REU Day 4
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1) RSK & plactic monoid review

Insert a word w to get an
insertion tableau $P(w)$,
a semistandard Young tableau
(SSYT)
and a recording tableau $Q(w)$,
a standard Young tableau
(SYT)

EXAMPLE: $w = 61325$

P	Q
$\begin{array}{\|c\|}\hline 6 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline\end{array}$
$\begin{array}{\|c\|}\hline 1 \\ \hline 6 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline 2 \\ \hline\end{array}$
$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 6 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$
$\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$
$\begin{array}{\|c\|c\|c\|}\hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array} = P(w)$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 3 & 5 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} = Q(w)$

Q is simply recording the order in which the boxes are added to the shape.

(see Stanley p. 113)

<u>THM</u>: The map

$$\rho : \text{words} \longrightarrow \left\{ \left(\underset{\text{SSYT}}{P}, \underset{\text{SYT}}{Q} \right) : \begin{array}{c} \text{shape } P \\ = \text{shape } Q \end{array} \right\}$$

$$w \longmapsto (P(w), Q(w))$$

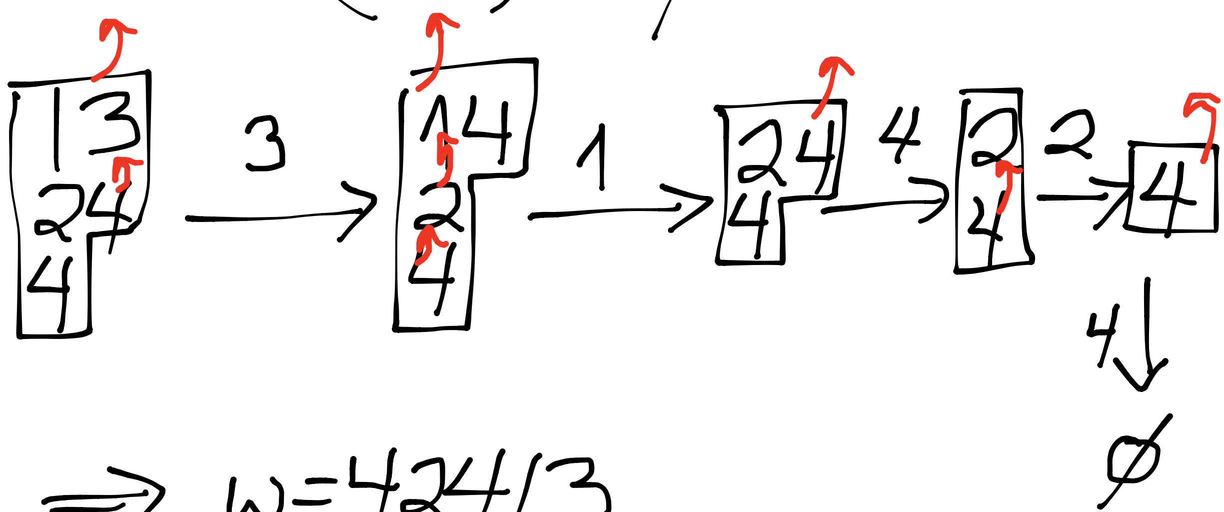
is a <u>bijection</u>

(called the R-S-K correspondence)

Robinson-Schensted-Knuth

<u>Proof</u>: ρ^{-1} can be constructed using Q to tell where to start <u>reverse</u>-<u>bumping</u>...

$$(P,Q) = \left(\begin{matrix} 1\,3 \\ 2\,4 \\ 4 \end{matrix} \;,\; \begin{matrix} 1\,3 \\ 2\,5 \\ 4 \end{matrix} \right)$$



$\Longrightarrow w = 42413$

Q: When does $P(w) = P(u)$?

Knuth relations:

- $xzy \equiv zxy$ if $x \le y < z$
- $yxz \equiv yzx$ if $x < y \le z$

Q: Given a tabeau T, which w have $P(w) = T$?

We can list all possible recording tableaux Q to pair with T, and reverse insert to recover the words w.

EXAMPLE: $T = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \cline{1-1} \end{array}$

$Q_1 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}$ $\qquad$ $Q_2 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array}$

$w_1 = 121$ $\qquad$ $w_2 = 211$

FACT 1: $w \equiv \mathrm{row}(P(w))$

FACT 2: Each plactic class contains exactly one reading word (of a SSYT).

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2) The Poirier-Reutenauer bialgebra (PR)

- What's a bialgebra?

A vector space V with...

$$\begin{cases} \cdot \text{ Product: } m: V \otimes V \longrightarrow V \\ \cdot \text{ Coproduct: } \Delta: V \longrightarrow V \otimes V \end{cases}$$

must be compatible:

$$\Delta(X \cdot Y) = \Delta(X) \cdot \Delta(Y)$$

For a SYT T, let

$$\Pi := \sum_{w \equiv \mathrm{row}(T)}{}' \, w = \sum_{w : P(w) = T} w$$

<u>EXAMPLE</u>: $T = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$ has

$$\Pi = 2413 + 2143$$

Let PR be the $\mathbb{R}$-vector space generated by $\{\mathbb{T} : T \text{ a SYT}\}$.

<u>Product for PR</u>

Start with 2 words

w_1 a word on letters $\{1,2,\dots,n\}$ using each letter exactly once

w_2 a word on $\{1,2,\dots,m\}$ similarly

$w_2[n] := w_2$ with each letter incremented by $+n$

<u>EXAMPLE</u>: $w_1 = 1324$ $n=4$

$w_2 = 21$

$w_2[4] = 65$

Define $w_1 * w_2 := w_1 ⧢ w_2[n]$

shuffle product of words: sum of all ways to shuffle w_1 and $w_2[n]$.

EXAMPLE: $w_1 = 312$ $w_2 = 12$

$$312 * 12 = 312 ⧢ 45$$

$$= 31245 + 31425 + 34125$$

$$+ 43125 + 31452 + 34152$$

$$+ 43152 + 34512 + 43152$$

$$+ 45312$$

Extend this product by linearity to a product on PR

EXAMPLE:

$$T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array} \qquad T_2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$$

$$\Pi_1 * \Pi_2 = (312 + 132) * (12)$$

$$= 312 ⧢ 45 + 132 ⧢ 45$$

$$= (\ldots \text{missing shuffles written out} \ldots)$$

$$= \Pi_3 + \Pi_4 + \Pi_5 + \Pi_6$$

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$$

Coproduct on PR:

DEF'N: The *standardization* of w is the unique permutation of $\{1,2,\ldots,|w|\}$ having letters in the same relative order as w.

EXAMPLE: $st(25)=12$

$st(1426)=1324$

Define $\Delta(w) = \sum_{u \cdot v = w} st(u) \otimes st(v)$

concatenation

EXAMPLE: $w = 312$

u	v
$\emptyset$	312
3	12
31	2
312	$\emptyset$

$$\Rightarrow \Delta(312) = \emptyset \otimes 312 + 1 \otimes 12 + 21 \otimes 1 + 312 \otimes \emptyset$$

We can extend this linearly to a coproduct on PR.

EXAMPLE: $T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$

$$\Delta(\Pi_1) = \Delta(312 + 132) = \Delta(312) + \Delta(132)$$

$$= \emptyset \otimes 312 + 1 \otimes 12 + 21 \otimes 1 + 312 \otimes \emptyset$$
$$+ \emptyset \otimes 312 + 1 \otimes 21 + 12 \otimes 1 + 132 \otimes \emptyset$$

$$= \emptyset \otimes (312 + 132) + 1 \otimes 12 + 21 \otimes 1 + 12 \otimes 1 + (312 + 132) \otimes \emptyset$$

$$= T_{\emptyset} \otimes T_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} + T_{\begin{smallmatrix}1\end{smallmatrix}} \otimes T_{\begin{smallmatrix}1&2\end{smallmatrix}} + T_{\begin{smallmatrix}1\\2\end{smallmatrix}} \otimes T_{\begin{smallmatrix}1\end{smallmatrix}} + T_{\begin{smallmatrix}1&2\end{smallmatrix}} \otimes T_{\begin{smallmatrix}1\end{smallmatrix}} + T_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} \otimes T_{\emptyset}.$$

FACT: There is a bialgebra morphism $T \longmapsto \underbrace{s_{shape(T)}}_{\text{Schur function}}$

So $T_1 * T_2$ tells us how to multiply $s_{\lambda_1} s_{\lambda_2}$ i.e. Littlewood-Richardson rule.

EXERCISE 12:

a) Show $\Delta(12 * 21)$

$$= \Delta(12) \cdot \Delta(21)$$

b) Prove $\Delta(X * Y)$

$$= \Delta(X) * \Delta(Y)$$

for 2 permutations X, Y.

(<u>Note</u>: in $PR \otimes PR$ on RHS's above $(12 \otimes 1) * (21 \otimes 1)$

$$:= (12 * 21) \otimes (1 * 1)$$

)

3) K-theoretic version

Hecke insertion (BKSTY):

We want insertion tableau to be an increasing tableau,

i.e. rows and columns strictly increasing

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

✓ OK

✓ OK

✗ BAD; not increasing

Hecke insertion: (see instructions on sheet...)

$$\begin{array}{|c|}\hline 1 \\ \hline \end{array} \xleftarrow{H} 2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \xleftarrow{H} 2 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline \end{array} \xleftarrow{H} 2 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}$$

$$\begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline \end{array} \xleftarrow{H} 2 = \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 3 \\ \cline{1-1} \end{array}$$

Let's insert a full word...

EXAMPLE: 12125354

$$\emptyset \xrightarrow{1} \begin{array}{|c|}\hline 1 \\ \hline\end{array} \xrightarrow{2} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \xrightarrow{1} \begin{array}{|c|c}\hline 1 & \multicolumn{1}{c|}{2} \\ \hline 2 \\ \cline{1-1}\end{array} \xrightarrow{2} \begin{array}{|c|c}\hline 1 & \multicolumn{1}{c|}{2} \\ \hline 2 \\ \cline{1-1}\end{array} \xrightarrow{5} \begin{array}{|c|c|c}\hline 1 & 2 & \multicolumn{1}{c|}{5} \\ \hline 2 \\ \cline{1-1}\end{array} \xrightarrow{3}$$

$$\begin{array}{|c|c|c}\hline 1 & 2 & \multicolumn{1}{c|}{3} \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \xrightarrow{5} \begin{array}{|c|c|c|c}\hline 1 & 2 & 3 & \multicolumn{1}{c|}{5} \\ \hline 2 & 5 \\ \cline{1-2}\end{array} \xrightarrow{4} \begin{array}{|c|c|c|c}\hline 1 & 2 & 3 & \multicolumn{1}{c|}{4} \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$$

EXAMPLE: 2131

$$\emptyset \xrightarrow{2} \begin{array}{|c|}\hline 2 \\ \hline\end{array} \xrightarrow{1} \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array} \xrightarrow{3} \begin{array}{|c|c}\hline 1 & \multicolumn{1}{c|}{3} \\ \hline 2 \\ \cline{1-1}\end{array} \xrightarrow{1} \begin{array}{|c|c}\hline 1 & \multicolumn{1}{c|}{3} \\ \hline 2 \\ \cline{1-1}\end{array}$$

NOTE that the shape doesn't necessarily change after insertion!

K-Knuth equivalence

1) $pp \overset{K}{\equiv} p$
2) $pqp \overset{K}{\equiv} qpq$
3) $xzy \overset{K}{\equiv} zxy$ if $x<y<z$
   $yxz \overset{K}{\equiv} yzx$ if $x<y<z$

EXAMPLE:

$312 \overset{K}{\equiv} 3312 \overset{K}{\equiv} 3132 \overset{K}{\equiv} 1312$

$\overset{K}{\equiv} 13312 \overset{K}{\equiv} 13132 \equiv \ldots$

- Every equivalence class has infinitely many elements
- $w \overset{K}{\equiv} \mathrm{row}\left(\underbrace{P_H(w)}_{\text{the Hecke insertion tableau of } w}\right)$
- WARNING: It is <u>not</u> true that each $\overset{K}{\equiv}$-class contains exactly one reading word of a tableau.

<u>EXAMPLE</u>: $13424 \overset{K}{\equiv} 13242$

$$P_H(13424) = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \neq P_H(13242) = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$

Motivated by this deficiency....

<u>DEF'N</u>: Say two increasing tableaux T_1, T_2 are <u>equivalent</u> $T_1 \overset{K}{\equiv} T_2$ if $\mathrm{row}(T_1) \overset{K}{\equiv} \mathrm{row}(T_2)$

e.g. $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \overset{K}{\equiv} \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$

(For more on this, see the summer 2014 REU project and arXiv preprint 1409....)

EXERCISE 13: Group these by their $\overset{K}{\equiv}$-class:

$$
\begin{array}{|c|c|c|}
\hline
1 & 2 & 4 \\
\hline
3 \\
\cline{1-1}
5 \\
\cline{1-1}
\end{array}
\qquad
\begin{array}{|c|c|c|}
\hline
1 & 2 & 4 \\
\hline
3 & 4 \\
\cline{1-2}
5 \\
\cline{1-1}
\end{array}
\qquad
\begin{array}{|c|c|c|}
\hline
1 & 2 & 4 \\
\hline
3 & 5 \\
\cline{1-2}
5 \\
\cline{1-1}
\end{array}
\qquad
\begin{array}{|c|c|c|}
\hline
1 & 2 & 4 \\
\hline
3 & 4 & 5 \\
\hline
5 \\
\cline{1-1}
\end{array}
$$

$$
\begin{array}{|c|c|c|c|}
\hline
1 & 2 & 4 & 5 \\
\hline
3 \\
\cline{1-1}
\end{array}
$$

(see Section 2 of the K-PR paper on the extra sheet)

We can again construct a bialgebra on the $\mathbb{R}$-vector space generated by the K-Knuth classes, similar to PR

- infinite sums!
- more than one reading word in a class.

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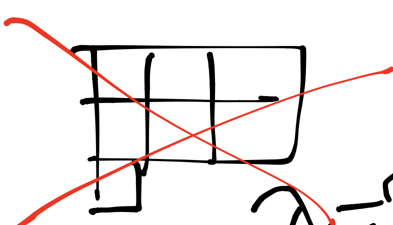
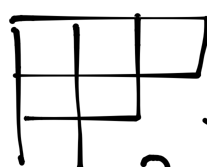
4) The shifted setting (see Sagan '87)

A partition is *strict* if

$$\lambda = (\lambda_1 > \lambda_2 > \dots > \lambda_k)$$



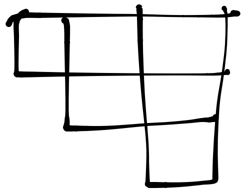
$\lambda_1 = \lambda_2$

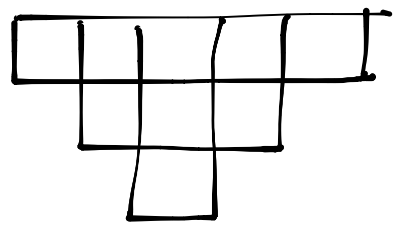


strict

$3 > 2 > 1$

Strict partitions can be represented by <u>shifted</u> shapes:

$(3,2,1) \longleftrightarrow$ 

$(5,3,1) \longleftrightarrow$ 

We can fill them the same way to get tableaux, and read row-words:

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline & 3 & 5 \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 6 \\ \cline{3-3}\end{array} \qquad \mathrm{row}(T) = 635/24$$

There is a shifted version of the RSK correspondance, called Sagan-Worley insertion.

EXAMPLE: 215436

$$\begin{array}{|c|}\hline 2 \\ \hline\end{array} \xrightarrow{1} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \xrightarrow{5} \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline\end{array}$$

$$\xrightarrow{4} \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline \multicolumn{1}{c|}{} & 5 & \multicolumn{1}{|c}{} \\ \cline{2-2}\end{array} \xrightarrow{3} \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 \\ \cline{2-3}\end{array}$$

$$\xrightarrow{6} \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 6 \\ \hline \multicolumn{1}{c|}{} & 4 & 5 & \multicolumn{1}{|c}{} \\ \cline{2-3}\end{array} = P_{SW}(215436)$$

Shifted Knuth equivalence:

- $xzy \overset{s}{\equiv} zxy$ if $x<y<z$
- $yxz \overset{s}{\equiv} yzx$ if $x<y<z$
- $xy \overset{s}{\equiv} yx$ if x, y are the first two letters in the word.

e.g. $12 \to \boxed{1|2}$, $21 \to \boxed{1|2}$

FACT: $P_{sw}(u) = P_{sw}(v)$ if and only if $u \overset{s}{\equiv} v$

We can form a shifted P-R bialgebra (Jing-Li)

REU Problem 4:

There is a shifted K-theoretic RSK insertion (see the arXiv paper 1410...).

Construct a shifted K-theoretic PR-bialgebra.