignoring this relation will not affect our discussion

Lemma 3.22. *$K(\vec{A},\vec{B})K(\vec{C},\vec{D})$ is TNN and can always be written in a factorization that uses the same number of parameters or fewer, such that the factorization only contains one instance of K.*

Proof. Since the only minor that is nonnegative is the full determinant, the product of two Ks must be TNN. Further, we can use the description, along with Cauchy-Binet, to show that the product of two Ks is in the cell given by $((\sigma(1),\sigma(2),1,2,\ldots,n-2),(\omega(1),\omega(2),1,2,\ldots,n-2))$. Thus, the corresponding reduced word in e_is and f_is has length between $4n-8$ and $4n-6$. KK has length $4n-6$, so we can always find a shorter word using only Chevalley generators. □

3.3 $k = n - 2$

Analogously to the previous section, we notice that 2.23 gives us that all of the generators we must add are the pentadiagonal unitriangular matrices previously described. Again, these are easy to parametrize in this special case; our parameter family is as follows:

$$T(\vec{a},\vec{b}) = \begin{bmatrix} 1 & a_1 & a_1b_1 & & & & \\ & 1 & a_2+b_1 & a_2b_2 & & & \\ & & 1 & \ddots & \ddots & & \\ & & & \ddots & a_{n-3}+b_{n-4} & a_{n-3}b_{n-3} & \\ & & & & 1 & b_{n-3} & b_{n-2}Y \\ & & & & & 1 & b_{n-2}X \\ & & & & & & 1 \end{bmatrix}$$

where $a_1, \ldots, a_{n-3}, b_1, \ldots, b_{n-2}$ are positive numbers,

$$Y = \prod_{i=1}^{n-3} b_i$$

$$X = \left|T_{[2,n-3],[3,n-2]}\right| = \sum_{k=1}^{n-3}\left(\prod_{\ell=2}^{k} b_{\ell-1} \prod_{\ell=k+1}^{n-3} a_\ell\right)$$

We could describe the solid minors as we did in 3.21, but they are nearly identical, since $T(\vec{a},\vec{b})_{[1,n-1],[2,n]} = K(\vec{a},\vec{b})$. The only difference between our generators is the extra row and column added.

We now turn our interest to relations involving generators of the form $T(\vec{a},\vec{c})$. It can be seen by direct computation that the following relations hold:

$$e_i(x)T(\vec{a},\vec{b}) = T(\vec{A},\vec{B})e_{i+2}(x'), \text{ where } 1 \le i \le n-3 \tag{3.3.1}$$

$$e_{n-2}(x)T(\vec{a},\vec{b}) = T(\vec{A},\vec{B})e_1(x') \tag{3.3.2}$$

$$e_{n-1}(x)T(\vec{a},\vec{b}) = T(\vec{A},\vec{B})e_2(x') \tag{3.3.3}$$

And finally, a slightly different relation:

$$e_{n-1}e_{n-2}T = e_{n-2}e_{n-1}T \sqcup e_{n-2}\cdots e_1e_{n-1}\cdots e_2 \sqcup e_{n-2}\cdots e_1e_{n-1}\cdots e_1$$

Notice that the fourth relation splits one "Bruhat cell" into three based on the value of the $[1, n-1], [2, n]$ determinant; negative, zero, and positive, respectively. This may represent splitting the open ball of the cell into two open balls, separated by a ball of lower dimension.

To prove that Equation 3.3 holds, we first observe that the product $e_{n-2}e_{n-1}T(\vec{a},\vec{b})$ always gives rise to a maximally $n-2$-nonnegative matrix. The product $e_{n-1}e_{n-2}T(\vec{a},\vec{b})$, on the other hand, can give rise to a maximally $n-2$-nonnegative matrix or to a totally nonnegative matrix. When the resulting matrix is totally nonnegative, it can be easily verified by computations that its factorization is one of the two factorizations entirely in terms of Chevalley generators above. The parameters of this relation can be found in Appendix B.

Analogously to the $(n-1)$ case, we can prove that one more relation exists between products of Ts, and that it can be safely ignored.

Lemma 3.23. *$T(\vec{A}, \vec{B})T(\vec{C}, \vec{D})$ can always be written in a factorization that uses the same number of parameters or fewer, such that the factorization only contains one instance of T.*

Proof. This proof is similar to before. The general description of Bruhat cells, along with Cauchy-Binet, indicate that if we want to give a factorization that does not use two copies of T, we would use at most $4n-10$ letters to do so. Since TT is length $4n-10$, we have our statement. □

3.4 Bruhat Cells

The semigroup of $(n-1)$-nonnegative invertible matrices and the semigroup of $(n-2)$-nonnegative upper (resp. lower) unitriangular invertible matrices can both be partitioned into *cells* based on their factorizations. In this section, we will describe these cells by *reduced words* and study their topology.

3.4.1 Background

We first describe the basic theory of the standard conception of Bruhat cells. We will work in $G = GL_n(\mathbb{R})$. The following will come primarily from [10] § 4. Many of the results here (as well as notions of total nonnegativity and total positivity) hold in the broader context where G is any semisimple group.

Let us establish some notation. Let B^+ (resp. B^-) be the subgroup of upper-triangular (resp. lower-triangular) matrices in G. We can identify $W = S_n$ with a subgroup of G in an obvious way: identify an $\omega \in W$ with the matrix sending the basis vector e_i to the basis vector $e_{\omega(i)}$ (i.e. the permutation matrix corresponding to ω).

For any $u \in W$, let B^+uB^+ (resp. B^-uB^-) denote the corresopnding double coset. We have decompositions

$$G = \bigcup_{u \in W} B^+uB^+ = \bigcup_{v \in W} B^-vB^-$$

We call a particular double coset B^+uB^+ or B^-vB^- a *Bruhat cell* of G. We then define *double Bruhat cells* as

$$B_{u,v} = B_u^+ \cap B_v^- := B^+uB^+ \cap B^-vB^-$$

so that G is partitioned into these $B_{u,v}$ for $(u, v) \in W \times W$.

As discussed in 2.1, the elementary Jacobi matrices generate the semigroup of totally nonnegative matrices. This leads to a factorization scheme for totally nonnegative matrices in a particular double Bruhat cell. Let

$$\mathcal{A} = \{1, \ldots, n-1, \tilde{1}, \ldots, \tilde{n}, \overline{1}, \ldots, \overline{n-1}\}$$

and let

$$\begin{aligned} x_{\tilde{i}}(t) &= h_i(t) \\ x_i(t) &= e_i(t) \\ x_{\bar{i}}(t) &= f_i(t) \end{aligned}$$

Then, for any word (i.e. an ordered sequence) $\mathbf{i} := (i_1, \ldots, i_\ell)$ of elements of $\mathcal{A}$, there is a map $x_{\mathbf{i}} : \mathbb{R}^\ell_{>0} \to G$ defined by

$$x_{\mathbf{i}}(t_1, \ldots, t_\ell) := x_{i_1}(t_1) \cdots x_{i_\ell}(t_\ell)$$

With some conditions imposed on $\mathbf{i}$, it turns out that the image of this map describes precisely the totally nonnegative matrices in a particular double Bruhat cell, allowing us to parametrize the double Bruhat cell and, consequently, the semigroup of totally nonnegative matrices. We describe this in more detail by introducing a definition:

Definition. Let $u, v \in W$. A *factorization scheme* of type (u, v) is a word $\mathbf{i}$ of length $n + \ell(u) + \ell(v)$ (where $\ell(u)$ denotes the Bruhat length of u in S_n) in the alphabet $\mathcal{A}$ such that the subword of barred (resp. unbarred) entries of $\mathbf{i}$ form a reduced word for u (resp. v) and such that each tilded entry $\tilde{i}$ is contained exactly once in $\mathbf{i}$.

Next, we have the main result which allows us to parametrize totally nonnegative matrices.

Theorem 3.24 (Theorems 4.4 and 4.12 in [10])**.** *If $\boldsymbol{i} = (i_1, \ldots, i_\ell)$ is a factorization scheme of type (u, v), then the product map $x_{\boldsymbol{i}}$ is a bijection between ℓ-tuples of positive real numbers and totally nonnegative matrices in the double Bruhat cell $B_{u,v}$.*

For the case of upper unitriangular matrices, we have that it suffices to consider Bruhat cells:

Theorem 3.25 (Theorems 2.2.3, 5.1.1, 5.1.4, and 5.4.1 of [1])**.** *Let $N_{\geq 0}$ be the set of $n \times n$ upper unitriangular totally-nonnegative matrices. Then, $N_{\geq 0} \cap B^-_w$ partition $N_{\geq 0}$ as w ranges over S_n. Furthermore, each $N_{\geq 0} \cap B^-_w$ is in bijective correspondence with an $\ell(w)$-tuple of positive real numbers via the map $(t_1, \ldots, t_{\ell(w)}) \mapsto e_{h_1}(t_1) \cdots e_{h_{\ell(w)}}(t_{\ell(w)})$ where $(h_1, \ldots, h_{\ell(w)})$ is a reduced word for w.*

In particular, the Bruhat cells give a stratification of the semigroup of upper unitriangular totally nonnegative matrices. The corresponding poset of closure relations is isomorphic to the poset induced by the Bruhat order on S_n. As a result, many of the nice properties of the Bruhat poset transfer to the Bruhat decomposition of unitriangular totally nonnegative matrices. We give a brief summary of some important results which will serve useful in our derivations of analagous results for the decomposition of unitriangular $(n-2)$-nonnegative matrices. These first three results allow us to think about parametrizations purely in terms of reduced words.

Lemma 3.26 (Subword Property, Theorem 2.2.2 of [2])**.** *Let $w = s_1 s_2 \ldots s_q$ be a reduced expression for $w \in S_n$. Then,*

$$u \leq w \iff \textit{there exists a reduced expression } u = s_{i_1} s_{i_2} \ldots s_{i_k},\ 1 \leq i_1 < \cdots < i_k \leq q$$

Theorem 3.27 (Corollary 2.2.3 of [2]). *For $u, w \in S_n$, the following are equivalent:*

- $u \leq w$
- *Every reduced expression for w has a subword that is reduced for u*
- *Some reduced expression for w has a subword that is reduced for u.*

Theorem 3.28 (Exchange Property, Theorem 1.5.1 of [2]). *Suppose $w = s_1 s_2 \dots s_k \in S_n$ be reduced. Then, if $\ell(sw) \leq \ell(w)$, then $sw = s_1 \dots \hat{s}_i \dots s_k$ for some $i \in [k]$ where*

This next result is less obvious. The following is a somewhat technical characterization of the Bruhat order. For $w \in S_n$, define

$$w[i,j] := |\{a \in [i] : w(a) \geq i\}|$$

for $i, j \in [n]$ (i.e. $w[i,j]$ counts the number of non-zero entries in the northeast corner above the entry (i,j) in the permutation matrix for w).

Lemma 3.29 (Theorem 2.1.5 of [2]). *Let $x, y \in S_n$. Then, the following are equivalent:*

- $x \leq y$
- $x[i,j] \leq y[i,j]$ *for all* $i, j \in [n]$

Usefully, Bruhat cells have good descriptions. We define everything here for the B^- decomposition, but taking the transpose will give everything analogously for the B^+ decomposition, and taking both conditions will give descriptions for the double Bruhat cells.

Definition. For an $\omega \in S_n$, let $X[I,J]$ be a ω-NE-ideal if $I = \omega(J)$ and $(\omega(i), i) \in (I, J) \implies (\omega(j), j) \in (I, J)$ for j such that $j > i$ and $\omega(j) < \omega(i)$.

Call $X[I,J]$ a shifted ω-NE-ideal if $I \leq I'$ and $J' \leq J$ in termwise order for some ω-NE-ideal (I', J') where $I, J \neq I', J'$.

Essentially we choose some set of entries that have ones in the permutation matrix ω, and have our ideal be those rows and columns, along with the rows and columns of any ones to the NE of any of our existing ones. Shifted ideals are submatrices that are further to the NE than the ideals.

Definition. Call a matrix X ω-NE-bounded if the following two conditions hold:

- $X[I,J] \neq 0$ for I, J ω-NE-ideal.
- $X[I,J] = 0$ for I, J shifted ω-NE-ideal.

For B^+, the analogous definitions will be called ω-SW-ideals and ω-SW-bounded matrices.

Lemma 3.30. *M is in B^-_w iff it is w-NE-bounded.*

Proof. From inspection the w is w-NE-bounded. Further, Cauchy-Binet gives us that multiplying by elements in B^- preserve this. ☐

3.4.2 Cells of $(n-2)$-nonnegative matrices

Given any matrix in the semigroup of invertible $(n-2)$-nonnegative unitriangular matrices (call this semigroup G), we can write it as a product of Chevalley generators of the form $e_i(a)$ (1-parameter families) and T-generators (a $(2n-5)$-parameter family).

Thus, we can associate factorizations of matrices in G to a word in the alphabet $A = \{e_1, \ldots, e_{n-1}, T\}$ For a word w, let $\ell(w)$ be the number of parameters in the product corresponding to the word (so $\ell(e_3e_2Te_2) = 2n-2$). Call this the *length* of the word.

We can associate a word in the alphabet to a subset of G through its associated map

$$x_w : \mathbb{R}_{>0}^{\ell(w)} \to G \qquad (t_1, \ldots, t_{\ell(w)}) \mapsto w_1(t_1)w_2(t_2)\cdots w_{\ell(w)}(t_{\ell(w)})$$

where we treat letters in the word as its corresponding generator (and of course, T is allocated $2n-5$ parameters). Let $V(w)$ be the matrices that have a factorization corresponding to the word w; this is equivalent to the image of x_w. Notice that any element of G is in some $V(w)$. Also notice that the relations given by 3.3.1 allows us to move between factorizations, and because the relations only contain subtraction-free rational expressions, we know that the same relations can be performed on all matrices with the same factorization, regardless of parameters. Thus, we can consider factorizations without concern for parameters.

Because of the above reasoning, the relations in 3.3.1 define an equivalence relation on words in A. We will define equality via equality mod this equivalence relation. If a word w has minimal length among all equal words, then we say that the word $w = i_1 \ldots i_l T$ is *reduced*.

As is clear by definition, if $u = w$ according to our relations, then $V(u) = V(w)$. However, we want this to be an if and only if statement, so we can say that two cells are disjoint exactly when the corresponding words are unequal, and thus enumerate the cells via reduced words. To do this, we need to use 3.3 and 3.23.

First, we will consider words with more than one T generator. From the relations, there exists a reduced word where the Ts are consecutive letters, and as a result, 3.23 tells us that there exists an equal reduced word with at most one T generator. Of course, the 3.3 gives an explicit example where the relations we already have are incomplete, but in fact, if we add this relation, we do have a complete list.

The question of how to resolve the relation is interesting; obviously, we can only include some of the cells, since otherwise we will get overlap. We will call choosing the cells on the right and disallowing the cells on the left the *fine* choice, and vice versa the *coarse* choice, mirroring the fact that one cell refines into three.

To give such a complete list, we will introduce the *Bruhat order*, which does, and will, hold in a more general setting than the one we described for S_n in the previous section. This is a partial order structure that arises in many places in algebra and geometry, but it plays an especially interesting role in the study of Coxeter groups. More details about this can be found in [2]. Here we will give a very brief description of the Bruhat order that relies on the subword property, not on the original definition that arose in the context of Coxeter groups and cell decompositions.

Let $w = w_1 \dots w_q$ be a reduced word. Then $u \leq w$ in the *strong* Bruhat ordering if there exists a reduced word $u = w_{i_1} \dots w_{i_r}$ where $1 \leq i_1 < \dots < i_r \leq q$. We say $u \leq w$ in the *weak* left Bruhat ordering if there exists a reduced word such that $w = w_1 \cdots w_k u$. We will eventually extend this ordering to a subword ordering on words including T. However, in following lemmas, we will only use the Bruhat order on words that do not involve T.

We are now ready to enumerate the cells that, as we will later show, partition the set of $(n-2)$-nonnegative unitriangular matrices.

Theorem 3.31. *Let A be the alphabet given by $\{1, \dots, n-1, T\}$, subject to the relations in 3.3.1, 3.3, and 2.6.*

Further, let $\alpha = (n-2)\cdots(1)(n-1)\cdots(1)$. Let $\beta = (n-2)\cdots(1)(n-1)\cdots(2)$. Let $w_{0,[n-2]} = (n-2, n-3, \dots, 1, n-1, n)$.

Then all words with at most one T are equal to one of the following distinct reduced coarse *words:*

$$w \in \begin{cases} w'U & w' \leq w_{0,[n-2]}, U \in \{T, (n-1)T, (n-2)T, (n-1)(n-2)T\} \\ w' & w' \not\geq \beta \end{cases}$$

And the following is a complete list of distinct reduced fine *words:*

$$w \in \begin{cases} w'U & w' \leq w_{0,[n-2]}, U \in \{T, (n-1)T, (n-2)T, (n-2)(n-1)T\} \\ w' & \end{cases}$$

where w' does not involve T.

Proof. First, notice that for words with a T, $n-1$ and $n-2$ commute with everything except for each other. Further, consider the following lemma:

Lemma 3.32.

- $k\ \alpha = \alpha\ (k+2 \mod n-1)$ *for all k.*
- *If $k \neq j$ then $k\ \alpha \neq j\ \alpha$ unless $k, j = n-2, n-1$, in which case they are all equal to α.*
- *$k\ \alpha$ is a reduced word unless $k = n-2, n-1$, in which case we have α as a reduced word.*

Proof. Examine the number of inversions of all of these. When the number of inversions increases, we know we have a reduced word, and so our permutations can be distinguished simply by value. Using this, showing all of the above requires minimal computation. □

Lemma 3.33. *Let φ_U be the map taking reduced words $w\alpha$ to wU, where $U \in \{T, (n-1)T, (n-2)T, (n-1)(n-2)T, (n-2)(n-1)T\}$, and w does not include $n-1$ or $n-2$. Then φ_U is a bijection.*

Proof. First, notice that this is a well-defined map. Second, this map is bijective, since the relations of $w\alpha$ and wU are exactly equal when we restrict to words without $n-1$ or $n-2$. □

Finally, notice that anything with more than one $(n-2)$ and $(n-1)$ is not a reduced word. Using 3.1.6 of [2], we get the statement. □

3.4.3 Topology of Cells

To begin proving topological properties about our cells, we will need the following technical lemma.

Lemma 3.34. *Let w be some reduced word of $\sigma \in S_n$ suffixed by some word α (that is, such that $\alpha \leq w$ in the weak left Bruhat order). Then, for $M \in B_\alpha$, $U(w \setminus \alpha)M \subset B_w$.*

Proof. Proof by induction. The base case is obviously true, since $M \in B_\alpha$. Now suppose we are taking some $N \in B_\beta$ and considering $e_i(c)N$ such that $s_i\beta$ is a reduced word. This occurs precisely when $s_i\beta$ has more inversions than β. Considering β in terms of one-line notation, we see this can only happen when we have i before $i+1$.

Now, we consider the $s_i\beta$-NE-ideals, and compare them to the β-NE-ideals. The ideals that do not contain rows i and $i+1$ are exactly the same, as are the ideals that contain both. In both cases, the corresponding minors are unaffected by the e_i. When a $s_i\beta$-NE-ideal contains row $i+1$, it must contain row i, so the only remaining case are the $s_i\beta$-NE-ideals that contain only i. These are in bijection to the β-NE-ideals containing i but not $i+1$. We know from an above lemma that for an $s_i\beta$-NE-ideal I, J,

$$\det(e_i(c)M)_{I,J} = \det M_{I,J} + cM_{I\setminus i \cup i+1, J} > 0$$

since the right hand side is the sum of a β-NE-ideal and a shifted β-NE-ideal.

Now, consider a shifted $s_i\beta$-NE-ideal I, J. We consider the I', J' from the definition (that is, the $s_i\beta$-NE-ideal such that $I \leq I'$ and $J' \leq J$). If I' does contains neither i nor $i+1$ or contains both, then I', J' is a β-NE-ideal as well, and I, J must be a sum of shifted ideals which I', J' apply for. If I' contains i but not $i+1$, then the ideal swapping out i for $i+1$ is a β-NE-ideal. This ideal shows that I, J can be expressed as a sum of shifted β-NE-ideal minors. □

Theorem 3.35. *For reduced words u and w as qualified by Theorem* 3.31*, if $u \neq w$ then $V(u)$ and $V(w)$ are disjoint (for both the coarse and fine cells).*

Proof. It is enough to show for the fine cells, since no two coarse cells contain the same fine cell. If the cell does not have a T this is a known result. Next, notice that $w'U$ is in the Bruhat cell given by the elements $w'\alpha \in G$. Thus, 3.34 tells us that different w' words give cells in different Bruhat cells, and so all of our cells are distinct, up to containing $n-2$ and $n-1$. But we know how to distinguish these: they appear precisely when $[1, n-1], [1, n-1]$ and $[2, n], [2, n]$ are nonzero, respectively. So a matrix cannot be in more than two cells with different elements in U, and we are done. □

Each of the cells $V(w)$ is homeomorphic to an open ball, as is proved in the following lemma. We take the standard topology on $GL_n(\mathbb{R})$.

Lemma 3.36. *Given a reduced word $w = w_1 \dots w_k$, the map from the parameters $(a_1 \dots a_k)$ to $e_{w_1}(a_1) \dots e_{w_k}(a_k)$ is a homeomorphism when its image is restricted to $V(w)$. (As always, one of the w_i's could be a T, in which case it has $2n-5$ parameters.)*

Proof. We first show that the map is a bijection; suppose that it is not injective. Since this result is already known for words not involving T, we may assume that w has a T and that it occurs at the end of the word. Thus, we have

$$e_{w_1}(a_1) \dots e_{w_i}(a_i) T(a_{i+1} \dots a_{i+2n-4}) = e_{w_1}(a'_1) \dots e_{w_i}(a'_i) T(a'_{i+1} \dots a'_{i+2n-4})$$

Then we cannot have the parameters of T, i.e. $a_{i+1} \cdots a_{i+2n-4}$, be different while the parameters $a_1 \cdots a_i$ are the same. In this case the superdiagonal values are guaranteed to be different, since they are nondegenerate in the parameters of T. So there must be an e-parameter that is different. Suppose without loss of generality that it is a_1 that is different from a'_1, say $a_1 > a'_1$. Then we will have:

$$e_{w_1}(a_1 - a'_1) \dots e_{w_i}(a_i) T(a_{i+1} \dots a_{i+2n-4}) = e_{w_2}(a'_2) \dots e_{w_i}(a'_i) T(a'_{i+1} \dots a'_{i+2n-4})$$

By Theorem 3.35 this is a contradiction, as we now have two different reduced words for the same element.

We now only need to show that the map and its inverse is continuous. Clearly, the forward map is continuous, since we can express the matrix entries as polynomials in the parameters.

For the inverse map, first note that T is a homeomorphism, since we can give an explicit rational inverse map. We consider the functions that give the parameters of the factorization based on the word w from the matrix entries. If $w = w_1 \cdots w_k$, then we first determine the parameter a_1 of e_{w_1}. This must be the maximum value of a_1 that will leave the matrix $n-2$-nonnegative, since otherwise this would violate 3.30. Thus, from 2.21, a_1 will be the minimum value of the set of a's that make any minor zero. Since a_1 is the minimum of a number of continuous functions, a_1 is itself determined by a continuous function. We can then recurse on the resulting matrix to get the pre-image. □

A conjecture of Fomin, proved by Hersh in [12], is that the closure of the cells $U(w)$ where the w consists of Chevalley generators alone, is homeomorphic to a closed ball. In this section we will try to understand the structure of the closure of $V(w)$. Most of the following results follow very closely from propositions and proofs in Pylyavskyy's lecture notes [18]. The following lemma describes the closure of the cell $V(T)$.

Lemma 3.37. *By setting parameters of T to zero, we get matrices that correspond to permutations that are below at least one of the permutations described below in the Bruhat order.*

(a) $T_1^i = e_{n-3} \cdots e_1 \, e_{n-1} \cdots \hat{e_i} \cdots e_2$, *where* $2 \le i \le n-2$.

(b) $T_2^i = e_{n-2} \cdots \hat{e_i} \cdots e_1 \, e_{n-1} \cdots \hat{e_{i+1}} \cdots e_2$, *where* $1 \le i \le n-2$.

Generators with a cap represent missing generators.

Proof. If one of the b_i's is 0, then it is straightforward to verify by computation that the word

$$e_{n-3}(a_{n-3})\cdots e_1(a_1)\ e_{n-1}(X)e_{n-2}(b_{n-3})\cdots \widehat{e_{i+1}(b_i)}\cdots e_3(b_2)e_2(b_1)$$

describes a factorization for the matrix $T(\vec{a},\vec{b})$. Note that X can be expressed in terms of $\vec{a}$ and $\vec{b}$ and $Y=0$ in this case. Conversely, every matrix of such a factorization, for $2\leq i\leq n-2$, is in the closure of T and corresponds to the matrix $T(\vec{a},\vec{b})$ where $b_{i-1}=0$.

If one of the a_i's are 0, then the factorization is

$$e_{n-2}(X')e_{n-3}(a_{n-3})\cdots \widehat{e_i(a_i)}\cdots e_1(a_1)\ e_{n-1}(b_{n-2})\cdots \widehat{e_{i+1}(b_{i+1})}\cdots e_2(b_1),$$

where X' is given by X/b_{n-2}, and X is as usual a polynomial in components of $\vec{a}$ and $\vec{b}$. Conversely, every matrix of such a factorization with $1\leq i\leq n-3$ corresponds to the the matrix where a_i is set to 0. □

The statement above inspires us to extend the partial Bruhat ordering on words to another partial ordering on all words, including those with a T-generator. We define the subwords of T to be the reduced words of the cells described in Lemma 3.37. This naturally extends to a general subword order: we say that $u\leq w$ if there exist reduced word representations of u and w where u is a subword of w (although for Coxeter groups, the expression for w does not matter, as we will see, it does in this case). Notice that this has a geometric interpretation, since $w_0\leq w_1 \implies V(w_0)\subset\overline{V(w_1)}$. This is because setting parameters to zero is equivalent to considering the closure of the parameter space, which maps inside the closure of the cell. Further, every element of the cell can be achieved by setting parameters to zero, which follows from 3.37.

We can actually prove that this subword order exactly describes the closures of cells. To prove this, we will describe the closure of $U=\{T,(n-1)T,(n-2)T,(n-2)(n-1)T\}$ in two ways, through subwords and through determinants, and together these will give a simple, straightforward characterization.

First, we give a lemma that will be used throughout.

Lemma 3.38. *Let S be contained in a classical Bruhat cell $U(w)$. Then $\overline{S}$ is contained in the disjoint union of the cells $U(w')$, where $w'\leq w$.*

Proof. Using the language of Lemma 3.29, define for all $(i,j)\in[n]\times[n]$, $N_w(i,j)=\mid\{k\mid k\leq i, w(k)\geq j)\}\mid$. Then $u\leq w$ if and only if for every (i,j), we have $N_u(i,j)\leq N_w(i,j)$. Thus, if $u\not\leq w$, there exists (i,j) with $N_u(i,j)>N_w(i,j)$. Consider the minimal u-NE-ideal X_c containing cell (i,j). Then $|X_C|=\det X_C\neq 0$ for $X\in V(u)$, by Lemma 3.30. But if $X_C\in V(w)$, then X_C is not of full rank, because it is obtained by performing row operations on a matrix of rank less than $N_u(i,j)$. Thus $\det X_C=0$, which means $X_C\notin V(w)$. □

Proposition 3.39. *Let $U=\{T,(n-1)T,(n-2)T,(n-2)(n-1)T,(n-1)(n-2)T\}$.*

1. *If a matrix m is in the closure of an element $u\in U$, then the cell γ corresponding to m must satisfy $\gamma\leq\alpha$.*

2. *Further, if $u = T$, $(n-1)T$, $(n-2)T$, or $(n-2)(n-1)T$, and m is TNN, then we also require that, respectively, $\gamma(1) \neq n-1, n$ and $\gamma(2) \neq n$; $\gamma(1) \neq n-1, n$; $\gamma(1), \gamma(2) \neq n$; or $\gamma(1) \neq n$.*

3. *The closure of $u \in U$ contains the TNN cells below u in the subword order.*

4. *The TNN cells below $u \in U$ in the subword order are exactly the ones that satisfy the conditions in 1.*

Together, this tells us that the subword order reflects the topological structure via the closure order, at least up to elements in U.

Proof.

1. This follows from 3.38.

2. In all of the described cells, the $[2, n], [1, n-1]$ minor is negative. Thus, for a TNN matrix to be in the closure, this minor is zero, unless we are in the cell $(n-1)(n-2)T$, in which case it can be arbitrary, so we do not need the condition. By 3.30, the requirements $\gamma(1) \neq n$, $\gamma(1) \neq n-1, \gamma(2) = n$ are equivalent to, respectively, $m_{[2,n],[1,n-1]}$, $m_{[2,n-1],[1,n-2]}$, $m_{[3,n],[2,n-1]}$ having determinant zero for every m in $V(\gamma)$. Depending on the cell, elements in the cell may have these conditions hold true, and it occurs precisely in the manner described above.

3. This follows immediately from 3.37.

4. This is an elementary argument, using the fact that the elements below γ in the Bruhat order are precisely the subwords of some particular reduced expression for γ (Theorem 3.27).

   First, we notice that if σ satisfies the set of conditions corresponding to any particular $u \in U$, then so does everything below σ. This can be seen from Lemma 3.29, and matches with what we expect. We will use this fact throughout. We begin with $(n-2)(n-1)T$, which has the fewest conditions, and work our way downward.

   Consider $\alpha = (n-2)\cdots(1)(n-1)\cdots(1)$, as we have previously defined it. Notice that it sends 1 to n (it is easy to see this by considering (i) as the function swapping i with $i+1$). Now, consider some subword $\beta \leq \alpha$. In order for β to satisfy $\beta(1) \neq n$, we must remove at least one letter from the $(n-1)\cdots(1)$ portion of the word. If we remove the (1), then the result is a reduced word. If we remove something else, such as (i), then the reduced expression for β is below the expression we get when we just remove (1). Thus, the β must be below

   $$(n-2)\cdots(1)(n-1)\cdots(2)$$

   This satisfies the condition that $\beta(1) = n$, so we have shown that these exactly define our desired cells. This can be formed from a subword:

   $$(n-2)\cdots(1)(n-1)\cdots(2) = (n-2)(n-1)T_2^{n-2}$$

Now, consider adding the condition that $\gamma(1) \neq n-1$. The top element does not satisfy this condition, so we must consider proper subwords. Obviously, we must remove a letter from the first descending sequence. From computation it turns out that this is all we need, and we get that any reduced subword that satisfies our conditions are below one of

$$(n-2)\cdots(\hat{i})\cdots(1)(n-1)\cdots(2) = (n-1)T_2^i$$

Similarly, if we have the condition that $\gamma(1), \gamma(2) \neq n$, we simply need to remove a letter from the second descending sequence, and reduced subwords must be beow one of

$$(n-2)\cdots(1)(n-1)\cdots(\hat{i})\cdots(2) = (n-2)T_1^i$$

Finally, if we add both conditions, we can consider the words above, except removing one of the letters in the first descending sequence, we get precisely the words below T. We can see this from rewriting the above as

$$(n-2)\cdots(i)(n-1)\cdots(i+1)(i-1)\cdots(1)(i-1)\cdots(2)$$

We must remove a letter in the first or third descending sequence; if we remove one from the first sequence, then the first half is a subword of the word achieved from removing $(n-2)$ (as in T_1). If we remove a letter from the third sequence, then the second half is a subword of the word achieved from removing $(i-1)$ (as in T_2).

Finally, $\alpha = (n-1)(n-2)T_2^{n-2}$, so we have all cells below α as expected.

□

Now, we will consider the order induced by closure: that is, $u \leq w$ precisely when $V(u) \subset \overline{V(w)}$. Note that any element in the closure of a cell must be $(n-2)$-nonnegative and unitriangular, so we know the closure must be contained in the disjoint union of some set of cells. We will show that the cells of subwords are enough, and because we know these are contained in the closure, we will get that the closure is precisely this union of cells. That is, the poset given by subword order is equal to the poset given by closure order.

Proposition 3.40. *The closure $\overline{V(w)}$ consists of all $V(u)$ for all $u \leq w$ in the Bruhat order, for both coarse and fine cells.*

Proof. It should be clear that if $u \leq w$ in the subword order, then $u \leq w$ in the closure order, since setting parameters to zero (which is what gives us the subword order), as we have shown, surjects onto the resulting cells, and the continuous parameter map maps the closure of the parameter space into the closure of the image.

So all we need to do is show that if $u \leq w$ in the closure order, then $u \leq w$ in the subword order. If both u and w are TNN, then this is a known result. The situation that u is not TNN but w is cannot occur, since an element in the closure of w must be TNN.

Now, consider w not TNN, so w can be written as $w_1 w_2$ a reduced word where w_1 does not include $(n-1)$ or $(n-2)$ and $w_2 \in U$. When u is also not TNN (and split into $u_1 u_2$

in a similar way, a necessary condition is that $u_1\alpha \leq w_1\alpha$, from 3.34 and properties of the standard Bruhat decomposition. The non-TNN cells in Bruhat cells below $w'\alpha$ are precisely the sU, where $s \leq w'$ and $U \in \{T, (n-1)T, (n-2)T, (n-1)(n-2)T, (n-2)(n-1)T\}$. It is easy to see that the subword order mirrors the closure order up to the elements in U from the locations of zero minors. For example, the closure $(n-1)T$ cannot intersect with $(n-1)(n-2)T$, since elements of $(n-1)T$, and thus its closure, must have the top-left large minor be zero, which is always positive in $(n-1)(n-2)T$. The same argument also works to show that we must have u_2 be a subword of w_2. From here, the possible cells that could intersect with the closure of w are precisely the subwords of w.

When w is TNN, then notice that the determinantal identities in 3.38 must also apply to u, because u_1 cannot affect these. Thus, we must take a subword of α where these identities hold. By 3.38, these are exactly the subwords of T. □

All of the above work gives us enough structure to say the following:

Corollary 3.41. *Both the coarse and the fine cells form a CW complex of the space of $(n-2)$-nonnegative unitriangular matrices.*

Specifically, the closure poset corresponding to the CW complex is given by the subword order.

3.4.4 Further Comments on the Poset

We know that the poset in the TNN case has many special properties: it has a top and bottom element (Proposition 2.3.1 of [2]), it is ranked (Theorem 2.2.6 of [2]), and it is Eulerian. We will show how far these properties extend.

First, notice that the coarse cells have a top and bottom element, and the fine cells have a bottom element and two top elements (one for the TNN matrices, and one for the rest).

In this section we will prove that the poset on cells $V(w)$ is a graded poset. The choice of coarse or fine cells does not matter here, since the proof is based on the fact that the vast majority of the poset is based on the standard Bruhat order.

Lemma 3.42 (Exchange Property for new relations)**.** *If w is a word in $\Lambda = \{1, \dots, n-1\}$ subject to Chevalley relations; that is, the following relations hold:*

- $i\ i \leftrightarrow i$ *(the shortening relation)*
- $i\ j\ i \leftrightarrow j\ i\ j$ *if* $|i-j| = 1$ *(the adjacent relation)*
- $i\ j \leftrightarrow j\ i$ *if* $|i-j| > 1$ *(the nonadjacent relation)*

Then if w is a reduced word, for $t \in \Lambda$, exactly one of the following is true:

- *tw is reduced, so $\ell(tw) = \ell(w) + 1$;*
- *$tw = w$, so $\ell(tw) = \ell(w)$.*

Proof. Suppose tw is not reduced. Let $M = \{m_1, m_2, \ldots, m_r\}$ be a sequence with $tw = m_1$, m_r a reduced word, each one at most one local move away from the previous, with no $i \to i\, i$ moves.

To see that there always exists such a sequence, consider this. We know such a sequence exists for Coxeter groups from [2] (see section 3.3.1). Mimic this sequence until we hit a shortening relation. Use this relation, and continue with the resulting smaller word.

Define $\varphi : M \to [\ell(w)+1] \cup \varnothing$ recursively in the following way (we want it to indicate a sort of location for our t when we are performing the local moves):

$$\varphi(m_1) = 1$$

$$\varphi(m_i) = \begin{cases} \varnothing & \text{in shortening relation or } \varphi(m_{i-1}) = \varnothing \\ \varphi(m_{i-1}) \pm 1 & \text{in left/right position of nonadjacent relation} \\ \varphi(m_{i-1}) \pm 2 & \text{in left/right position of adjacent relation} \\ \varphi(m_{i-1}) - 1 & \text{shortening relation earlier in word} \\ \varphi(m_{i-1}) & \text{otherwise} \end{cases}$$

Lemma 3.43. *The following properties are true:*

(a) φ *is well-defined.*

(b) There are no length-shortening moves that don't involve t

(c) $m_r = w$

Proof.

(a) Because we chose the sequence such that we never get a longer word than tw, our function remains in the codomain. Thus, the only thing to check for well-definedness is whether $\varphi(m_{i-1})$ can ever be in the middle of an adjacent relation.

Let $n_i = m_i \setminus m_{i,\varphi(m_i)}$, where we take out nothing if $\varphi(m_i) = \varnothing$; that is, we take out the t from the word, and $\varnothing$ signifies that the t no longer exists. Then notice that $w = n_1$, and each n_i is at most one local move away from n_{i-1}. The reason for this is that removing t does not affect any local moves not involving t, and the local moves that do involve t don't affect anything except t. Suppose we do have a move where the location of the t is in the center. Then if we consider the n_i up to that point, we get that there is an n_i with two adjacent identical letters:

$$m_i = \cdots i\; t\; i \cdots \implies n_i = \cdots i\; i \cdots$$

However, this would imply that w can be reduced to something of smaller length. This is a contradiction.

(b) Same reasoning; consider the n_i. If there was a length-shortening move then obviously we would get that n_i is a series of moves that shortens w, which is not possible.

(c) We must have a shortening relation to get a reduced word. This relation must contain t, so there must be exactly one. Notice that once $\varphi(m_i) = \varnothing$, $m_i = n_i$. We know that $w = n_i$ for all i. Thus, $m_i = w$.

□

This gives us the statement. □

Lemma 3.44. *The order restricted to $w\ T$ is graded.*

Proof. We know that when we restrict to $w\ T$, we get a poset that is isomorphic to the product poset of an interval in the strong Bruhat poset with a binary four-element poset (corresponding to containment of $(n-1)$ and $(n-2)$). Both of these are graded, and it is easy to see that the rank function is the equivalent to the sum of the rank functions of the individual posets. Thus, the Bruhat poset being graded implies that the poset we are interested in is graded. □

Theorem 3.45. *The Bruhat poset is graded.*

Proof. For anything not containing a T, this is well-known, one proof being Theorem 2.2.6 of [2]. For T, this is known by the lemma. We must consider words $w\ T$.

Now, suppose that wT is reduced but reducing T to t makes wt not reduced. We want to show that there is a chain between wT and wt that behaves correctly with respect to our rank function.

Let $w = w_1 \cdots w_a$. Then consider $w_i \cdots w_a t$, starting from $i = a$ to $i = 1$. Using the exchange property, we can see that this reduces to some $w't$, where w' is a subword of t. Thus, this has the intermediary $w'T$, and from the lemma we get intermediaries as desired. □

However, we do not get the niceness of Eulerian-ness.

Remark 3.46. For $n \leq 4$, the poset is Eulerian. For $n > 4$, the poset is not Eulerian: by computation using Lemma 3.29, the interval $[(n-2)\cdots(3)(n-1)\cdots(3), T]$ has only three elements, the middle one being T_2^1.

Future avenues to explore are whether we still get nice properties like shellability or semi-Eulerianness, which we suspect might hold, at least in the interval below T.

As another thing to consider, recall that we can make the choice of whether to take $n-2\ n-1$ or $n-1\ n-2$. If we choose the coarse option at a certain level, it must be consistent with all the options above. Thus, we can consider taking refinements between coarse and fine. This is a topic for future research, and we have no current results about these.

3.4.5 Cells of $(n-1)$-nonnegative matrices

Cells of $n-1$-nonnegative matrices are easier to describe than those of $n-2$-nonnegative matrices, as the following theorem will help to show.

Theorem 3.47. *The k-nonnegative matrices consist of two path-connected components, those with negative determinant and those with positive determinant.*

Proof. Take some k-nonnegative matrix M. Consider some minor that is 0. If we cannot affect this minor with Chevalleys then it must be the case that that M is not invertible. Thus, we can always make minors nonzero. Thus, we can always bring M to a totally positive matrix, except for the determinant itself which must have fixed sign. We know that totally positive matrices are homeomorphic to a ball, so we are done for that component. We can reduce any $k-1$-nonnegative matrix to a G, which is a ball by the above lemma. Thus, this is path-connected as well. □

Thus, we essentially have two separate components.

Proposition 3.48. *Let B be the alphabet given by $\{1, \ldots, n-1, \overline{1}, \ldots, \overline{n-1}, K, \tilde{1}, \ldots, \tilde{n}\}$, subject to the relations in 3.2.1 and 2.6.*

Further, let $w_{0,[i,j]}$ denote the longest length word in the set of permutations of $\{i, i+1, \ldots, j\}$. For example, $w_0 = w_{0,[1,n]}$, and $w_{0,[1,n-1]} = (n-1, n-2, \ldots, 1, n)$.

Then all words with at most one K are equal to one of the following distinct reduced words, up to existence of $\tilde{i}$:

(a) $(\sigma)(\overline{\omega})(\tilde{1})\cdots(\tilde{n})$, where $\sigma, \omega \in S_n$ are represented by an arbitrarily chosen reduced word factorization. (Note that any shuffle of the permutations, and any permutation of the $\tilde{i}$ will also work.)

(b) $(\sigma)(\overline{\omega})(Z)(\tilde{1})\cdots(n \tilde{-} 1)$ with $\sigma \leq w_{0,[1,n-1]}$, $\omega \leq w_{0,[2,n]}$, and $Z \in \{K, (\overline{1})K, (n-1)K, (\overline{1})(n-1)k\}$

Proof. It suffices to only consider reduced words, and we can safely ignore $\tilde{i}$s in our relations. Clearly, when our reduced word contains no K, we have the result (namely, the reduced words in the (a)).

Now, we consider words with a K. Notice that, without loss of generality we can move K to the end of the word. Further, our relations give us that $\overline{1}$ and $(n-1)$ commute with everything, since moving the letter to the other side of K flips es to fs and allows these to commute. As such, a reduced word can only have at most one of each of these; we account for this with our Z word, and consider the resulting e_is and f_is. Call what we have left w.

Notice that when we only consider words without $(n-1)$ or $\overline{1}$, Z (or more specifically, K) obeys the precisely the same relations as the word

$$\alpha = (n-1)(n-2)\cdots(1)(\overline{1})\cdots(\overline{n-2})(\overline{n-1})$$

Thus, wZ is reduced if and only if $w\alpha$ is reduced; in fact, we do not need to restrict to words without $(n-1)$ or $\overline{1}$, since words with them can never be reduced anyways. We can consider the e_is and f_is individually, and split α into a and $\overline{a}$ and split w into v and $\overline{v}$. To enumerate the distinct v such that va is reduced, we simply need to find the interval above a in the weak Bruhat order. By Proposition 3.1.6 of [2], this occurs precisely when v is in the interval given in the problem statement, and the analogous statement works for the f_is. Distinctness of the words follows from the distinctness of the corresponding permutations. □

Now, we can show that the above words reflect the topological space it comes from. Let $V(w)$ be as defined previously, only slightly changed for a different alphabet.

Theorem 3.49. *For reduced words u, w given by 3.48, if $u \neq w$ then $V(u)$ and $V(w)$ are disjoint. As a result, these $V(u)$ partition the semigroup of $(n-1)$-nonnegative invertible matrices.*

Proof. Using 3.34, this is easy to see, since K is in the cell corresponding to α. To distinguish cells containing $(n-1)$ and $\overline{1}$, notice that these are in the factorization precisely when the $[1, n-1], [1, n-1]$ and $[2, n], [2, n]$ are nonzero, respectively. □

Proposition 3.50. *The closure $\overline{V(w)}$ is exactly the union of $V(u)$ for $u \leq w$ in the Bruhat order.*

Proof. Because we have no connection between the "negative" part and the "positive" part of the poset, proving this is trivial, since the Bjorner-Brenti characterization gives us the statement immediately. □

We can again use the fact that both parts of the poset are isomorphic to intervals of Bruhat posets to say the following:

Proposition 3.51. *Both parts of the poset are graded, and have a top and bottom element. They are also Eulerian (making the poset as a whole trivially semi-Eulerian).*

4 Testing

4.1 Definitions

We start by giving a brief overview of relevant background on cluster algebras. This background is given for the sake of completeness, but only the combinatorial properties are used. For more detailed and general discussion, see [9], [15], and [8]. These definitions are reproduced in a slightly modified form below.

Definition. A *quiver* is a directed multigraph with no loops or two-cycles. The vertices are labeled with elements of $[m]$. A directed edge (i, j) will be denoted $i \to j$. A *mutation* of a quiver Q at vertex j is a process, defined as follows, that produces another quiver $\mu_j(Q)$.

1. For all pairs of vertices i, k such that $i \to j \to k$, create an arrow $i \to k$.
2. Reverse all arrows adjacent to j.
3. Delete all two cycles.

If two quivers are related by a sequence of mutations, we say they are *mutation equivalent.*

Definition. Let $\mathcal{F} = \mathbb{C}(x_1, \ldots, x_m)$ be the field of rational functions over $\mathbb{C}$ in m independent variables (this is our *ambient field*). A *labeled seed* of geometric type in $\mathcal{F}$ is a pair $(\tilde{\boldsymbol{x}}, Q)$ where $\tilde{\boldsymbol{x}} = (x_1, \ldots, x_m)$ is an algebraically independent set of $\mathcal{F}$ and Q is a quiver on m vertices such that vertices in $[n]$ are *mutable* and vertices in $[n+1, m]$ are *frozen* (unable to be mutated at). We call $\tilde{\boldsymbol{x}}$ the labeled *extended cluster* of the seed; $\boldsymbol{x} = (x_1, \ldots, x_n)$ the *cluster* with elements $x_1, \ldots, x_n$ the *cluster variables*; and remaining elements $x_{n+1}, \ldots, x_m$ the *frozen variables*.

Definition. A *seed mutation* at index $j \in [n]$ satisfies $\mu_j((\tilde{\boldsymbol{x}}, Q)) = (\tilde{\boldsymbol{x}}', \mu_j(Q))$, where $x'_i = x_i$ if $i \neq j$ and x'_j satisfies the following *exchange relation*:

$$x_j x'_j = \prod_{i \to j} x_i + \prod_{j \to k} x_k,$$

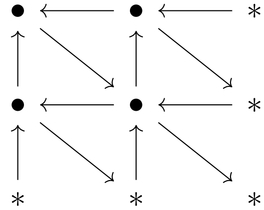
where arrows are counted with multiplicity. The right hand side is also referred to as the *exchange polynomial*.

Definition. For some starting seed $(\boldsymbol{x}, Q)$, let χ be the union of all cluster variables over seeds which are mutation equivalent. Let $R = \mathbb{C}[x_{n+1}, \ldots, x_m]$. Then the *cluster algebra* of rank n over R is $\mathcal{A} = R[\chi]$ together with the seeds generating it.

We do not care about the algebraic structure of a cluster algebra so much as the combinatorial objects behind it: the clusters and seeds themselves.

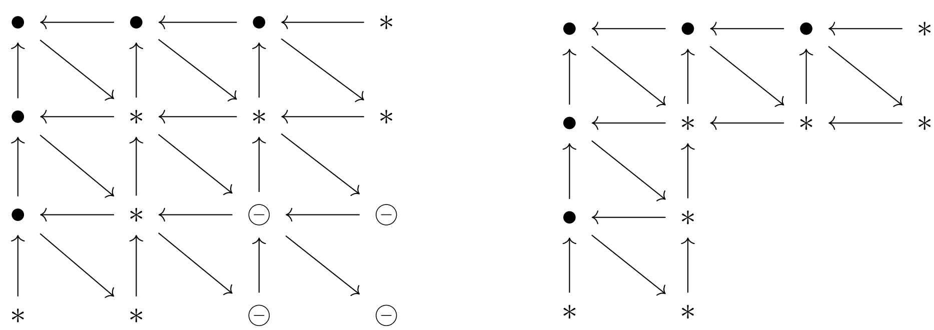
Definition. We consider two clusters *equivalent* if they share the same variables, up to permutation. Then the *exchange graph* has equivalence classes of clusters as vertices and an edge if two clusters are connected via a quiver mutation.

Total positivity tests for $n \times n$ matrices form a cluster algebra of rank $(n-1)^2$. All cluster variables correspond to rational functions in the matrix entries of a matrix of indeterminates $X = (x_{ij})_{i,j \in [n]}$. This in effect treats each cluster variable as a function, and testing total positivity of a particular matrix M is done by evaluating each cluster variable function, plugging the corresponding matrix entry M_{ij} in for x_{ij}. The *initial minors quiver*, $Q_i(n)$, has n^2 vertices, labeled by $(i, j) \in [n] \times [n]$. The variable associated with (i, j) is $\left| X_{[i-m+1,i],[j-m+1,j]} \right|$ where $m = \min(i, j)$. In other words, the variables are the entries of the initial minor matrix for X. There are arrows $(i, j+1) \to (i, j)$, $(i+1, j) \to (i, j)$, and $(i, j) \to (i+1, j+1)$. The vertices (n, j) and (i, n) are frozen for all $i, j \in [n]$. Below is an example for $n = 3$. On the left is the initial minors quiver, where frozen vertices are denoted with $*$. Note that edges between frozen vertices never affect any exchange relations or mutations and hence can be disregarded. On the right is a table containing the cluster variables corresponding to each vertex.



	col 1	col 2	col 3
row 1	x_{11}	x_{12}	x_{13}
row 2	x_{21}	$x_{11}x_{22} - x_{12}x_{21}$	$x_{12}x_{23} - x_{13}x_{22}$
row 3	x_{31}	$x_{21}x_{32} - x_{22}x_{31}$	det

This framework leads to a natural set of sub-cluster algebras when looking for k-positivity tests. Because every exchange polynomial is subtraction free, using an exchange relation preserves positivity of the old cluster variable as long as all variables used in the mutation are positive. So for any quiver corresponding to a cluster in the totally positive cluster algebra whose variables are all minors, we take the sub-cluster algebra generated by the subquiver formed by freezing all vertices adjacent to minors of order greater than k, and then deleting all vertices whose variables are minors of order greater than k. These deleted vertices will sometimes be referred to as *dead vertices*. This ensures that mutations will preserve positivity. It will also be interesting at some points to consider the quiver this comes from, as it shows how the sub-cluster algebra embeds into the total positivity one, so we will move freely between these interpretations. When restricted, we will refer to the subquiver as the *k-quiver*, and when looking at how it's embedded we will refer to it as the *full quiver*. In the following example of the full quiver for $n = 3$, $k = 2$ with the initial minors quiver, frozen vertices are represented with $*$ and dead vertices are represented with $\ominus$. The k-quiver is depicted on the right.



This still does not quite give a k-positivity test: for general k the minimal size of a test is n^2 (which follows from the corresponding result for $k = n$ in Example 3.1.8 of [5]). The solution is to define a *test cluster*: we append to the extended cluster polynomials in the matrix entries until the size is n^2; these variables will be a potential k-positivity test and stay constant across the entire sub-cluster algebra. These extra variables will sometimes be referred to as *test variables*. For example, all test clusters for $k = n$ are in fact extended clusters. The cluster for the k-quiver initial minors quiver can be extended by adding all the missing solid minors of order k as test variables, giving the k-initial minors test. Not all choices of extra variables will give a valid k-positivity test, and in fact not all clusters can even be extended to k-positivity test of minimal size, as we shall discuss in Section 4.2. Although we do know which to add in specific cases (see Sections 4.3 and 4.4), as of now we lack a proof for the general method. With this setup, proving that a single test cluster in such a sub-cluster algebra is a k-positivity test proves that all clusters are: we can go between the variables in the extended clusters using subtraction-free rational expressions, and the rest of the variables in the test cluster stay the same.

For any k, the exchange graphs for these restricted sub-cluster algebras break the total positivity exchange graph into connected components. This is because the freezing of a vertex corresponds to deleting all edges corresponding to mutation there from the graph, and likewise for marking a vertex as dead. We can relate these components by looking at quivers for the sub-cluster algebras.

Definition. Two clusters from different sub-cluster algebras have a *bridge* between them if they have the same test cluster and there is a quiver mutation connecting them which occurs at a vertex which is frozen in the k-quiver.

See Figure 2 for an example. We can think of a bridge as swapping a cluster variable for a test variable. Mutation at dead vertices provides another method of jumping between sub-cluster algebras, as both the resulting test cluster and k-quiver are identical. In fact, mutation at a dead vertex produces a completely identical sub-cluster algebra, but one which is embedded differently in the total positivity cluster algebra. If one sub-cluster algebra provides k-positivity tests, then so do any connected via a bridge, as the "starting" test cluster which is bridged to is the same as one in the old sub-cluster algebra and hence also provides a k-positivity test.

4.2 The $n = 3$, $k = 2$ case

For 3×3 matrices, we'll label the entries as shown below:

$$M := \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix}$$

An uppercase letter will denote the 2×2 minor formed by picking rows and columns that do not contain the lowercase version of the letter. For example, $A := ej - fh$. We further define $K := aA - \det M$ and $L := jJ - \det M$. From Exercise 1.4.4 of [8], we know that the only possible cluster variables correspond to minors and the two extra polynomials K and L. For a matrix which is totally positive, K and L must also be positive since they occur in clusters (and hence can be written as subtraction-free rational expressions in the initial minors). For a matrix which is maximally 2-positive, K and L are also positive as they are both differences of a positive term and a negative one. Therefore we can further generalize the natural sub-cluster algebras to starting quivers which also contain K or L, where still only vertices adjacent to the determinant are frozen. The exchange graphs for the 8 sub-cluster algebras are depicted in Figure 1. The vertices are labeled by the cluster variables corresponding to vertices of the quiver which are mutable in the total positivity algebra, so that the extended cluster contains the listed variables plus $cgCG$. The two large associahedra both generate 2-positivity tests: the left contains $Afhj$ and so extending the test cluster with J creates the anti-diagonal k-initial test; the right contains $Jabd$ and so extending the test cluster with A creates the k-initial test. None of the other components can give 2-positivity tests (at least, not of size n^2): all are missing both of the minors A and J, but the extended cluster can only have one test variable added to it. Why must the test cluster contain both A and J? Consider the matrix

$$\begin{bmatrix} \epsilon & 1 & \epsilon^2 \\ 1 & \epsilon & 1 \\ \epsilon^2 & 1 & \epsilon^{-2} \end{bmatrix}$$

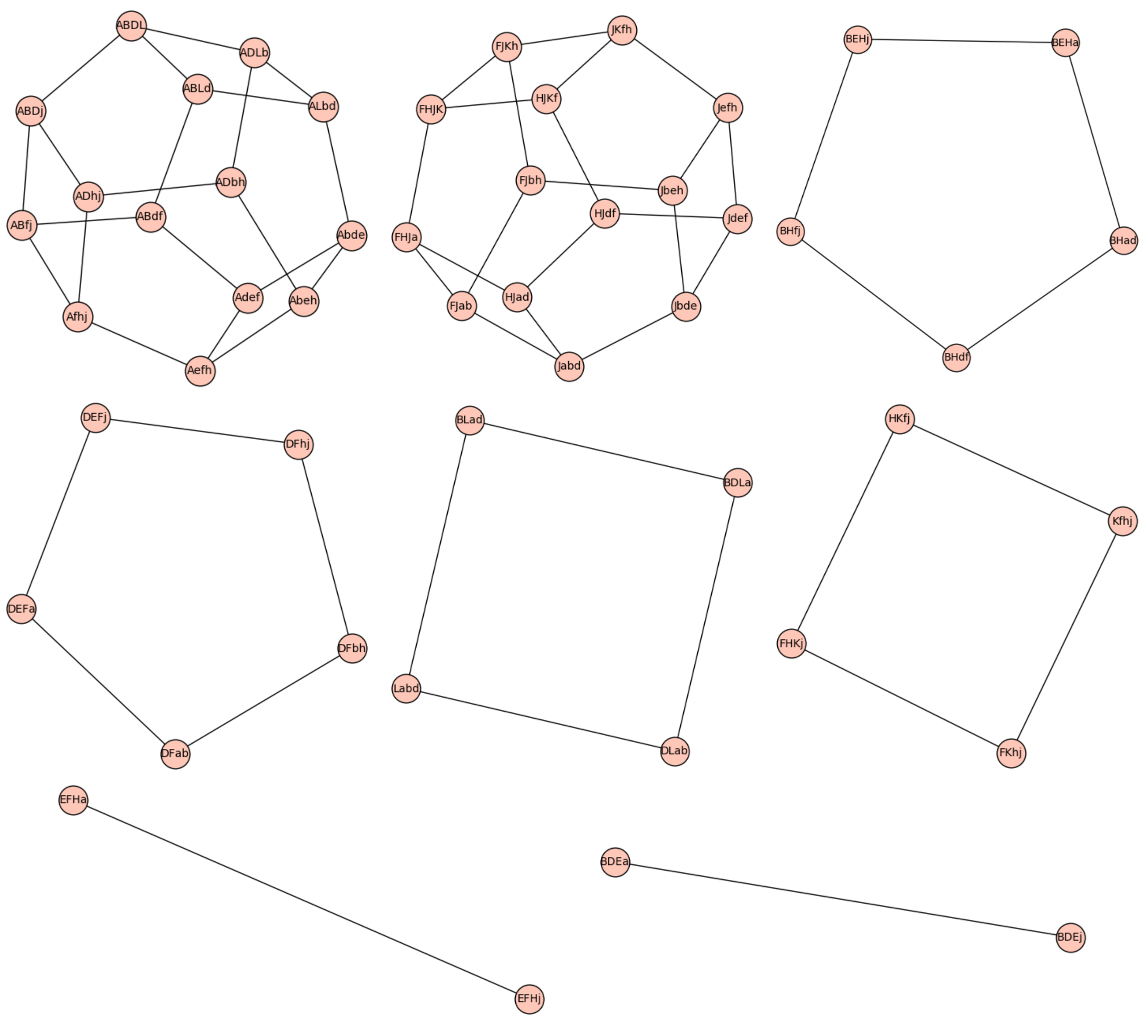


Figure 1: The connected components of a 2-positivity test graph derived from the 3×3 exchange graph.

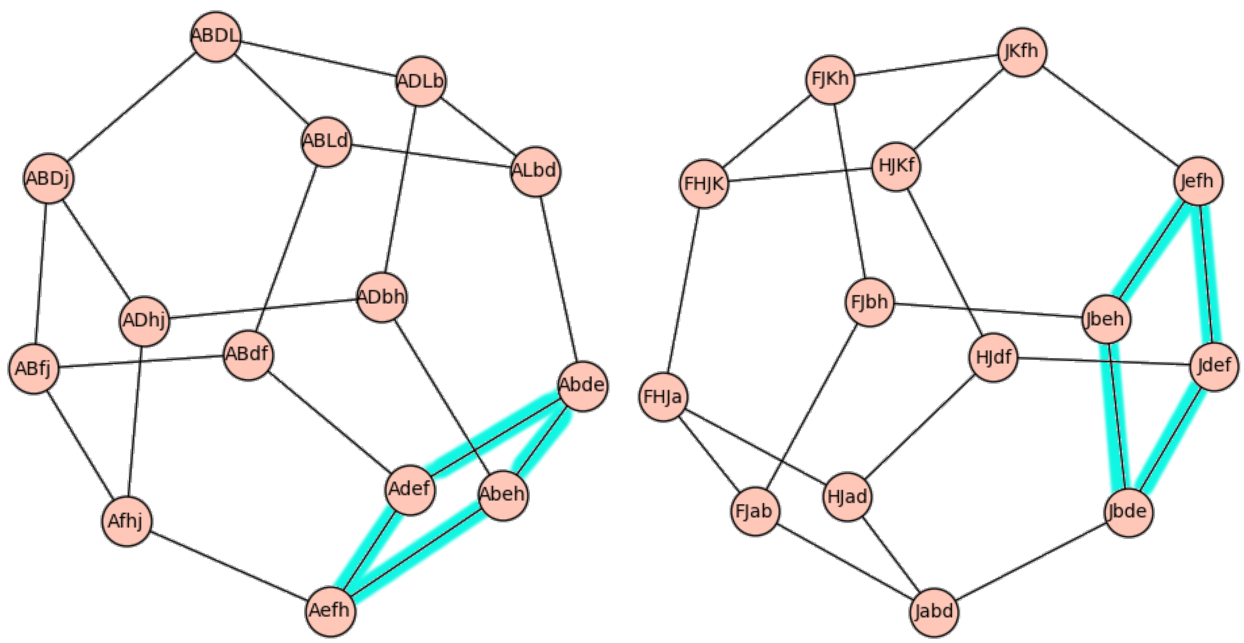


Figure 2: The bridges between the two associahedra. The left has test variable J and the right has test variable A. The two cyan squares pair up to give 4 bridges, by matching $Adef$-$Jdef$, $Aefh$-$Jefh$, $Abeh$-$Jbeh$, and $Abde$-$Jbde$ (i.e. those with the same test cluster).

for some small positive constant ϵ. Note that every minor except J is positive, as well as the non-minors K and L. Thus, the positivity of J is not implied by the positivity of any other cluster variables involved in this example, and thus it must appear in every 2-positivity test. The same applies to A using the anti-diagonal flip of this matrix.

The bridging is depicted in Figure 2. The two highlighted cyan squares have the same test clusters (though different extended clusters), and we get 4 bridges between them by swapping A and J in and out of the clusters.

Note that in general, the subquiver induced by mutable vertices of the 2-quiver of the initial minors quiver is an orientation of the Dynkin diagram A_{2n-3}. From Theorem 5.1.3 of [15] and the discussion in Chapter 6, the exchange graph is then the corresponding associahedron of Cartan type A_{2n-3}.

4.3 k-essential minors

To help determine which of these components provide tests, we define the following: a minor $|X_{I,J}|$ is *k-essential* if there exists a matrix M such that $|M_{I,J}| \leq 0$, but $\forall (I', J') \neq (I, J)$, $|I'| = |J'| \leq k$, we have $|M_{I',J'}| > 0$. That is to say, a k-essential minor appears in all possible tests for k-positivity consisting only of minors, although based on general behavior we expect them to be present universally. Throughout the rest of the paper, we use the terms *k-minor* and *minor of order k* interchangeably.

Proposition 4.1. *Solid k-minors are k-essential for $k \leq 3$.*

Proof. The $k = 1$ case is trivial, as there is exactly one test for 1-positivity: that consisting of all n^2 elements of the matrix. Explicitly, we can let $x_{i,j} = -1, x_{i',j'} = 1$ for $(i, j) \neq (i', j')$.

For $k = 2$, let $I = \{i_1, i_1 + 1\}$, $J = \{j_1, j_1 + 1\}$ and consider the matrix

$$M_2 := \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \epsilon^{-5} & \epsilon^{-3} & 1 & \epsilon^3 & \epsilon^7 & \epsilon^{10} & \cdots \\ \cdots & \epsilon^{-3} & \epsilon^{-2} & 1 & \epsilon^2 & \epsilon^5 & \epsilon^7 & \cdots \\ \cdots & 1 & 1 & \epsilon & 1 & \epsilon^2 & \epsilon^3 & \cdots \\ \cdots & \epsilon^3 & \epsilon^2 & 1 & \epsilon & 1 & 1 & \cdots \\ \cdots & \epsilon^7 & \epsilon^5 & \epsilon^2 & 1 & \epsilon^{-2} & \epsilon^{-3} & \cdots \\ \cdots & \epsilon^{10} & \epsilon^7 & \epsilon^3 & 1 & \epsilon^{-3} & \epsilon^{-5} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

It is defined so that

$$(M_2)_{I,J} = \begin{bmatrix} \epsilon & 1 \\ 1 & \epsilon \end{bmatrix}$$

where ϵ is a sufficiently small positive constant, and the powers of ϵ throughout the rest of the matrix are inductively chosen so that all minors crossing neither the rows nor columns of the central minor with that entry in the outermost (i.e. farthest from the center) position are positive. A submatrix "crossing" a row indexed by r means that $\ell < r < h$, where ℓ is the lowest row index in the submatrix and h the highest. Crossing a column is defined similarly. To show that this matrix is 2-positive everywhere except at the central minor, first note that we already know that all the 1-minors (entries of the matrix) are positive. Furthermore, if we look at the 2-positive matrix formed by switching the 1's and ϵ's in the I, J minor, we see that we only need to look at minors whose values are smaller in M_2, i.e. those with a 1 from the central minor off the diagonal, or an ϵ from the central minor on the diagonal. These are positive by construction, as none of these cross the central minor.

The $k = 3$ case has more cases, but is roughly analogous. Let

$$M_3 := \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \epsilon^{-1} & 1 & \epsilon^2 & \epsilon^4 & \epsilon^8 & \cdots \\ \cdots & 1 & 1+\epsilon & 1+\epsilon & \epsilon & \epsilon^4 & \cdots \\ \cdots & \epsilon^2 & 1+\epsilon & 1+2\epsilon & 1+\epsilon & \epsilon^2 & \cdots \\ \cdots & \epsilon^4 & \epsilon & 1+\epsilon & 1+\epsilon & 1 & \cdots \\ \cdots & \epsilon^8 & \epsilon^4 & \epsilon^2 & 1 & \epsilon^{-1} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

again making all minors of size 3 or less not crossing the central minor positive by construction. As before, we can make the whole matrix 3-positive by replacing the central minor with

$$\begin{bmatrix} 1 & \epsilon & \epsilon^2 \\ \epsilon & 1 & \epsilon \\ \epsilon^2 & \epsilon & 1 \end{bmatrix}$$

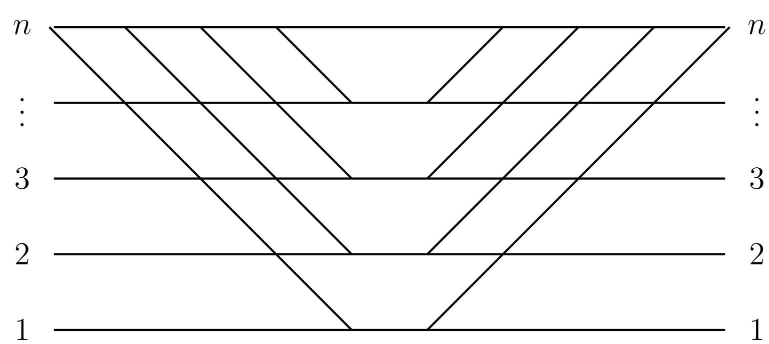
so minors not intersecting the central minor need not be considered either. This covers all 1- and 2-minors. 3-minors that cross the central 3 rows but not the central 3 columns (or the reverse) have a 1 in the upper left and the lower right, and term of order 1 in the center. All other terms in such a minor are smaller than this one by a factor of ϵ, so the minor is positive. Now consider a minor crossing both the center rows and columns. The upper left and lower right entries are negative powers of ϵ, while the upper right and lower left entries are positive powers of ϵ. The middle entry is on the scale of either 1 or ϵ, and all other terms are of order 1 at most. Thus, all other terms aside from the main diagonal term are smaller by a factor of ϵ, and thus the minor is positive. □

Providing a constructive proof for the general case has proved difficult, as the central minors were constructed from maximally k-positive matrices consisting only of 1s and 0s, which do not exist for $k \geq 3$. Nevertheless, this is expected to generalize:

Conjecture 4.2.

- *Solid k-minors are k-essential.*
- *k-essential minors are present in every k-positivity test.*

By the combinatorial proof of Theorem 3.1.10 of [5] and the discussion following it, all corner minors are n-essential. That proof motivates an interesting classification of totally nonnegative k-positive matrices in terms of planar networks. For convenience, the following definitions are repeated from [11]. The planar network Γ_0 is



where all edges are directed rightwards and sources on the left side and sinks on the right side are labeled top to bottom from n to 1. We can think of this as being composed of n "tracks", where the i-th track is the path of horizontal edges connecting source i and sink i. The *weight matrix* has (i, j)-th entry the sum of weights of all paths from source i to sink j, where the weight of a path is the product of the weights of its edges. An edge is *essential* if it is slanted or is one of the n horizontal edges in the middle. A weighting is *semi-essential* if every essential edge has weight ≥ 0 and every other edge has weight 1. Then any invertible totally nonnegative matrix can be written as the weight matrix of some semi-essential weighting of Γ_0 (though perhaps not uniquely).

Proposition 4.3. *A semi-essential weighting yields a k-positive weight matrix if and only if every horizontal edge in the lowest k tracks is > 0, and the first k downward slanted and last*

k upward slanted edges between tracks i and $i+1$ are > 0.

Proof. Suppose we have such a weighting. Then for any $I, J \subset [n]$, $|I| = |J| \leq k$, there is a vertex disjoint path connecting sources in I to sinks in J which doesn't go through any 0 edges. Specifically, the ℓ-th source in I takes the ℓ-th downward-slanted edge and follows the slanted edges until the the ℓ-th track, then takes the ℓ-th from the right upward-slanted path until the ℓ-th sink in J. Thus the appropriate minor is > 0 by Lindström's Lemma.

Conversely, suppose the weight matrix is k-positive. Then all the corner minors are positive up to order k. The only path from n to 1 goes down all the first downward slants, and so by positivity of this corner, all are positive. In general, the only collection of vertex disjoint paths from $[n-\ell+1, n]$ to $[\ell]$ takes the first ℓ downward slants all the way down and then goes across. By positivity of that corner minor, all of these edges must be positive. The same argument applies to paths from $[\ell]$ to $[n-\ell+1, n]$ and upward slanted edges. □

This has as a corollary a weaker version of Corollary 2.20, where $k+1$-nonnegativity is replaced with total nonnegativity.

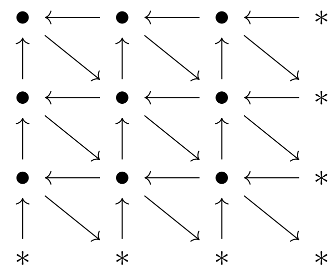
4.4 Path between Tests

In the general case, we would like to use our two known tests to find more. We do this by explicitly constructing a path between the initial minors quiver and its opposite quiver (the same quiver with all arrows reversed) which corresponds to the anti-diagonal flip test. A path here means a sequence of mutations such that every seed found corresponds to a valid k-positivity test.

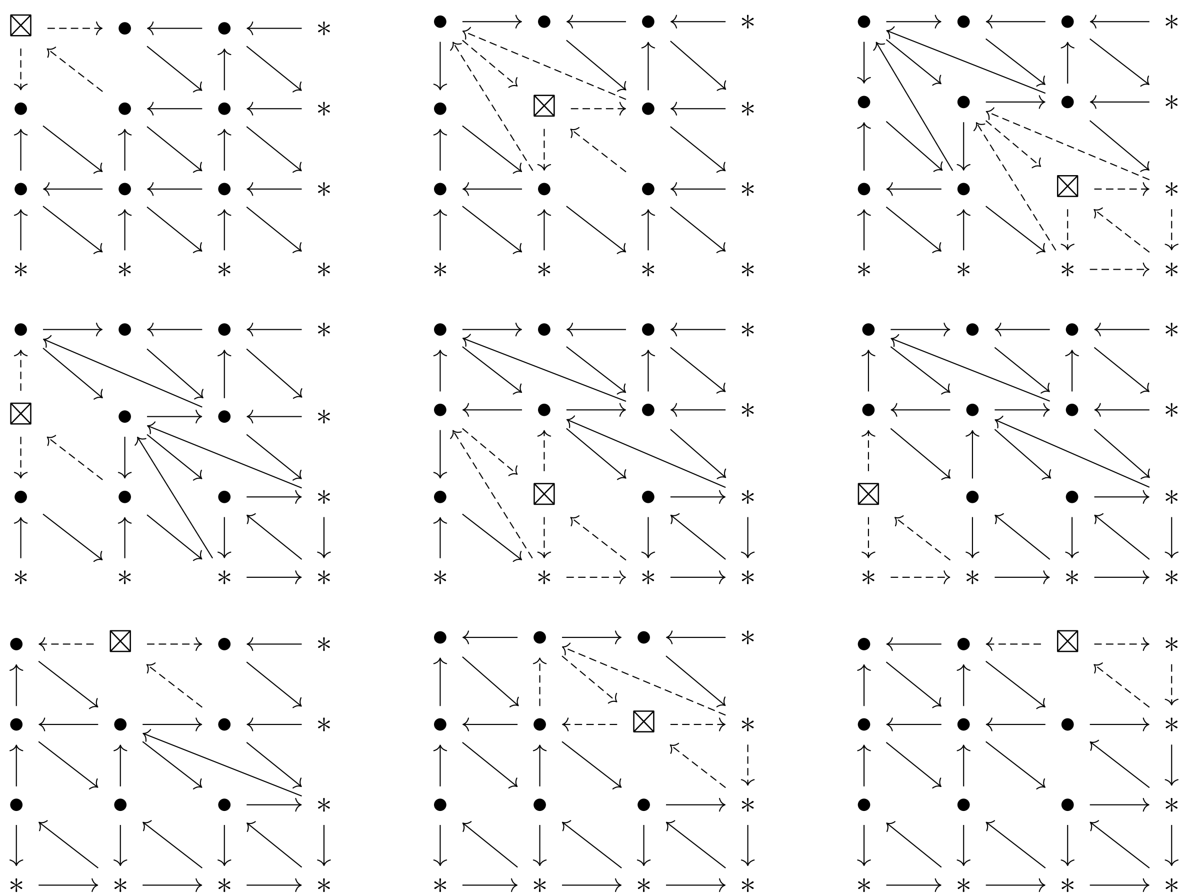
Proposition 4.4. *The following path connects the quivers corresponding to the initial test of Theorem 2.14 and its antidiagonal flip. Every edge in the path is a valid determinant-avoiding mutation, with the exception of a set of bridges and mutations at dead vertices. The path is as follows: mutate down the main diagonal, then along each sub- and superdiagonal (always skipping the last element, which lies in the last row or column). Repeat in the top left $m \times m$ submatrix as m ranges from $n-1$ to 1. Or more algorithmically:*

```
Q := Q_i(n)
For m in n, n-1, ..., 1:
    For i in [m-1]: # mutate down the diagonal
        Q := μ_(i,i)(Q)
    For r in [2,m-1]: # mutate down subdiagonals
        For i in [0,m-r-1]:
            Q := μ_(r+i,i+1)(Q)
    For c in [2,m-1]: # mutate down superdiagonals
        For i in [0,m-r-1]:
            Q := μ_(r+i,i+1)(Q)
```

Note that this requires $O(n^3)$ mutations. We now work through an example. The initial minors quiver is depicted below, with $*$ marking the frozen vertices.



Below is the first round of mutations, arranged in normal reading order. The $\boxtimes$ represents the vertex which was mutated at to get from the previous diagram, and arrows are dashed if they have changed.

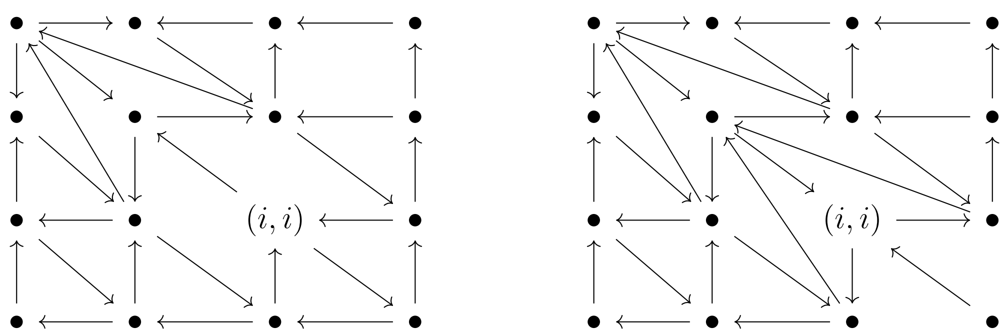


At the end of this round, we see that the upper right 3×3 subquiver is in fact the $n = 3$ initial minors quiver, and so the next round commences. We also see that the arrows not in that 3×3 subquiver have reversed directions, setting the outer portion up to be the antidiagonal flip of the initial minors quiver.

Proof. We do this via induction. After the ℓ-th round of this algorithm (which occurs in the $(n-\ell+1) \times (n-\ell+1)$ submatrix), we show that the variable at (i,j) corresponds to minor $\left|X_{[i-m+\ell+1,i+\ell],[j-m+\ell+1,j+\ell]}\right|$ where $m = \min(i,j)$. and that the subquiver obtained from the top left $(n-\ell) \times (n-\ell)$ submatrix (the "square" subquiver) has the form of the initial minors quiver, the subquiver obtained by eliminating that (the "L" subquiver) has the form of the opposite quiver, except that any edges on the boundary of these two subquivers are missing. Note in particular that each vertex always corresponds to a solid minor, the size of the submatrix to which it corresponds never changes, and the bottom left corner of the submatrix shifts one down its subdiagonal each time. Additionally, the test variables are always all of the solid minors of order k which are not already present in the extended cluster

Initially, we have $\ell = 0$ and the variables are in fact $\left|X_{[i-m+1,i],[j-m+1,j]}\right|$, the L subquiver is only the outer right and left edges which do have the horizontal and vertical arrows missing, and the test variables are the missing order k solid minors, as this is the k-initial minors test.

We now address what happens during the ℓ-th round. First we check the mutation down the diagonal. Mutating at (i,i) for $i > 1$ transforms the quiver as in the before and after images below (which each depict a subquiver). Note that if $i = n - \ell$, then the last row and column of arrows are not present in the initial image, and the after image has an up and right arrow in place of the missing ones.



In the base case of the upper left corner, one can check the forms of the quiver, and the exchange polynomial is

$$x' \cdot x_{\ell,\ell} = x_{\ell+1,\ell} \cdot x_{\ell,\ell+1} + \left|X_{[\ell,\ell+1],[\ell,\ell+1]}\right|$$

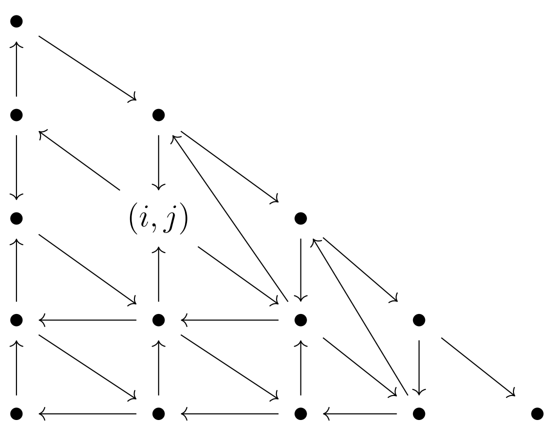
and thus $x' = x_{\ell+1,\ell+1}$. When mutating at (i,i) for $i > 1$, the exchange polynomial is

$$x' \cdot \left|X_{[\ell,i+\ell-1],[\ell,i+\ell-1]}\right| = \left|X_{[\ell+1,i+\ell-1],[\ell+1,i+\ell-1]}\right| \cdot \left|X_{[\ell,i+\ell],[\ell,i+\ell]}\right| + \left|X_{[\ell+1,i+\ell],[\ell,i+\ell-1]}\right| \cdot \left|X_{[\ell,i+\ell-1],[\ell+1,i+\ell]}\right|.$$

By Lewis Carroll's identity, this gives $x' = \left|X_{[\ell+1,i+\ell],[\ell+1,i+\ell]}\right|$. It is not hard to check the new quiver has the correct form as well. In particular, mutating at (k,k) exchanges $\left|X_{[\ell,k+\ell-1],[\ell,k+\ell-1]}\right|$ for $\left|X_{[\ell+1,k+\ell],[\ell+1,k+\ell]}\right|$. But the latter was already in the test cluster since it's a k-initial minor, and so this mutation is a bridge. Mutating at (i,i) for $i > k$ doesn't actually affect the test cluster at all, since these are dead vertices.

The previous paragraph gives the form of the quiver after the diagonal mutations. Now restrict to the subquiver using vertices on the diagonal and below (the case of above diagonal is symmetric). Inducting down the subdiagonal, one can check that mutating at (i,j) takes the quiver between the before and after subquivers depicted below. The case when $j = 1$

or $i = n - \ell$ have slightly different form, but one can check that mutating at $j = 1$ gives the correct setup for the general case (and clears the "extra" arrow), and that mutating at $i = \ell - 1$ also leaves the correct form, particularly in the arrows coming into and out of (i, j) from below, satisfying that part of the inductive hypothesis.



We now address the variables, again by induction on round. The diagonal case is addressed above. Without loss of generality, we travel down a subdiagonal (so that for any (i, j) we have $j = \min(i, j)$). As the formula for diagonal variables is in the same form as subdiagnal variables, the base case of the longest subdiagonal behaves the same as the general case, so we candeal with them together. When inducting down a particular subdiagonal, for $j = 1$, the exchange equation gives

$$x' \cdot x_{i+\ell-1,\ell} = x_{i-1+\ell,\ell+1} \cdot x_{i+\ell,\ell} + \left|X_{[i+\ell-1,i+\ell],[\ell,\ell+1]}\right|$$

and thus

$$x' = x_{i+\ell,\ell+1}.$$

Otherwise we then get an exchange equation of the form

$$\begin{aligned} x' \cdot \left|X_{[i-j+\ell,i+\ell-1],[\ell,j+\ell-1]}\right| &= \left|X_{[i-j+\ell,(i-1)+\ell],[\ell+1,j+\ell]}\right| \cdot \left|X_{[(i+1)-j+\ell,i+\ell],[\ell,j+\ell-1]}\right| \\ &\quad + \left|X_{[i-j+\ell+1,(i-1)+\ell],[\ell+1,(j-1)+\ell]}\right| \cdot \left|X_{[i-j+\ell,i+\ell],[\ell,j+\ell]}\right|. \end{aligned}$$

Using Lewis Carroll's identity on the submatrix with rows $[i - j + \ell, i + \ell]$ and columns $[\ell, j + \ell]$, this gives

$$x' = \left|X_{[i-j+\ell+1,i+\ell],[\ell+1,j+\ell]}\right|.$$

This proves the form of the variables.

Now we confirm the validity of these mutations in preserving k-positivity tests. Mutating at (i, k) turns $\left|X_{[i-k+\ell,i+\ell-1],[\ell,k+\ell-1]}\right|$ into $\left|X_{[i-k+\ell+1,i+\ell],[\ell+1,k+\ell]}\right|$. The latter is a k-initial minor from the test cluster and so this is an allowed exchange. As before, (i, j) for $j > k$ is a dead vertex and such mutations don't affect the test. Based on the form of the quiver, no other mutations go through larger submatrices. These are the only mutations we need to worry about, as one can check that any arrow added by an arbitrary mutation along the path only goes to the previous row and/or column. □

From the proof of this proposition, we can also easily prove the following fact:

Proposition 4.5. *Each sub-cluster algebra found along the path described in Proposition 4.4 has rank* $(n-1)^2 - (n-k)^2$.

Proof. The rank of the subcluster algebra is the number of active vertices in its quivers. The initial quiver has $(n-1)^2 - (n-k)^2$ active vertices: the bottom right $(n-k)^2$ are ignored as they correspond to minors of size $> k$, and the W of frozen vertices adjacent to this square contains $2n-1$ elements. This gives the correct rank. As discussed in the above proof, no mutation at a dead vertex in the ignored square affects any of the active vertices, and a mutation at a frozen vertex (which occurs when jumping between subalgebras) never adds edges between active and dead vertices, and always keeps the frozen vertex adjacent to a dead one. Therefore the number of active vertices is the same. □

In fact, there is some choice in the order to do these mutations. Consider permutations of the path in which mutations on any particular sub- or super diagonal occur sequentially. Just as in the original path, the ℓ-th time mutating on a particular diagonal does not mutate at the last ℓ vertices. In other words, the only change is how these (sub-/super)diagonals are interleaved with each other. We use words in an alphabet $A = \{d, b_i, p_i \,|\, i \in [n]\}$ to keep track of this interleaving, where the letter in the j-th position denotes the j-th series of mutations: a d is mutating the diagonal, a b_i is mutating the i-th subdiagonal, and p_i is mutating the i-th superdiagonal. A word is turned into a diagram as follows: attach all x's (for any fixed letter x) into a diagonal chain of boxes with that many elements. Then attach them so that the first d box is anchored in the upper left corner, the first p_1 box (if any) is to the right, and the first p_i box is to the right of the first p_{i-1} box (if any, leaving a gap if it is not present). The same rule holds for the b_i, but these get attached to the bottom instead. For example, the word $dp_1p_2dp_1b_2$ becomes

d	p_1	p_2
	d	p_1
b_2		

A *valid path variant* is one which at every step looks locally like the original path, both in shape of quiver and variables. Specifically, after the ℓ-th mutation at any vertex, the variable and local quiver are the same as in the original path. These diagram transformations determine when some variants are valid.

Lemma 4.6. *Let* w *be a word with* $n-1$ d*'s,* $n-1-i$ b_i*'s, and* $n-1-i$ p_i*'s. If the diagram formed from every initial subword of* w *is a Young diagram, then this sequence of mutations gives a valid path variant.*

For example, the word $dp_1p_2b_1dp_1b_2b_1d$ is valid for $n=4$, but $dp_1p_2db_1p_1b_2b_1d$ is not because the initial subword dp_1p_2d has diagram

d	p_1	p_2
	d	

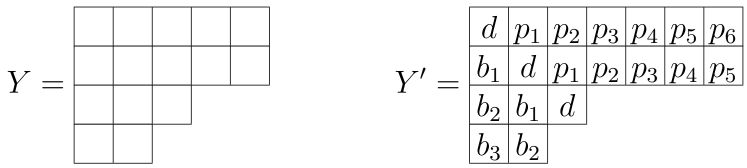
Proof. Since mutations on sub- and superdiagonals are isolated from each other, we can freely commute mutations above and mutations below. We note that at any point along the path, any vertex in any quiver can only be adjacent to a subset of 8 different vertices: those above, below, left, right, above-left, below-right, as well as two more for the "extra slanted" edges (with specific direction depending on whether the vertex is diagonal, above, or below). Thus the mutation is indeed only affected by the variables and shape of the quiver locally, and we now proceed to confirm that the details of the proof of the variables in the path applies for such words.

All of the following statements can be verified inductively. For the local area to have the right shape to apply the ℓ-th mutation at a vertex on the main diagonal, it's good enough to have cleared the "extra" edges by mutating above and to the left $\ell - 1$ times. To get the same exchange relation, the vertices below and to the right must also have been mutated at exactly $\ell - 1$ times. The equivalent condition for the constructed diagram is that when placing the ℓ-th d box, it has a b_1 to the left and a p_1 above, and no extra b_1 or p_1 boxes to the right or below. Next look at the ℓ-th mutation of subdiagonal vertices (superdiagonal are symmetric). For the exchange relation to have the right form, the higher diagonal must have been mutated ℓ times, and the lower diagonal only $\ell - 1$. This same condition gives the quivers the right form. This corresponds to having a b_{i-1} above and a b_{i+1} to the right, but no extra below or to the left. Therefore since w gives a Young diagram at every step, the proof of Proposition 4.4 extends and for any k, these mutations preserve k-positivity tests. □

Such Young diagram words give valid choices of paths between components of the exchange graph. Observe that two different such words which give the same final Young diagram both end in the same component. Observe also that any boxes outside of an $(n - k) \times (n - k)$ square are all mutations at mutable vertices in the k-quiver (since the length down which we mutate the diagonal decreases by one each round). Thus one gets a correspondence between Young diagrams contained in an $(n - k) \times (n - k)$ square and these components found along the path. Using such a Young diagram, one can also recover an explicit test cluster as discussed below.

In order to catalog the connected components of the exchange graph found along this family of paths, we define the *bridge graph* as $G_b = (V_b, E_b)$, where the elements of V_b are test cluster sets of the connected components on the path and we assign an edge between two components if some mutation on some path in the family connects them. We then label these edges by the elements that differ between them. If the lower-right corners of the exchanged minors lie on the center diagonal, we use the label d; if it lies on the i-th superdiagonal (resp. subdiagonal) we use p_i (resp. b_i); this then gives a labeling of the components by the Young diagram which is built from these blocks as in the above discussion.

Given a Young diagram Y contained in a $(n - k) \times (n - k)$ box, we can give its corresponding test cluster, arranged as entries of an $n \times n$ matrix M. First we construct a related Young diagram Y' by taking Y and for every row of length $n - k$, appending $k - 1$ boxes the right, and for every column of length $n - k$, appending $k - 1$ boxes below. Note that Y' is now contained in a $(n - 1) \times (n - 1)$ box. Additionally, if Y filled its entire box, add boxes to Y' until it does as well. In terms of path mutations, these additions correspond to adding the extra p_i, b_i, and d whose mutations occur entirely at mutable vertices. Now label all the boxes in Y' with the same alphabet from before For example, if $n = 7$ and $k = 2$, and $Y = (5, 5, 3, 2)$, then $Y' = (7, 7, 3, 2)$ and the following diagram labels each box according to which component it belongs to.



The entries of M are similar to the entries of the k-initial minor matrix, except that the lower right corner of each minor is shifted down the appropriate sub/super diagonal by the number of boxes in that component, but the entry in (i, j) can be shifted at most $\min(n - i, n - j)$ places. More formally, if $i < j \leq k$, the entries of M are as follows:

$$
\begin{aligned}
m_{ii} &= \left| X_{[1+D,i+D],\,[1+D,i+D]} \right| \\
m_{ij} &= \left| X_{[1+P_{j-i},i+P_{j-i}],\,[j-i+P_{j-i}+1,j+P_{j-i}]} \right| \\
m_{ji} &= \left| X_{[j-i+B_{j-i}+1,j+B_{j-i}],\,[1+B_{j-i},i+B_{j-i}]} \right|
\end{aligned}
$$

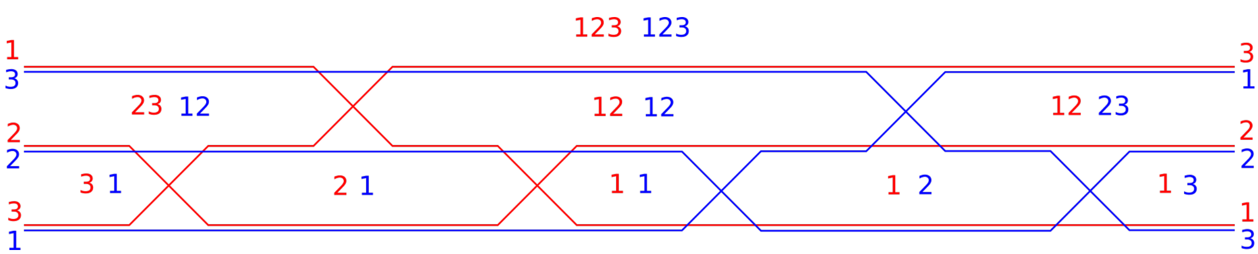
where $d = \min(n - i, \#\{d \in Y'\})$, $p_{j-i} = \min(n - j, \#\{p_{j-i} \in Y'\})$, and $b_{j-i} = \min(n - j, \#\{b_{j-i} \in Y'\})$ (and for the diagonal entry, we can take $i = k$ as well). Otherwise, if $k \leq i < j$, the entries of M are all $k \times k$ minors. These can be filled in as desired, though it is convenient to have the lower right corners on the correct diagonal.

4.5 Double Wiring Diagrams

We start by recalling the appropriate definitions from [11].

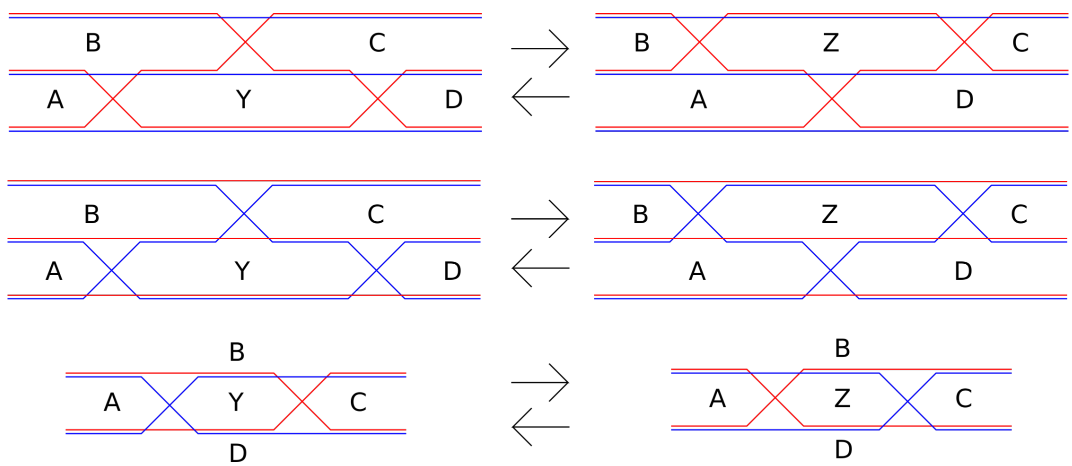
Definition. A *wiring diagram* consists of a family of n piecewise straight lines, all of the same color, such that each line intersects every other line exactly once. A *double wiring diagram* is two wiring diagrams of different color which are overlaid.

We will color our diagrams red and blue, and number the strings such that the left endpoint of the reds go down, and the left endpoints of the blue go up. Each diagram has n^2 "chambers". We can label a chamber by the tuple (r, b), where r is the indices of all red strings passing below it, and b is the indices of all blue strings passing below it. For example,



We can associate each chamber with the minor of the correspondingly indexed submatrix $|X_{r,b}|$. With this correspondence, every double wiring diagram gives a total positivity test.

Additionally, double wiring diagrams can be transformed into a quiver giving the corresponding test (see [8]) and there is also a method for transforming double wiring diagrams via braid relations (see [11]). These are depicted below.



A braid move only changes a single minor (in each case, minors Y and Z are interchanged), and in all cases the exchange relation $YZ = AC + BD$.

An alternative way to conceptualize this process is as follows: To describe a diagram, it is sufficient to describe the relative positions of all of the crossings. We can think of the diagram having n "tracks", where track i has all the chambers for $i \times i$ submatrices, and each crossing occurs in one of the first $n-1$ tracks. We label a red crossing in the i-th track as e_i, and a blue crossing in the i-th track as f_i. Then a sequence of crossings describing a double wiring diagram is a reduced word for the element (w_0, w_0) of the Coxeter group $S_n \times S_n$, where w_0 is the longest word, the order reversing permutation. This choice of variable names is not coincidental, double wiring diagrams and those with the weaker condition that every pair of same colored strings intersects at most once corresponds to factorizations, see [11]. We note that the braid moves of $e_\ell e_{\ell+1} e_\ell \leftrightarrow e_{\ell+1} e_\ell e_{\ell+1}$ and $f_\ell f_{\ell+1} f_\ell \leftrightarrow f_{\ell+1} f_\ell f_{\ell+1}$ have exchange relations which use minors of order ℓ and $\ell+1$, The move $e_\ell f_\ell \leftrightarrow f_\ell e_\ell$ uses minors of order $\ell-1$, ℓ, $\ell+1$.

The Young diagram correspondence from the previous section can now be extended one step further to result in a double wiring diagram. First, some useful notation. Let $r_i = e_{n-i} \cdots e_1$ for $1 \leq i < n-k$, and let $r_{n-k} = \prod_{i=1}^{k} e_i \cdots e_1$, ordered such that $i = 1$ is on the left and $i = k$ is on the right. Let $b_i = f_1 \cdots f_{n-i}$ for $1 \leq i < n-k$ and let $b_{n-k} = \prod_{i=k}^{1} f_1 \cdots f_i$, where now the concatenation runs in the opposite order: $i = k$ is on the left and $k = 1$ is on the right. For example, the lexicographically minimal diagram is the word $r_{n-k} \cdots r_1 b_1 \cdots b_{n-k}$, which corresponds to the initial minors test, and the word $b_1 \cdots b_{n-k} r_{n-k} \cdots r_1$ is the anti-diagonoal flip of the initial minors test. The order of the red wires between r_i and r_{i-1} (where r_0 and r_{n-k+1} correspond to no crossings) for $i \leq n-k$ is $i(i+1)\cdots(n-1)n(i-1)\cdots 21$, reading from bottom to top. This is proven via induction on i. If $i = 1$, the wires are already in order. In the general case, moving from the right of r_i to the left brings the bottom wire up to the $n-i$-th level, and all wires originally in the $(n-i)$-th row or lower are shifted down by one. To the left of r_{n-k}, the wires are ordered $n(n-1)\cdots 21$. The blue case behaves symmetrically, and the order of the blue wires between b_{i-1} and b_i for $i \leq n-k$ is

$i(i+1)\cdots(n-1)n(i-1)\cdots 21$ and right of b_{n-k} is $n(n-1)\cdots 21$.

Theorem 4.7. *Let Y be a Young diagram which fits in a $(n-k)\times(n-k)$ box. Construct the corresponding double wiring diagram as follows: start with the word $b_1\cdots b_{n-k}$. For $i\in[n-k]$, insert r_i between b_ℓ and $b_{\ell+1}$ where ℓ is the number of boxes in the i-th row of Y. If there is already an r_j in that position, insert r_i to the left if and only if $i>j$, otherwise insert it to the right. The result is an interleaving of the words $b_1\cdots b_{n-k}$ and $r_{n-k}\cdots r_1$ which gives the total positivity test corresponding to that path component. To turn it into the correct k-positivity test, disregard all chambers above the k-th track and add in the remaining solid minors.*

Proof. We do prove this by showing that the full quiver given by this Young diagram corresponds to this double wiring diagram, and then the correct k-positivity test statement will follow. When doing this, we for simplicity work with Young diagrams contained in an $(n-1)\times(n-1)$ box, i.e. the $k=1$ case. Here, r_{n-k} behaves exactly like the rest of the letters because it is only composed of a single decreasing chain of e_i; in fact $r_{n-1}=e_1$. The other cases will follow since grouping more decreasing chains together into r_{n-k} just corresponds to skipping the intermediate diagrams which go with those independent moves, and this grouping corresponds to the extra row or column which is appended in the Young diagram correspondence described at the end of Section 4.4. We proceed by induction on the number of boxes in the diagram. The base case is the lexicographically minimal diagram, already discussed. Assume the statement holds for diagrams with j boxes. Now add an ℓ-th box to the i-th row, where i is a row such that this is a valid addition. This changes the word from $\cdots r_i b_\ell\cdots$ to $\cdots b_\ell r_i\cdots$. The chambers which change are in tracks $\min(n-i,n-\ell)$ and lower. The chamber in track j goes from $([i,i+j-1],[\ell,\ell+j-1])$ to $([i+1,i+j],[\ell+1,\ell+j])$. The added box is on the main diagonal if $i=\ell$, is in component $P_{\ell-i}$ if $\ell>i$, and otherwise in $B_{i-\ell}$. The number of boxes in this component in the new diagram is $\min(i,\ell)$. By the inductive hypothesis and the original chambers, we see that if the minors are arranged in a matrix (as in the construction of the Young diagram test), it is the correct diagonal which is being changed, and the resulting chambers are correct as well. □

As when working with quivers, we can restrict the allowed braid moves so that k-positivity tests are preserved. As before, we eliminate mutations at minors of order $\le k$ (meaning in track k or lower) whose exchange relations involve minors of order $>k$. Mutations at higher order minors are allowed as such minors are disregarded in the test anyway. By looking at the exchange relations for the braid moves, the disallowed mutations are of the form $e_kf_k\leftrightarrow f_ke_k$, $e_ke_{k+1}e_k\leftrightarrow e_{k+1}e_ke_{k+1}$, and $f_kf_{k+1}f_k\leftrightarrow f_{k+1}f_kf_{k+1}$. Although going "forwards" in the second two mutations is technically a mutation at a higher order minor, it introduces an order k minor and so can't be simply disregarded.

A Code

All code used can be found at https://github.com/ewin-t/k-nonnegativity. In particular, we have code for:

1. Generating shapes of op-irreducible matrices (through a somewhat-optimized brute force technique).
2. Generating k-nonnegative matrices (slowly and through brute force).
3. Generating the exchange graphs of the sub-cluster algebras for $k \leq 2$ or $n \leq 3$.

B Relations in Full

First, the $k = n - 1$ relations (that is, for $K(\vec{a}, \vec{b})$):

In these relations, the variables on the right-hand side are expressed in terms of the variables on the left-hand side.

(1) $e_i(x)K(\vec{a}, \vec{b}) = K(\vec{A}, \vec{B})e_{i+1}(x'),$ where $1 \leq i \leq n-2$:

The following equalities hold for $i < n - 2$.

$$\vec{A} = \left(a_1, \dots, a_{i-1}, a_i + x, \frac{a_i a_{i+1}}{a_i + x}, a_{i+2}, \dots, a_{n-2}\right)$$
$$\vec{B} = \left(b_1, \dots, b_{i-1}, b_i + \frac{x a_{i+1}}{a_i + x}, \frac{b_i b_{i+1}(a_i + x)}{b_i(a_i + x) + x a_{i+1}}, b_{i+2}, \dots, b_{n-1}\right)$$
$$x' = \frac{b_{i+1} a_{i+1} x}{b_i(a_i + x) + x a_{i+1}}.$$

and in the other direction,

$$\vec{a} = \left(A_1, \dots, A_{i-1}, \frac{A_i A_{i+1} B_{i+1} + A_i A_{i+1} x'}{A_{i+1} B_{i+1} + A_{i+1} x' + B_i x'}, A_{i+1} + \frac{B_i x'}{B_{i+1} + x'}, A_{i+2}, \dots, A_{n-3}\right)$$
$$\vec{b} = \left(B_1, \dots, B_{i-1}, \frac{B_i B_{i+1}}{B_{i+1} + x'}, B_{i+1} + x', B_{i+2}, \dots, B_{n-2}\right)$$
$$x = \frac{x' A_i B_i}{A_{i+1} B_{i+1} + A_{i+1} x' + B_i x'}$$

and when $i = n - 2$, we have

$$\vec{A} = (a_1, \dots, a_{n-3}, a_{n-2} + x)$$
$$\vec{B} = \left(b_1, \dots, b_{n-2}, \frac{b_{n-1} a_{n-2}}{a_{n-2} + x}\right)$$
$$x' = \frac{b_{n-1} \cdots b_2 b_1 x}{b_{n-2}(a_{n-2} + x)}$$

and in the other direction,

$$\vec{a} = \left(A_1, \ldots, A_{n-3}, \frac{A_{n-2}B_1 \cdots B_{n-1}}{B_1 \cdots B_{n-1} + x'B_{n-2}}\right)$$
$$\vec{b} = \left(B_1, \ldots, B_{n-2}, B_{n-1} + \frac{x'}{B_1 \cdots B_{n-3}}\right)$$
$$x = \frac{A_{n-2}B_{n-2}x'}{B_1 \cdots B_{n-1} + x'B_{n-2}}$$

(2) $e_{n-1}(x)K(\vec{a},\vec{b}) = K(\vec{A},\vec{B})f_{n-1}(x')h_{n-1}(c)$:

$$c = \frac{Y}{Y + xX} = \frac{1}{1 + x'X}$$
$$\vec{A} = \vec{a}$$
$$\vec{B} = \left(b_1, \ldots, b_{n-3}, \frac{b_{n-2}}{c}, b_{n-1}\right)$$
$$x' = \frac{x}{Y}$$

(3) $f_{i+1}(x)K(\vec{a},\vec{b}) = h_{i+2}(1/w)K(\vec{A},\vec{B})f_i(x)h_i(w)$, where $1 \leq i \leq n-2$:

when $1 \leq i < n-2$, we have:

$$w = \frac{1}{1 + xa_{i+1} + xb_i}$$
$$\vec{A} = \left(a_1, \ldots, a_{i-2}, a_{i-1}, a_i(xa_{i+1} + 1), \frac{a_{i+1}(xa_{i+1} + xb_i + 1)}{1 + xa_{i+1}}, \frac{a_{i+2}}{xa_{i+1} + xb_{i+1} + 1}, a_{i+3}, \ldots, a_{n-2}\right)$$
$$\vec{B} = \left(b_1, \ldots, b_{i-2}, b_{i-1}(xa_{i+1} + xb_i + 1), \frac{b_i}{xa_{i+1} + 1}, \frac{b_{i+1}(1 + xa_{i+1})}{xa_{i+1} + xb_i + 1}, b_{i+2}, \ldots, b_{n-1}\right)$$

and for the other direction:

$$w = \frac{1}{1 + xA_{i+1} + xB_i}$$
$$\vec{a} = \left(A_1, \ldots, A_{i-2}, A_{i-1}, \frac{A_i(1 + xB_i)}{xA_{i+1} + xB_i + 1}, \frac{A_{i+1}}{1 + xB_i}, A_{i+2}(xA_{i+1} + xB_i + 1), A_{i+3}, \ldots, A_{n-2}\right)$$
$$\vec{b} = \left(B_1, \ldots, B_{i-2}, \frac{B_{i-1}}{xA_{i+1} + xB_i + 1}, \frac{B_i(xA_{i+1} + xB_i + 1)}{1 + xB_i}, B_{i+1}(1 + xB_i), B_{i+2}, \ldots, B_{n-1}\right)$$

and when $i = n-2$:

$$w = \frac{1}{1 + xb_{n-2}} \text{ and } \vec{A} = \vec{a}$$
$$\vec{B} = \left(b_1, \ldots, b_{n-4}, b_{n-3}(xb_{n-2} + 1), b_{n-2}, \frac{b_{n-1}}{xb_{n-2} + 1}\right)$$

(4) $f_1(x)K(\vec{a},\vec{b}) = K(\vec{A},\vec{B})e_1(x')h_1(c)$:

$$\vec{A} = \left(\frac{a_1}{1+xa_1}, a_2, \ldots, a_{n-2}\right)$$
$$\vec{B} = \vec{b}$$
$$x' = xb_1a_1$$
$$c = \frac{1}{1+xa_1}$$

For the other direction, we use $a_1 = A_1 + \frac{A_1x'}{B_1}$ and $c = \frac{B_1}{B_1+x'}$.

(5) $h_i(x)K(\vec{a},\vec{b}) = K(\vec{A},\vec{B})h_{i-1}(x)$, where $2 \leq i \leq n$:

$$\vec{A} = \left(a_1, \ldots, a_{i-1}, xa_i, \frac{a_{i+1}}{x}, a_{i+2}, \ldots, a_{n-3}\right)$$
$$\vec{B} = \left(b_1, \ldots, xb_{i-1}, \frac{b_i}{x}, b_{i+1}, b_{i+2}, \ldots, b_{n-2}\right)$$

(6) $h_1(x)K(\vec{a},\vec{b}) = K(\vec{A},\vec{B})$, where $\vec{A} = (xa_1, a_2, \ldots, a_{n-3})$ and $\vec{B} = \vec{b}$.

(7) $K(\vec{a},\vec{b})h_n(x) = K(\vec{A},\vec{B})$, where $\vec{A} = \vec{a}$ and $\vec{B} = (b_1, \ldots, b_{n-3}, xb_{n-2})$.

Now, we list the $k = n-2$ relations (that is, for the $T(\vec{a},\vec{b})$ parameter family):

(1) $e_i(x)T(\vec{a},\vec{b}) = T(\vec{A},\vec{B})e_{i+2}(x')$, where $1 \leq i \leq n-3$.

$$\vec{A} = \left(a_1, \ldots, a_{i-1}, a_i + x, \frac{a_ia_{i+1}}{a_i+x}, a_{i+2}, \ldots, a_{n-3}\right)$$
$$\vec{B} = \left(b_1, \ldots, b_{i-1}, b_i + \frac{xa_{i+1}}{x+a_i}, \frac{b_ib_{i+1}(x+a_i)}{b_i(a_i+x)+xa_{i+1}}, b_{i+2}, \ldots, b_{n-2}\right)$$
$$x' = \frac{b_{i+1}a_{i+1}x}{b_i(a_i+x)+xa_{i+1}}$$

In the other direction, we have:

$$\vec{a} = \left(A_1, \ldots, A_{i-1}, \frac{A_iA_{i+1}B_{i+1} + A_iA_{i+1}x'}{A_{i+1}B_{i+1} + A_{i+1}x' + B_ix'}, A_{i+1} + \frac{B_ix'}{B_{i+1}+x'}, A_{i+2}, \ldots, A_{n-3}\right)$$
$$\vec{b} = \left(B_1, \ldots, B_{i-1}, \frac{B_iB_{i+1}}{B_{i+1}+x'}, B_{i+1} + x', B_{i+2}, \ldots, B_{n-2}\right)$$
$$x = \frac{x'A_iB_i}{A_{i+1}B_{i+1} + A_{i+1}x' + B_ix'}$$

(2) $e_{n-2}(x)T(\vec{a},\vec{b}) = T(\vec{A},\vec{B})e_1(x')$.

Here $\vec{A}$ and $\vec{B}$ satisfy the following recurrence:

$$\begin{aligned} B_{n-3} &= b_{n-3} + x \\ A_i &= (a_i \cdot b_i)/B_i, \text{ where } 1 \le i \le n-3 \\ B_i &= a_{i+1} + b_i - A_{i+1}, \text{ where } 1 \le i \le n-4 \\ x' &= a_1 - A_1. \end{aligned}$$

(Note that $B_{n-3} > b_{n-3}$, and consequently $A_{n-3} < a_{n-3}$. In turn, $B_{n-2} > b_{n-2}$, etc, so that in general $B_i > b_i$ and $A_i < a_i$.) In the other direction,

$$\begin{aligned} a_1 &= x' + A_1 \\ c_i &= A_i C_i / a_i \text{ where } 1 \le i \le n-3 \\ a_i &= A_i + C_{i-1} - c_{i-1}, \text{ where } 1 \le i \le n-4 \\ x &= C_{n-3} - c_{n-3}. \end{aligned}$$

(Similarly, $a_1 > A_1$, consequently $c_2 < C_2$. In turn, $a_2 > A_2$, etc, so that in general $a_i > A_i$ and $c_i < C_i$.)

(3) $e_{n-1}(x)T(\vec{a},\vec{b}) = T(\vec{A},\vec{B})e_2(x')$

$$\begin{aligned} \vec{A} &= \vec{a} \\ \vec{B} &= \left(b_1, \ldots, b_{n-3}, b_{n-2} + \frac{b_{n-2}}{b_1 x},\right) \\ x' &= \frac{x}{\left|T(\vec{a},\vec{b})_{[3,n-3],[4,n-2]}\right|} \end{aligned}$$

In the other direction,

$$\begin{aligned} \vec{a} &= \vec{A} \\ \vec{b} &= \left(B_1, \ldots, B_{n-3}, \frac{B_{n-2}B_1}{B_1 + x'},\right) \\ x &= x' \left|T(\vec{A},\vec{B})_{[3,n-3],[4,n-2]}\right| \end{aligned}$$

(4) $e_{n-1}e_{n-2}T = e_{n-2}e_{n-1}T \sqcup e_{n-2}\cdots e_1 e_{n-1}\cdots e_2 \sqcup e_{n-2}\cdots e_1 e_{n-1}\cdots e_1$.

The three factorizations on the right hand side of the equation arise from three possible values of the minor $\left|M_{[2,n][1,n-1]}\right|$, where $M = e_{n-1}(u)e_{n-2}(v)T(\vec{a},\vec{b})$.

(a) When the minor is negative, then we have:

$$\begin{aligned} (a_1 \ldots a_{n-3}) \cdot v \cdot (X + u) &< (a_1 \ldots a_{n-3}) \cdot (b_{n-2}Y + vX) \\ \Rightarrow\ v \cdot (X + u) &< (b_{n-2}Y + vX) \\ \Rightarrow\ vu &< b_{n-2}Y \end{aligned}$$

Then the matrix M can be factored as follows.

$$e_{n-1}(u)e_{n-2}(v)T(\vec{a},\vec{b}) = e_{n-2}(v)e_{n-1}(u)T(\vec{A},\vec{B}),$$

where $\vec{A} = \vec{a}$ and $\vec{B} = (b_1, \ldots, b_{n-3}, b_{n-2} - uv/Y)$.

(b) When the minor is zero, then we have:

$$\begin{aligned}(a_1 \ldots a_{n-3}) \cdot v \cdot (X+u) &= (a_1 \ldots a_{n-3}) \cdot (b_{n-2}Y + vX) \\ \Rightarrow\ v \cdot (X+u) &= b_{n-2}Y + vX\end{aligned}$$

It follows that the matrix is totally nonnegative and can be factored as shown below.

$$e_{n-1}(u)e_{n-2}(v)T(\vec{a},\vec{b}) = e_{n-2}(v)e_{n-3}(a_{n-3}) \cdots e_1(a_1)e_{n-1}(X+u)e_{n-2}(b_{n-3}) \cdots e_2(b_1)$$

(c) When the minor is positive, then we have:

$$\begin{aligned}(a_1 \ldots a_{n-3}) \cdot v \cdot (X+u) &> (a_1 \ldots a_{n-3}) \cdot (b_{n-2}Y + vX) \\ \Rightarrow\ v \cdot (X+u) &> b_{n-2}Y + vX\end{aligned}$$

It follows that the matrix is totally nonnegative and can be factored as written below.

$$e_{n-1}(u)e_{n-2}(v)T(\vec{a},\vec{b}) = e_{n-2}(v')e_{n-3}(A_{n-3}) \cdots e_1(A_1)e_{n-1}(X+u)e_{n-2}(B_{n-3}) \cdots e_1(B_0),$$

where $\vec{A}$, $\vec{B}$ and v' can be determined from $\vec{a}$, $\vec{b}$, u and v, by using the following recursive formulas.

$$\begin{aligned}v' &= \frac{b_{n-2}Y + vX}{X+u} \\ B_{n-3} &= b_{n-3} + v - v' \\ A_i &= (a_i \cdot b_i)/B_i, \text{ where } 1 \le i \le n-3 \\ B_i &= a_{i+1} + b_i - A_{i+1}, \text{ where } 0 \le i \le n-4\end{aligned}$$

Note that our calculations above show that $v > v'$, and this will show that $a_i > A_i$ and $b_i < B_i$ for all i.

C Op-irreducible shapes for 5×5 matrices

We give enumerate all op-irreducible shapes for 5×5 matrices, from fewest nonzero entries to most nonzero entries. This was computed through brute-force code. In the matrices below, the asterisks denote the locations of nonzero entries. We only list the shapes up to twelve nonzero entries, since we can establish a bijection between shapes with k nonzero entries and shapes with $25 - k$ nonzero entries, by swapping the locations of zero and nonzero entries. Thus, we only need to list the shapes with at most half of the entries nonzero. These are permuted in what we perceive to be the neatest form; we have not found a consistent order that allows for clear patterns in the shapes.

The observant reader will note that if there is a row with 1 or $n-1$ nonzero entries, then there must be a column with 1 or $n-1$ nonzero entries, respectively. Removing the row and column gives us an op-irreducible size $n-1$ shape.

$$\begin{bmatrix} * & & & & \\ & * & & & \\ & & * & & \\ & & & * & \\ & & & & * \end{bmatrix} \begin{bmatrix} * & & & & \\ & * & & & \\ & & * & * & \\ & & * & & * \\ & & & * & * \end{bmatrix} \begin{bmatrix} * & & & & \\ & * & * & & \\ & * & & * & \\ & & * & & * \\ & & & * & * \end{bmatrix} \begin{bmatrix} * & & & & \\ & * & * & & \\ & * & & * & \\ & * & & & * \\ & & * & * & * \end{bmatrix} \begin{bmatrix} * & * & & & \\ * & & * & & \\ & * & & * & \\ & & * & & * \\ & & & * & * \end{bmatrix}$$

$$\begin{bmatrix} * & * & & & \\ * & & * & & \\ * & & & * & \\ & * & & & * \\ & & * & * & * \end{bmatrix} \begin{bmatrix} * & * & & & \\ * & & * & & \\ & * & & * & \\ & & * & & * \\ * & & & * & * \end{bmatrix} \begin{bmatrix} * & * & & & \\ * & & * & & \\ & * & & * & \\ & * & * & & * \\ * & & & * & * \end{bmatrix} \begin{bmatrix} * & * & & & \\ * & & * & & \\ & * & * & * & \\ & * & * & & * \\ & & & * & * \end{bmatrix} \begin{bmatrix} * & * & & & \\ * & & * & & \\ * & & & * & \\ * & & & & * \\ & * & * & * & * \end{bmatrix}$$

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