Monotone Paths in Zonotopes (Ref: PhD Thesis of R. Edman)

① Polytopes (Ref: Ziegler's "Lectures on Polytopes")
② Functionals & monotone paths (motivation from LP)
  - flip graphs
③ Zonotopes
④ coherence
⑤ REU Problem
⑥ Tools: deletion/contraction
  duality

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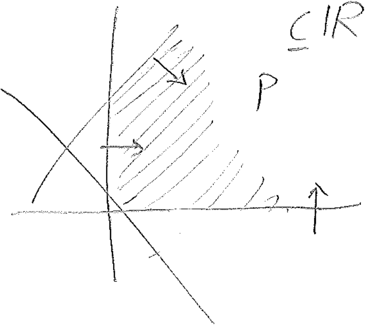
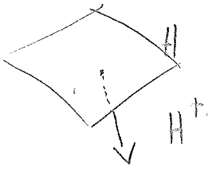
① A polyhedron $P \subset \mathbb{R}^d$ is a finite intersection $\bigcap_{i=1}^{t} H_i^+$

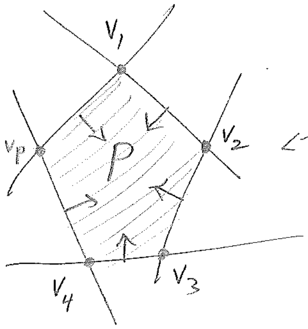
H_i^+ = a halfspace $= \{\underline{x} \in \mathbb{R}^d : f_i(\underline{x}) \le c_i\}$

$f_i(\underline{x}) = a_1x_1 + \cdots + a_dx_d$ $\quad a_i \in \mathbb{R}$ $\quad c_i \in \mathbb{R}$ $\quad f \in (\mathbb{R}^d)^*$

e.g.



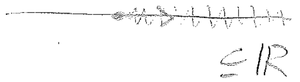


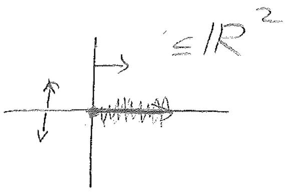


a polytope P is a bounded polyhedron

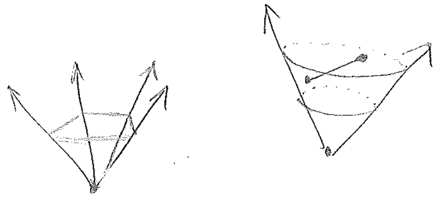
$\overset{\text{thm}}{\Longleftrightarrow}$ $P :=$ convex hull of a finite set $\{v_1, \ldots, v_p\} \subseteq \mathbb{R}^d$

more polyhedra

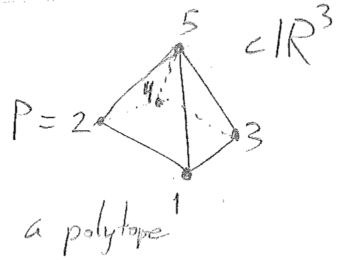




Note: not every convex cone is a polyhedron



(cf. Mike's problem)



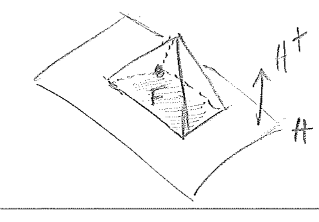
A (proper) face F of P is an intersection

$F = P \cap H$ where

$\{f(\underline{x}) \le c\} = H^+$ is a half space supporting P

$\cup$ $\quad (P \subseteq H^+)$

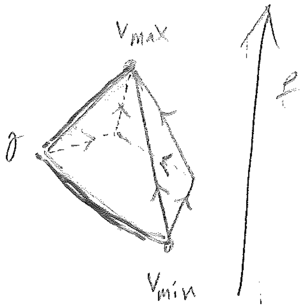
$\{f(\underline{x}) = c\} = H$

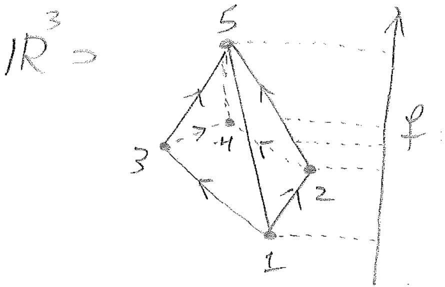


② Functionals & Monotone paths

$a_1x_1+\cdots+a_dx_d$

Def: say a (linear) functional $\overset{\shortparallel}{f} \in (\mathbb{R}^d)^*$ is (edge-)generic on P if it's never constant on any edge of P so it orients all edges:

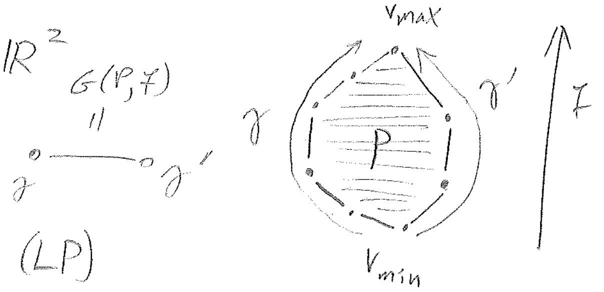




γ an f-monotone path is a path in P from v_{min} to v_{max} along edges (increasing f)

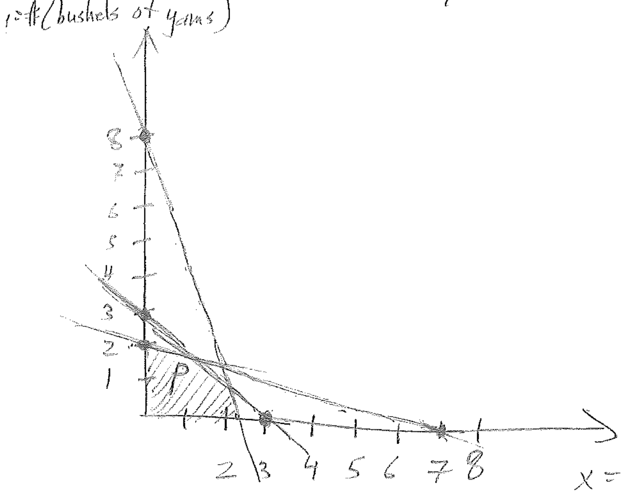
Example: $d=2$: P a polygon $\subseteq \mathbb{R}^2$

$G(P,f) \overset{\shortparallel}{} \quad \gamma$ ——— γ'



Fake Motivation: Linear Programming (LP)

solves an optimization problem using monotone paths



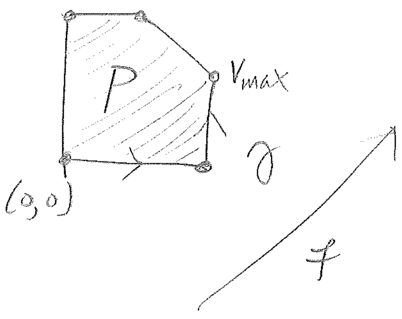
$x \geq 0$, $y \geq 0$

$x+y \leq 3$ = total # of bushels possible

$3x+y \leq 8$ =water use constraint

$x+3y \leq 8$ = seed cost constraint

Feasible polyhedron



maximize profit:

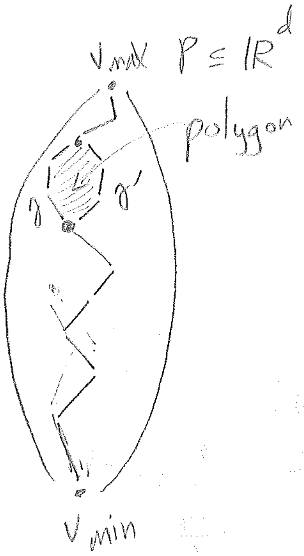
$$f(x,y) = \underbrace{c_1}_{\text{\$ of profit per bushel of xiqua}} x + \underbrace{c_2}_{\text{same for yams}} y$$

Def: The flip graph $G(P,f)$ P a polytope

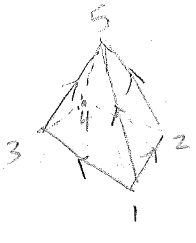
$= (V, E)$ f a (edge) generic f on P

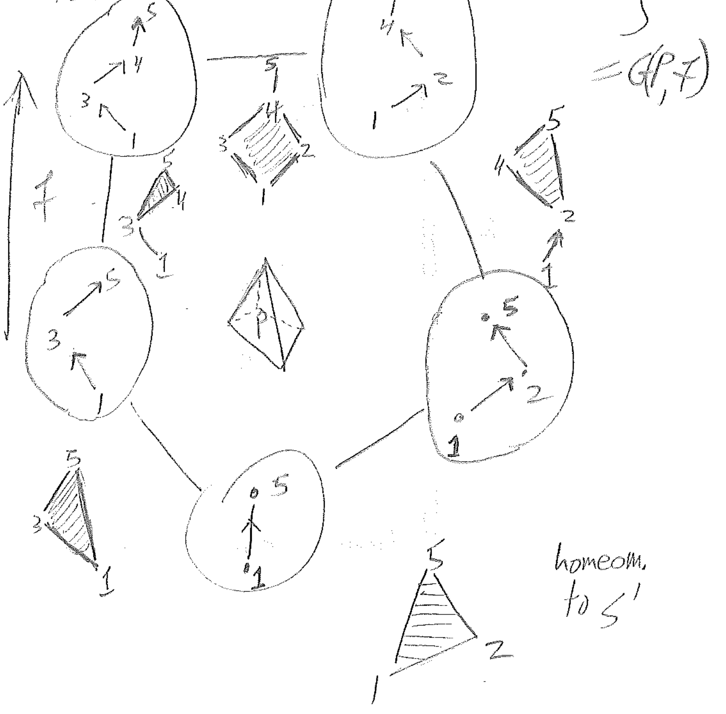
$V = \{f\text{-monotone paths in } P\}$

$E = \{\{\gamma,\gamma'\},\ \gamma, \gamma'$ differ by a flip across a 2-dim'l face of $P\}$



Examples





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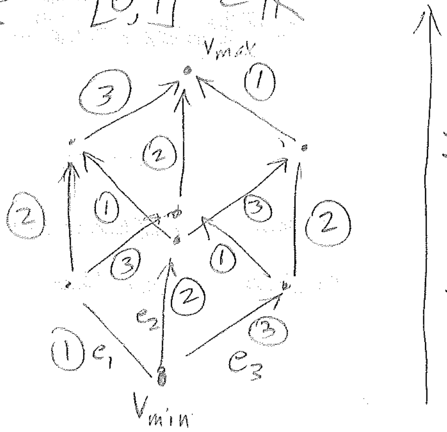
③ n-cubes $[0,1]^n \subseteq \mathbb{R}^n$

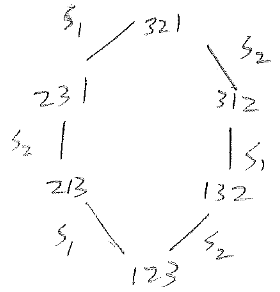
$n=3$



$f = c_1x_1 + c_2x_2 + c_3x_3$

$c_1, c_2, c_3 > 0$

$f(e_i) > 0$



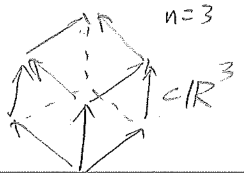
$G(P,f) = (V, E)$ = Cayley graph for $W = S_n$ w.r.t. Coxeter generators

$V = S_n$, $E = \{\{w, ws\} : s \in S = \{s_1, \ldots, s_{n-1}\}\}$

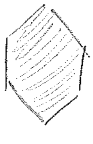
③ Zonotopes

Def: The zonotope $P = Z(A)$ generated by a matrix $A = d\left\{\underbrace{\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}}_{n}\right. \in \mathbb{R}^{d\times n}$ $a_1, \ldots, a_n \in \mathbb{R}^d$

is the image of n-cube $[0,1]^n \twoheadrightarrow Z(A)$



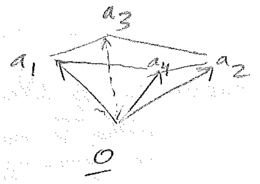
$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "A"] & \mathbb{R}^d \\ e_i \arrow[r, mapsto] & a_i \end{tikzcd}$$



Example $d=3$ $n=4$

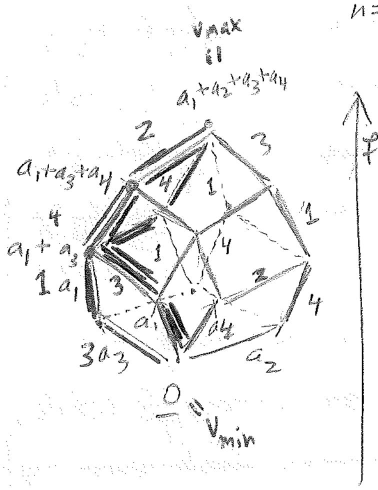
$$A = d{=}3\left\{ \underbrace{\begin{array}{c|cccc|} & a_1 & a_2 & a_3 & a_4 \\ x_1 & 1 & 0 & 0 & +1 \\ x_2 & 0 & 1 & 0 & +1 \\ x_3 & 0 & 0 & 1 & -1 \end{array}}_{n=4} \right.$$

$a_1 + a_2 = a_3 + a_4$



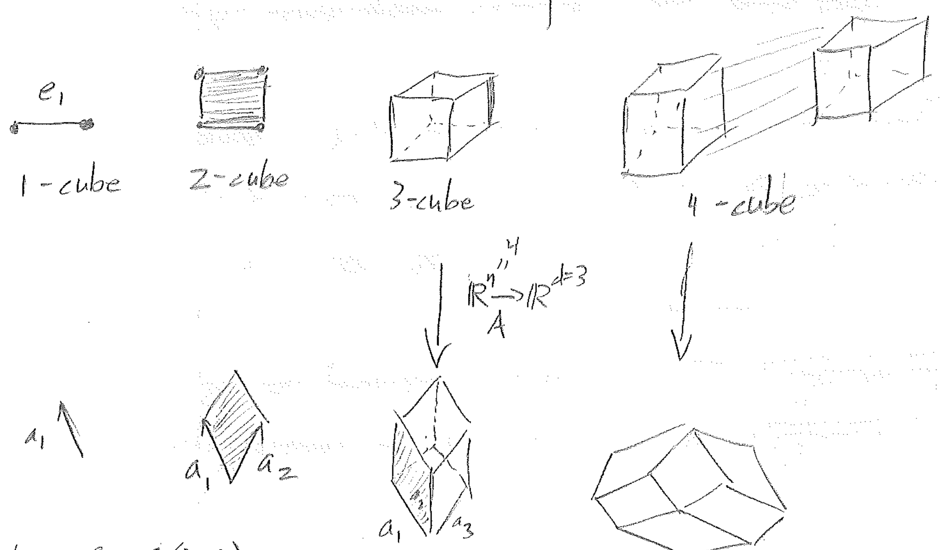
(drawn in $\mathbb{R}^3$)

$Z(A) =$



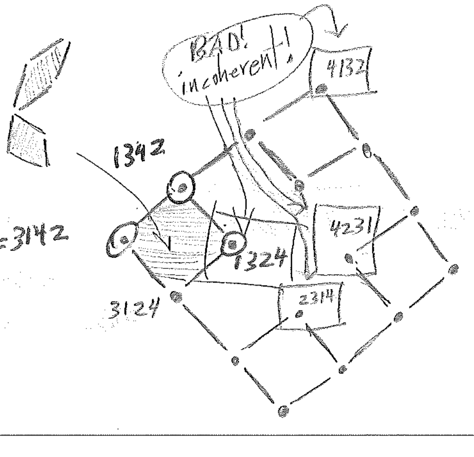
$\in (\mathbb{R}^3)^*$ with $f(a_1), f(a_2), f(a_3), f(a_4) > 0$

$\subset \mathbb{R}^{\overset{d}{\overset{\|}{3}}}$



Example of $G(P, f)$ ④ $\quad P = \overset{\|}{Z(A)}$

Edman's Thesis Examples 3.3, 5.6
pp. 26, 52
Fig, 3.1, 5.1

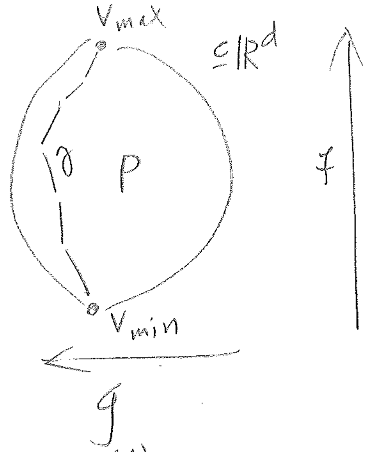


For $f(x_1, x_2, x_3) = 3x_1 + 2x_2 + 3x_3$

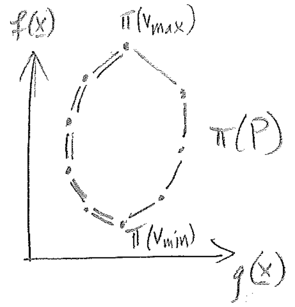
Who are the bad, twisted, incoherent γ we should remove?

Def: Fixing (P, f), say γ an f-monotone path is incoherent if $\exists g \in (\mathbb{R}^d)^*$ s.t. that $\pi(\gamma)$ is on the boundary of $\pi(P)$ for $\mathbb{R}^d \longrightarrow \mathbb{R}^2$

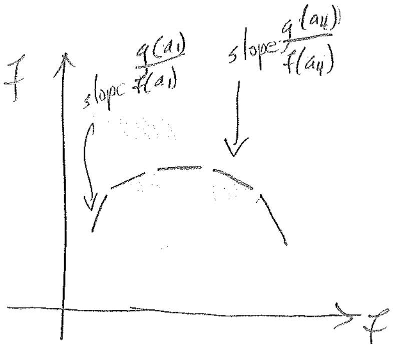
$$\underline{x} \longmapsto \begin{bmatrix} f(\underline{x}) \\ g(\underline{x}) \end{bmatrix}$$



$\mathbb{R}^d \longrightarrow \mathbb{R}^2$



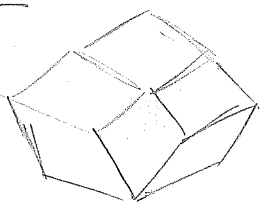
If $\overset{= Z(A)}{P}$ is a zonotope want g s.t. $g = (a_1, a_2, \dots, a_n)$ $\quad \frac{g(a_1)}{f(a_1)} < \frac{g(a_2)}{f(a_2)} < \dots < \frac{g(a_n)}{f(a_n)}$



<u>Thm</u> (Billera-Sturmfels 1992) "Fiber Polytopes"
For any edge-generic f on a dim'l polytope P
G(P, f) restricted to the coherent γ
is the 1-skeleton of a (d-1)-dim'l polytope
(vertices, edges)

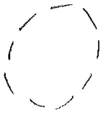
Examples

①



$\mathbb{R}^{3=d}$
$\cup$
Z(A) $\quad f = 3x_1 + 2x_2 + x_3$ has all but
4132
4231
2314
1324
coherent

and removing them gives a 12-gon, the boundary of the (d-1)-polytope



② $Z(A) \subset \mathbb{R}^n$

n-cube has all γ coherent; (n-1)-polytope is permutohedron

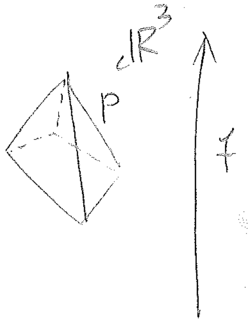
③ permutohedron with f gives $G(Z(A_{n-1}), f)$

$Z(A_{n-1})$

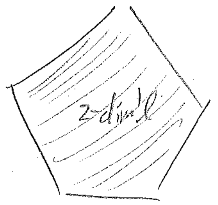
$$\begin{array}{c} G(Z(A_{n-1}), f) \\ \| \\ (V\ ,\ E) \end{array}$$

$V = \underbrace{\mathcal{R}}_{?}\, (\underset{\underset{S_n}{\uparrow}}{n\,n-1 \cdots 321})$

$E =$ Coxeter relations $ss's\ldots = s'ss'\ldots$.

④ 

had all γ coherent.



$G(P, f)$

Too hard a PROBLEM?

Which (P, f) have no incoherent γ (i.e. all γ coherent)?

Not too hard

<u>REU Problem 9</u>: Which zonotopes $(Z(A), f)$ have all f-monotone paths γ coherent?

R. Edman's thesis gives nice (!) easy answers for

$n-d=0$ (silly; n-cube)

$n-d=1$

$n-d=2$

Contraction $\hat{A} = \begin{bmatrix} | & & | \\ \pi(a_1) & \cdots & \pi(a_3) \\ | & & | \end{bmatrix}$

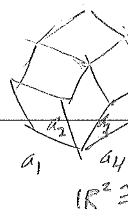
Tools: (A) Deletion-Contraction

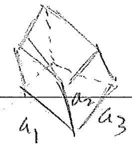
$$A = \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 & a_4 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array} \xrightarrow{\text{deletion of } a_4} A^- = \begin{array}{c} \begin{matrix} a_1 & a_2 & a_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

incoherents in A^- $\overset{\text{Lemma (Edman)}}{\Longrightarrow}$ incoherents in A

$\overset{\text{Lemma (Edman)}}{\nearrow}$

incoherents in $\hat{A}$

$Z(A)$ 

$\rightsquigarrow Z(A^-) =$ 

$\rightsquigarrow$ contraction on a_4

$\mathbb{R}^2 \cong$ 

project a_1, a_2, a_3 onto $\mathbb{R}^d / \mathbb{R}a_4 \cong \mathbb{R}^{d-1}$

So we only need to find the minor-minimal obstructions that have incoherent g's

minimal w.r.t. deletion-contraction

(B) Duality

$$A = d\left\{ \underbrace{\begin{bmatrix} | & & | \\ a_1 & & a_n \\ | & & | \end{bmatrix}}_{n} \right.$$

full rank $d \leq n$

$a_1 + a_2 = a_3 + a_4$

$$= \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \end{array}$$

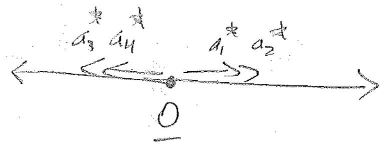
are dual if $\underbrace{Row(A)^{\perp}}_{\ker A} = Row(A^*)$ inside $\mathbb{R}^n$

('Gale dual')

$$A^* = n-d\left\{ \underbrace{\begin{bmatrix} | & | & & | \\ a_1^* & a_2^* & \cdots & a_n^* \\ | & | & & | \end{bmatrix}}_{n} \right.$$

$$\underbrace{\ker A}_{nullspace(A)} = \begin{array}{c} \begin{array}{cccc} a_1^* & a_2^* & a_3^* & a_4^* \end{array} \\ \begin{bmatrix} +1 & +1 & -1 & -1 \end{bmatrix} \end{array}$$

$A^* =$



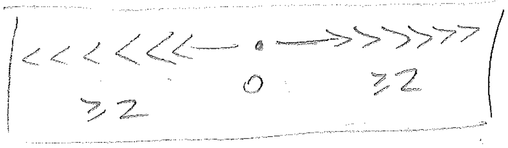
$\mathbb{R}^1$

corank $n-d$

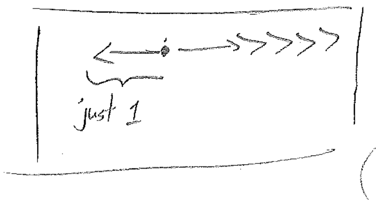
0 n-cube $A^* \subseteq \mathbb{R}^0$ all coherent ✓

1 $A^* \subseteq \mathbb{R}^1$

Thm (Edman).

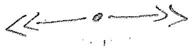


=> incoherents

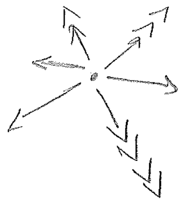


=> all incoherent

minimal-obstruction



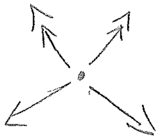
$n-d=2$: $A^* \subset \mathbb{R}^2$



Thm (Edman): $(Z(A), f)$ has all coherent $\delta \iff A^*$ avoids as a minor

either

or



Duality exercises:

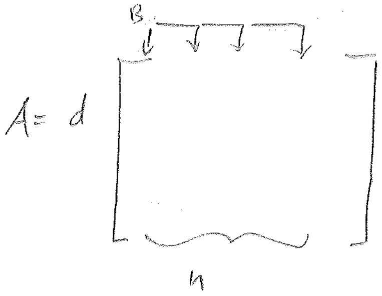
Exercise 25: Given $A \in \mathbb{R}^{d \times n}$ of full rank

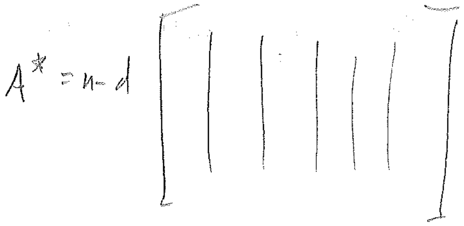
$A^* \in \mathbb{R}^{(n-d)\times n}$ with $Row(A)^\perp = Row(A^*)$ in $\mathbb{R}^n$

show there is some constant $c \in \mathbb{R}-\{0\}$ with this property:

$\forall B \in \binom{[n]}{d}$ ($[n] := \{1,2,\ldots,n\}$) $|B|=d$

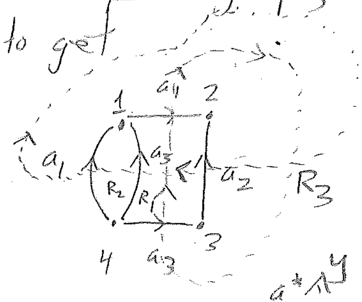
$$c \cdot \det(A|_{\text{columns of } \underbrace{B}_{d\times d}}) = \det(A^*|_{\text{columns } [n]\setminus B})$$

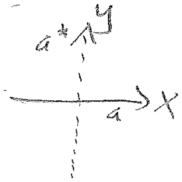




EXERCISE 26: $G=(V,E)$ a connected plane graph, orient it arbitrarily to get

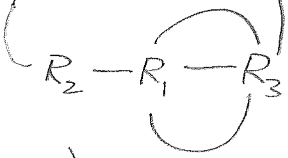
$$A=\begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & +1 & +1 \\ +1 & +1 & 0 & 0 & 0 \\ 0 & -1 & +1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \end{bmatrix} \end{array}$$





and form its plane dual $G^*=(V^*, E^*)$

$V^* = \{\text{regions of } G\}$, $E^* = $ oriented crossing edges



$$A^*=\begin{array}{c|ccccc} & a_1^* & a_2^* & a_3^* & a_4^* & a_5^* \\ \hline R_1 & -1 & +1 & +1 & 0 & -1 \\ R_2 & 0 & 0 & 0 & -1 & +1 \\ R_3 & +1 & -1 & -1 & +1 & 0 \end{array}$$

show (a) $Row(A)^\perp = Row(A^*)$

show (b) a subset $T \subset E$ forms a spanning tree in G $\iff \{a_1^*, \ldots, a_n^*\} \setminus T^*$ forms a spanning tree in G^*.