The Sandpile Group of Cayley Graphs

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Introduction

The **sandpile group** of a graph is a finite abelian group with combinatorial, algebraic, and geometric interpretations. We are interested in the sandpile group of hypercube graphs and their generalizations, Cayley graphs of the group $\mathbb{F}_2^r$ with arbitrary generating sets. While the Sylow-p component of these sandpile groups has been classified for $p \neq 2$ [1], the Sylow-2 subgroup remains a mystery. In this project, we use linear algebra and ring theory to achieve the following results:

- A sharp upper bound for the largest Sylow-2 cyclic factor in the sandpile group of an arbitrary Cayley graph;
- An exact formula for the largest few Sylow-2 cyclic factors of the sandpile group of a hypercube graph;
- A full classification of the sandpile group for $r = 2$ and other enlightening results for small r.

Definitions

The **Laplacian** of an undirected graph G, denoted $L(G)$, has entries

$$L(G)_{i,j} = \begin{cases} \deg(v_i) & i = j, \\ -\#\text{edges from } v_i \text{ to } v_j & i \neq j. \end{cases}$$

If $|V(G)| = k$, $L(G)$ is an integral $k \times k$ matrix, so we can view it as an endomorphism of $\mathbb{Z}$-modules $\mathbb{Z}^k \to \mathbb{Z}^k$. When G is connected, the kernel is $\mathrm{span}(\mathbf{1})$, so $\mathrm{coker}\, L(G) \cong \mathbb{Z} \oplus K(G)$ where $K(G)$ is a finite abelian group. We call $K(G)$ the **sandpile group** of G. For example, it is well known that complete graphs have $K(K_n) \cong (\mathbb{Z}_n)^{n-2}$.

Fixing $\Gamma = \mathbb{F}_2^r$, and a set of generators $M = \{v_1, \dots, v_n\}$ for Γ, we define the **Cayley graph** $G(\mathbb{F}_2^r, M)$ with vertices $V(G) = \mathbb{F}_2^r$ and $u, w \in V(G)$ share an edge if $u - w = v_i$ for a generator $v_i \in M$. Multiple edges are allowed. When $n = r$ and $M = \{e_1, \dots, e_n\}$, $G(\mathbb{F}_2^r, M) = Q_n$, is called the **hypercube graph**.

For Cayley graphs, $L(G(\mathbb{F}_2^r, M))$ has a natural eigenbasis indexed by the vectors $u \in \mathbb{F}_2^r$. Namely, let $f_u = \sum_{v \in \mathbb{F}_2^r} (-1)^{u \cdot v} e_v$. Then the f_u, f_v are pairwise orthogonal and f_u is an eigenvector with eigenvalue $\lambda_{u,M} := n - \sum_{v \in M} (-1)^{u \cdot v}$.

Previous Results

Previously, it has been proven:

1. For $p \neq 2$ we have
$$\mathrm{Syl}_p(K(G)) \simeq \mathrm{Syl}_p\left(\oplus_{u \in \mathbb{F}_2^r} \mathbb{Z}_{\lambda_{u,M}}\right)$$
2. [2] Let $c_k(G)$ denote the exponent of the kth largest cyclic factor in $\mathrm{Syl}_2(K(G))$. Then $c_1(Q_n) \leq n + \lfloor \log_2 n \rfloor$

Main Results I: Largest Sylow-2 Factor

First, we generalize the Anzis-Prasad bound [2]:

Theorem (General Sharp Upper Bound): Given an arbitrary Cayley graph $G(\mathbb{F}_2^r, M)$, we have

$$c_1(G) \leq \lfloor \log_2(n) \rfloor + r - 1,$$

which is sharp when $G = Q_{2^k}, Q_{2^k+1}$.

This is proven by using the technique in Anzis-Prasad [2] to view $K(G)$ as a ring

$$\mathbb{Z} \oplus K(G) \cong \mathbb{Z}[x_1, \dots, x_r] / \left(x_i^2 - 1, n - \sum_{i=1}^{n} \prod_j x_j^{(v_i)_j} \right)$$

Then $c_1(G)$ is the additive order of some $x_i - 1$ in the ring. By translating $x_i - 1$ back as vector in $\mathbb{Z} \oplus K(G)$, we have the following lemma

Lemma: $c_1(G)$ is the smallest C such that for any $S \subseteq [n], |S| \geq 2, d \in \mathbb{F}_2^{|S|} \setminus \{\mathbf{0}\}$,

$$\frac{C}{2^{r-|S|}} \sum_{u_S = d} \frac{1}{\lambda_{u,M}} \in \mathbb{Z}.$$

Note that when $G = Q_n$, the values of $\lambda_{u,M}$ are $n - 2k$ with multiplicity $\binom{n}{k}$. By fully exploiting the lemma above, we determine the exact value of $c_1(Q_n)$ using elementary number theory:

Theorem (Exact Max Factor for Hypercubes): For $G = Q_n$, we have the exact formula

$$c_1(Q_n) = \max\{\max_{x<n}\{v_2(x) + x\}, v_2(n) + n - 1\}.$$

We further generalized this result and proved that

Theorem (More Max Factors for Hypercubes): For hypercube graph Q_n and $2 \leq k \leq n - 1$, we have the exact formula

$$c_k(Q_n) = \max_{1 \leq x < n} \{v_2(x) + x\}.$$

The reason is $c_k(Q_n)$ is the additive order of $x_k - x_1$ for $2 \leq k \leq n - 1$, and they are all the same due to symmetries.

Limits of Main Results I

We are unable to determine $c_k(Q_n)$ for $k \geq n$ because $c_n(Q_n)$ is the additive order of some $x_1 x_2 - 1 + t_1(x_1 - 1) + t_2(x_2 - 1)$, and its value is more difficult to determine number-theoretically. However, based on data, we conjecture that for $n \geq 3$, $c_n(Q_n) = \max\{\max_{x<n-1}\{v_2(x) + x\}, v_2(n-1) + n - 3\}$. and for $n \geq 4$ that $c_{n+1}(Q_n) = \max_{x<n-1}\{v_2(x) + x\}$. We have no conjectures for further factors.

Main Results II: Sandpile group for small r

We can use the interpretation of $c_1(Q_n)$ as a minimal C to completely determine the sandpile group in small cases. For example, when $r = 2$, we have a complete classification:

Theorem (Classification for $r = 2$): If $M = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}^a, \begin{bmatrix} 0 \\ 1 \end{bmatrix}^b, \begin{bmatrix} 1 \\ 1 \end{bmatrix}^c \right)$ with $\gcd(a, b, c) = 1$ then

$$\mathrm{Syl}_2 K(G) = \begin{cases} \mathbb{Z}_{2^{v_2(b+c)+1}} & a \text{ odd}, b, c \text{ even} \\ \mathbb{Z}_{2^{v_2(a+b)+1}} & a, b \text{ odd}, c \text{ even} \\ \mathbb{Z}_{2^e} \times \mathbb{Z}_{2^f} & a, b, c \text{ odd} \end{cases}$$

where $f = \max(v_2(a+b), v_2(b+c), v_2(a+c)) + 1$ and $e = v_2(|K(G)|) - f$.

The Sylow-2 group for $r = 2$ has at most 2 factors, so it suffices to compute $c_1(G)$, which is what the proof amounts to.

For $r = 3$, $\mathrm{Syl}_2(K(G))$ can have many more cyclic factors, but when M is **generic**, i.e. when $\sum_{v \in M} v \neq \vec{0}$, it turns out there are only 3 cyclic factors, which is the smallest number of factors. In this case,

Theorem ($\mathbf{r = 3}$ 'generic case'): Suppose that the sum $\sum_{\vec{v} \in M} \vec{v} \neq \vec{0}$, and let $d_1 \leq \cdots \leq d_7$ be the powers of 2 in the eigenvalues $\lambda_{u,M}$. Then

$$\mathrm{Syl}_2(K(G)) = \begin{cases} \mathbb{Z}_{2^{d_5 - 1}} \times (\mathbb{Z}_{2^{d_7+1}})^2 & d_6 = d_7 \\ \mathbb{Z}_{2^{d_5}} \times \mathbb{Z}_{2^{d_6}} \times \mathbb{Z}_{2^{d_7+1}} & d_6 < d_7 \end{cases}$$

The proof of this theorem involves an explicit computation of the largest and second largest cyclic factors for each of these sandpile groups, applying the similar techniques used for the hypercube. These computations uniquely determine the 3rd factor since we already know the order of $\mathrm{Syl}_2(K(G))$.

Limits of Main Results II

Other $r = 3$ cases and $r = 4$ appear far less tractable, since in these cases $\mathrm{Syl}_2(K(G))$ has at least 5 cyclic factors, so we would need to explicitly compute 4 factors by hand. This was reasonable when we had the symmetry of the hypercube, but in general appears to be difficult.

Conclusion and Remaining Questions

1. The Sylow 2 component of the Sandpile group appears to be extremely complex, based on our results about the top factors for the hypercube.
2. We conjecture that the eigenvalues of the Laplacian (given that the graph is reduced) uniquely determine the group.
3. One mysterious conjecture is that
$$\mathrm{Syl}_2(K(Q_{2^k})) \cong \mathrm{Syl}_2(K(Q_{2^k-1}))^2 \times \mathbb{Z}2^{2^k+k-1}$$
This could fit into an interpretation via graph coverings.
4. Potential future approaches to this problem include Grobner bases, matroid deletion and contraction, and graph coverings.

References

[1] Joshua E. Ducey and Deelan M. Jalil.
Integer invariants of abelian cayley graphs.
Linear Algebra and its Applications, 445:316–325, 2014.

[2] Ben Anzis and Rohil Prasad.
On the critical groups of cubes, 2016.

[3] Hua Bai.
On the critical group of the n-cube.
Linear Algebra and its Applications, 369:251–26, 2003.

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