TORIC MUTATIONS IN THE DP$_2$ QUIVER AND SUBGRAPHS OF THE DP$_2$ BRANE TILING

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ABSTRACT. Brane tilings are infinite, bipartite, periodic, planar graphs that are dual to quivers. In this paper, we examine the del Pezzo 2 (dP$_2$) quiver and its brane tiling, which arise from the physics literature, in terms of toric mutations on its corresponding cluster. Specifically, we give explicit formulas for all cluster variables generated by toric mutation sequences. Moreover, for each such variable, we associate a subgraph of the dP$_2$ brane tiling to it such that its weight matches the variable.

1. INTRODUCTION

Cluster algebras are a class of commutative rings generated by cluster variables, which are partitioned into sets called clusters. Given an initial seed, an operation known as seed mutation can be applied iteratively to generate all cluster variables. The concept of cluster algebras was first introduced by Fomin and Zelevinsky [FZ02] as a tool to study total positivity and dual canonical bases in Lie theory. They have rich applications in different branches of mathematics, including algebraic combinatorics, tropical geometry, Teichmuller theory, and representation theory.

It is common to picture a cluster as a quiver with a cluster variable on each vertex. Some special quivers have planar duals, known as brane tilings, which are doubly-periodic, bipartite, planar graphs. The notion of brane tilings is first introduced in theoretical physics [FHV+06]. For such quivers, combinatorial interpretations of the cluster variables have been obtained by associating a subgraph of the brane tiling to each cluster variable such that the Laurent polynomial of the cluster variable is recoverable from a weighting scheme applied to the subgraph. See [MS10], [Mus11], and [LS13]. In particular, the quiver and brane tiling of the third del Pezzo (dP$_3$) surface [HS12] has been studied widely by [Zha], [LMNT14], and [LM15]. In this paper, we will generalize the techniques utilized in these papers and focus on the second del Pezzo (dP$_2$) surface. Specifically, we will classify all cluster variables generated by toric mutations and give combinatorial interpretations for them.

2. PRELIMINARIES

2.1. Quiver and Cluster Mutations.

Definition 2.1 (Quiver and Cluster). A *quiver* is a finite directed graph Q with a set of vertices V and a set of edges E. We can associate a cluster variable x_i to the vertex labeled i. The *cluster* is the ordered set of cluster variables $\{x_1, \ldots, x_n\}$ at each vertex, assuming $|V| = n$. For a cluster $S = \{x_1, \ldots, x_n\}$, let $S[i]$ refer to the ith cluster variable.

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In this paper, we allow quivers to have multiple edges connecting two vertices but there can be no 2-cycles or 1-cycles (loops).

Definition 2.2 (Quiver Mutation). Mutating at a vertex i in Q is denoted by μ_i and corresponds to the following actions on the quiver:

- For every 2-path through i (e.g. $j \to i \to k$), add an edge from j to k.
- Reverse the directions of the arrows incident to i.
- Delete any 2-cycles created from the previous two steps.

Correspondingly, the cluster variable at vertex i is updated and all other cluster variables stay the same. The update follows this binomial exchange relation:

$$x'_i x_i = \prod_{i \to j \text{ in } Q} x_j^{a_{i \to j}} + \prod_{j \to i \text{ in } Q} x_j^{b_{j \to i}},$$

where x'_i is the new cluster variable at vertex i and $a_{i \to j}$ is the number of edges from i to j.

The binomial exchange relation replaces $S[i]$ by the new cluster variable x'_i. We denote this replacement by

$$S[i] \leftarrow \frac{\prod\limits_{i \to j \text{ in } Q} x_j^{a_{i \to j}} + \prod\limits_{j \to i \text{ in } Q} x_j^{b_{j \to i}}}{x_i}.$$

2.2. **The Del Pezzo 2 Quiver and its Brane Tiling.** In this paper, we will study a special quiver associated to the second del Pezzo surface (dP$_2$) [BP01] and its brane tiling, as seen in Figure 1.
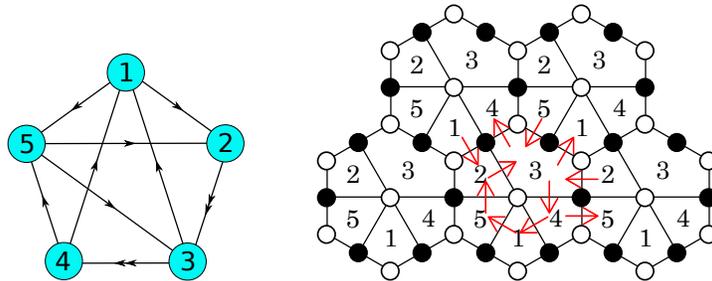

FIGURE 1. dP$_2$ quiver Q and its associated brane tiling $\mathcal{T}$ (Figure 30 of [HS12])

To get from a brane tiling to the corresponding quiver, we look at each edge e up to translation, noticing that any brane tiling is periodic, bipartite and planar. Assume that e borders block i and j such that as we go across from block i to block j, the black end point of e is on the left and the white end point of e is on the right. For this edge e, we add an edge in the quiver that goes from i to j. The red arrows in Figure 1 show this process.

We use Q to denote the dP$_2$ quiver and $\mathcal{T}$ to denote its associated brane tiling .

2.3. **Toric Mutation and Two Models of Quivers.**

Definition 2.3 (Toric Vertex and Toric Mutation). We say that a vertex in a quiver is *toric* if it has in-degree 2 and out-degree 2. A *toric mutation* is a cluster mutation at a toric vertex.

Definition 2.4 (Model). We say that two quivers Q_1 and Q_2 are of the same *model* if they are isomorphic as directed graphs (there exists a bijection between their vertices that preserves edges), or if Q_1 is isomorphic as graph to Q_2 with all edges in Q_2 reversed.

It is easy to check that the dP$_2$ quiver Q has two models that can be reached from the original quiver by toric mutations. Use **Model 1** to denote the original quiver Q and **Model 2** to denote the quiver obtained from Q by mutating at vertex 2. Figure 2 shows these two models. As a side note, the word "model" is also seen as "phase" in the literature [HS12].
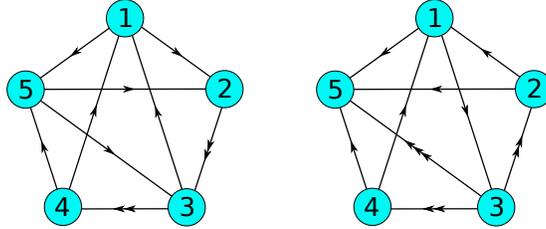

FIGURE 2. model 1 and model 2 of the dP$_2$ quiver (Figure 30 and 31 of [HS12])
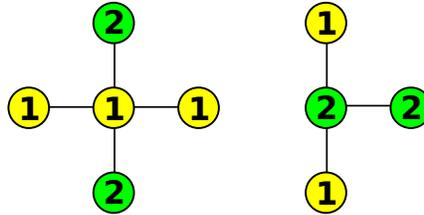
Transitions between these two models are shown in Figure 3.

FIGURE 3. Adjacency between different models (Figure 18 of [EF12])

3. CLASSIFICATION OF TORIC MUTATION SEQUENCES

Definition 3.1 (ρ-mutation sequences). We define the following operation sequences consisting of mutations and permutations, where concatenation of operations is done from left to right. A permutation permutes the vertices and their associated cluster variables accordingly.

$$\rho_1 = \mu_1 \circ (54321), \quad \rho_2 = \mu_5 \circ (12345), \quad \rho_3 = \mu_2 \circ \mu_4 \circ (24),$$

$$\rho_4 = \mu_2 \circ \mu_1 \circ \mu_4 \circ (531), \quad \rho_5 = \mu_4 \circ \mu_5 \circ \mu_2 \circ (351),$$

$$\rho_6 = \mu_2 \circ \mu_1 \circ \mu_2 \circ (531)(24), \quad \rho_7 = \mu_4 \circ \mu_5 \circ \mu_4 \circ (135)(24).$$

We call each ρ_i a $\rho-$mutation and any concatenation of ρ_i's a $\rho-$mutation sequence.

As a side note, it is technically more correct to name "ρ-mutation" as "ρ-operation". However, we follow conventions set in [LMNT14] and [LM15] and thus choose the name "ρ-mutation".

These ρ-mutations all fix the quiver (but not the cluster variables), that is, $\rho_i(Q) = Q$, for $i = 1, \ldots, 7$. Notice that in the original quiver Q, there are no edges connecting vertex 2 and 4. This means mutation at 2 and mutation at 4 commute, so ρ_3 can also be written as $\rho_3 = \mu_4 \circ \mu_2 \circ (24)$.

It is easy to construct Figure 4, which shows all possible toric mutation sequences that start from the original dP_2 quiver and return to model 1 the first time, from Figure 3. In this way, it is clear that combinations of these seven $\rho-$mutations give us all possible toric mutation sequences that start in model 1 and end in model 1 up to a permutation of vertices.
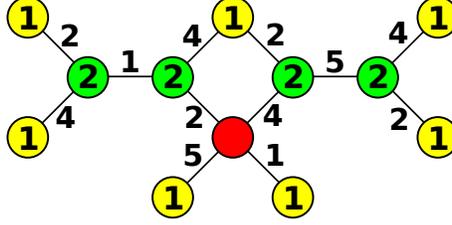

FIGURE 4. All possible toric mutation sequences that start from model 1 and return to model 1 the first time. The red circle represents the initial quiver Q. Numbers on edges represent the vertices mutated.

Proposition 3.2 (Relations of ρ-mutations).

$$\rho_4\{x_1, x_2, x_3, x_4, x_5\} = \rho_1^2\rho_3\{x_1, x_2, x_3, x_4, x_5\},$$
$$\rho_5\{x_1, x_2, x_3, x_4, x_5\} = \rho_2^2\rho_3\{x_1, x_2, x_3, x_4, x_5\},$$
$$\rho_6\{x_1, x_2, x_3, x_4, x_5\} = \rho_1^2\{x_1, x_2, x_3, x_4, x_5\},$$
$$\rho_7\{x_1, x_2, x_3, x_4, x_5\} = \rho_2^2\{x_1, x_2, x_3, x_4, x_5\}.$$

$$\rho_1\rho_2\{x_1, \ldots, x_5\} = \rho_2\rho_1\{x_1, \ldots, x_5\} = \rho_3^2\{x_1, \ldots, x_5\} = \{x_1, x_2, x_3, x_4, x_5\},$$
$$\rho_1^2\rho_3\{x_1, \ldots, x_5\} = \rho_3\rho_1^2\{x_1, \ldots, x_5\}, \quad \rho_2^2\rho_3\{x_1, \ldots, x_5\} = \rho_3\rho_2^2\{x_1, \ldots, x_5\},$$
$$\rho_1\rho_3\rho_2\{x_1, \ldots, x_5\} = \rho_2\rho_3\rho_1\{x_1, \ldots, x_5\}.$$

Note that it suffices to define ρ_1, ρ_2, ρ_3 because ρ_4, ρ_5, ρ_6, ρ_7 can be written in terms of the previous three.

Theorem 3.3. *Any toric mutation sequence in dP_2 quiver that starts and ends at model 1 can be written, up to a permutation of cluster variables, as $\rho_t^k(\rho_3\rho_1)^m\rho_3^w$, where $k, m \in \mathbb{Z}_{\geq 0}$, $t \in \{1, 2\}$ and $w \in \{0, 1\}$.*

Proof. This theorem is essentially saying that all ρ-mutation sequences can be written in a certain form. Fix a generic ρ-mutation sequence.

Since $\rho_1\rho_2 = \rho_2\rho_1 = \rho_3^2 = 1$, we can assume that this sequence does not contain consecutive ρ_3's and does not contain adjacent ρ_1 and ρ_2. Therefore, we can write it as $\rho_{j_1}^{\alpha_1}\rho_3\rho_{j_2}^{\alpha_2}\rho_3\cdots\rho_{j_N}^{\alpha_N}$ with possibly a ρ_3 at the beginning and a ρ_3 at the end, where $j_i \in \{1, 2\}$ and $\alpha_i \in \mathbb{Z}_{>0}$.

Notice that by Proposition 3.2, ρ_1^2 and ρ_2^2 will commute with everything. So whenever we see two consecutive ρ_1's or consecutive ρ_2's, we can pull them to the front. As a result, we can further simplify this sequence as $\rho_t^n\rho_3\rho_{\ell_1}\rho_3\rho_{\ell_2}\cdots\rho_{\ell_s}$ with possibly a ρ_3 at the end, where $t, \ell_1, \ldots, \ell_s \in \{1, 2\}$ and $n \in \mathbb{Z}_{\geq 0}$.

Proposition 3.2 gives $\rho_1\rho_3\rho_2 = \rho_2\rho_3\rho_1$, which means ρ_1 and ρ_2 "commute" with a ρ_3 in between. Therefore, in $\rho_3\rho_{\ell_1}\rho_3\rho_{\ell_2}\cdots\rho_{\ell_s}$ (with possibly a ρ_3 in the end), we are able to put

all ρ_1's in front of ρ_2's. The sequence now has the form $\rho_t^n(\rho_3\rho_1)^r(\rho_3\rho_2)^s$, with possibly a ρ_3 in the end.

Take a sufficiently large M and write the sequence as $\rho_t^n\rho_2^M\rho_1^M(\rho_3\rho_1)^r(\rho_3\rho_2)^s$. Since ρ_1^2 commute with everything, we will take ρ_1^2 in the term ρ_1^M to cancel all the ρ_2's in $(\rho_3\rho_2)^s$, since M is sufficiently large. Finally, we naturally merge the remaining ρ_1's in the previous ρ_1^M with ρ_t^n, ρ_2^M and get $\rho_t^k(\rho_3\rho_1)^m$ with possibly a ρ_3 in the end, with $t \in \{1,2\}$ and $m, k \in \mathbb{Z}_{\geq 0}$, as desired. $\qquad\square$

Remark 3.4. Figure 5 gives a way to visualize the $\rho-$mutation sequences as an analog of alcove walk discussed in the dP3 case [LM15]. In Figure 5, each vertex corresponds to a cluster with a model 1 quiver. We can arbitrarily select one as the initial cluster. A horizontal step to the right is ρ_1; a horizontal step to the left is ρ_2; and a vertical step is ρ_3.
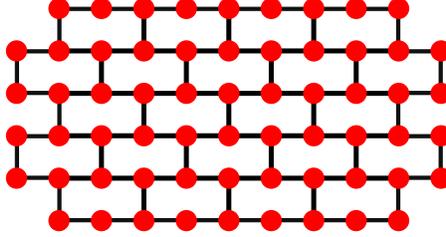

FIGURE 5. Visualization of $\rho-$mutation sequences

4. EXPLICIT FORMULAS FOR CLUSTER VARIABLES

In this section, we will give explicit formulas for all cluster variables that can be generated by toric mutations for the dP$_2$ quiver.

Suppose that the cluster variables are initialized as $\{x_1, x_2, x_3, x_4, x_5\}$.

Definition 4.1 (Laurent Polynomial for Somos-5 Sequence)**.**
For $n \geq 6$, define recursively

$$x_n := \frac{x_{n-1}x_{n-4} + x_{n-2}x_{n-3}}{x_{n-5}}.$$

For $n \leq 0$, define recursively

$$x_n := \frac{x_{n+1}x_{n+4} + x_{n+2}x_{n+3}}{x_{n+5}}.$$

Remark 4.2. For each $n \in \mathbb{Z}$, Definition 4.1 gives us a way to define x_n as a rational function in x_1, x_2, x_3, x_4, x_5. Moreover, the equation

$$(1) \qquad\qquad x_nx_{n+5} = x_{n+1}x_{n+4} + x_{n+2}x_{n+3}$$

is satisfied for each $n \in \mathbb{Z}$. Therefore, it is clear that if we assign 1 to $x_1, \ldots, x_5$, then both $\{x_n\}_{n=1,2,\ldots}$ and $\{x_{6-n}\}_{n=1,2,\ldots}$ give us the Somos-5 sequence.

Definition 4.3. Define the following constants

$$A := \frac{x_1x_5 + x_3^2}{x_2x_4}, \qquad B := \frac{x_2x_6 + x_4^2}{x_3x_5} \left(= \frac{x_1x_4^2 + x_2x_3x_4 + x_2^2x_5}{x_1x_3x_5} \right).$$

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Lemma 4.4. *For each $n \in \mathbb{Z}$,*

$$A = \frac{x_{2n-1}x_{2n+3} + x_{2n+1}^2}{x_{2n}x_{2n+2}}, \qquad B = \frac{x_{2n}x_{2n+4} + x_{2n+2}^2}{x_{2n+1}x_{2n+3}}.$$

Proof. The lemma is correct when $n = 1$ by definition. By an inductive argument, it suffices to show that, for each $m \in \mathbb{Z}$,

$$\frac{x_m x_{m+4} + x_{m+2}^2}{x_{m+1}x_{m+3}} = \frac{x_{m+2}x_{m+6} + x_{m+4}^2}{x_{m+3}x_{m+5}}.$$

According to Equation (1), we have

$$
\begin{aligned}
\frac{x_{m+2}x_{m+6} + x_{m+4}^2}{x_{m+3}x_{m+5}} &= \frac{x_{m+2}\dfrac{x_{m+2}x_{m+5} + x_{m+3}x_{m+4}}{x_{m+1}} + x_{m+4}^2}{x_{m+3}x_{m+5}}\\[2mm]
&= \frac{x_{m+2}^2}{x_{m+1}x_{m+3}} + \frac{x_{m+4}\big(x_{m+2}x_{m+3} + x_{m+1}x_{m+4}\big)}{x_{m+1}x_{m+3}x_{m+5}}\\[2mm]
&= \frac{x_{m+2}^2}{x_{m+1}x_{m+3}} + \frac{x_{m+4}x_m x_{m+5}}{x_{m+1}x_{m+3}x_{m+5}}\\[2mm]
&= \frac{x_m x_{m+4} + x_{m+2}^2}{x_{m+1}x_{m+3}}
\end{aligned}
$$

as desired. $\qquad\square$

Theorem 4.5. *Define $\rho_1^k := \rho_2^{-k}$ for $k < 0$. Define $g(s,k) := \left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{s+1}{2} \right\rfloor$ if k is even and $g(s,k) := \left\lfloor \frac{s-1}{2} \right\rfloor \left\lfloor \frac{s}{2} \right\rfloor$ if k is odd. Then we have, for $k \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geq 0}$,*

$$
\begin{aligned}
\rho_1^k(\rho_3\rho_1)^s\{x_1, x_2, x_3, x_4, x_5\} = \{ &A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+1},\\
&A^{g(s,k)}B^{g(s,k+1)}x_{k+s+2},\\
&A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+3},\\
&A^{g(s,k)}B^{g(s,k+1)}x_{k+s+4},\\
&A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+5}\}.
\end{aligned}
$$

Proof. We will divide our toric mutation sequence into two steps: ρ_1^k and $(\rho_3\rho_1)^s$. Then we will use straightforward induction.

Step 1: $\rho_1^k\{x_1, x_2, x_3, x_4, x_5\} = \{x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}\}$ for $k \in \mathbb{Z}$.

This is true for $k = 0$. Let us suppose that this holds for some $k \geq 0$. Then

$$
\begin{aligned}
\rho_1^{k+1}\{x_1, x_2, x_3, x_4, x_5\} &= \rho_1\{x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}\}\\
&= (54321)\big(\mu_1\{x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}\}\big)\\
&= (54321)\{\frac{x_{k+2}x_{k+5} + x_{k+3}x_{k+4}}{x_{k+1}}, x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}\}\\
&= (54321)\{x_{k+6}, x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}\}\\
&= \{x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}, x_{k+6}\}.
\end{aligned}
$$

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By induction, this proves the claim for $k \geq 0$. The proof for $k \leq 0$ can be done the same way.

Before doing the next step, we first show that if $k + s$ is odd, then

(2)
$$g(s+1, k) = 2g(s, k) - g(s-1, k) + 1$$
$$g(s+1, k+1) = 2g(s, k+1) - g(s-1, k+1).$$

If k is even, then s is odd and these two equations become

$$\left\lfloor \frac{s+1}{2} \right\rfloor \left\lfloor \frac{s+2}{2} \right\rfloor = 2 \left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{s+1}{2} \right\rfloor - \left\lfloor \frac{s-1}{2} \right\rfloor \left\lfloor \frac{s}{2} \right\rfloor + 1$$

$$\Leftrightarrow \left(\frac{s+1}{2} \right)^2 = 2 \left(\frac{s-1}{2} \right) \left(\frac{s+1}{2} \right) - \left(\frac{s-1}{2} \right)^2 + 1$$

and

$$\left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{s+1}{2} \right\rfloor = 2 \left\lfloor \frac{s-1}{2} \right\rfloor \left\lfloor \frac{s}{2} \right\rfloor - \left\lfloor \frac{s-2}{2} \right\rfloor \left\lfloor \frac{s-1}{2} \right\rfloor$$

$$\Leftrightarrow \left(\frac{s-1}{2} \right) \left(\frac{s+1}{2} \right) = 2 \left(\frac{s-1}{2} \right)^2 - \left(\frac{s-3}{2} \right) \left(\frac{s-1}{2} \right)$$

which are clearly correct.

If k is odd, then s is even and these two equations become

$$\left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{s+1}{2} \right\rfloor = 2 \left\lfloor \frac{s-1}{2} \right\rfloor \left\lfloor \frac{s}{2} \right\rfloor - \left\lfloor \frac{s-1}{2} \right\rfloor \left\lfloor \frac{s-2}{2} \right\rfloor + 1$$

$$\Leftrightarrow \left(\frac{s}{2} \right)^2 = 2 \left(\frac{s}{2} \right) \left(\frac{s-2}{2} \right) - \left(\frac{s-2}{2} \right)^2 + 1$$

and

$$\left\lfloor \frac{s+1}{2} \right\rfloor \left\lfloor \frac{s+2}{2} \right\rfloor = 2 \left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{s+1}{2} \right\rfloor - \left\lfloor \frac{s-1}{2} \right\rfloor \left\lfloor \frac{s}{2} \right\rfloor$$

$$\Leftrightarrow \left(\frac{s}{2} \right) \left(\frac{s+2}{2} \right) = 2 \left(\frac{s}{2} \right)^2 - \left(\frac{s-2}{2} \right) \left(\frac{s}{2} \right)$$

which are clearly correct.

With the same argument, we can show that if $k + s$ is even, then

(3)
$$g(s+1, k) = 2g(s, k) - g(s-1, k)$$
$$g(s+1, k+1) = 2g(s, k+1) - g(s-1, k+1) + 1.$$

Step 2: Calculate $\rho_1^k (\rho_3 \rho_1)^s \{x_1, x_2, x_3, x_4, x_5\}$.

From step 1, $\rho_1^k (\rho_3 \rho_1)^s \{x_1, x_2, x_3, x_4, x_5\} = (\rho_3 \rho_1)^s \{x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}\}$.

Since $g(0, k) = g(1, k) = 0$, no matter the parity of k, when $s = 0$, the theorem holds. Now assume that the theorem holds for some $s - 1 \geq 0$. It suffices to show

$$(\rho_3 \rho_1) \{ A^{g(s,k)} B^{g(s,k+1)} x_{k+s}, A^{g(s-1,k)} B^{g(s-1,k+1)} x_{k+s+1}, A^{g(s,k)} B^{g(s,k+1)} x_{k+s+2},$$

$$A^{g(s-1,k)} B^{g(s-1,k+1)} x_{k+s+3}, A^{g(s,k)} B^{g(s,k+1)} x_{k+s+4} \}$$

$$= \{ A^{g(s+1,k)} B^{g(s+1,k+1)} x_{k+s+1}, A^{g(s,k)} B^{g(s,k+1)} x_{k+s+2}, A^{g(s+1,k)} B^{g(s+1,k+1)} x_{k+s+3},$$

$$A^{g(s,k)} B^{g(s,k+1)} x_{k+s+4}, A^{g(s+1,k)} B^{g(s+1,k+1)} x_{k+s+5} \}.$$

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Denote $\rho_1^k(\rho_3\rho_1)^{s-1}\{x_1,\ldots,x_5\}$ as S, and let $S[i]$ be the ith element of S. Recall that to apply $\rho_3\rho_1$ to S, we will first do $\rho_3 = \mu_2 \circ \mu_4 \circ (24)$. As we mutate vertex 2, the new cluster variable at vertex 2 is updated as

$$S[2] \leftarrow \frac{S[1]S[5] + S[3]^2}{S[2]}$$

$$= \frac{A^{2g(s,k)} B^{2g(s,k+1)} (x_{k+s} x_{k+s+4} + x_{k+s+2}^2)}{A^{g(s-1,k)} B^{g(s-1,k+1)} x_{k+s+1}}$$

According to Lemma 4.4 and Equation (2), if $k + s$ is odd, the above expression becomes

$$\frac{A^{2g(s,k)} B^{2g(s,k+1)} A x_{k+s+3}}{A^{g(s-1,k)} B^{g(s-1,k+1)}} = A^{g(s+1,k)} B^{g(s+1,k+1)} x_{k+s+3}.$$

Similarly by Lemma 4.4 and Equation (3), if $k + s$ is even, we have

$$\frac{A^{2g(s,k)} B^{2g(s,k+1)} B x_{k+s+3}}{A^{g(s-1,k)} B^{g(s-1,k+1)}} = A^{g(s+1,k)} B^{g(s+1,k+1)} x_{k+s+3}.$$

Then as we mutate vertex 4, with the same argument, we can show that

$$S[4] \leftarrow A^{g(s+1,k)} B^{g(s+1,k+1)} x_{k+s+1}.$$

So if we let $S' = \rho_3 S$, then S' and S differ only in the 2$^{\text{nd}}$ and the 4$^{\text{th}}$ coordinate. Specifically,

$$S'[2] = A^{g(s+1,k)} B^{g(s+1,k+1)} x_{k+s+1}, \quad S'[4] = A^{g(s+1,k)} B^{g(s+1,k+1)} x_{k+s+3}.$$

Finally, we mutate at vertex 1 in S' and get

$$S'[1] \leftarrow \frac{S'[2]S'[5] + S'[3]S'[4]}{S'[1]}$$

$$= \frac{A^{g(s+1,k)+g(s,k)} B^{g(s+1,k+1)+g(s,k+1)} (x_{k+s+1} x_{k+s+4} + x_{k+s+2} x_{k+s+3})}{A^{g(s,k)} B^{g(s,k+1)} x_{k+s}}$$

$$= A^{g(s+1,k)} B^{g(s+1,k+1)} x_{k+s+5}.$$

After applying a permutation (54321), we obtain the desired identity, which completes the induction step. $\qquad\square$

Corollary 4.6. *All cluster variables that may appear through toric mutation sequences can be written in the forms*

$$A^{n^2} B^{n(n-1)} x_{2m}, \quad A^{n(n-1)} B^{n^2} x_{2m-1} \qquad \text{where } m, n \in \mathbb{Z}.$$

Proof. We will first explain that all cluster variables that appear from toric mutations can be achieved by $\rho-$mutation sequences in the form of $\rho_1^k(\rho_3\rho_1)^s$, for some $k \in \mathbb{Z}$ and $s \in \mathbb{Z}\geq 0$. According to Theorem 3.3, every toric mutation sequence from model 1 to model 1 can be written as $\rho_1^k(\rho_3\rho_1)^s$ or $\rho_1^k(\rho_3\rho_1)^s\rho_3$ for some $k \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geq 0}$. The proof for Theorem 4.5 shows that cluster variables of $\rho_1^k(\rho_3\rho_1)^s\rho_3\{x_1,\ldots,x_5\}$ are included in $\rho_1^k(\rho_3\rho_1)^s\{x_1,\ldots,x_5\}$ and $\rho_1^k(\rho_3\rho_1)^{s+1}\{x_1,\ldots,x_5\}$. Now we consider any toric mutation sequence that takes the original model 1 quiver to some model 2 quiver. According to Figure 4, this model 2 quiver can reach two different model 1 quivers in one step of toric mutation. So the cluster variables corresponding to this specific toric mutation sequence that ends on a model 2 quiver are included in the cluster variables that are generated by these two model 1 quivers.

Then we can take a closer look at the cluster variables shown in Theorem 4.5. Since $g(s,k)$ depends on the value of s and the parity, but not the actual value, of k, it is easy to see that all cluster variables that appear can be written as $A^{g(s,k)}B^{g(s,k+1)}x_{k+s}$ for some $k \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geq 0}$. Conversely, for any $k \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geq 0}$, $A^{g(s,k)}B^{g(s,k+1)}x_{k+s}$ can be generated by a toric mutation sequence according to Theorem 4.5. To look at this term closely, we consider the following four cases according to the parity of s and k.

Case 1: s is even and k is even. Let $s = 2n$ and $k + s = 2m$. We have $n \geq 0$. Then

$$A^{g(s,k)}B^{g(s,k+1)}x_{k+s} = A^{\lfloor \frac{s}{2} \rfloor \lfloor \frac{s+1}{2} \rfloor}B^{\lfloor \frac{s-1}{2} \rfloor \lfloor \frac{s}{2} \rfloor}x_{2m} = A^{n^2}B^{n(n-1)}x_{2m}.$$

Case 2: s is odd and k is odd. Let $s = 2n + 1$ and $k + s = 2m$. We have $n \geq 0$. Then

$$A^{g(s,k)}B^{g(s,k+1)}x_{k+s} = A^{\lfloor \frac{s-1}{2} \rfloor \lfloor \frac{s}{2} \rfloor}B^{\lfloor \frac{s}{2} \rfloor \lfloor \frac{s+1}{2} \rfloor}x_{2m} = A^{n^2}B^{n(n+1)}x_{2m}.$$

Case 3: s is even and k is odd. Let $s = 2n$ and $k + s = 2m - 1$. We have $n \geq 0$. Then

$$A^{g(s,k)}B^{g(s,k+1)}x_{k+s} = A^{\lfloor \frac{s-1}{2} \rfloor \lfloor \frac{s}{2} \rfloor}B^{\lfloor \frac{s}{2} \rfloor \lfloor \frac{s+1}{2} \rfloor}x_{2m-1} = A^{n(n-1)}B^{n^2}x_{2m-1}.$$

Case 4: s is odd and k is even. let $s = 2n + 1$ and $k + s = 2m - 1$. We have $n \geq 0$. Then

$$A^{g(s,k)}B^{g(s,k+1)}x_{k+s} = A^{\lfloor \frac{s}{2} \rfloor \lfloor \frac{s+1}{2} \rfloor}B^{\lfloor \frac{s-1}{2} \rfloor \lfloor \frac{s}{2} \rfloor}x_{2m-1} = A^{n(n+1)}B^{n^2}x_{2m-1}.$$

Cases 1 and 2 can be merged by letting $n \in \mathbb{Z}$ instead of just $n \in \mathbb{Z}_{\geq 0}$. Similarly cases 3 and 4 can be merged. Finally, we conclude that all cluster variables generated by toric mutations can be written as either

$$A^{n^2}B^{n(n-1)}x_{2m}, \quad A^{n(n-1)}B^{n^2}x_{2m-1} \qquad \text{where } m, n \in \mathbb{Z}.$$

$\square$

5. Subgraphs of the Brane Tiling

For our purpose, every graph we consider will be a subgraph of the dP_2 brane tiling so it is bipartite, planar and weighted. For such a graph G, which is bipartite, let V_1 and V_2 be its corresponding vertex sets. For any vertex set $V_0 \subset V_1 \cup V_2$, define $G - V_0$ to be the graph obtained by removing each vertex in V_0, as well as the edges that are incident to it, from G. These notations will be used for the rest of the paper.

We want to find a subgraph for each cluster variable that appears through toric mutations, such that the subgraph's weight equals the cluster variable. We will use the weighting scheme utilized in [LM15], [LMNT14], [Spe07], [Zha], and etc.

Definition 5.1 (Weight of Subgraphs). We associate a weight $\frac{1}{x_i x_j}$ to each edge bordering block labeled i and j. For a set of edges M, define its weight $w(M)$ to be the product of the weights of the edges. For a subgraph G of the brane tiling, let $\mathcal{M}(G)$ be the collection of its perfect matchings where each perfect matching is represented as a set of edges. Then, we define the weight of G as

$$w(G) = \sum_{M \in \mathcal{M}(G)} w(M).$$

In order to get recursive relations on the variables which correspond to subgraphs, we will need lemmas that help us represent the weight of a large graph in terms of the weights of smaller graphs. Below we state Kuo's condensation theorems [Kuo06], [Kuo04].

Lemma 5.2 (Balanced Kuo Condensation; Theorem 5.1 in [Kuo04]). *Let G be a weighted planar bipartite graph discussed above with $|V_1| = |V_2|$. Assume that p_1, p_2, p_3, p_4 are four vertices appearing in a cyclic order on a face of G with $p_1, p_3 \in V_1$ and $p_2, p_4 \in V_2$. Then*

$$w(G)w(G - \{p_1, p_2, p_3, p_4\}) = w(G - \{p_1, p_2\})w(G - \{p_3, p_4\})$$
$$+ w(G - \{p_1, p_4\})w(G - \{p_2, p_3\}).$$

Lemma 5.3 (Unbalanced Kuo Condensation; Theorem 5.2 in [Kuo04]). *Let G be a weighted planar bipartite graph discussed above with $|V_1| = |V_2| + 1$. Assume that p_1, p_2, p_3, p_4 are four vertices appearing in a cyclic order on a face of G with $p_1, p_2, p_3 \in V_1$ and $p_4 \in V_2$. Then*

$$w(G - \{p_2\})w(G - \{p_1, p_3, p_4\}) = w(G - \{p_1\})w(G - \{p_2, p_3, p_4\})$$
$$+ w(G - \{p_3\})w(G - \{p_1, p_2, p_4\}).$$

Lemma 5.4 (Non-alternating Kuo Condensation; Theorem 5.3 in [Kuo04]). *Let G be a weighted planar bipartite graph discussed above with $|V_1| = |V_2|$. Assume that p_1, p_2, p_3, p_4 are four vertices appearing in a cyclic order on a face of G with $p_1, p_2 \in V_1$ and $p_3, p_4 \in V_2$. Then*

$$w(G - \{p_1, p_4\})w(G - \{p_2, p_3\}) = w(G)w(G - \{p_1, p_2, p_3, p_4\})$$
$$+ w(G - \{p_1, p_3\})w(G - \{p_2, p_4\}).$$

6. Contours for Cluster Variables

In this section, we describe a method to get the subgraph corresponding to any cluster variable obtained by toric mutations for the dP_2 quiver. Specifically, we use 5-sided contours to cut our brane tiling. We will define the rules to cut the subgraphs, and the formula of the contours.

6.1. **Graphs from Contours.** Given a 5-tuple $(a, b, c, d, e) \in \mathbb{Z}^5$ with $a + b = d$ and $a + e = c$ (see Figure 6 right for those relations), we consider a 5-sided **contour** whose side-lengths are a, b, c, d, e in clockwise order, starting from the upper right corner. Figure 6 (left) shows the fundamental shape of the contour, with each length being positive. In the case of negative side-lengths, we will draw the corresponding side in the opposite direction.

See Figure 7 (left) for an example of a 5-tuple and its contour. We will abuse notation and denote a geometric contour by its corresponding 5-tuple.
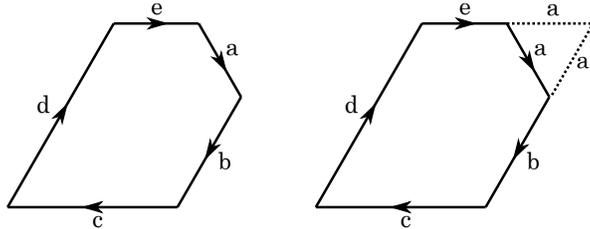

FIGURE 6. **Left**: 5-sided fundamental shape; **Right**: relations between side lengths.

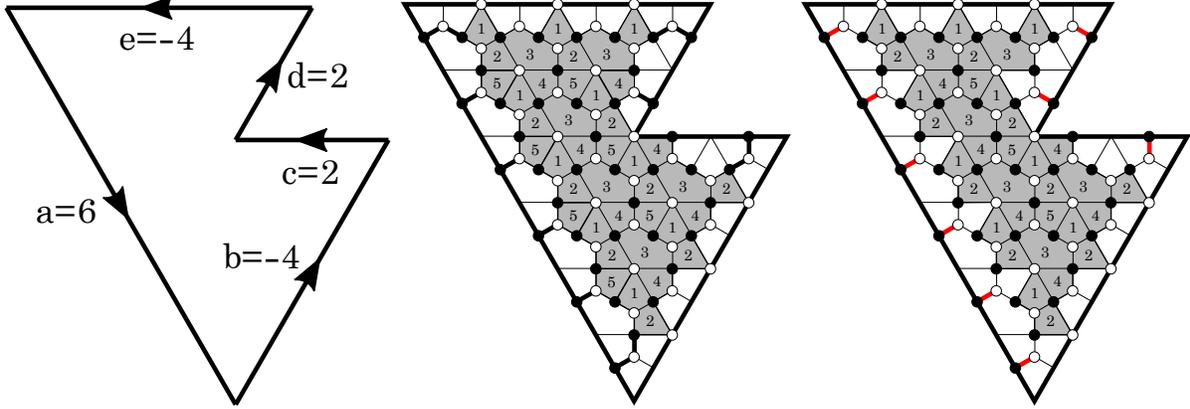Now we define the rule to get a subgraph from a contour.

FIGURE 7. Example of a contour $C = (6, -4, 2, 2, -4)$ and its subgraphs $\mathcal{G}(C)$ (shaded region and darkened edges) and $\widehat{\mathcal{G}}(C)$ (shaded region).

Definition 6.1 (Rules to Get Subgraph).

The white vertex between edges c and d is called the **special vertex**.

Step 1: Given a 5-sided contour $C = (a, b, c, d, e) \in \mathbb{Z}^5$, we superimpose the contour on the brane tiling $\mathcal{T}$ such that the vertex between side a and e sits on any white vertex of degree 5, while each side follows.

Step 2: On each side of positive length, we keep the black points while removing the white points; on each side of negative length, we keep the white points while removing the black points; on each side of zero length, we remove the single white point if it is not the special vertex.

Step 3: Each corner vertex will be white. If the two adjacent sides of a corner vertex are both non-positive, we keep the vertex; otherwise, we remove it. As for the special vertex, if a is even, we keep the special vertex; if a is odd, we remove the special vertex. Call the graph that remains inside the contour $\mathcal{G}(C)$.

Step 4: In the resulting graph, we connect any vertex of valence 1 to its adjacent vertex. Call the edge of this connection a **forced matching**. Then delete these two vertices from the graph. Repeat this step until every vertex in the subgraph has valence at least 2.

Step 5: Call the resulting graph $\widehat{\mathcal{G}}(C)$ the **subgraph** of contour C. Often we may refer to $\widehat{\mathcal{G}}(C)$ as either $\widehat{\mathcal{G}}(a, b, c, d, e)$ or simply $\widehat{\mathcal{G}}$.

Definition 6.2. For any graph G, let $\widehat{G}$ denote the graph obtained by removing all forced matchings.

Remark 6.3. Note that our notation of graphs $\mathcal{G}$ and $\widehat{\mathcal{G}}$ for a contour is the opposite of the notation in [LM15].

We have already defined the weighting $w(G)$ of a graph G in Definition 5.1. To fully recover the cluster variables from graphs, we define covering monomials for this specific brane tiling. The covering monomial has a more general definition in [JMZ00] and [Jeo11].

Definition 6.4 (Covering Monomial). For this definition, we think of every block labeled 3 as two separate blocks labeled 3. Given a contour C, let a_j be the number of blocks labeled j enclosed in C. Let b_j be the number of blocks labeled j adjacent to a forced

matching in C. If the special vertex is kept (i.e. if a is even) and the contour passes through the middle of a 3-block near the special vertex (see Figure 8), let $c_3 = 1$. Otherwise, let $c_3 = 0$. The covering monomial of graph $\mathcal{G}(C)$, denoted as $m(\mathcal{G}(C))$, is the product $x_1^{a_1} x_2^{a_2} x_3^{a_3+c_3} x_4^{a_4} x_5^{a_5}$. The covering monomial of graph $\widehat{\mathcal{G}}(C)$, denoted as $m(\widehat{\mathcal{G}}(C))$, is the product $x_1^{a_1-b_1} x_2^{a_2-b_2} x_3^{a_3-b_3+c_3} x_4^{a_4-b_4} x_5^{a_5-b_5} = \frac{m(\mathcal{G}(C))}{x_1^{b_1} x_2^{b_2} x_3^{b_3} x_4^{b_4} x_5^{b_5}}$.
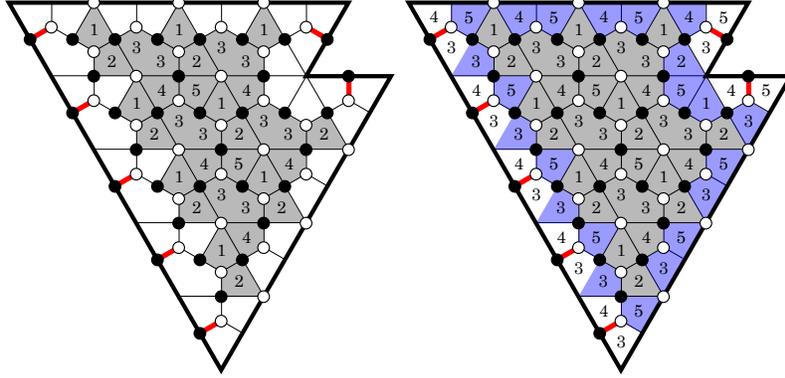
FIGURE 8. Example of a subgraph $\widehat{\mathcal{G}}(5, -4, 1, 1, -4)$ and its covering monomial $m(\widehat{\mathcal{G}}(5, -4, 1, 1, -4))$. The gray and purple blocks are included in the covering monomial.

Remark 6.5. Our definitions of weight and covering monomial remain unchanged if we think of each six sided 3-block as two separate four sided blocks without an edge between them. Each 3-block will be drawn as two separate 3-blocks if they appear on the boundary of our contour for sake of visualizing weight and covering monomial.

For any graph G with an associated contour, denote the product of its weight and its covering monomial as

$$c(G) := w(G)m(G).$$

6.2. **Contours of Cluster Variables.** By Corollary 4.5, we have that all the cluster variables are of the form $A^{n^2} B^{n^2-n} x_{2k}$ or $A^{n^2+n} B^{n^2} x_{2k-1}$ where $n, k \in \mathbb{Z}$. Now we state the main result of this section that gives a formula of the contours of these two families.

Theorem 6.6. *For $k \geq 2$, we associate the following contours to the cluster variables such that if C is the contour associated with a cluster variable, then $c(\widehat{\mathcal{G}}(C))$ equals the Laurent polynomial of that cluster variable.*

$$A^{n^2} B^{n^2-n} x_{2k} = c\left(\widehat{\mathcal{G}}\left(k-2+n, -\left\lceil\frac{k-4+5n}{2}\right\rceil, 2n-1, \left\lfloor\frac{k-3n}{2}\right\rfloor, 1+n-k\right)\right),$$

$$A^{n^2+n} B^{n^2} x_{2k-1} = c\left(\widehat{\mathcal{G}}\left(k-2+n, -\left\lceil\frac{k-2+5n}{2}\right\rceil, 2n, \left\lfloor\frac{k-2-3n}{2}\right\rfloor, 2+n-k\right)\right).$$

Notice that when $k \leq 1$, we can reflect the subgraph of $A^p B^q x_{6-2k}$ $(p, q \in \mathbb{Z})$ along x_3, which means we replace x_2 with x_4 and x_1 with x_5 to get the subgraph of $A^p B^q x_{2k}$ since block 2 and block 4, and block 1 and block 5 are symmetric with respect to x_3 in the brane

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tiling and A, B are also fixed if we interchange x_2 with x_4 or/and x_1 with x_5. Therefore we only need to consider the situation where $k \geq 2$.

Before proving this result, we first look at the six possible shapes of the contours based on the relationship between n and k, as is shown in Figure 9.
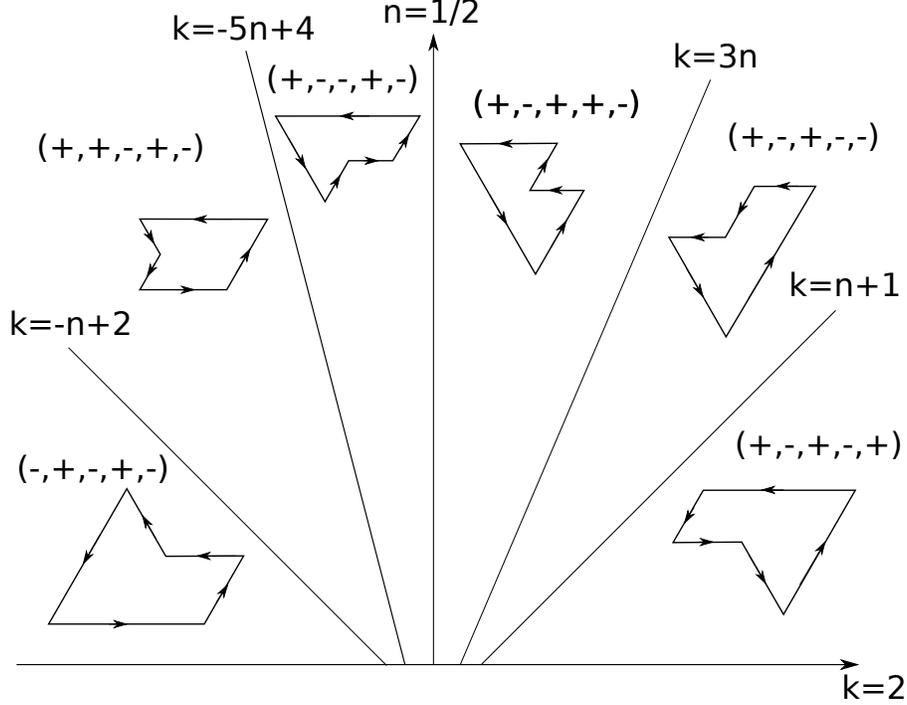

FIGURE 9. Possible shapes of 5-sided contour

We will use the entire next section to prove the main theorem.

7. PROOF OF MAIN THEOREM (THEOREM 6.6)

7.1. **Overview of induction procedure.** We use Kuo's condensation to inductively prove that multiplying the weight and covering monomial of these contours yields the Laurent polynomials of our cluster variables. First we show that the weights satisfy the desired recurrences. Then we show that for any form of recurrence, the covering monomials will be correct in the sense that multiplying the weight and covering monomials of our subgraphs gives the Laurent polynomial. We abuse notation by saying a graph G equals a cluster variable when we mean the weight of G, $w(G)$, will give us the cluster variable's Laurent polynomial with the appropriate covering monomial.

The base case is $n = 0$, which is proved in Section 7.3. Notice that when $n = 0$, our formula for the contour in the main theorem contains two families: $\{x_{2k-1}\}_{k \geq 2}$ and $\{x_{2k}\}_{k \geq 2}$. So essentially, there are two different cases here.

After proving the base case when $n = 0$, we split our way to consider the families of variables with $n > 0$ and families of variables with $n < 0$ separately.

For $n \geq 1$, by induction hypothesis, assume that we already have the contours for variables $A^{m^2} B^{m(m-1)} x_{2k}$ and $A^{m(m+1)} B^{m^2} x_{2k-1}$ for all $k \geq 2$ and $0 \leq m \leq n-1$. Then for each $k \geq 2$,

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consider the following identity (recurrence):

$$(A^{n^2}B^{n(n-1)}x_{2k})(A^{(n-1)^2}B^{(n-1)(n-2)}x_{2k+2})$$
$$=(A^{(n-1)n}B^{(n-1)^2}x_{2k-1})(A^{(n-1)n}B^{(n-1)^2}x_{2k+3})+(A^{(n-1)n}B^{(n-1)^2}x_{2k+1})^2.$$

It is clear that among all the five terms appeared above, $A^{n^2}B^{n(n-1)}x_{2k}$ is the only term that we do not have already. Therefore, it suffices to find some graph G and points p_1, p_2, p_3, p_4 and use some version of Kuo's condensation theorem on it to prove that this term actually equals the weight of the subgraph that we described in the main theorem, correspondingly. Note that the graph G we use and the points p_1, p_2, p_3, p_4 we choose depend on some relations between n and k. Now that we have the terms $\{A^{n^2}B^{n(n-1)}x_{2k-1}\}$ for $k \geq 2$, consider the following identity (recurrence) for each $k \geq 3$:

$$(A^{n(n+1)}B^{n^2}x_{2k-1})(A^{(n-1)^2+(n-1)}B^{(n-1)^2}x_{2k+1})$$
$$=(A^{n^2}B^{n(n-1)}x_{2k-2})(A^{n^2}B^{n(n-1)}x_{2k+2})+(A^{n^2}B^{n(n-1)}x_{2k})^2.$$

Similarly, there is only one term $A^{n(n+1)}B^{n^2}x_{2k-1}$ that we do not currently have. And by some Kuo's condensation theorem, we will get the desired result. One thing to notice here is that the above recurrence cannot be applied to $k = 2$ since we do not have the term $A^{n^2}B^{n(n-1)}x_2$ in our theorem. To solve this problem and get the contour formula for $A^{n(n+1)}B^{n^2}x_3$, we use the following recurrence:

$$(A^{n(n+1)}B^{n^2}x_3)(A^{n^2}B^{n(n-1)}x_8)$$
$$=(A^{n(n+1)}B^{n^2}x_5)(A^{n^2}B^{n(n-1)}x_6)+(A^{n(n+1)}B^{n^2}x_7)(A^{n^2}B^{n(n-1)}x_4).$$

Once this step is done, our inductive step is finished.

For $n \leq -1$, the argument is very similar. By induction hypothesis, assume that we already have the contours for variables $A^{m^2}B^{m(m-1)}x_{2k}$ and $A^{m(m+1)}B^{m^2}x_{2k-1}$ for all $n+1 \leq m \leq 0$. The recurrence

$$(A^{n^2}B^{n(n-1)}x_{2k})(A^{(n+1)^2}B^{n(n+1)}x_{2k+2})$$
$$=(A^{n(n+1)}B^{n^2}x_{2k-1})(A^{n(n+1)}B^{n^2}x_{2k+3})+(A^{n(n+1)}B^{n^2}x_{2k+1})^2$$

will give us contours for all variables $A^{n^2}B^{n(n-1)}x_{2k}$ for all $k \geq 2$. After that, the recurrence

$$(A^{n(n+1)}B^{n^2}x_{2k-1})(A^{(n+1)^2+(n+1)}B^{(n+1)^2}x_{2k+1})$$
$$=(A^{(n+1)^2}B^{(n+1)n}x_{2k-2})(A^{(n+1)^2}B^{(n+1)n}x_{2k+2})+(A^{(n+1)^2}B^{(n+1)n}x_{2k})^2$$

will give us contours for all variables $A^{n(n+1)}B^{n^2}x_{2k-1}$ for all $k \geq 3$. For the missing variables in the form of $A^{n(n+1)}B^{n^2}x_3$, we use the recurrence

$$(A^{n(n+1)}B^{n^2}x_3)(A^{(n+1)^2}B^{n(n+1)}x_8)$$
$$=(A^{n(n+1)}B^{n^2}x_5)(A^{(n+1)^2}B^{n(n+1)}x_6)+(A^{n(n+1)}B^{n^2}x_7)(A^{(n+1)^2}B^{n(n+1)}x_4).$$

This completes the inductive step.

Section 7.3 will prove the base case ($n = 0$) and Section 7.4 will prove one case of the inductive step. Notice that for the inductive step, we have 28 cases in total and we will not present explicit proofs for all cases. The cases are divided, generally speaking, by whether side lengths of the contour are greater or smaller than 0 and by some parity conditions on n

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and k. Section 7.2 will give a summary of the techniques used to prove the remaining cases. All of them can be proved in exactly the same format, and in Appendix 9, we provide the necessary data for readers to verify the correctness of these remaining cases.

7.2. **Overview of Proof Techniques.** We have different cases to prove depending on the relations between n and k since different relations will lead to different shapes of our contour. In this section, we will introduce the general format while the details are given in Appendix 9.

Step 1. Consider a contour $C = (a, b, c, d, e)$ with the special vertex kept or removed, and 4 points p_1, p_2, p_3, p_4 inside the contour. Depending on the whether the graph $\mathcal{G}(C)$ is balanced or not and depending on the colors and positions of p_1, p_2, p_3, p_4, we will use a particular version of Kuo's condensation theorems which will always be of the form:

$$w\left(\mathcal{G}(C) - S_1\right) w(\mathcal{G}(C) - S_2) = w\left(\mathcal{G}(C) - S_3\right) w(\mathcal{G}(C) - S_4) + w\left(\mathcal{G}(C) - S_5\right) w(\mathcal{G}(C) - S_6),$$

where each S_i is a subset of $\{p_1, p_2, p_3, p_4\}$. Notice that in this step, $\mathcal{G}(C) - S_i$ may include many forced matchings. Now we multiply both sides of the equation by $m(\mathcal{G}(C))^2$, the square of the covering monomial of the graph G. Each term in the equation is then of the form $m(\mathcal{G}(C))w(\mathcal{G}(C) - S_i)$.

Step 2. For each $i = 1, \ldots, 6$, we find a contour C_i inside C such that $\mathcal{G}\widehat{(C) - S_i} = \widehat{\mathcal{G}}(C_i)$. Recall that $\widehat{G}$ is graph G with all forced matchings removed. We find C_i by first describing points p_1, p_2, p_3, p_4 and how removing each point separately will change the contour C. Then we can add these effects together to get the total effect of removing S_i. Notice that the additivity of such effects is not trivial in general, but it is easy to verify for each of our cases.

This is the core step of our proof. The effects of removing each point p_i from $\widehat{\mathcal{G}}(C)$ will be stated and justified through diagrams.

Step 3. Now we want to relate $m(\mathcal{G}(C))w(\mathcal{G}(C) - S_i)$ to $c(\widehat{\mathcal{G}}(C_i))$. Consider $\mathcal{G}(C_i)$. By definition, we know that $\mathcal{G}(C_i)$ and $\mathcal{G}(C) - S_i$ only differ by a set of forced matchings of $\mathcal{G}(C) - S_i$ inside contour C and outside contour C_i. Meanwhile, $m(\mathcal{G}(C))$ and $m(\mathcal{G}(C_i))$ differ by a factor of the product of all the blocks (the product of variables corresponding to the blocks) inside C but outside C_i. As each block can be in only one forced matching (otherwise the matching would not be forced), the quotient

$$\frac{m(\mathcal{G}(C))w(\mathcal{G}(C) - S_i)}{m(\mathcal{G}(C_i))w(\mathcal{G}(C_i))}$$

is the product of all the blocks inside C and outside C_i that are not adjacent to any forced matchings inside C and outside C_i. Let these blocks form set T_i. We are also using the notation $T(S_i)$ with T_i interchangeably. For each case, we will explicitly provide $T_1, \ldots, T_6$ for a choice of points p_1, p_2, p_3, p_4 and check that

$$(4) \qquad \left(\prod_{j \in T_1} x_j\right)\left(\prod_{j \in T_2} x_j\right) = \left(\prod_{j \in T_3} x_j\right)\left(\prod_{j \in T_4} x_j\right) = \left(\prod_{j \in T_5} x_j\right)\left(\prod_{j \in T_6} x_j\right).$$

Also, notice that

$$m(\mathcal{G}(C_i))w(\mathcal{G}(C_i)) = m(\widehat{\mathcal{G}}(C_i))w(\widehat{\mathcal{G}}(C_i)) =: c(\widehat{\mathcal{G}}(C_i))$$

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since by definition, both $m(\mathcal{G}(C_i))/m(\widehat{\mathcal{G}}(C_i))$ and $w(\widehat{\mathcal{G}}(C_i))/w(\mathcal{G}(C_i))$ equals the product of blocks adjacent to the forced matchings of $\mathcal{G}(C_i)$. Combining these arguments, we conclude that

$$c(G_1)c(G_2) = c(G_3)c(G_4) + c(G_5)c(G_6)$$

where $G_i = \mathcal{G}(C_i)$.

In this step, we are essentially checking that the covering monomials match up with the weights used in Kuo's condensation theorems to give the correct Laurent polynomials.

Step 4. Using the induction hypothesis, we can identify five of the expressions $c(G_i)$ as the Laurent polynomials of cluster variables. Therefore, the sixth expression is the Laurent polynomial of the next cluster variable in the sequence.

We provide the details of these steps in Section 7.3 and Section 7.4.

Definition 7.1 (Notation)**.** We establish the following notations before presenting the proof.

Let $(a, b, c, d, e) - K$ be the contour of side lengths a, b, c, d, e with the special vertex kept and $(a, b, c, d, e) - R$ be the contour of side lengths a, b, c, d, e with the special vertex removed. We will write $\mathcal{G}(a, b, c, d, e,) - K$ (resp. $-R$) to denote the subgraph obtained from contour $(a, b, c, d, e) - K$ (resp. $-R$). Similarly for $\widehat{\mathcal{G}}$.

We say that point p_i is a white (or black) point on edge a (or b, c, d, e) if it is one of the white (or black) points on the boundary of $\widehat{\mathcal{G}}(C)$ facing edge a, where C is some contour. This notation follows from [LM15] and it does not necessarily mean that p_i is on edge a (or b, c, d, e) of the contour.

7.3. Base case $(n = 0)$**.** When $n = 0$, the cluster variables $A^{n^2} B^{n^2-n} x_{2k}$ and $A^{n^2+n} B^{n^2} x_{2k-1}$ where $n \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}$ become the terms $\{x_m\}_{m \in \mathbb{Z}}$ of the Somos-5 sequence.

For $1 \leq i \leq 5$, let C_i be the contour defined in Theorem 6.6 for the initial cluster variable x_i. We verify the weights and covering monomials of these contours. As shown in Figure 10, the subgraphs for these cluster variables are empty so they have weight 1. Recall that by definition, the covering monomials for C_3 and C_4 have an additional x_3 term. We can see that $c(\widehat{\mathcal{G}}(C_i)) = x_i$ for $1 \leq i \leq 5$.
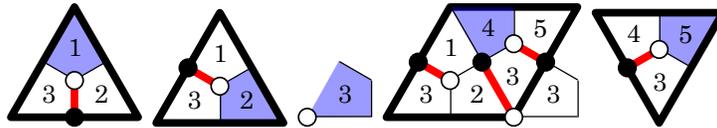

FIGURE 10. For $1 \leq i \leq 5$, we give contours C_i for terms x_i of the Somos-5 sequence. The purple blocks are what remain after multiplying the weights and covering monomials of these graphs.

Now assume the contours for x_i for all $i \leq m - 1$ give the correct Laurent polynomials for our cluster variables. We show the contour defined in Theorem 6.6 for x_m is correct.

Case 1: $m = 2k - 1$. We take the following contour

$$C = (a, b, c, d, e) = \left(k - 2, -\left\lceil \frac{k-2}{2} \right\rceil, 0, \left\lfloor \frac{k-2}{2} \right\rfloor, 2 - k \right).$$

Since $k > 3$, we will have $a > 0, b < 0, d \geq 0$ and $e < 0$.

Let $G = \widehat{\mathcal{G}}(C)$. We will then follow the steps shown in Section 7.2.

Step 1. We apply balanced Kuo's condensation theorem (Lemma 5.2) and write down

$$w(\mathcal{G}(C))w(\mathcal{G}(C) - \{p_1, p_2, p_3, p_4\}) = w(\mathcal{G}(C) - \{p_1, p_2\})w(\mathcal{G}(C) - \{p_3, p_4\})$$
$$+ w(\mathcal{G}(C) - \{p_1, p_4\})w(\mathcal{G}(C) - \{p_2, p_3\}).$$

where we let $S_1 = \emptyset$, $S_2 = \{p_1, p_2, p_3, p_4\}$, $S_3 = \{p_1, p_2\}$, $S_4 = \{p_3, p_4\}$, $S_5 = \{p_1, p_4\}$, $S_6 = \{p_2, p_3\}$. Then we multiply both sides by $m(\mathcal{G}(C))^2$.

Step 2. We define the black points p_1, p_3 and white points p_2, p_4 as follows.
- Let p_1 be any black point on edge e.
- Let p_2 be any white point on edge a.
- Let p_3 be any black point on edge b.
- Let p_4 be a white point near edge c defined as follows:
  - If $k \equiv 0 \pmod 2$, then $a \equiv 0 \pmod 2$ so the special vertex is kept. Let p_4 be the kept special white point between edges c and d.
  - If $k \equiv 1 \pmod 2$, then $a \not\equiv 0 \pmod 2$ so the special vertex is removed. Let p_4 be the other white point on the 5-block which contains the removed white point between edges c and d.

We also give the effects of removing each point separately:
- The effect of removing p_1 is $(a, b, c, d, e) \to (a - 1, b, c, d - 1, e + 1)$. We may also write this succinctly as $-\{p_1\} = (-1, 0, 0, -1, 1)$. This effect is equivalent to deleting a trapezoid along edge e of the original contour.
- The effect of removing p_2 is $(a, b, c, d, e) \to (a - 1, b + 1, c, d, e + 1)$. It is equivalent to deleting a trapezoid along edge a.
- The effect of removing p_3 is $(a, b, c, d, e) \to (a - 1, b + 1, c - 1, d, e)$.
- The effect of removing p_4 is $(a, b, c, d, e) - K \to (a, b, c, d, e) - R$ and $(a, b, c, d, e) - R \to (a, b + 1, c, d + 1, e) = K$ depending on the parity of k.

The position of each point and the effect of removing each point can be seen in Figure 11 (special point kept) and Figure 12 (special point removed). In the figures, we use big red dots to indicate point p_i and red edges to indicate forced matchings. The shadowed region is what's removed from the original contour after deleting the corresponding point. We also use black letters K/R to indicate whether the special point is kept or removed in the original contour and use blue letters for the new contour.
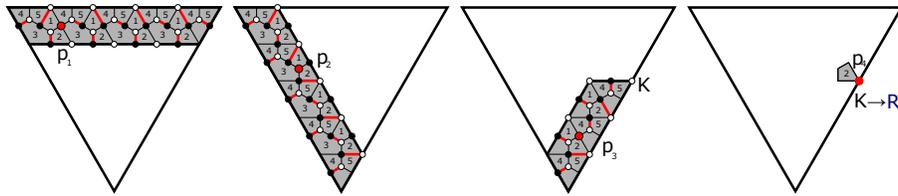

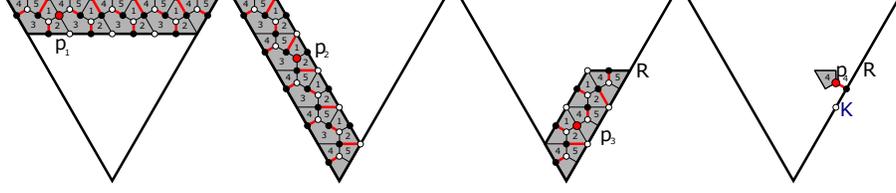FIGURE 11. Effects of removing points for x_{2k-1}, k even.

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FIGURE 12. Effects of removing points for x_{2k-1}, k odd.

Below, we explicitly write down the contour C_i satisfying $\mathcal{G}(\widehat{C}) - S_i = \widehat{\mathcal{G}}(C_i)$ for each S_i, with the corresponding cluster variable, followed from induction hypothesis. Here, we omit the details of doing explicit calculation of adding and subtracting 1's.

Subcase 1: k is even, i.e. $m \equiv 3 \pmod 4$. We have $C = (a, b, c, d, e) - K$.

$$G - \{\widehat{p_1, p_2}, p_3, p_4\} - K = \widehat{\mathcal{G}}(a - 3, b + 2, 0 - 1, d - 1, e + 2) - R$$
$$= \widehat{\mathcal{G}}(C_2), \text{ graph of } x_{2k-6}$$

$$G - \{\widehat{p_1, p_2}\} - K = \widehat{\mathcal{G}}(a - 2, b + 1, 0, d - 1, e + 2) - K$$
$$= \widehat{\mathcal{G}}(C_3), \text{ graph of } x_{2k-5}$$

$$G - \{\widehat{p_3, p_4}\} - K = \widehat{\mathcal{G}}(a - 1, b + 1, c - 1, d, e) - R$$
$$= \widehat{\mathcal{G}}(C_4), \text{ graph of } x_{2k-2}$$

$$G - \{\widehat{p_1, p_4}\} - K = \widehat{\mathcal{G}}(a - 1, b, c, d - 1, e + 1) - R$$
$$= \widehat{\mathcal{G}}(C_5), \text{ graph of } x_{2k-3}$$

$$G - \{\widehat{p_2, p_3}\} - K = \widehat{\mathcal{G}}(a - 2, b + 2, 0 - 1, d, e + 1) - K$$
$$= \widehat{\mathcal{G}}(C_6), \text{ graph of } x_{2k-4}$$

Subcase 2: k is odd, i.e. $m \equiv 1 \pmod 4$. We have $C = (a, b, c, d, e) - R$.

$$G - \{\widehat{p_1, p_2}, p_3, p_4\} - R = \widehat{\mathcal{G}}(a - 3, b + 3, c - 1, d, e + 2) - K$$
$$= \widehat{\mathcal{G}}(C_2), \text{ graph of } x_{2k-6}$$

$$G - \{\widehat{p_1, p_2}\} - R = \widehat{\mathcal{G}}(a - 2, b + 1, c, d - 1, e + 2) - R$$
$$= \widehat{\mathcal{G}}(C_3), \text{ graph of } x_{2k-5}$$

$$G - \{\widehat{p_3, p_4}\} - R = \widehat{\mathcal{G}}(a - 1, b + 2, c - 1, d + 1, e) - K$$
$$= \widehat{\mathcal{G}}(C_4), \text{ graph of } x_{2k-2}$$

$$G - \{\widehat{p_1, p_4}\} - R = \widehat{\mathcal{G}}(a - 1, b + 1, c, d, e + 1) - K$$
$$= \widehat{\mathcal{G}}(C_5), \text{ graph of } x_{2k-3}$$

$$G - \{\widehat{p_2, p_3}\} - R = \widehat{\mathcal{G}}(a - 2, b + 2, c - 1, d, e + 1) - R$$
$$= \widehat{\mathcal{G}}(C_6), \text{ graph of } x_{2k-4}$$

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By the Somos-5 recurrence $x_{2k-1}x_{2k-6} = x_{2k-5}x_{2k-2} + x_{2k-3}x_{2k-4}$ we conclude that $G = \widehat{\mathcal{G}}(C_1)$ is the graph of x_{2k-1} (after verifying step 3).

Step 3. Now that we have all of the contours C_i, we specify the sets T_i (defined in Section 7.2) for a specific choice of p_1, p_2, p_3, p_4.

$G - K$ (Special vertex kept): let p_1 be the rightmost (B) point on edge e (not in a forced matching), p_2 be the topmost (W) point on edge a (not in a forced matching), p_3 be the bottommost (B) point on edge b (in a forced matching), p_4 be the special vertex. See Figure 13.

$$T(\emptyset) = 1, \quad T(\{p_1, p_2, p_3, p_4\}) = x_3 x_3 x_3 x_4, \quad T(\{p_1, p_2\}) = x_3 x_4,$$
$$T(\{p_3, p_4\}) = x_3 x_3, \quad T(\{p_2, p_3\}) = x_3 x_3, \quad T(\{p_1, p_4\}) = x_3 x_4.$$
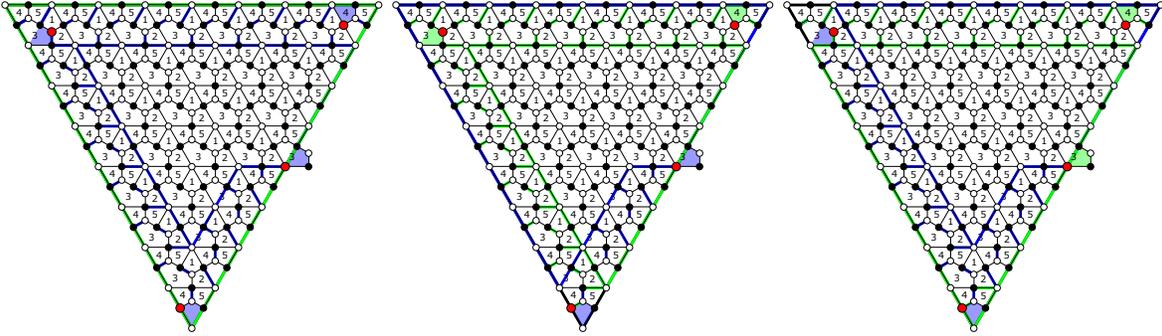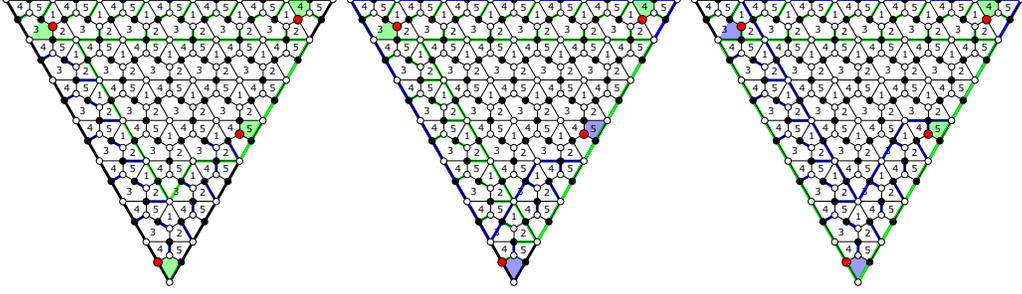

FIGURE 13. Covering monomial for x_{2k-1}, k even. Left: $T(\emptyset)$ and $T(\{p_1, p_2, p_3, p_4\})$. Middle: $T(\{p_1, p_2\})$ and $T(\{p_3, p_4\})$. Right $T(\{p_2, p_3\})$ and $T(\{p_1, p_4\})$.

$G - R$ (Special vertex removed): let p_1 be the rightmost (B) point on edge e (not in a forced matching), p_2 be the topmost (W) point on edge a (not in a forced matching), p_3 be the bottommost (B) point on edge b (in a forced matching), p_4 be the other white vertex on the 5-block below the special vertex. See Figure 14.

$$T(\emptyset) = 1, \quad T(\{p_1, p_2, p_3, p_4\}) = x_3 x_3 x_3 x_4, \quad T(\{p_1, p_2\}) = x_3 x_4,$$
$$T(\{p_3, p_4\}) = x_3 x_5, \quad T(\{p_2, p_3\}) = x_3 x_3, \quad T(\{p_1, p_4\}) = x_4 x_4.$$

We see that equation 4 is satisfied:

$$\left(\prod_{j \in T_1} x_j\right)\left(\prod_{j \in T_2} x_j\right) = \left(\prod_{j \in T_3} x_j\right)\left(\prod_{j \in T_4} x_j\right) = \left(\prod_{j \in T_5} x_j\right)\left(\prod_{j \in T_6} x_j\right).$$

Finally, we conclude that $c(\widehat{\mathcal{G}}(C_1))$ is the Laurent polynomial of x_{2k-1}, as desired.

Case 2: $m = 2k$. Consider the following contour

$$C = (a, b, c, d, e) = \left(k - 2, -\left\lceil\frac{k-4}{2}\right\rceil, -1, \left\lfloor\frac{k}{2}\right\rfloor, 1 - k\right).$$

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FIGURE 14. Covering monomial for x_{2k-1}, k even. Left: $T(\emptyset)$ and $T(\{p_1, p_2, p_3, p_4\})$. Middle: $T(\{p_1, p_2\})$ and $T(\{p_3, p_4\})$. Right $T(\{p_2, p_3\})$ and $T(\{p_1, p_4\})$.

Since $k \geq 3$, we have $a > 0, b \geq 0, d > 0$ and $e < 0$. The proof is similar to the first case. In Step 1 we again use balanced Kuo's condensation on $\mathcal{G}(C)$ and use the same notation for each S_i. In Step 2 we define the four points as follows.

- Let p_1 be any white point on edge a.
- Let p_2 be any black point on edge e.
- Let p_3 be any white point on edge d.
- Let p_4 be a black point near edge c on edge d defined as follows:
    - If $k \equiv 0 \pmod 2$, then $a \equiv 0 \pmod 2$ so the special point is kept. Let p_4 be the black point on the edge between the 4-block and 5-block above the special point.
    - If $k \equiv 1 \pmod 2$, then $a \not\equiv 0 \pmod 2$ so the special point is removed. Let p_4 be the lowest black point on edge d.

We also give the effects of removing each point separately:

- The effect of removing p_1 is $(a, b, c, d, e) \to (a - 1, b + 1, c, d, e + 1)$.
- The effect of removing p_2 is $(a, b, c, d, e) \to (a - 1, b, c, d - 1, e + 1)$.
- The effect of removing p_3 is $(a, b, c, d, e) - K \to (a, b - 1, c + 1, d - 1, e + 1) - R$ and $(a, b, c, d, e) - R \to (a, b, c + 1, d, e + 1) - K$ depending on the parity of k.
- The effect of removing p_4 is $(a, b, c, d, e) - K \to (a, b - 1, c, d - 1, e) - R$ and $(a, b, c, d, e) - R \to (a, b, c, d, e) - K$ depending on the parity of k.

The position of each point and the effect of removing each point is shown in Figure 15 (special point kept) and Figure 16 (special point removed).
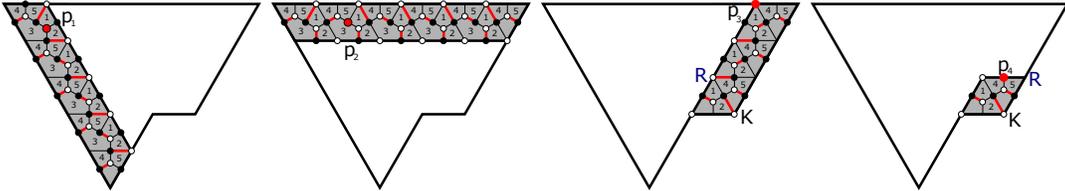

FIGURE 15. Effects of removing points for x_{2k}, k even.

Below, we explicitly write down the contour C_i satisfying $\widehat{\mathcal{G}(C)} - S_i = \widehat{\mathcal{G}}(C_i)$ for each S_i, with the corresponding cluster variable (after verifying step 3).
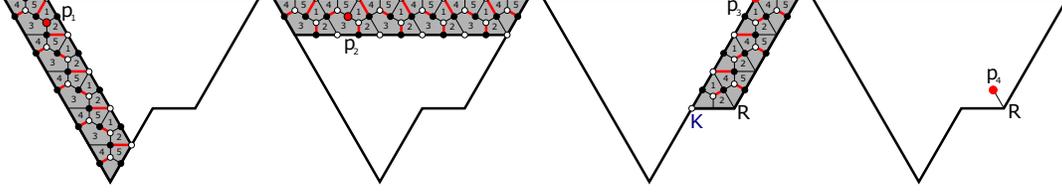
FIGURE 16. Effects of removing points for x_{2k}, k odd.

Subcase 1: k is even, i.e. $m \equiv 0 \pmod 4$. We have $C = (a, b, c, d, e) - K$.

$$G - \{\widehat{p_1, p_2}, p_3, p_4\} - K = \widehat{\mathcal{G}}(a - 2, b, c + 1, d - 2, e + 3) - K$$
$$= \widehat{\mathcal{G}}(C_2), \text{ graph of } x_{2k-5}$$

$$G - \widehat{\{p_1, p_2\}} - K = \widehat{\mathcal{G}}(a - 2, b + 1, c, d - 1, e + 2) - K$$
$$= \widehat{\mathcal{G}}(C_3), \text{ graph of } x_{2k-4}$$

$$G - \widehat{\{p_3, p_4\}} - K = \widehat{\mathcal{G}}(a, b - 1, c + 1, d - 1, e + 1) - K$$
$$= \widehat{\mathcal{G}}(C_4), \text{ graph of } x_{2k-1}$$

$$G - \widehat{\{p_1, p_4\}} - K = \widehat{\mathcal{G}}(a - 1, b - 1, c + 1, d - 2, e + 2) - R$$
$$= \widehat{\mathcal{G}}(C_5), \text{ graph of } x_{2k-3}$$

$$G - \widehat{\{p_2, p_3\}} - K = \widehat{\mathcal{G}}(a - 1, b, c, d - 1, e + 1) - R$$
$$= \widehat{\mathcal{G}}(C_6), \text{ graph of } x_{2k-2}$$

Subcase 2: k is odd, i.e. $m \equiv 2 \pmod 4$. We have $C = (a, b, c, d, e) - R$.

$$G - \{\widehat{p_1, p_2}, p_3, p_4\} - R = \widehat{\mathcal{G}}(a - 2, b, c + 1, d - 2, e + 3) - R$$
$$= \widehat{\mathcal{G}}(C_2), \text{ graph of } x_{2k-5}$$

$$G - \widehat{\{p_1, p_2\}} - R = \widehat{\mathcal{G}}(a - 2, b + 1, c, d - 1, e + 2) - R$$
$$= \widehat{\mathcal{G}}(C_3), \text{ graph of } x_{2k-4}$$

$$G - \widehat{\{p_3, p_4\}} - R = \widehat{\mathcal{G}}(a, b - 1, c + 1, d - 1, e + 1) - R$$
$$= \widehat{\mathcal{G}}(C_4), \text{ graph of } x_{2k-1}$$

$$G - \widehat{\{p_1, p_4\}} - R = \widehat{\mathcal{G}}(a - 1, b, c + 1, d - 1, e + 2) - K$$
$$= \widehat{\mathcal{G}}(C_5), \text{ graph of } x_{2k-3}$$

$$G - \widehat{\{p_2, p_3\}} - R = \widehat{\mathcal{G}}(a - 1, b + 1, c, d, e + 1) - K$$
$$= \widehat{\mathcal{G}}(C_6), \text{ graph of } x_{2k-2}$$

By the Somos-5 recurrence $x_{2k}x_{2k-5} = x_{2k-4}x_{2k-1} + x_{2k-2}x_{2k-3}$ we conclude that $\widehat{\mathcal{G}}(C_1)$ is the graph of x_{2k}.

In Step 3 we specify the sets T_i and verify equation 4.

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$G - K$ (Special vertex kept): let p_1 be the bottommost (W) point on edge a (not in a forced matching), p_2 be the leftmost (B) point on edge e (not in a forced matching), p_3 be the topmost (W) point on edge d, p_4 be the (B) point on the edge between the 4-block and 5-block above the special vertex. See Figure 18.

$$T(\emptyset) = 1, \quad T(\{p_1, p_2, p_3, p_4\}) = x_3 x_4 x_4 x_5, \quad T(\{p_1, p_2\}) = x_3 x_5,$$
$$T(\{p_3, p_4\}) = x_4 x_4, \quad T(\{p_2, p_3\}) = x_4 x_5, \quad T(\{p_1, p_4\}) = x_3 x_4.$$
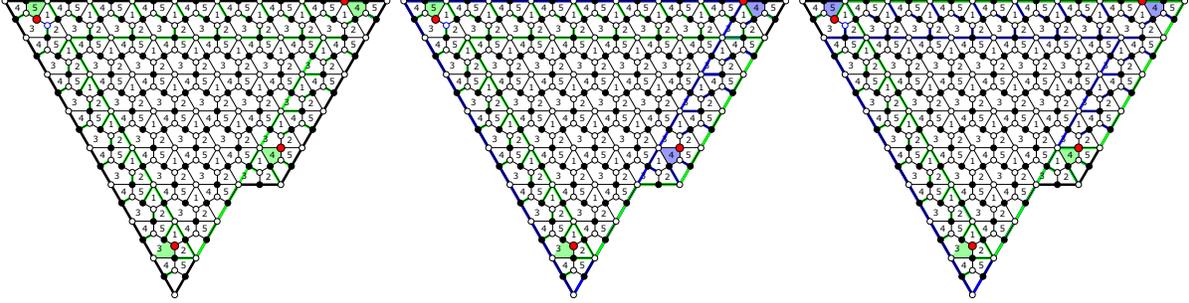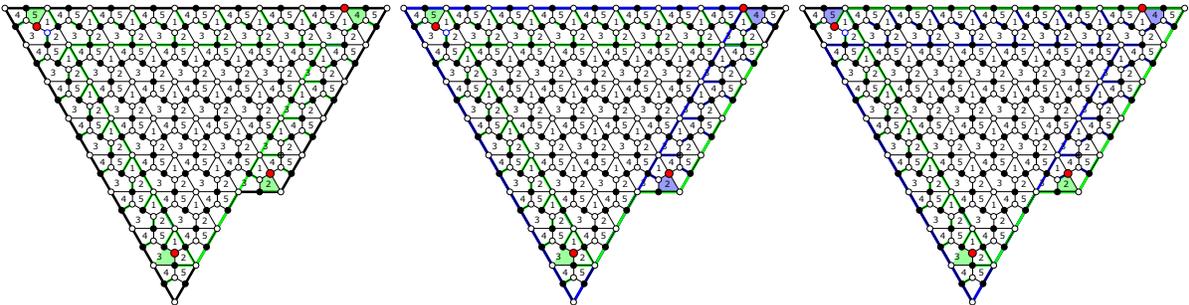

FIGURE 17. Covering monomial for x_{2k}, k even. Left: $T(\emptyset)$ and $T(\{p_1, p_2, p_3, p_4\})$. Middle: $T(\{p_1, p_2\})$ and $T(\{p_3, p_4\})$. Right $T(\{p_2, p_3\})$ and $T(\{p_1, p_4\})$.

$G - R$ (Special vertex removed): let p_1 be the bottommost (W) point on edge a (not in a forced matching), p_2 be the leftmost (B) point on edge e (not in a forced matching), p_3 be the topmost (W) point on edge d, p_4 be the (B) point on the edge between the 2-block and 3-block above the special vertex.

$$T(\emptyset) = 1, \quad T(\{p_1, p_2, p_3, p_4\}) = x_2 x_3 x_4 x_5, \quad T(\{p_1, p_2\}) = x_3 x_5,$$
$$T(\{p_3, p_4\}) = x_2 x_4, \quad T(\{p_2, p_3\}) = x_4 x_5, \quad T(\{p_1, p_4\}) = x_2 x_3.$$

FIGURE 18. Covering monomial for x_{2k}, k odd. Left: $T(\emptyset)$ and $T(\{p_1, p_2, p_3, p_4\})$. Middle: $T(\{p_1, p_2\})$ and $T(\{p_3, p_4\})$. Right $T(\{p_2, p_3\})$ and $T(\{p_1, p_4\})$.

7.4. Inductive Step for $A^{n^2}B^{n(n-1)}x_{2k}$, $n \geq 1$, $k \geq 3n - 1$.

As we have explained in Section 7.1, we will only show the inductive step for the case $A^{n^2}B^{n(n-1)}x_{2k}$. All of the other cases will be proved in the same way and we provide the data in Appendix 9 for doing so.

Assume the contours of $A^{m^2}B^{m(m-1)}x_{2k}$ and $A^{m(m+1)}B^{m^2}x_{2k+1}$, as defined in Theorem 6.6, give the correct cluster variables for any $m \leq n - 1$ and $k \geq 3n - 1$. Now we want to show that the contour of $A^{m^2}B^{m(m-1)}x_{2k}$ is correct for any $k \geq 3n - 1$ and $m = n$.

The recurrence we will use is

$$(5) \qquad (A^{n^2}B^{n(n-1)}x_{2k})(A^{(n-1)^2}B^{(n-1)(n-2)}x_{2k+2})$$

$$(6) \qquad =(A^{(n-1)n}B^{(n-1)^2}x_{2k-1})(A^{(n-1)n}B^{(n-1)^2}x_{2k+3}) + (A^{(n-1)n}B^{(n-1)^2}x_{2k+1})^2$$

where by the induction hypothesis, we have the correctness of the contours for cluster variables $A^{(n-1)^2}B^{(n-1)(n-2)}x_{2k+2}$, $A^{(n-1)n}B^{(n-1)^2}x_{2k-1}$, $A^{(n-1)n}B^{(n-1)^2}x_{2k+1}$ and $A^{(n-1)n}B^{(n-1)^2}x_{2k+3}$.

For this case, let contour C be the following:

$$C = (a, b, c, d, e) = \left(k - 1 + n, -\left\lceil \frac{k+5n-5}{2} \right\rceil, 2n - 2, \left\lfloor \frac{k-3n+3}{2} \right\rfloor, n - k - 1 \right).$$

Since $k \geq 3n - 1$, we have $a > 0, b < 0, c \geq 0, d > 0, e < 0$. Again, we will use the steps described in Section 7.2. Let $G = \widehat{\mathcal{G}}(C)$.

Step 1: We use non-alternating Kuo Condensation theorem (Lemma 5.4) and write down

$$w(G - \{p_1, p_2\})w(G - \{p_3, p_4\}) = w(G)w(G - \{p_1, p_2, p_3, p_4\})$$
$$+ w(G - \{p_1, p_3\})w(G - \{p_2, p_4\}).$$

where we let $S_1 = \{p_1, p_2\}$, $S_2 = \{p_3, p_4\}$, $S_3 = \emptyset$, $S_4 = \{p_1, p_2, p_3, p_4\}$, $S_5 = \{p_1, p_3\}$, $S_6 = \{p_2, p_4\}$. Then we multiply both sides by $m(\mathcal{G}(C))^2$.

Step 2. We define the four points p_1, p_2, p_3, p_4 on edge d, e, b, c respectively, where p_1, p_4 are white, while p_2, p_3 are black. We list the effect of each removal as follows.

$$-\{p_1\} = \begin{cases} (0, -1, 1, -1, 1) - R, & \text{if } G = (a, b, c, d, e) - K \\ (0, 0, 1, 0, 1) - K, & \text{if } G = (a, b, c, d, e) - R \end{cases}$$

$$-\{p_2\} = (-1, 0, 0, -1, 1)$$

$$-\{p_3\} = (-1, 1, -1, 0, 0)$$

$$-\{p_4\} = \begin{cases} (0, 0, 0, 0, 0) - R, & \text{if } G = (a, b, c, d, e) - K \\ (0, 1, 0, 1, 0) - K, & \text{if } G = (a, b, c, d, e) - R \end{cases}$$

The position of these points and the effects of removing each point, when the special point is kept, is shown in Figure 19 (p_1) and Figure 20 (p_2, p_3, p_4). Notice that after we remove p_1, as shown in Figure 19, some area gets deleted (grey) and some area gets added (pink). The position of these points and effects of removing each point, when the special point ire removed, is shown in Figure 21.

Below, we explicitly write down the contour C_i satisfying $\widehat{\mathcal{G}(C) - S_i} = \widehat{\mathcal{G}}(C_i)$ for each S_i, with the corresponding cluster variable, followed from induction hypothesis.
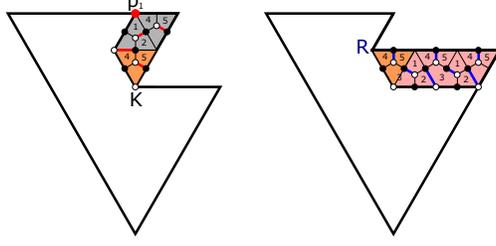
FIGURE 19. Effect of removing p_1 for $A^{n^2}B^{n(n-1)}$ with $n \geq 1$, $k \geq 3n-1$ and the special point kept. Left: before removal. Right: after removal.
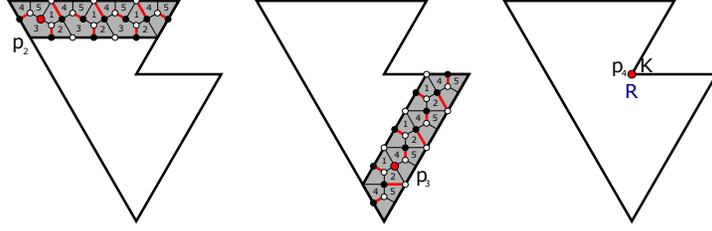

FIGURE 20. Effects of removing p_2, p_3, p_4 for $A^{n^2}B^{n(n-1)}$ with $n \geq 1$, $k \geq 3n-1$ and the special point kept.
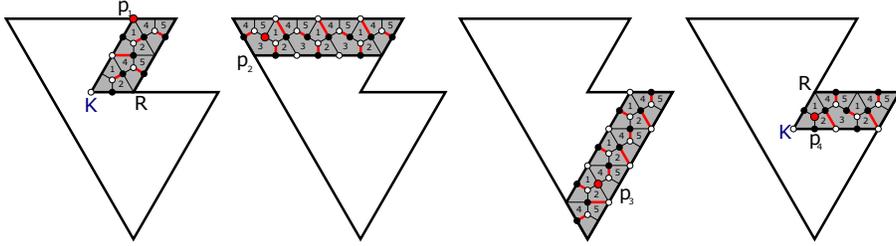

FIGURE 21. Effects of removing p_i's for $A^{n^2}B^{n(n-1)}$ with $n \geq 1$, $k \geq 3n-1$ and the special point removed.

Case 1: $n + k$ is odd. Thus, the special vertex is kept, and $C = (a, b, c, d, e) - K$.

$$
\begin{aligned}
G - \widehat{\{p_1, p_2\}} &= \widehat{\mathcal{G}}(a - 1, b - 1, c + 1, d - 2, e + 2) - R \\
&= \widehat{\mathcal{G}}(C_1) \\
G - \widehat{\{p_3, p_4\}} &= \widehat{\mathcal{G}}(a - 1, b + 1, c - 1, d, e) - R \\
&= \widehat{\mathcal{G}}(C_2), \text{ graph of } A^{(n-1)^2}B^{(n-1)(n-2)}x_{2k+2} \\
\widehat{G} &= \widehat{\mathcal{G}}(a, b, c, d, e) - K \\
&= \widehat{\mathcal{G}}(C_3), \text{ graph of } A^{n(n-1)}B^{(n-1)^2}x_{2k+3} \\
G - \widehat{\{p_1, p_2, p_3, p_4\}} &= \widehat{\mathcal{G}}(a - 2, b + 1, c, d - 1, e + 2) - K \\
&= \widehat{\mathcal{G}}(C_4), \text{ graph of } A^{n^2-n}B^{(n-1)^2}x_{2k-1}
\end{aligned}
$$

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$$G - \widehat{\{p_1, p_3\}} = \widehat{\mathcal{G}}(a - 1, b, c, d - 1, e + 1) - R$$
$$= \widehat{\mathcal{G}}(C_5), \text{ graph of } A^{n^2 - n} B^{(n-1)^2} x_{2k+1}$$
$$G - \widehat{\{p_2, p_4\}} = \widehat{\mathcal{G}}(a - 1, b, c, d - 1, e + 1) - R$$
$$= \widehat{\mathcal{G}}(C_6), \text{ graph of } A^{n^2 - n} B^{(n-1)^2} x_{2k+1}$$

Case 2: $n + k$ is even. Thus, the special vertex is removed, and $C = (a, b, c, d, e) - R$.

$$G - \widehat{\{p_1, p_2\}} = \widehat{\mathcal{G}}(a - 1, b, c + 1, d - 1, e + 2) - K$$
$$= \widehat{\mathcal{G}}(C_1)$$
$$G - \widehat{\{p_3, p_4\}} = \widehat{\mathcal{G}}(a - 1, b + 2, c - 1, d + 1, e) - K$$
$$= \widehat{\mathcal{G}}(C_2), \text{ graph of } A^{(n-1)^2} B^{(n-1)(n-2)} x_{2k+2}$$
$$\widehat{G} = \widehat{\mathcal{G}}(a, b, c, d, e) - R$$
$$= \widehat{\mathcal{G}}(C_3), \text{ graph of } A^{n(n-1)} B^{(n-1)^2} x_{2k+3}$$
$$G - \widehat{\{p_1, p_2, p_3, p_4\}} = \widehat{\mathcal{G}}(a - 2, b + 1, c, d - 1, e + 2) - R$$
$$= \widehat{\mathcal{G}}(C_4), \text{ graph of } A^{n^2 - n} B^{(n-1)^2} x_{2k-1}$$
$$G - \widehat{\{p_1, p_3\}} = \widehat{\mathcal{G}}(a - 1, b + 1, c, d, e + 1) - K$$
$$= \widehat{\mathcal{G}}(C_5), \text{ graph of } A^{n^2 - n} B^{(n-1)^2} x_{2k+1}$$
$$G - \widehat{\{p_2, p_4\}} = \widehat{\mathcal{G}}(a - 1, b + 1, c, d, e + 1) - K$$
$$= \widehat{\mathcal{G}}(C_6), \text{ graph of } A^{n^2 - n} B^{(n-1)^2} x_{2k+1}$$

By recurrence 5, we conclude that $\widehat{\mathcal{G}}(C_1)$ is the graph corresponding to $A^{n^2} B^{n^2 - n} x_{2k}$.

Step 3. In this step we specify the sets T_i for a choice of p_1, p_2, p_3, p_4 and verify equation 4. In each of the diagrams in Figures 22 and 23, p_1, p_2, p_3, p_4 are the red points. Each figure shows the new contours C_i and C_{i+1} in green and in blue.

There is a bijection between perfect matchings of $\mathcal{G}(C_i)$ and perfect matchings of $\mathcal{G}(C) - S_i$. Let M be any perfect matching of $\mathcal{G}(C_i)$. Essentially, the weight of the blocks in T_i is exactly what we need to multiply $m(\mathcal{G}(C_i)) w(M)$ by so that it corresponds to a term of $m(\mathcal{G}(C)) w(\mathcal{G}(C) - S_i)$.

We explain how Figure 22(Left) allows us to determine that the weight of T_1 is $x_3 x_3 x_4 x_4$. Let us start with the green contour C_1. A perfect matching of $\mathcal{G}(C_1)$ corresponds to a perfect matching of $G - \{p_1, p_2\}$ if we remove the red matchings and add in the green matchings. Algebraically, this corresponds to multiplying by the weight of these matchings. The covering monomial of $\mathcal{G}(C_1)$ must be multiplied by the weight of all blocks that are outside the green contour C_1 and within the largest contour C. Note that the weight of these blocks are divided out by many of the green matchings and only the two 4-blocks (green) along edge e and the single 3-block (cyan) near the special vertex remain.

In this particular case, the contour C_1 is not completely contained in C, so we must also divide by the weight of all blocks within C_1 and outside C. Again, note that these weight of these blocks divide out all but one of the red matchings. So overall, the weight of T_1 includes $x_3 x_4 x_4$ from the covering monomial of C, part of the weight $\frac{1}{x_3 x_4 x_5}$ of two green matchings (shaded), and the weight $x_4 x_5$ of the single red matching within C_1 (green). So the weight of T_1 is $x_4 x_4$. Similarly, we find the weight of T_2 is $x_3 x_3$ since we simply need to multiply by the weight of blocks outside C_2 within C and the only blocks that are not divided out by forced matchings are the 3-block near the special vertex (cyan) and the 3-block near p_3 (blue).
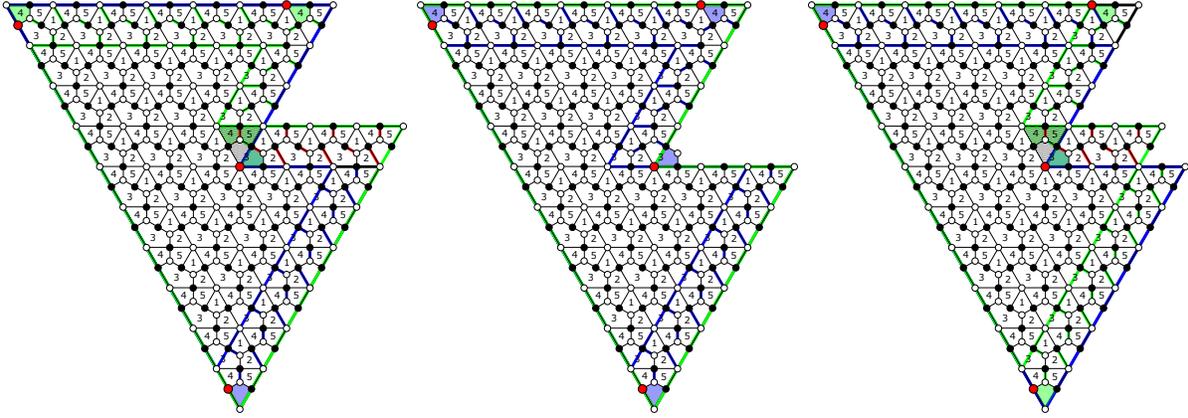
Case 1: Special vertex kept. See Figure 22.

FIGURE 22. Covering monomial for the case of $A^{n^2} B^{n(n-1)}$ with $n \geq 1$, $k \geq 3n - 1$ and the special point kept. Left: $T(\{p_1, p_2\})$ and $T(\{p_3, p_4\})$. Middle: $T(\{p_1, p_2, p_3, p_4\})$ and $T(\emptyset)$. Right: $T(\{p_1, p_3\})$ and $T(\{p_2, p_4\})$.

For this choice of p_1, p_2, p_3, p_4, we have

$$\prod_{j \in T_1} x_j = x_3 x_4 x_4 \frac{1}{x_3 x_4 x_5} x_4 x_5 = x_4 x_4, \qquad \prod_{j \in T_2} x_j = x_3 x_3,$$

$$\prod_{j \in T_3} x_j = 1, \qquad \prod_{j \in T_4} x_j = x_3 x_3 x_4 x_4,$$

$$\prod_{j \in T_5} x_j = x_3 x_3 x_4 \frac{1}{x_3 x_4 x_5} x_4 x_5 = x_3 x_4, \qquad \prod_{j \in T_6} x_j = x_3 x_4.$$

Case 2: Special vertex removed. See Figure 23.
For this choice of p_1, p_2, p_3, p_4, we have

$$\prod_{j \in T_1} x_j = x_4 x_4, \qquad \prod_{j \in T_2} x_j = x_1 x_3,$$

$$\prod_{j \in T_3} x_j = 1, \qquad \prod_{j \in T_4} x_j = x_3 x_3 x_4 \frac{x_3 \cdot x_1 x_3 x_4 x_5}{x_3 x_3 x_4 x_5} = x_1 x_3 x_3 x_4,$$
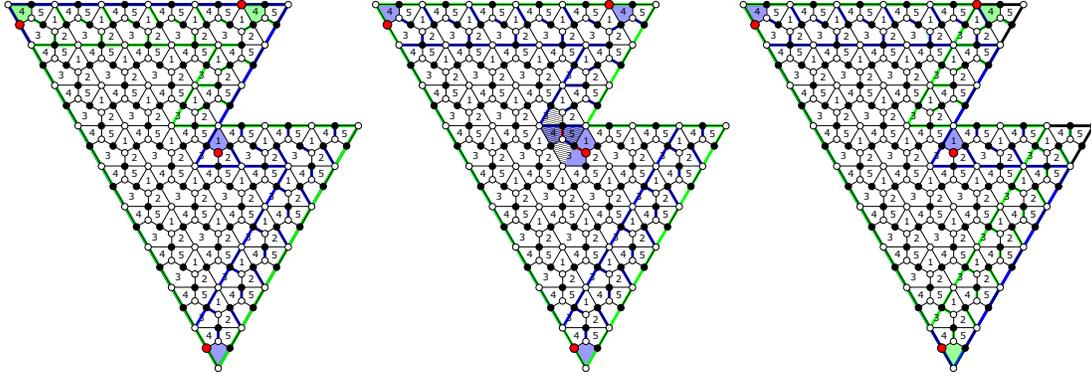
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FIGURE 23. Covering monomial for the case of $A^{n^2}B^{n(n-1)}$ with $n \geq 1$, $k \geq 3n - 1$ and the special point removed. Left: $T(\{p_1, p_2\})$ and $T(\{p_3, p_4\})$. Middle: $T(\{p_1, p_2, p_3, p_4\})$ and $T(\emptyset)$. Right: $T(\{p_1, p_3\})$ and $T(\{p_2, p_4\})$.

$$\prod_{j \in T_5} x_j = x_3 x_4, \qquad \prod_{j \in T_6} x_j = x_1 x_4.$$

We see that equation 4 holds in both cases.

Remark 7.2. As long as we fix the side and the color of a point p_i, the effect of removing p_i will be the same regardless of the shape of the contour, i.e. regardless of the signs of the other side lengths. For instance, as shown in Figure 24, the effects of removing p_4 in shapes $(+, -, +, +, -)$ and $(+, -, +, -, +)$ are the same.
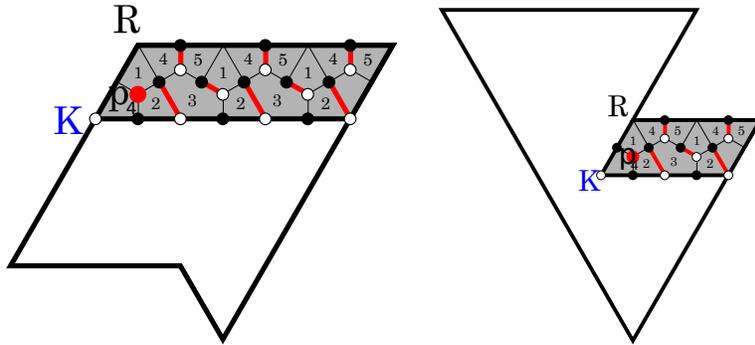

FIGURE 24. The effects of removing p_4 in shape $(+, -, +, -, +)$ and $(+, -, +, +, -)$.

Remark 7.3. The subgraphs in dP_2 quiver can look significantly different from those in dP_3 quiver. When side c is long, we will have a lot of forced edges, which results in different shapes. See Figure 25.

8. Comparison with the Octahedron Recurrence

David Speyer has given another combinatorial interpretation for the Laurent polynomial of the Somos-5 sequence in terms of the weight of some subgraphs of another brane tiling [Spe07]. See Figure 26 for the brane tiling and its corresponding quiver.
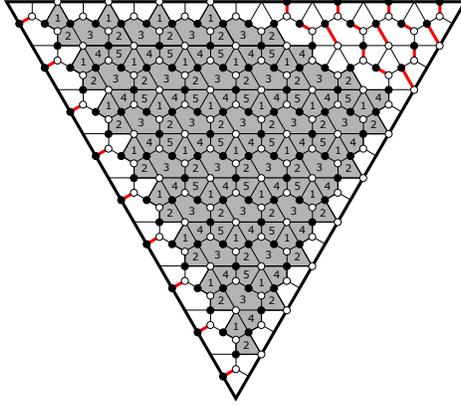
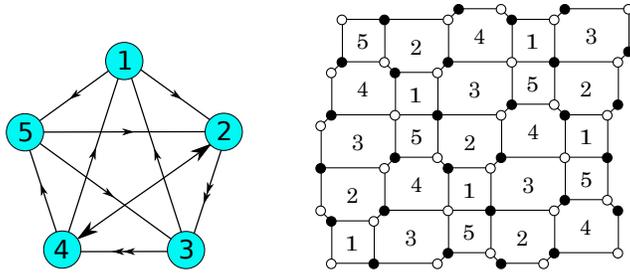FIGURE 25. Graph for $A^6B^4x_{17}$. Long edge c results many in forced edges.

FIGURE 26. The quiver and the brane tiling studied in [Spe07]

Notice that if we add a 2-cycle between vertex 2 and vertex 4 in our dP$_2$ quiver, we will obtain the quiver shown in Figure 26. However, it is hard to describe the transformation of these two brane tilings in a simple way.
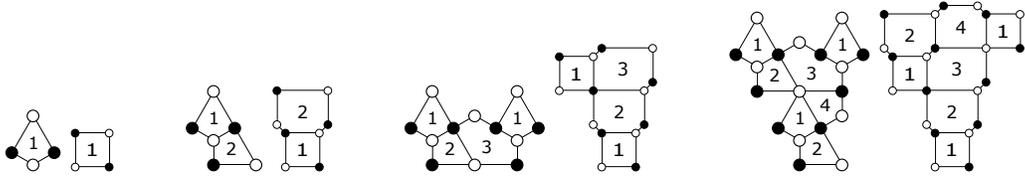

FIGURE 27. subgraphs corresponding to terms x_6, x_7, x_8, x_9 in two different tilings

We provide a few terms of the Laurent polynomial of the Somos-5 sequence written as subgraphs of these two different brane tilings in Figure 27 and Figure 28. As we can see, the blocks in each pair of subgraphs are similar but not exactly the same. Moreover, the subgraphs corresponding to x_n in the dP$_2$ brane tiling are growing in two different directions (upper right and lower right) but subgraphs in the tiling considered by Speyer are growing in only one direction (up), as n increases. Therefore, we believe that these two problems regarding the two different tilings are sufficient different. There clearly exists some bijection between these subgraphs as we know how to generate them given x_n but as these two tilings are very different despite the similarity in the corresponding quivers, we believe such bijection is not worth considering in great details.
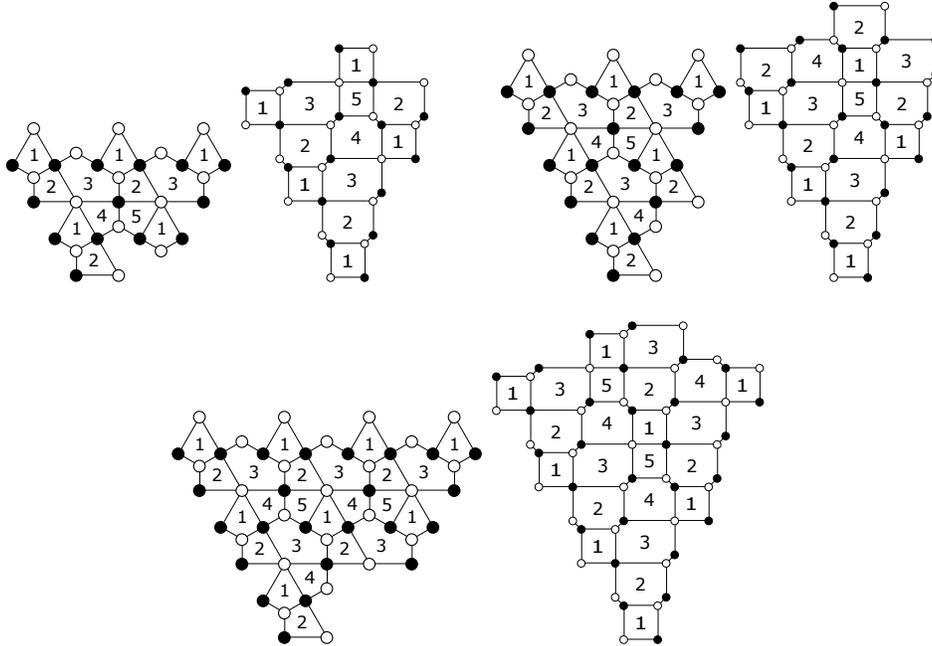
FIGURE 28. subgraphs corresponding to terms x_{10}, x_{11}, x_{12} in two different tilings

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9. Appendix

Here are the data for other cases of Theorem 6.6. We group these cases by the form of cluster variables. The way to use the appendix is shown in Section 7.1 and Section 7.2.

9.1. $A^{n^2} B^{n(n-1)} x_{2k}$, $n \geq 1$, $k \geq 2$. Recurrence we use:

$$(A^{n^2} B^{n(n-1)} x_{2k})(A^{(n-1)^2} B^{(n-1)(n-2)} x_{2k+2})$$
$$=(A^{(n-1)n} B^{(n-1)^2} x_{2k-1})(A^{(n-1)n} B^{(n-1)^2} x_{2k+3}) + (A^{(n-1)n} B^{(n-1)^2} x_{2k+1})^2$$

Kuo's four points: p_1, p_2, p_3, p_4 are on edge d, e, b, c respectively.

9.1.1. *Case 1.* $k \geq 3n - 1$. Non-alternating Kuo. Shape $(+, -, +, +, -)$.

$$G := \widehat{\mathcal{G}}\left(k - 1 + n, -\left\lceil \frac{k + 5n - 5}{2} \right\rceil, 2n - 2, \left\lfloor \frac{k - 3n + 3}{2} \right\rfloor, n - k - 1\right)$$

$$-\{p_1\}(W) : K \to (0, -1, +1, -1, +1) - R$$
$$R \to (0, 0, +1, 0, +1) - K$$
$$-\{p_2\}(B) : \to (-1, 0, 0, -1, +1)$$
$$-\{p_3\}(B) : \to (-1, +1, -1, 0, 0)$$
$$-\{p_4\}(W) : K \to (0, 0, 0, 0, 0) - R$$
$$R \to (0, +1, 0, +1, 0) - K$$

$$G = A^{(n-1)n} B^{(n-1)^2} x_{2k+3}$$
$$G - \{p_1, p_2, p_3, p_4\} = A^{(n-1)n} B^{(n-1)^2} x_{2k-1}$$
$$G - \{p_1, p_3\} = A^{(n-1)n} B^{(n-1)^2} x_{2k+1}$$
$$G - \{p_2, p_4\} = A^{(n-1)n} B^{(n-1)^2} x_{2k+1}$$
$$G - \{p_1, p_2\} = A^{n^2} B^{n^2 - n} x_{2k}$$
$$G - \{p_3, p_4\} = A^{(n-1)^2} B^{(n-1)(n-2)} x_{2k+2}$$

$G - K$ (Special vertex kept): let p_1 be the topmost (W) point on edge d, p_2 be the leftmost (B) point on edge e, p_3 be the bottommost (B) point on edge b, p_4 be the special vertex.

$$T(\emptyset) = 1, \quad T(\{p_1, p_2, p_3, p_4\}) = x_3 x_3 x_4 x_4, \quad T(\{p_1, p_3\}) = x_3 x_4,$$
$$T(\{p_2, p_4\}) = x_3 x_4, \quad T(\{p_1, p_2\}) = x_4 x_4, \quad T(\{p_3, p_4\}) = x_3 x_3.$$

$G - R$ (Special vertex removed): let p_1 be the topmost (W) point on edge d, p_2 be the leftmost (B) point on edge e, p_3 be the bottommost (B) point on edge b, p_4 be the (W) vertex bordering the 1-block below the special vertex.

$$T(\emptyset) = 1, \quad T(\{p_1, p_2, p_3, p_4\}) = x_1 x_3 x_4 x_4, \quad T(\{p_1, p_3\}) = x_3 x_4,$$
$$T(\{p_2, p_4\}) = x_1 x_4, \quad T(\{p_1, p_2\}) = x_4 x_4, \quad T(\{p_3, p_4\}) = x_1 x_3.$$

9.1.2. *Case 2.* $n + 1 \leq k \leq 3n - 2$. Unbalanced Kuo. Shape $(+, -, +, -, -)$.
When $(k + 2) + (n - 1)$ is odd, let

$$G := \widehat{\mathcal{G}}\left(k - 1 + n, -\left\lceil \frac{k + 5n - 5}{2} \right\rceil, 2n - 2 + 1, \left\lfloor \frac{k - 3n + 3}{2} \right\rfloor, n - k - 1 + 1\right) - K$$

When $(k + 2) + (n - 1)$ is even, let

$$G := \widehat{\mathcal{G}}\left(k - 1 + n, -\left\lceil \frac{k + 5n - 5}{2} \right\rceil - 1, 2n - 2 + 1, \left\lfloor \frac{k - 3n + 3}{2} \right\rfloor - 1, n - k - 1 + 1\right) - R$$

$$
\begin{aligned}
-\{p_1\}(B) : K &\to (0, 0, -1, 0, -1) - R \\
R &\to (0, +1, -1, +1, -1) - K \\
-\{p_2\}(B) : &\to (-1, 0, 0, -1, +1) \\
-\{p_3\}(B) : &\to (-1, +1, -1, 0, 0) \\
-\{p_4\}(W) : K &\to (0, 0, 0, 0, 0) - R \\
R &\to (0, +1, 0, +1, 0) - K
\end{aligned}
$$

$$
\begin{aligned}
G - \{p_1\} &= A^{(n-1)n} B^{(n-1)^2} x_{2k+3} \\
G - \{p_2, p_3, p_4\} &= A^{(n-1)n} B^{(n-1)^2} x_{2k-1} \\
G - \{p_3\} &= A^{(n-1)n} B^{(n-1)^2} x_{2k+1} \\
G - \{p_1, p_2, p_4\} &= A^{(n-1)n} B^{(n-1)^2} x_{2k+1} \\
G - \{p_2\} &= A^{n^2} B^{n^2 - n} x_{2k} \\
G - \{p_1, p_3, p_4\} &= A^{(n-1)^2} B^{(n-1)(n-2)} x_{2k+2}
\end{aligned}
$$

$G - K$ (Special vertex kept): let p_1 be the bottommost (B) point on edge d, p_2 be the leftmost (B) point on edge e, p_3 be the bottommost (B) point on edge b, p_4 be the special vertex.

$$T(\{p_1\}) = x_3, \quad T(\{p_2, p_3, p_4\}) = x_3 x_3 x_4, \quad T(\{p_3\}) = x_3,$$
$$T(\{p_1, p_2, p_4\}) = x_3 x_3 x_4, \quad T(\{p_2\}) = x_4, \quad T(\{p_1, p_3, p_4\}) = x_3 x_3 x_3.$$

$G - R$ (Special vertex removed): let p_1 be the bottommost (B) point on edge d, p_2 be the leftmost (B) point on edge e, p_3 be the bottommost (B) point on edge b, p_4 be the (W) vertex bordering the 1-block below the special vertex.

$$T(\{p_1\}) = x_4, \quad T(\{p_2, p_3, p_4\}) = x_1 x_3 x_4, \quad T(\{p_3\}) = x_3 x_3,$$
$$T(\{p_1, p_2, p_4\}) = x_3 x_3 x_3 x_4, \quad T(\{p_2\}) = x_4, \quad T(\{p_1, p_3, p_4\}) = x_1 x_3 x_3.$$

9.1.3. *Case 3.* $k \leq n$. Balanced Kuo. Shape $(+, -, +, -, +)$.
When $(k + 2) + (n - 1)$ is odd, let

$$G := \widehat{\mathcal{G}}\left(k - 1 + n - 1, -\left\lceil \frac{k + 5n - 5}{2} \right\rceil, 2n - 2 + 1, \left\lfloor \frac{k - 3n + 3}{2} \right\rfloor - 1, n - k - 1 + 2\right) - K$$

When $(k+2) + (n-1)$ is even, let

$$G := \widehat{\mathcal{G}}\left(k-1+n-1, -\left\lceil\frac{k+5n-5}{2}\right\rceil - 1, 2n-2+1, \left\lfloor\frac{k-3n+3}{2}\right\rfloor - 2, n-k-1+2\right) - R$$

$$
\begin{aligned}
-\{p_1\}(B) :&K \to (0,0,-1,0,-1) - R \\
&R \to (0,+1,-1,+1,-1) - K \\
-\{p_2\}(W) :& \to (+1,0,0,+1,-1) \\
-\{p_3\}(B) :& \to (-1,+1,-1,0,0) \\
-\{p_4\}(W) :&K \to (0,0,0,0,0) - R \\
&R \to (0,+1,0,+1,0) - K
\end{aligned}
$$

$$
\begin{aligned}
G - \{p_1, p_2\} &= A^{(n-1)n} B^{(n-1)^2} x_{2k+3} \\
G - \{p_3, p_4\} &= A^{(n-1)n} B^{(n-1)^2} x_{2k-1} \\
G - \{p_2, p_3\} &= A^{(n-1)n} B^{(n-1)^2} x_{2k+1} \\
G - \{p_1, p_4\} &= A^{(n-1)n} B^{(n-1)^2} x_{2k+1} \\
G &= A^{n^2} B^{n^2-n} x_{2k} \\
G - \{p_1, p_2, p_3, p_4\} &= A^{(n-1)^2} B^{(n-1)(n-2)} x_{2k+2}
\end{aligned}
$$

$G - K$ (Special vertex kept): let p_1 be the second from top (B) point on edge d, p_2 be the second from left (W) point on edge e, p_3 be the second from bottom (B) point on edge b, p_4 be the special vertex.

$$T(\{p_1, p_2\}) = x_1 x_3, \quad T(\{p_3, p_4\}) = x_3 x_5, \quad T(\{p_2, p_3\}) = x_3 x_5,$$
$$T(\{p_1, p_4\}) = x_1 x_3, \quad T(\emptyset) = 1, \quad T(\{p_1, p_2, p_3, p_4\}) = x_1 x_3 x_3 x_5.$$

$G - R$ (Special vertex removed): let p_1 be the second from bottom (B) point on edge d, p_2 be the second from right (W) point on edge e, p_3 be the second from bottom (B) point on edge b, p_4 be the (W) vertex bordering the 1-block below the special vertex.

$$T(\{p_1, p_2\}) = x_1 x_3, \quad T(\{p_3, p_4\}) = x_1 x_5, \quad T(\{p_2, p_3\}) = x_3 x_5,$$
$$T(\{p_1, p_4\}) = x_1 x_1, \quad T(\emptyset) = 1, \quad T(\{p_1, p_2, p_3, p_4\}) = x_1 x_1 x_3 x_5.$$

9.2. $A^{n^2+n} B^{n^2} x_{2k-1}$, $n \geq 1$, $k \geq 3$. Recurrence we use:

$$(A^{n^2+n} B^{n^2} x_{2k-1})(A^{(n-1)^2+(n-1)} B^{(n-1)^2} x_{2k+1})$$
$$= (A^{n^2} B^{n(n-1)} x_{2k-2})(A^{n^2} B^{n(n-1)} x_{2k+2}) + (A^{n^2} B^{n(n-1)} x_{2k})^2$$

Kuo's four points: p_1, p_2, p_3, p_4 are on edge d, e, b, c respectively.
The effect of removing p_1, p_2, p_3, p_4 and the sets used in the proof of covering monomial are the same as in Section 9.1.

9.2.1. *Case 1.* $k \geq 3n + 1$. Non-alternating Kuo. Shape $(+, -, +, +, -)$.

$$G := \widehat{\mathcal{G}} \left(k - 1 + n, -\left\lceil \frac{k + 5n - 3}{2} \right\rceil, 2n - 1, \left\lfloor \frac{k - 3n + 1}{2} \right\rfloor, n - k \right).$$

$$G = A^{n^2} B^{n^2 - n} x_{2k+2}$$

$$G - \{p_1, p_2, p_3, p_4\} = A^{n^2} B^{n^2 - n} x_{2k-2}$$

$$G - \{p_1, p_3\} = A^{n^2} B^{n^2 - n} x_{2k}$$

$$G - \{p_2, p_4\} = A^{n^2} B^{n^2 - n} x_{2k}$$

$$G - \{p_1, p_2\} = A^{n^2 + n} B^{n^2} x_{2k-1}$$

$$G - \{p_3, p_4\} = A^{(n-1)^2 + (n-1)} B^{(n-1)^2} x_{2k+1}$$

9.2.2. *Case 2.* $n + 2 \leq k \leq 3n$. Unbalanced Kuo. Shape $(+, -, +, -, -)$.
When $(k + 1) + n$ is odd, let

$$G := \widehat{\mathcal{G}} \left(k - 1 + n, -\left\lceil \frac{k + 5n - 3}{2} \right\rceil, 2n - 1 + 1, \left\lfloor \frac{k - 3n + 1}{2} \right\rfloor, n - k + 1 \right) - K$$

When $(k + 1) + n$ is even, let

$$G := \widehat{\mathcal{G}} \left(k - 1 + n, -\left\lceil \frac{k + 5n - 3}{2} \right\rceil - 1, 2n - 1 + 1, \left\lfloor \frac{k - 3n + 1}{2} \right\rfloor - 1, n - k - 1 + 1 \right) - R$$

$$G - \{p_1\} = A^{n^2} B^{n^2 - n} x_{2k+2}$$

$$G - \{p_2, p_3, p_4\} = A^{n^2} B^{n^2 - n} x_{2k-2}$$

$$G - \{p_3\} = A^{n^2} B^{n^2 - n} x_{2k}$$

$$G - \{p_1, p_2, p_4\} = A^{n^2} B^{n^2 - n} x_{2k}$$

$$G - \{p_2\} = A^{n^2 + n} B^{n^2} x_{2k-1}$$

$$G - \{p_1, p_3, p_4\} = A^{(n-1)^2 + (n-1)} B^{(n-1)^2} x_{2k+1}$$

9.2.3. *Case 3.* $3 \leq k \leq n + 1$. Balanced Kuo. Shape $(+, -, +, -, +)$.
When $(k + 1) + n$ is odd, let

$$G := \widehat{\mathcal{G}} \left(k - 1 + n - 1, -\left\lceil \frac{k + 5n - 3}{2} \right\rceil, 2n - 1 + 1, \left\lfloor \frac{k - 3n + 1}{2} \right\rfloor - 1, n - k + 2 \right) - K$$

When $(k + 1) + n$ is even, let

$$G := \widehat{\mathcal{G}} \left(k - 1 + n - 1, -\left\lceil \frac{k + 5n - 3}{2} \right\rceil - 1, 2n - 1 + 1, \left\lfloor \frac{k - 3n + 1}{2} \right\rfloor - 2, n - k + 2 \right) - R$$

$$G - \{p_1, p_2\} = A^{n^2} B^{n^2 - n} x_{2k+2}$$

$$G - \{p_3, p_4\} = A^{n^2} B^{n^2 - n} x_{2k-2}$$

$$G - \{p_2, p_3\} = A^{n^2} B^{n^2 - n} x_{2k}$$

$$G - \{p_1, p_4\} = A^{n^2} B^{n^2-n} x_{2k}$$

$$G = A^{n^2+n} B^{n^2} x_{2k-1}$$

$$G - \{p_1, p_2, p_3, p_4\} = A^{(n-1)^2+(n-1)} B^{(n-1)^2} x_{2k+1}$$

9.3. $A^{n^2} B^{n(n-1)} x_{2k}$, $n \leq -1$, $k \geq 2$. Recurrence we use:

$$(A^{n^2} B^{n(n-1)} x_{2k})(A^{(n+1)^2} B^{n(n+1)} x_{2k+2})$$
$$= (A^{n(n+1)} B^{n^2} x_{2k-1})(A^{n(n+1)} B^{n^2} x_{2k+3}) + (A^{n(n+1)} B^{n^2} x_{2k+1})^2$$

Kuo's four points: p_1, p_2, p_3, p_4 are on edge d, a, b, c respectively.

9.3.1. *Case 1.* $k \geq 1 - 5n$. Non-alternating Kuo. Shape $(+, -, -, +, -)$.

$$G := \widehat{\mathcal{G}}\left(k + n, -\left\lceil \frac{k+5n}{2} \right\rceil, 2n, \left\lfloor \frac{k-3n}{2} \right\rfloor, n - k \right)$$

$$-\{p_1\}(W) : K \to (0, -1, +1, -1, +1) - R$$
$$R \to (0, 0, +1, 0, +1) - K$$
$$-\{p_2\}(W) : \to (-1, +1, 0, 0, +1)$$
$$-\{p_3\}(B) : \to (-1, +1, -1, 0, 0)$$
$$-\{p_4\}(B) : K \to (0, -1, 0, -1, 0) - R$$
$$R \to (0, 0, 0, 0, 0) - K$$

$$G = A^{n^2+n} B^{n^2} x_{2k+3}$$
$$G - \{p_1, p_2, p_3, p_4\} = A^{n^2+n} B^{n^2} x_{2k-1}$$
$$G - \{p_1, p_3\} = A^{n^2+n} B^{n^2} x_{2k+1}$$
$$G - \{p_2, p_4\} = A^{n^2+n} B^{n^2} x_{2k+1}$$
$$G - \{p_1, p_4\} = A^{(n+1)^2} B^{(n+1)^2-(n+1)} x_{2k+2}$$
$$G - \{p_2, p_3\} = A^{n^2} B^{n^2-n} x_{2k}$$

$G - K$ (Special vertex kept): let p_1 be the topmost (W) point on edge d, p_2 be the topmost (W) point on edge a (not in a forced matching), p_3 be the bottommost (B) point on edge b (not in a forced matching), p_4 be the (B) point on the edge between the 4-block and 5-block above the special vertex.

$$T(\emptyset) = 1, \quad T(\{p_1, p_2, p_3, p_4\}) = x_4 x_4 x_5 x_5, \quad T(\{p_1, p_3\}) = x_4 x_5,$$
$$T(\{p_2, p_4\}) = x_4 x_5, \quad T(\{p_1, p_4\}) = x_4 x_4, \quad T(\{p_2, p_3\}) = x_5 x_5.$$

$G - R$ (Special vertex removed): let p_1 be the topmost (W) point on edge d, p_2 be the topmost (W) point on edge a (not in a forced matching), p_3 be the bottommost (B) point on edge b (in a forced matching), p_4 be the (B) point on the edge between the 2-block and 3-block above the special vertex.

$$T(\emptyset) = 1, \quad T(\{p_1, p_2, p_3, p_4\}) = x_2 x_3 x_4 x_5, \quad T(\{p_1, p_3\}) = x_3 x_4,$$

$$T(\{p_2, p_4\}) = x_2 x_5, \quad T(\{p_1, p_4\}) = x_2 x_4, \quad T(\{p_2, p_3\}) = x_3 x_5.$$

9.3.2. *Case 2.* $2 - n \leq k \leq -5n$. Unbalanced Kuo. Shape $(+, +, -, +, -)$.

$$G := \widehat{\mathcal{G}}\left(k + n - 1, -\left\lceil \frac{k + 5n}{2} \right\rceil + 1, 2n - 1, \left\lfloor \frac{k - 3n}{2} \right\rfloor, n - k\right)$$

$$-\{p_1\}(W) : K \to (0, -1, +1, -1, +1) - R$$
$$R \to (0, 0, +1, 0, +1) - K$$
$$-\{p_2\}(W) : \to (-1, +1, 0, 0, +1)$$
$$-\{p_3\}(W) : \to (+1, -1, +1, 0, 0)$$
$$-\{p_4\}(B) : K \to (0, -1, 0, -1, 0) - R$$
$$R \to (0, 0, 0, 0, 0) - K$$

$$G - \{p_3\} = A^{n^2+n} B^{n^2} x_{2k+3}$$
$$G - \{p_1, p_2, p_4\} = A^{n^2+n} B^{n^2} x_{2k-1}$$
$$G - \{p_1\} = A^{n^2+n} B^{n^2} x_{2k+1}$$
$$G - \{p_2, p_3, p_4\} = A^{n^2+n} B^{n^2} x_{2k+1}$$
$$G - \{p_2\} = A^{n^2} B^{n^2-n} x_{2k}$$
$$G - \{p_1, p_3, p_4\} = A^{(n+1)^2} B^{(n+1)^2-(n+1)} x_{2k+2}$$

$G - K$ (Special vertex kept): let p_1 be the topmost (W) point on edge d, p_2 be the topmost (W) point on edge a, p_3 be the bottommost (W) point on edge b, p_4 be the (B) point on the edge between the 4-block and 5-block above the special vertex.

$$T(\{p_3\}) = 1, \quad T(\{p_1, p_2, p_4\}) = x_4 x_4 x_5 x_5, \quad T(\{p_1\}) = x_4 x_5,$$
$$T(\{p_2, p_3, p_4\}) = x_4 x_5, \quad T(\{p_2\}) = x_4 x_4, \quad T(\{p_1, p_3, p_4\}) = x_5 x_5.$$

$G - R$ (Special vertex removed): let p_1 be the topmost (W) point on edge d, p_2 be the topmost (W) point on edge a, p_3 be the bottommost (B) point on edge b, p_4 be the (B) point on the edge between the 2-block and 3-block above the special vertex.

$$T(\{p_3\}) = x_2, \quad T(\{p_1, p_2, p_4\}) = x_4 x_4 x_5, \quad T(\{p_1\}) = x_4,$$
$$T(\{p_2, p_3, p_4\}) = x_2 x_4 x_5, \quad T(\{p_2\}) = x_5, \quad T(\{p_1, p_3, p_4\}) = x_2 x_4 x_4.$$

9.3.3. *Case 3.* $2 \leq k \leq 1 - n$. Balanced Kuo. Shape $(-, +, -, +, -)$.

$$G := \widehat{\mathcal{G}}\left(k + n - 2, -\left\lceil \frac{k + 5n}{2} \right\rceil + 2, 2n - 1, \left\lfloor \frac{k - 3n}{2} \right\rfloor, n - k + 1\right)$$

$$-\{p_1\}(W) : K \to (0, -1, +1, -1, +1) - R$$
$$R \to (0, 0, +1, 0, +1) - K$$
$$-\{p_2\}(B) : \to (+1, -1, 0, 0, -1)$$

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$$-\{p_3\}(W): \to (+1, -1, +1, 0, 0)$$
$$-\{p_4\}(B): K \to (0, -1, 0, -1, 0) - R$$
$$R \to (0, 0, 0, 0, 0) - K$$

$$G - \{p_2, p_3\} = A^{n^2+n} B^{n^2} x_{2k+3}$$
$$G - \{p_1, p_4\} = A^{n^2+n} B^{n^2} x_{2k-1}$$
$$G - \{p_1, p_2\} = A^{n^2+n} B^{n^2} x_{2k+1}$$
$$G - \{p_3, p_4\} = A^{n^2+n} B^{n^2} x_{2k+1}$$
$$G - \{p_1, p_2, p_3, p_4\} = A^{(n+1)^2} B^{(n+1)^2-(n+1)} x_{2k+2}$$
$$G = A^{n^2} B^{n^2-n} x_{2k}$$

$G-K$ (Special vertex kept): let p_1 be the topmost (W) point on edge d, p_2 be the bottommost (B) point on edge a, p_3 be the topmost (W) point on edge b (not in a forced matching), p_4 be the (B) point on the edge between the 4-block and 5-block above the special vertex.

$$T(\{p_2, p_3\}) = x_2 x_2, \quad T(\{p_1, p_4\}) = x_4 x_4, \quad T(\{p_1, p_2\}) = x_2 x_4,$$
$$T(\{p_3, p_4\}) = x_2 x_4, \quad T(\{p_1, p_2, p_3, p_4\}) = x_2 x_2 x_4 x_4, \quad T(\emptyset) = 1.$$

$G - R$ (Special vertex removed): let p_1 be the topmost (W) point on edge d, p_2 be the bottommost (B) point on edge a, p_3 be the topmost (W) point on edge b (not in a forced matching), p_4 be the (B) point on the edge between the 2-block and 3-block above the special vertex.

$$T(\{p_2, p_3\}) = x_2 x_2, \quad T(\{p_1, p_4\}) = x_2 x_4, \quad T(\{p_1, p_2\}) = x_2 x_4,$$
$$T(\{p_3, p_4\}) = x_2 x_2, \quad T(\{p_1, p_2, p_3, p_4\}) = x_2 x_2 x_2 x_4, \quad T(\emptyset) = 1.$$

9.4. $A^{n^2+n} B^{n^2} x_{2k-1}$, $n \leq -1$, $k \geq 3$. Recurrence we use:

$$(A^{n^2+n} B^{n^2} x_{2k-1})(A^{(n+1)(n+2)} B^{(n+1)^2} x_{2k+1})$$
$$= (A^{(n+1)^2} B^{(n+1)n} x_{2k-2})(A^{(n+1)^2} B^{(n+1)n} x_{2k+2}) + (A^{(n+1)^2} B^{(n+1)n} x_{2k})^2$$

Kuo's four points: p_1, p_2, p_3, p_4 are on edge d, a, b, c respectively.
The effect of removing p_1, p_2, p_3, p_4 and the sets used in the proof of covering monomial are the same as in Section 9.3.

9.4.1. *Case 1.* $k \geq -1 - 5n$. Non-alternating Kuo. Shape $(+, -, -, +, -)$.

$$G := \widehat{\mathcal{G}}\left(k+n, -\left\lceil \frac{k+5n+2}{2} \right\rceil, 2(n+1)-1, \left\lfloor \frac{k-3n-2}{2} \right\rfloor, 1+n-k\right)$$

$$G = A^{(n+1)^2} B^{(n+1)n} x_{2k+2}$$
$$G - \{p_1, p_2, p_3, p_4\} = A^{(n+1)^2} B^{(n+1)n} x_{2k-2}$$
$$G - \{p_1, p_3\} = A^{(n+1)^2} B^{(n+1)n} x_{2k}$$
$$G - \{p_2, p_4\} = A^{(n+1)^2} B^{(n+1)n} x_{2k}$$

$$G - \{p_1, p_4\} = A^{(n+1)^2 + (n+1)} B^{(n+1)^2} x_{2k+1}$$
$$G - \{p_2, p_3\} = A^{n^2 + n} B^{n^2} x_{2k-1}$$

9.4.2. *Case 2.* $2 - n \le k \le -2 - 5n$. Unbalanced Kuo. Shape $(+, +, -, +, -)$.

$$G := \widehat{\mathcal{G}}\left(k + n - 1, -\left\lceil \frac{k + 5n + 2}{2} \right\rceil + 1, 2(n+1) - 1 - 1, \left\lfloor \frac{k - 3n - 2}{2} \right\rfloor, 1 + n - k\right)$$

$$G - \{p_3\} = A^{(n+1)^2} B^{(n+1)n} x_{2k+2}$$
$$G - \{p_1, p_2, p_4\} = A^{(n+1)^2} B^{(n+1)n} x_{2k-2}$$
$$G - \{p_1\} = A^{(n+1)^2} B^{(n+1)n} x_{2k}$$
$$G - \{p_2, p_3, p_4\} = A^{(n+1)^2} B^{(n+1)n} x_{2k}$$
$$G - \{p_1, p_3, p_4\} = A^{(n+1)^2 + (n+1)} B^{(n+1)^2} x_{2k+1}$$
$$G - \{p_2\} = A^{n^2 + n} B^{n^2} x_{2k-1}$$

9.4.3. *Case 3.* $2 \le k \le 1 - n$. Balanced Kuo. Shape $(-, +, -, +, -)$.

$$G := \widehat{\mathcal{G}}\left(k + n, -\left\lceil \frac{k + 5n + 2}{2} \right\rceil, 2(n+1) - 1, \left\lfloor \frac{k - 3n - 2}{2} \right\rfloor, 1 + n - k\right)$$

$$G - \{p_2, p_3\} = A^{(n+1)^2} B^{(n+1)n} x_{2k+2}$$
$$G - \{p_1, p_4\} = A^{(n+1)^2} B^{(n+1)n} x_{2k-2}$$
$$G - \{p_1, p_3\} = A^{(n+1)^2} B^{(n+1)n} x_{2k}$$
$$G - \{p_2, p_4\} = A^{(n+1)^2} B^{(n+1)n} x_{2k}$$
$$G - \{p_1, p_2, p_3, p_4\} = A^{(n+1)^2 + (n+1)} B^{(n+1)^2} x_{2k+1}$$
$$G = A^{n^2 + n} B^{n^2} x_{2k-1}$$

9.5. $A^{n^2 + n} B^{n^2} x_3$, $n \ge 1$. Recurrence we use:

$$(A^{n(n+1)} B^{n^2} x_3)(A^{n^2} B^{n(n-1)} x_8)$$
$$= (A^{n(n+1)} B^{n^2} x_5)(A^{n^2} B^{n(n-1)} x_6) + (A^{n(n+1)} B^{n^2} x_7)(A^{n^2} B^{n(n-1)} x_4).$$

Kuo's four points: p_1, p_2, p_3, p_4 are on edge e, a, c, d respectively.
When $n = 1$: can verify the contour match the graph using Balanced Kuo or just directly verify the matching polynomial.
Let $n \ge 2$. Unbalanced Kuo. Shape $(+, -, +, -, +)$.
When $n + 3$ is odd, let

$$G := \widehat{\mathcal{G}}\left(3 - 2 + n, -\left\lceil \frac{3 - 2 + 5n}{2} \right\rceil, 2n, \left\lfloor \frac{3 - 3n - 2}{2} \right\rfloor, n - 3 + 2\right) - R$$

When $n + 3$ is even, let

$$G := \widehat{\mathcal{G}}\left(3 - 2 + n, -\left\lceil\frac{3 - 2 + 5n}{2}\right\rceil - 1, 2n, \left\lfloor\frac{3 - 3n - 2}{2}\right\rfloor - 1, n - 3 + 2\right) - K$$

$$-\{p_1\}(W) : \rightarrow (+1, 0, 0, +1, -1)$$
$$-\{p_2\}(W) : \rightarrow (-1, +1, 0, 0, +1)$$
$$-\{p_3\}(W) :K \rightarrow (0, 0, 0, 0, 0) - R$$
$$R \rightarrow (0, +1, 0, +1, 0) - K$$
$$-\{p_4\}(B) :K \rightarrow (0, 0, -1, 0, -1) - R$$
$$R \rightarrow (0, +1, -1, +1, -1) - K$$

$$G - \{p_1\} = A^{n^2+n}B^{n^2}x_7$$
$$G - \{p_2, p_3, p_4\} = A^{n^2}B^{n(n-1)}x_4$$
$$G - \{p_3\} = A^{n^2+n}B^{n^2}x_5$$
$$G - \{p_1, p_2, p_4\} = A^{n^2}B^{n(n-1)}x_6$$
$$G - \{p_2\} = A^{n^2+n}B^{n^2}x_3$$
$$G - \{p_1, p_3, p_4\} = A^{n^2}B^{n(n-1)}x_8$$

$G - K$ (Special vertex kept): let p_1 be the leftmost (W) point on edge e (the bottommost point of edge d), p_2 be the bottommost (W) point on edge a (not in a forced matching), p_3 be the special vertex, p_4 be the (B) point on the edge between the 1-block and 4-block below the special vertex.

$$T(\{p_1\}) = x_1, \quad T(\{p_2, p_3, p_4\}) = x_1x_3x_5, \quad T(\{p_3\}) = x_3,$$
$$T(\{p_1, p_2, p_4\}) = x_1x_1x_5, \quad T(\{p_2\}) = x_5, \quad T(\{p_1, p_3, p_4\}) = x_1x_1x_3.$$

$G - R$ (Special vertex removed): let p_1 be the leftmost (W) point on edge e (the bottommost point of edge d), p_2 be the bottommost (W) point on edge a (not in a forced matching), p_3 be the (W) point below the special vertex, p_4 be the (B) point below p_3.

$$T(\{p_1\}) = x_1, \quad T(\{p_2, p_3, p_4\}) = x_1x_3x_5, \quad T(\{p_3\}) = x_1,$$
$$T(\{p_1, p_2, p_4\}) = x_1x_3x_5, \quad T(\{p_2\}) = x_5, \quad T(\{p_1, p_3, p_4\}) = x_1x_1x_3.$$

9.6. $A^{n^2+n}B^{n^2}x_3$, $n \le -1$. Recurrence we use:

$$(A^{n(n+1)}B^{n^2}x_3)(A^{(n+1)^2}B^{n(n+1)}x_8)$$
$$=(A^{n(n+1)}B^{n^2}x_5)(A^{(n+1)^2}B^{n(n+1)}x_6) + (A^{n(n+1)}B^{n^2}x_7)(A^{(n+1)^2}B^{n(n+1)}x_4)$$

Kuo's four points: p_1, p_2, p_3, p_4 are on edge a, e, c, b respectively.
When $n = -1$, can check directly to see contour for Bx_3 is correct.
Let $n \le -2$. Unbalanced Kuo. Shape $(-, +, -, +, -)$.

When $n + 3$ is odd, let

$$G := \widehat{\mathcal{G}}\left(3 - 2 + n, -\left\lceil\frac{3 - 2 + 5n}{2}\right\rceil + 1, 2n, \left\lfloor\frac{3 - 3n - 2}{2}\right\rfloor + 1, n - 3 + 2\right) - K$$

When $n + 3$ is even, let

$$G := \widehat{\mathcal{G}}\left(3 - 2 + n, -\left\lceil\frac{3 - 2 + 5n}{2}\right\rceil - 1, 2n, \left\lfloor\frac{3 - 3n - 2}{2}\right\rfloor - 1, n - 3 + 2\right) - R$$

$$-\{p_1\}(B) : \to (+1, -1, 0, 0, -1)$$
$$-\{p_2\}(B) : \to (-1, +1, 0, 0, +1)$$
$$-\{p_3\}(B) :K \to (0, -1, 0, -1, 0) - R$$
$$R \to (0, 0, 0, 0, 0) - K$$
$$-\{p_4\}(W) : \to (+1, -1, +1, 0, 0)$$

$$G - \{p_1\} = A^{n^2+n}B^{n^2}x_7$$
$$G - \{p_2, p_3, p_4\} = A^{n^2}B^{n(n-1)}x_4$$
$$G - \{p_3\} = A^{n^2+n}B^{n^2}x_5$$
$$G - \{p_1, p_2, p_4\} = A^{n^2}B^{n(n-1)}x_6$$
$$G - \{p_2\} = A^{n^2+n}B^{n^2}x_3$$
$$G - \{p_1, p_3, p_4\} = A^{n^2}B^{n(n-1)}x_8$$

$G-K$ (Special vertex kept): let p_1 be the topmost (B) point on edge a (in a forced matching), p_2 be the rightmost (B) point on edge e (in a forced matching), p_3 be the (B) point with 3 neighbors on the 3-block above the special vertex, p_4 be the bottommost (W) point on edge b.

$$T(\{p_1\}) = x_1, \quad T(\{p_2, p_3, p_4\}) = x_2 x_3 x_5, \quad T(\{p_3\}) = x_5,$$
$$T(\{p_1, p_2, p_4\}) = x_1 x_2 x_3, \quad T(\{p_2\}) = x_3, \quad T(\{p_1, p_3, p_4\}) = x_1 x_2 x_5.$$

$G - R$ (Special vertex removed): let p_1 be the topmost (B) point on edge a (in a forced matching), p_2 be the rightmost (B) point on edge e (in a forced matching), p_3 be the (B) point on the edge between the 2-block and 3-block above the special vertex, p_4 be the bottommost (W) point on edge b.

$$T(\{p_1\}) = x_1, \quad T(\{p_2, p_3, p_4\}) = x_2 x_2 x_5, \quad T(\{p_3\}) = x_2,$$
$$T(\{p_1, p_2, p_4\}) = x_1 x_2 x_5, \quad T(\{p_2\}) = x_5, \quad T(\{p_1, p_3, p_4\}) = x_1 x_2 x_2.$$

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