Applications of Polyhedral Geometry to Computational Representation Theory

By

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B.S. (University of California, Davis) 2001

DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

in the

OFFICE OF GRADUATE STUDIES

of the

UNIVERSITY OF CALIFORNIA

DAVIS

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2006

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Abstract

We investigate the consequences of applying the theoretical and algorithmic tools of polyhedral geometry to computational representation theory. The central problem motivating our study is that of computing tensor product multiplicities, also known as Clebsch–Gordan coefficients, for finite-dimensional complex semisimple Lie algebras. In addition to representation theory, the computation of these numbers has applications in algebraic geometry, quantum mechanics, and theoretical computer science.

Even though computing Clebsch–Gordan coefficients is $\#P$-hard in general, we show that, when the Lie algebra is fixed, there is a polynomial time algorithm based on counting the lattice points in polytopes. Moreover, we show that, for Lie algebras of type A, there is an algorithm to decide when the coefficients are nonzero in polynomial time for arbitrary rank based on Khachiyan's ellipsoid algorithm. Our experiments show that this polyhedral algorithm is superior in practice to the standard techniques for computing multiplicities when the weights have large entries but small rank. Using an implementation of this algorithm, we provide experimental evidence for two conjectured generalizations of the saturation theorem of Knutson and Tao (1999). One of these conjectures, which applies to all of the classical root systems, is an extension of earlier work by King, Tollu, and Toumazet (2004).

In pursuit of proofs of these conjectures, we turn to a theoretical study of stretched Clebsch–Gordan coefficients in the special case of stretched Kostka coefficients for type-A Lie algebras. We approach this problem via the geometry and combinatorics of Gelfand–Tsetlin polytopes, which encode the Kostka coefficients of $\mathfrak{gl}_n(\mathbb{C})$. We present a combinatorial structure on Gelfand–Tsetlin patterns, which constitute the polyhedral cone within which Gelfand–Tsetlin polytopes exist. This combinatorial structure, which we call a *tiling*, encodes both the combinatorics of the polytope and the geometry of its embedding with respect to the integer lattice.

We use tilings of Gelfand–Tsetlin patterns to give a combinatorial characterization of the vertices of Gelfand–Tsetlin polytopes and a method to calculate the dimension of the minimal face containing a given Gelfand–Tsetlin pattern. As an application, we settle a conjecture of Berenstein and Kirillov (1995) that the vertices of Gelfand–Tsetlin polytopes are integral. We prove the conjecture in $n \leq 4$, and we construct an example for each $n \geq 5$, with arbitrarily increasing denominators as n grows, of a non-integral vertex. This is the first infinite family of non-integral polyhedra for which the Ehrhart counting function is still a polynomial. We also derive a bound on the denominators for the non-integral vertices when n is fixed.

Continuing our application of the Gelfand–Tsetlin tiling machinery, we study the stretched Kostka coefficient $\mathcal{K}_{\lambda\beta}$, which is the map $n \mapsto K_{n\lambda,n\beta}$ sending each positive integer n to the Kostka coefficient indexed by $n\lambda$ and $n\beta$. Kirillov and Reshetikhin (1986) have shown that stretched Kostka coefficients are polynomial functions of n. King, Tollu, and Toumazet have conjectured that these polynomials always have nonnegative coefficients (2004), and they have given a conjectural expression for their degrees (2005). We prove the values conjectured by King, Tollu, and Toumazet for the degrees of stretched Kostka coefficients.

Acknowledgements

It is with great joy and gratitude that I take this opportunity to express my thanks to the many people who helped make it possible for me to reach this point. I must begin by thanking my advisor, Jesús De Loera. Many students of mathematics enter graduate school with only a dim conception of just how, exactly, one goes about being a mathematician. Unlike some of the physical sciences, ours is not a field in which undergraduates routinely work under a professor, assisting in original research. Most of us begin graduate school with a good idea of the *subject* of mathematics, but with only a vague notion of the *activity* of mathematics. Jesús encouraged me to remedy this by beginning research as soon as possible. For this reason alone, I count myself fortunate to have had him as my thesis advisor. Right from the beginning, I benefited tremendously from his keen sense of which open problems lie within the reach of an inexperienced researcher.

As I advanced in my work, I have continued to be the beneficiary of Jesús's exceptional generosity with his time and ideas. His enthusiasm for the process of discovery is always evident, and he has never failed to be supportive when I have encountered difficulties in my own research. He has also devoted a great deal of effort to helping me attain the less tangible, but equally important, benefits of meeting prominent mathematicians and showing them my work. He has served as the perfect counterbalance to my introverted nature. Indeed, he has provided me with an excellent example of all the qualities that lead to a successful career in mathematics, and for this I cannot thank him enough.

I would also like to thank Greg Kuperberg for introducing me to the amazing interplay between combinatorics and algebra. A visit to his office has never failed to enlighten me when something seemed obscure. Anne Schilling has always been generous with her advice, both mathematical and professional. Monica Vazirani and Eric Rains provided many helpful suggestions for both the content and the format of my presentations, to my lasting benefit. More generally, the research community at UC Davis has been exceptionally enriching, and the administrative staff has been unfailingly kind and helpful.

Kevin Woods has been a wonderful collaborator. I have been fortunate to benefit from his knowledge of combinatorics and discrete geometry. I would also like to extend my appreciation to Matthias Beck and Sinai Robins, who organized a very enjoyable and productive graduate summer school which brought many interesting avenues of possible future research to my attention. Charles Cochet was a great help as I was beginning to learn the subtleties of Lie algebras. Benjamin Howard showed me applications to algebraic geometry that I didn't know my work had.

The international mathematical community has also been very kind to me. Christophe Tollu kindly invited me to speak at the University of Paris, 13, where he and his collaborators shared some of their very interesting work and conjectures with me. Francisco Santos and Christos Athanasiadis gave me the wonderful opportunity to speak about my work in beautiful Crete. During the Spring of '05, I visited the University of Magdeburg, Germany, where Prof. Weismantel and his group made me feel very welcome, and Achill Schürmann and his family helped make this *ausländer* feel at home in a foreign land.

The work in this dissertation would have suffered greatly were it not for the many helpful comments and suggestions that I have received from Arkady Berenstein, Misha Kapovich, Allen Knutson, Peter Littelmann, Ezra Miller, Jörg Rambau, Etienne Rassart, Francisco Santos, and Michèle Vergne, as well as the anonymous referees of the articles in which some of these results have been published.

And finally, I must express my love and gratitude to my family, who provided me with the emotional and material support without which I could never have gotten this far. Mathematics is not a field that lends itself easily to explanation to the nonspecialist. Nonetheless, my family has always expressed their faith and confidence in the value of my work. My education would have been far more difficult without the sacrifices my grandparents made to provide for their grandchildren. My mother, father, and sisters have always been there when I needed them, and they have been boundlessly generous with their love and support.

CHAPTER 1

Introduction

1.1. Historical Overview

Over the course of the last sixty years, the theory of convex rational polyhedra has revealed itself to be a rich source of algorithmic tools for studying the representation theory of Lie algebras. Many aspects of representation theory have long lent themselves to combinatorial interpretations. Discrete structures such as weight lattices, semi-standard Young tableaux, Dynkin diagrams, and crystal graphs are central to understanding the representations of Lie algebras [**23, 27, 33, 48**].

Combinatorial interpretations are also indispensable if we wish to *compute* properties of representations. A fundamental problem in computational representation theory, and the principle motivation of the present study, is the so-called *Clebsch–Gordan* problem [**23**]: Given highest weights λ, μ, and ν for a semisimple Lie algebra $\mathfrak{g}$, compute the multiplicity of the irreducible representation V_ν with highest weight ν in the tensor product of V_λ and V_μ. We call this multiplicity the *Clebsch–Gordan* coefficient $C_{\lambda\mu}^{\nu}$ of $\mathfrak{g}$ associated with the highest weights λ, μ, and ν.

The concrete computation of Clebsch–Gordan coefficients has attracted much attention from not only representation theorists, but also from physicists, who employ them in the study of quantum mechanics (see, *e.g.*, [**4, 12, 71**]). The importance of these coefficients is also evident from their widespread appearance in other fields of mathematics besides representation theory. For example, the Littlewood–Richardson coefficients appear in combinatorics via symmetric functions and in enumerative algebraic geometry via Schubert varieties and Grassmannians (see, for instance, [**21, 47, 49, 62**]).

Clebsch–Gordan coefficients are manifestly nonnegative integers, a fact which has naturally motivated researchers to devise combinatorial interpretations of these numbers. The most famous of these is the Littlewood–Richardson rule (see [**23, 27, 48**]), which expresses $C_{\lambda\mu}^{\nu}$ for $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ as the number of skew Littlewood–Richardson tableaux with shape and content determined by the choice of highest weights. Combinatorial rules have also been introduced for arbitrary semisimple Lie algebras. One such rule employs the canonical bases of Lusztig [**45, 46**]. An equivalent rule, due to Kashiwara and Nakashima, rest upon the machinery of crystal graphs and crystal bases [**31, 33**]. In addition, Littelmann has presented a method that can be seen as a direct generalization of the Littlewood–Richardson rule. This method computes $C_{\lambda\mu}^{\nu}$ by enumerating certain piecewise linear paths in the weight lattice of $\mathfrak{g}$ [**43, 44**].

Concurrently with the development of these combinatorial descriptions of Clebsch–Gordan coefficients, another approach has opened the door to a vast store of theoretical and algorithm techniques. This approach is the encoding of Clebsch–Gordan coefficients as the number of integer solutions to a finite system of linear inequalities and equalities with integer coefficients. That is, in the language of discrete geometry, $C_{\lambda\mu}^{\nu}$ is the number of integer lattice points in some rational convex polytope.

The origins of the polyhedral approach go back over fifty years to the work of Gelfand and Tsetlin, who provided the first encoding of weight-space multiplicities for $\mathfrak{gl}_n(\mathbb{C})$ as the number of integer solutions to a finite set of linear inequalities [**24**]. Work accelerated in the 1980s, following the first encoding of Littlewood–Richardson coefficients as integral points in polytopes in the Ph.D. thesis of S. Johnson [**28**]. In 1986, Gelfand and Zelevinsky generalized the earlier work of Gelfand and Tsetlin to give another polyhedral encoding of Littlewood–Richardson coefficients [**25**]. In 1988, Berenstein and Zelevinsky expressed these polytopes in terms of weight partitions, and they conjectured that similar techniques could be used to enumerate Clebsch–Gordan coefficients for arbitrary semisimple Lie algebras [**6**]. In 1999, Knutson and Tao provided dramatic evidence for the usefulness of the polyhedral approach when they used a reformulation of the type-A versions of the Berenstein–Zelevinsky polytopes to prove the saturation theorem [**41**]. This had been a long-standing conjecture whose solution solved the problem posed by Hermann Weyl in 1912 of determining the eigenvalues of the sum of two Hermitian matrices given the eigenvalues of the summands [**70**] (see [**22**] for a history of this problem and its solution). Finally, in 2001, Berenstein and Zelevinsky completed their program of providing polyhedral descriptions of Clebsch–Gordan coefficients for all semisimple Lie algebras [**8**].

The development of the Berenstein–Zelevinsky polytopes has not been the only means by which mathematicians have been able to apply polyhedral techniques to representation theory with fruitful results. Billey, Guillemin, and Rassart have used the theory of vector partition functions to prove polynomiality results for the behavior of weight-space multiplicities for type A as the weights are varied [**9**]. Rassart has shown that similar polynomiality results extend to the Littlewood–Richardson coefficients [**59**]. King, Tollu, and Toumazet applied the theory of Ehrhart quasi-polynomials of polytopes to study so-called stretched Littlewood–Richardson coefficients [**35**] (about which we will have much more to say below). Baldoni, Beck, Cochet, and Vergne have used polyhedral interpretations of standard Lie algebraic formulations of Clebsch–Gordan coefficients to extend some of the results of Billey, Guillemin, and Rassart to arbitrary classical root systems [**1**]. Pak and Vallejo have studied the combinatorial complexity of the bijections defined by the polyhedral models and have analyzed the structure of the associated polyhedral cones, giving linearized versions of various discrete algorithms on tableaux [**56**].

Very recently, Narayanan proved that the computation of Clebsch–Gordan coefficients is in general a $\#P$-complete problem [**55**]. Nonetheless, we may still ask whether one can decide the positivity of a general Clebsch–Gordan coefficient in polynomial time. Here, we do not need to compute $C_{\lambda\mu}^{\nu}$ exactly. Rather, we are only interested in determining whether $C_{\lambda\mu}^{\nu} > 0$. This question takes on added interest in light of recent developments tying the computational complexity of Clebsch–Gordan coefficients to theoretical computer science. Mulmuley and Sohoni [**53, 54, 52**] have developed a program connecting the P vs. NP question to a class of problems known in geometric complexity theory as *subgroup restriction* *problems.* Suppose that G is a complex connected reductive group and that H is a connected reductive subgroup. Then there exists a decomposition of the irreducible representation $V_\lambda(G)$ of G with highest weight λ into irreducible H-submodules. The subgroup restriction problem is to answer the following question: *Given highest weights α and β for G and H respectively, does $V_\beta(H)$ appear in the decomposition of $V_\alpha(G)$?* The goals of the Mulmuley–Sohoni program are as follows: (1) find an algorithm to answer this question that is polynomial in the input sizes of α, β, and the rank of G; and (2) show that this result can be used to construct certain geometric obstructions that would establish that $P \neq NP$. In the context of representation theory, the subgroup restriction problem becomes the problem of establishing whether $C_{\lambda\mu}^{\nu} > 0$ for a given triple of highest weights.

As representation theorists were discovering how to encode Clebsch–Gordan coefficients and weight-space multiplicities as the number of lattice points in polytopes, convex geometers were developing theoretical results from which powerful consequences would follow in representation theory. The theory of the enumerative combinatorics of rational polyhedra in particular has made tremendous strides in the last forty years. Here we are interested in the number $\varphi_A(b)$ of integer solutions $x \in \mathbb{Z}^n$ to a system of linear inequalities $Ax \leq b$, where A is an $m \times n$ matrix and $b \in \mathbb{Z}^m$.

A seminal result of Ehrhart in 1962 [**20**] (see also [**63**, Chapter 4]) provides a beautiful description of the counting function $i_P(n) = \varphi_A(nb)$, which gives the number of integer lattice points in the nth dilation of the rational polytope $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. In 1995, Sturmfels incorporated the results of Ehrhart into a description of the analogous multivariate counting function corresponding to translating facets of the polytope independently of each other [**67**]. In 1997, Sturmfels and Thomas studied the continuity properties of $\varphi_A(b)$ [**68**]. This work served as one of the foundations of Knutson and Tao's proof of the saturation theorem using hive polytopes [**41**] (see also the excellent exposition in [**10**]).

Researchers have also gained a deeper understanding of the computational complexity of polytopes. In 1979, Khachiyan introduced the ellipsoid algorithm for performing linear programming on a rational polytope in time polynomial in the input size of the data defining the polytope (*e.g.*, a list of its vertices) [**34**]. In particular, Khachiyan's algorithm decides the feasibility of a linear system of inequalities in polynomial time. In 1994, Barvinok presented an algorithm that, for fixed dimension d, enumerates the lattice points in a rational polytope in $\mathbb{R}^d$ in time polynomial in its input size [**3**].

As mentioned above, Narayanan showed that the computation of Clebsch–Gordan coefficients is a $\#P$-complete problem [**55**]. Nonetheless, one can ask for an algorithm that behaves well when some parameter is fixed. Stembridge raised the challenge of crafting algorithms based on geometric ideas such as Littelmann's paths [**44**] or Kashiwara's crystal bases [**32**] (see comment on page 29, section 7, of [**66**]). As we show below, there is such an algorithm, based on the polyhedral geometry of the Clebsch–Gordan coefficients.

The geometric results achieved by the polyhedral researchers have profound implications for the structure of Lie algebras. Moreover, the algorithmic results of Barvinok and Khachiyan permit the computation of Clebsch–Gordan coefficients in contexts where it had previously been impossible. The experimental evidence we have produced with these computations has brought to light several fascinating properties of the representations of the classical groups. These properties motivate several conjectures that follow naturally from the polyhedral approach. We feel confident in predicting that, as the research community pursues the proofs of these and other conjectures, the polyhedral approach will continue to be a force in computational representation theory for years to come.

1.2. Mathematical Background and Notation

1.2.1. Representations of semisimple Lie algebras. We now briefly discuss the classification of the representations of semisimple Lie algebras. The results we discuss are well-known, and our notation is largely standard. See [**23**] or [**27**] for a full development of this fundamental theory.

Let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra, and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then every irreducible representation V of $\mathfrak{g}$ decomposes into *weight spaces* $V(\beta)$, $\beta \in \mathfrak{h}^*$, which are scaled under the action of $\mathfrak{h}$. More precisely, we have that

$$V = \bigoplus_{\beta \in \mathfrak{h}^*} V(\beta), \tag{1.1}$$

where $V(\beta) = \{v \in V : h \cdot v = \beta(h)v, \text{ for all } h \in \mathfrak{h}\}$. The elements $\beta \in \mathfrak{h}^*$ indexing the weight spaces $V(\beta)$ are the *weights* of the representation V. All weights of the representations of $\mathfrak{g}$ lie in a lattice $\Lambda_W \subset \mathfrak{h}^*$ called the *weight lattice* of $\mathfrak{g}$. The *rank* of $\mathfrak{g}$ is defined to be the rank of Λ_W as an abelian group.

When this process of decomposing into weight spaces is applied to the adjoint representation, we get a decomposition of $\mathfrak{g}$ itself into a finite number of *root spaces*:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where the direct sum is over the finite set $R \subset \Lambda_W$ of *roots*. The roots lie in a sublattice Λ_R of Λ_W with finite index. To each root α, we associate the subalgebra $\mathfrak{s}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ of $\mathfrak{g}$. In fact, $\mathfrak{s}_\alpha$ is an isomorphic copy of $\mathfrak{sl}_2(\mathbb{C})$ generated by *Chevalley generators* $e_\alpha \in \mathfrak{g}_\alpha$, $f_\alpha \in \mathfrak{g}_{-\alpha}$, and $\alpha^\vee \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Though e_α and f_α are not uniquely determined by this description, $\alpha^\vee$ is the unique element of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\alpha(\alpha^\vee) = 2$. We call $\alpha^\vee$ a *coroot* of $\mathfrak{g}$.

By fixing a generic linear functional l on $\mathfrak{h}^*$, we partition R into a set of *positive roots* $R^+ = \{\beta \in R : l(\beta) > 0\}$ and a set of *negative roots* $R_- = \{\beta \in R : l(\beta) < 0\}$. The reflections $s_\alpha \colon \mathfrak{h}^* \to \mathfrak{h}^*$, $\alpha \in R$, in the hyperplanes $\Omega_\alpha = \{\gamma \in \mathfrak{h}^* : \gamma(\alpha^\vee) = 0\}$ defined by

$$s_\alpha(\beta) = \beta - \frac{2\beta(\alpha^\vee)}{\alpha(\alpha^\vee)}$$

generate the *Weyl group* $\mathfrak{W}$ of $\mathfrak{g}$. The hyperplanes Ω_α, $\alpha \in R$, form a central hyperplane arrangement whose complement in $\Lambda_W \otimes \mathbb{R}$ is a union of open polyhedral cones with vertices at the origin. The *chambers* of $\mathfrak{g}$ are the closures of the full-dimensional components of the complement of this hyperplane arrangement. Among these chambers, we distinguish the *Weyl chamber* $\mathscr{W}$, defined by

$$\mathscr{W} = \left\{\beta \in \Lambda_W \otimes \mathbb{R} : \beta(\alpha^\vee) \geq 0 \text{ for all } \alpha \in R^+\right\}$$

In an arbitrary irreducible representation V, the root spaces $\mathfrak{g}_\alpha$ act by permuting the weight spaces according to addition of weights. That is, if $g \in \mathfrak{g}_\alpha$, then $g(V(\beta)) = V(\alpha + \beta)$. In particular, there is a unique weight space $V(\lambda)$ of V

such that $g(V(\lambda)) = 0$ whenever $g \in \mathfrak{g}_\alpha$ and $\alpha \in R$. The corresponding weight λ is called the *highest weight* of the irreducible representation V. If two irreducible representations have the same highest weight, then they are isomorphic. Thus, the irreducible representations of $\mathfrak{g}$ are indexed by their highest weights. These highest weights all lie in the Weyl chamber $\mathscr{W}$. Indeed, the highest weights of $\mathfrak{g}$ are precisely the elements of $\Lambda_W \cap \mathscr{W}$. We denote the irreducible representation with highest weight λ by V_λ.

The representation theory of semisimple Lie algebras reflects that of finite groups in that every representation may be decomposed into a direct sum of irreducible representations. In particular, given highest weights λ and μ, we have a decomposition

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu \in \Lambda_W \cap \mathscr{W}} V_\nu^{\oplus C_{\lambda\mu}^\nu} = \bigoplus_{\nu \in \Lambda_W \cap \mathscr{W}} C_{\lambda\mu}^\nu V_\nu. \tag{1.2}$$

We call the numbers $C_{\lambda\mu}^\nu$ appearing in the decomposition (1.2) *Clebsch–Gordan coefficients*. In the specific case of type-A Lie algebras, these values are also called *Littlewood–Richardson coefficients*. When we are specifically discussing the type-A case, we will adhere to convention and write $c_{\lambda\mu}^\nu$ for $C_{\lambda\mu}^\nu$.

A special case of Clebsch–Gordan coefficients are the *weight-space multiplicities*. These are the dimensions of the weight spaces arising in the decomposition (1.1). Given a highest weight λ and a weight β of a Lie algebra $\mathfrak{g}$, the weight-space multiplicity $K_{\lambda\beta}$ is defined by

$$K_{\lambda\beta} = \dim V_\lambda(\beta).$$

Though it is not immediately evident from this algebraic definition, every weight-space multiplicity of $\mathfrak{g}$ is a Clebsch–Gordan coefficient of $\mathfrak{g}$. In particular,

$$K_{\lambda\beta} = C_{\nu-\beta,\lambda}^\nu$$

if ν is a highest weight of $\mathfrak{g}$ chosen sufficiently far from the hyperplanes Ω_α, $\alpha \in R^+$, bounding the Weyl Chamber [**72**, §131]. See the discussion following Proposition 2.5 for an explicit bijection establishing this equality in type A, where the weight-space multiplicities are known as *Kostka coefficients*.

1.2.2. Enumeration of lattice points in polytopes. A *polyhedron* P is the set of solutions $x \in \mathbb{R}^n$ to a finite system of linear inequalities

$$\begin{gathered} a_{11}x_1 + \cdots + a_{1n}x_n \le b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \le b_m. \end{gathered}$$

We say that P is *rational* if the a_{ij}'s and the b_i's are all rational. We produce an equivalent system of inequalities if we multiply both sides of each inequality by the least common multiple of the denominators of the a_{ij}'s and the b_i's. Therefore, we may assume without loss of generality that each a_{ij} and b_i is an integer. We express such a system as a matrix inequality $Ax \le b$, where $A = (a_{ij}) \in \mathbb{Z}^{m\times n}$ and $b = (b_1, \ldots, b_m)^t \in \mathbb{Z}^m$. Given $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$, $H = \{x \in \mathbb{R}^n : c^t x = d\}$ is a *support hyperplane* of P if $c^t x \le d$ for all $x \in P$. A *face* of P is the intersection of P with a support hyperplane. Observe that, under this definition, the empty set and P itself are both faces of P. The *dimension* of a face F of P is the dimension of the affine linear subspace spanned by F. The *facets* of P are those faces that are not properly contained in any face of P other than P itself.

A polyhedron is called a *polytope* if it is bounded. It is a well-known (but not entirely trivial) theorem that the bounded rational polyhedra in $\mathbb{R}^n$ are precisely the convex hulls of finite sets of points in $\mathbb{Q}^n$ [**60, 73**]. An *integral* polytope is the convex hull of a finite set of points in $\mathbb{Z}^n$.

Given an integer matrix $A \in \mathbb{Z}^{m\times n}$, define the counting function $\varphi_A\colon \mathbb{Z}^m \to \mathbb{Z}$ by

$$\varphi_A(b) = \#\left\{x \in \mathbb{Z}^n : Ax \le b\right\}.$$

Hence, $\varphi_A(b)$ is the number of points in the integer lattice $\mathbb{Z}^n$ contained in the polytope $P = \{x \in \mathbb{R}^n : Ax \le b\}$. We call φ_A a *vector partition function* because it expresses the number of ways to write a vector b as a nonnegative linear combination of the columns of the matrix A with integer coefficients.

Observe that if

$$a_{i1}x_1 + \cdots + a_{in}x_n \le b_i$$

is one of the facet-defining inequalities of P, then varying b_i corresponds to a parallel translation of the facet on which this inequality is attained with equality. In particular, scaling the vector b by a factor of m corresponds to dilating the polytope P by a factor of m. We denote the dilated polytope by mP:

$$mP = \{x \in \mathbb{R}^n : \tfrac{1}{m}x \in P\}.$$

In 1962, Ehrhart proved the following remarkable result [**19, 20**]. If P is a d-dimensional rational polytope in $\mathbb{R}^n$, then there is a degree-d *quasi-polynomial* function $i_P\colon \mathbb{Z} \to \mathbb{Z}$ with rational coefficients such that the restriction of i_P to the positive integers counts the number of lattice points in integral dilations of P. That is, we have that

$$i_P(m) = |mP \cap \mathbb{Z}^n|, \qquad m = 1, 2, 3, \dots.$$

This means that there exist periodic functions $c_0, \dots, c_d\colon \mathbb{Z} \to \mathbb{Q}$ such that

$$i_P(m) = c_d(m)m^d + \cdots + c_1(m)m + c_0(m), \qquad m = 1, 2, 3, \dots.$$

Equivalently, there are polynomials $f_1(t), \dots, f_M(t) \in \mathbb{Q}[t]$ such that each $f_j(t)$ is of degree $\le d$, and

$$i_P(m) = \begin{cases} f_1(m) & \text{if } m \equiv 1 \bmod M, \\ \quad\vdots & \\ f_M(m) & \text{if } m \equiv M \bmod M. \end{cases}$$

If i_P can be expressed in terms of M polynomials in this fashion, we say that M is a *quasi-period* of i_P. (We do not assume that M is the minimum such number.) When P is an *integral* polytope, then i_P has a quasi-period of 1; that is, $i_P(m)$ is simply a polynomial function of m [**19**].

Several important pieces of geometric information are encoded in the coefficients of the Ehrhart quasi-polynomial of a rational polytope. First, as mentioned, the degree of the quasi-polynomial is the dimension of the polytope. Moreover, the coefficient of the leading term in each nonzero constituent $f_j(x)$ is the volume of the polytope, where the volume is measured with respect to the sublattice that is the intersection of the affine span of P with $\mathbb{Z}^n$. If, in addition, the polytope is integral, then the $(d-1)$-degree coefficient c_{d-1} is one-half the sum of the volumes of the facets, where the volume of each facet is measured with respect to the sublattice defined by its affine span. Finally, when the polytope is integral, the constant term c_0 is always 1. For an excellent introduction to Ehrhart quasi-polynomials that includes proofs of all these properties, see [**63**, Chapter 4].

Such concise geometric interpretations for the other coefficients of Ehrhart quasi-polynomials are not known. Even in the case of integral polytopes, the only expressions known are complicated alternating sums involving the toric geometry of the polytope (see, for example, [**57**]). It is possible to give elementary geometric interpretations for the terms in these alternating sums; A. Schürmann has expressed them in terms of the volumes of the intersections of the polytope and all lattice translates of the fundamental domains of the sublattices spanned by the faces [**61**]. Nonetheless, the alternating nature of these sums presents an obstacle to understanding the behavior of the coefficients. For example, we conjecture below that the coefficients of the Ehrhart quasi-polynomials of Berenstein–Zelevinsky polytopes are always nonnegative (Conjecture 3.8), but it is difficult to prove such bounds using the alternating sums currently available.

1.3. Polytopes for encoding Clebsch–Gordan coefficients

In this section, we present the polyhedral encodings of Clebsch–Gordan coefficients due to Berenstein and Zelevinsky [**8**]. Given highest weights λ, μ, and ν for a finite-dimensional complex semi-simple Lie algebra $\mathfrak{g}$, Berenstein and Zelevinsky define a corresponding polytope $BZ_{\lambda\mu}^{\nu}(\mathfrak{g})$. In the remaining chapters, we will be applying the polyhedral theory discussed above to these polytopes to derive theoretical and computational results about the behavior of Clebsch–Gordan coefficients as the parameterizing highest weights are varied. We refer to these polytopes as *BZ-polytopes*. Their crucial property is guaranteed by the following theorem.

THEOREM 1.1. [**8**, Theorems 2.3 and 2.4] *Given highest weights λ, μ, and ν for a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$, the number of integer lattice points in $BZ_{\lambda\mu}^{\nu}(\mathfrak{g})$ is the Clebsch–Gordan coefficient $C_{\lambda\mu}^{\nu}$ for $\mathfrak{g}$. That is*

$$C_{\lambda\mu}^{\nu} = |BZ_{\lambda\mu}^{\nu}(\mathfrak{g}) \cap \mathbb{Z}^d|$$

where d is the dimension of the ambient space containing $BZ_{\lambda\mu}^{\nu}(\mathfrak{g})$.

We now provide the defining inequalities of these polytopes in each of the classical root systems A_r, B_r, C_r, and D_r.

1.3.1. Type A_r. Littlewood–Richardson coefficients were shown in [**5**] to be equal to the number of integral lattice points in members of a particular family of polytopes. In 1999, Knutson and Tao [**41**] used these polytopes to prove the saturation theorem (Theorem 3.1 below). More precisely, Knutson and Tao applied a lattice-preserving linear map to the polytopes of Berenstein and Zelevinsky, producing what they call *hive polytopes* (Definition 1.3 below). These are a particularly symmetrical description of the Berenstein–Zelevinsky polytopes in type A, and we will be using them extensively in the following discussion. These polytopes exist in the polyhedral cone of *hive patterns*, which we now define.

DEFINITION 1.2. Fix $r \in \mathbb{Z}_{\geq 0}$ and let $\mathcal{H} = \{(i,j,k) \in \mathbb{Z}_{\geq 0}^3 : i+j+k = r\}$. A *hive pattern of size r* is a map

$$h \colon \mathcal{H} \to \mathbb{R}_{\geq 0}, \qquad (i,j,k) \mapsto h_{ijk},$$

satisfying the *rhombus inequalities*:

$$\begin{aligned}h_{i,j-1,k+1} + h_{i-1,j+1,k} &\le h_{ijk} + h_{i-1,j,k+1},\\ h_{ijk} + h_{i-1,j-1,k+2} &\le h_{i,j-1,k+1} + h_{i-1,j,k+1},\\ h_{i+1,j-1,k} + h_{i-1,j,k+1} &\le h_{ijk} + h_{i,j-1,k+1}.\end{aligned}$$

for $(i,j,k) \in \mathcal{H}$, $i,j \ge 1$.

We usually think of a hive pattern of size r as a triangular array of real numbers:

$$\begin{array}{ccccccccc} & & & & h_{0,0,r} & & & & \\ & & & h_{1,0,r-1} & & h_{0,1,r-1} & & & \\ & & h_{2,0,r-2} & & h_{1,1,r-2} & & h_{0,2,r-2} & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ h_{r,0,0} & & h_{r-1,1,0} & & \cdots & & h_{1,r-1,0} & & h_{0,r,0}\end{array}$$

In this representation, the rhombus inequalities state that, for each "little rhombus" of entries (which comes in one of three orientations), the sum of the entries on the long diagonal does not exceed the sum of the entries on the short diagonal. That is, when the entries a, b, c, and d of a hive pattern are in one of the three configurations

$$\begin{array}{ccc} c & & b \\ & & \\ & a & & d \end{array} \qquad \begin{array}{ccc} & c & \\ a & & b \\ & d & \end{array} \qquad \begin{array}{cccc} & a & & c \\ & & & \\ d & & b & \end{array}$$

we have that $a + b \ge c + d$. Here is an example of a hive pattern with $r = 4$:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 8 & & 5 & & & \\ & & 13 & & 12 & & 8 & & \\ & 18 & & 17 & & 15 & & 11 & \\ 20 & & 20 & & 18 & & 16 & & 12 \end{array}$$

Recall that a partition of length r is a sequence λ of integers $\lambda_1 \ge \cdots \ge \lambda_r \ge 0$. We write $|\lambda|$ for $\sum_{i=1}^r \lambda_i$, the *size* of the partition λ.

Definition 1.3. Given partitions $\lambda, \mu, \nu \in \mathbb{Z}^r_{\ge 0}$, the *hive polytope* $H^\nu_{\lambda\mu}$ is the set of hive patterns with boundary entries fixed as in Figure 1.1.

Knutson and Tao do not require in their definition that $\lambda, \mu, \nu \ge 0$. However, when considering Littlewood–Richardson coefficients, it suffices to restrict ourselves to the case in which λ, μ, and ν are partitions; in particular, their coordinates are nonnegative. This has the consequence that hive polytopes lie in the nonnegative orthant, which will be convenient in Section 3.1.

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & \nu_1 = \bullet & & \bullet = \lambda_1 & & & \\
 & & \nu_1+\nu_2 = \bullet & & \bullet & & \bullet = \lambda_1+\lambda_2 & & \\
 & \nu_1+\nu_2+\nu_3 = \bullet & & \bullet & & \bullet & & \bullet = \lambda_1+\lambda_2+\lambda_3 & \\
 & & & \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots & & & \\
|\nu| = \bullet & & \cdots & & \bullet & \bullet & \bullet & & \bullet = |\lambda|
\end{array}$$

$$= |\lambda| + |\mu| \qquad = |\lambda| + \mu_1 + \mu_2 + \mu_3 \qquad = |\lambda| + \mu_1 + \mu_2 \qquad = |\lambda| + \mu_1$$

FIGURE 1.1. The boundary entries of the hive patterns in the hive polytope $H^{\nu}_{\lambda\mu}$.

From the perspective of computational complexity, it is important to note that, for fixed r, the input size of a hive polytope $H^{\nu}_{\lambda\mu}$ grows linearly with the input sizes of the weights λ, μ, and ν. As we have indicated several times, our interest in hive polytopes arises from the following result:

LEMMA 1.4. [**5, 42**] *The Littlewood–Richardson coefficient $c^{\nu}_{\lambda\mu}$ equals the number of integer lattice points in the hive polytope $H^{\nu}_{\lambda\mu}$.*

Unfortunately, the descriptions of the BZ-polytopes for the other classical Lie algebras are more involved than that of the hive polytopes above. Nonetheless, Berenstein and Zelevinsky have provided completely explicit descriptions (see Theorems 2.5 and 2.6 of [**8**]), which we now repeat. In the case of each root system, we write $e_0, \dots, e_{r-1}$ for the standard basis of $\Lambda_R \otimes \mathbb{R}$.

1.3.2. Types B_r and C_r. Let $\mathfrak{g}$ be of type B_r (so that $\mathfrak{g} \cong \mathfrak{so}_{2r+1}$) or of type C_r (so that $\mathfrak{g} \cong \mathfrak{sp}_{2r}$). The Cartan matrices $(a_{ij})_{0\le i,j\le r-1}$ for types B_r and C_r are respectively

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & -2 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}.$$

Index the simple roots $\alpha_0, \dots, \alpha_{r-1}$ of $\mathfrak{g}$ so that $\alpha_1, \dots, \alpha_{r-1}$ form a root system of type A_{r-1}. More precisely, for type B_r, we put $\alpha_0 = e_0$ and $\alpha_i = e_i - e_{i-1}$, $1 \le i \le r-1$, and for type C_r, we put $\alpha_0 = 2e_r$ and $\alpha_i = e_i - e_{i-1}$, $1 \le i \le r-1$. Then the *simple coroots* $\alpha_0^\vee, \dots, \alpha_{r-1}^\vee$ of $\mathfrak{g}$ are given by $\alpha_j(\alpha_i^\vee) = a_{ij}$. Finally, let $a = |a_{10}|$.

DEFINITION 1.5. Let $\mathfrak{g}$ be a simple Lie algebra of type B_r or C_r. Given a triple of highest weights λ, μ, and ν for $\mathfrak{g}$, the BZ-polytope $BZ^{\nu}_{\lambda\mu}(\mathfrak{g})$ is the set of tuples $(t_i^{(j)} : 0 \le |i| \le j < r)$ in $\mathbb{R}^{r^2}$ that solve the following linear inequalities and equalities (with the convention that $t_i^{(j)} = 0$ unless $0 \le |i| \le j < r$):

(1) $2t_{-j}^{(j)} \ge \cdots \ge 2t_{-1}^{(j)} \ge at_0^{(j)} \ge 2t_1^{(j)} \ge \cdots \ge 2t_j^{(j)} \ge 0$, for $0 \le j < r$;

(2) $\sum_{0 \le |i| \le j < r} t_i^{(j)} \alpha_{|i|} = \lambda + \mu - \nu$;

(3) $\lambda(\alpha_0^\vee) \ge t_0^{(0)}$, and

$$\lambda(\alpha_j^\vee) \ge \max\{t_j^{(j)},\, at_0^{(j)} - t_1^{(j-1)} - t_{-1}^{(j)},\, t_1^{(j-1)} + t_{-1}^{(j)} - at_0^{(j-1)}\} \\ \cup \{\varphi_i^{(j)}(t) : 1 \le i < j\}$$

for $0 \le j < r$, where

$$\varphi_i^{(j)}(t) = \max\{t_i^{(j)} + t_{-i}^{(j)} - t_{i+1}^{(j-1)} - t_{-i-1}^{(j)}, \\ t_{i+1}^{(j-1)} + t_{-i-1}^{(j)} - t_{-i}^{(j-1)} - t_i^{(j-1)},\, t_{\pm i}^{(j)} - t_{\pm i}^{(j-1)}\};$$

(4) $\mu(\alpha_0^\vee) \ge \max_{j \ge 0} \,(t_0^{(j)} + \frac{2}{a}\sum_{k>j}(at_0^{(k)} - t_{-1}^{(k)} - t_1^{(k-1)}))$,

$$\mu(\alpha_1^\vee) \ge \max_{j\ge 1} \max\Big\{t_{-1}^{(j)} + \sum_{k=j+1}^{r} (2t_{-1}^{(k)} + 2t_1^{(k-1)} - at_0^{(k-1)} - t_{-2}^{(k)} - t_2^{(k-1)}), \\ t_1^{(j)} + \sum_{k=j+1}^{r} (2t_{-1}^{(k)} + 2t_1^{(k)} - at_0^{(k)} - t_{-2}^{(k)} - t_2^{(k-1)})\Big\}, \text{ and}$$

$$\mu(\alpha_i^\vee) \ge \max_{j\ge i} \max\Big\{t_{-i}^{(j)} + \sum_{k=j+1}^{r} (2t_{-i}^{(k)} + 2t_i^{(k-1)} - t_{-i+1}^{(k-1)} - t_{i-1}^{(k-1)} - t_{-i-1}^{(k)} - t_{i+1}^{(k-1)}), \\ t_i^{(j)} + \sum_{k=j+1}^{r} (2t_{-i}^{(k)} + 2t_i^{(k)} - t_{-i+1}^{(k)} - t_{i-1}^{(k)} - t_{-i-1}^{(k)} - t_{i+1}^{(k-1)})\Big\}$$

for $2 \le i < r$.

1.3.3. Type D_r. Let $\mathfrak{g}$ be of type D_r (so that $\mathfrak{g} \cong \mathfrak{so}_{2r}$). The Cartan matrix for $\mathfrak{g}$ is

$$(a_{ij})_{(i,j)\in\{-1\}\cup[r-1]} = \begin{bmatrix} 2 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}$$

Index the simple roots $\alpha_{-1}, \alpha_1, \ldots, \alpha_{r-1}$ of $\mathfrak{g}$ so that $\alpha_1, \ldots, \alpha_{r-1}$ form a root system of type A_{r-1}. More precisely, we put $\alpha_{-1} = e_1 + e_0$ and $\alpha_i = e_i - e_{i-1}$, $1 \le i \le r-1$. Then the *simple coroots* $\alpha_{-1}^\vee, \alpha_1^\vee, \ldots, \alpha_{r-1}^\vee$ of $\mathfrak{g}$ are given by $\alpha_j(\alpha_i^\vee) = a_{ij}$.

DEFINITION 1.6. Let $\mathfrak{g}$ be a simple Lie algebra of type D_r. Given a triple of highest weights λ, μ, and ν for $\mathfrak{g}$, the BZ-polytope $BZ^{\nu}_{\lambda\mu}(\mathfrak{g})$ is the set of tuples

$(t_i^{(j)} : 1 \le |i| \le j < r)$ in $\mathbb{R}^{r(r-1)}$ that solve the following linear inequalities and equalities (with the convention that $t_i^{(j)} = 0$ unless $1 \le |i| \le j < r$):

(1) $t_{-j}^{(j)} \ge \cdots \ge t_{-2}^{(j)} \ge \max(t_{-1}^{(j)}, t_1^{(j)}) \ge \min(t_{-1}^{(j)}, t_1^{(j)}) \ge t_2^{(j)} \ge \cdots \ge t_j^{(j)} \ge 0$ for $1 \le j < r$;

(2) $\sum_j (t_{-1}^{(j)}\alpha_{-1} + t_1^{(j)}\alpha_1) + \sum_{2 \le |i| \le j < r} t_i^{(j)}\alpha_{|i|} = \lambda + \nu - \mu$;

(3) $\lambda(\alpha_{\pm 1}^\vee) \ge t_{\pm 1}^{(1)}$, and

$$\begin{aligned}\lambda(\alpha_j^\vee) \ge \max\{&t_j^{(j)},\, t_{\pm 1}^{(j)} - t_{\mp 1}^{(j-1)},\, t_1^{(j)} + t_{-1}^{(j)} - t_{-2}^{(j)} - t_2^{(j-1)},\\ &t_{-2}^{(j)} + t_2^{(j-1)} - t_1^{(j-1)} - t_{-1}^{(j-1)} j)\} \cup \{\varphi_i^{(j)}(t) : 2 \le i < j\}\end{aligned}$$

for $2 \le j < r$, where $\varphi_i^{(j)}(t)$ is the same as in Definition 1.5;

(4) $\nu(\alpha_{\pm 1}^\vee) \ge \max_{j \ge 1} \, (t_{\pm 1}^{(j)} + \sum_{k > j}(2t_{\pm 1}^{(k)} - t_{-2}^{(k)} - t_2^{(k-1)}))$, and

$$\begin{aligned}\nu(\alpha_i^\vee) \ge \max_{j \ge i} \ \max &\Big\{ t_{-i}^{(j)} + \sum_{k=j+1}^{r} (2t_{-i}^{(k)} + 2t_i^{(k-1)} - t_{-i+1}^{(k-1)} - t_{i-1}^{(k-1)} - t_{-i-1}^{(k)} - t_{i+1}^{(k-1)}),\\ &t_i^{(j)} + \sum_{k=j+1}^{r} (2t_{-i}^{(k)} + 2t_i^{(k)} - t_{-i+1}^{(k)} - t_{i-1}^{(k)} - t_{-i-1}^{(k)} - t_{i+1}^{(k-1)}) \Big\}\end{aligned}$$

for $2 \le i < r$.

1.4. Polytopes for Kostka coefficients

For each $n \in \mathbb{N}$, let $\mathcal{I}_n = \{(i,j) \in \mathbb{Z}^2 : 1 \le i \le j \le n\}$. It will be convenient to arrange the entries of $\mathcal{I}_n$ in a triangular array as in Figure 1.2. We refer to the elements $(1,j), \ldots, (j,j)$ as the jth row of $\mathcal{I}_n$; that is, we enumerate the rows of $\mathcal{I}_n$ from the bottom. Let X_n be the set of maps $\mathbf{x} \colon \mathcal{I}_n \to \mathbb{R}$ assigning a number $x_{ij} \in \mathbb{R}$ to each $(i,j) \in \mathcal{I}_n$. Equivalently, the elements of X_n are triangular arrays $(x_{ij})_{1 \le i \le j \le n}$ with $x_{ij} \in \mathbb{R}$. We always depict such an array by arranging the entries as follows:

$$\begin{array}{ccccccccc} x_{1n} & & x_{2n} & & x_{3n} & & \cdots & & x_{nn} \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & x_{13} & & x_{23} & & x_{33} & & \\ & & & x_{12} & & x_{22} & & & \\ & & & & x_{11} & & & & \end{array}$$

Note that X_n inherits a normed vector space structure under the obvious isomorphism $X_n \cong \mathbb{R}^{n(n+1)/2}$. Therefore, we will be able to speak of cones, lattices, polyhedra, polytopes, and semigroups in X_n. The main region of interest for our study is the cone of *GT-patterns* in X_n.

Definition 1.7. A *Gelfand–Tsetlin pattern* (or *GT-pattern*) is an element $(x_{ij})_{1 \le i \le j \le n}$ of X_n satisfying the *Gelfand–Tsetlin inequalities*

$$x_{i,j+1} \ge x_{ij} \ge x_{i+1,j+1}, \quad \text{for } 1 \le i \le j \le n-1. \tag{1.3}$$

$$\begin{array}{ccccccccc} (1,n) & & (2,n) & & (3,n) & & \cdots & & (n,n) \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & (1,3) & & (2,3) & & (3,3) & & \\ & & & (1,2) & & (2,2) & & & \\ & & & & (1,1) & & & & \end{array}$$

FIGURE 1.2. The elements of $\mathcal{I}_n$ arranged in a triangular array.

If we arrange the entries x_{ij} as in the triangular array shown above, then these inequalities state that each entry not in the top row is weakly less than its upper-left neighbor and weakly greater than its upper-right neighbor.

The nonnegative solutions to the inequalities (1.3) define a polyhedral cone in $\mathbb{R}^{n(n+1)/2}$. See the top of Figure 1.3 for an example of a GT-pattern. The Gelfand–Tsetlin polytopes arise by intersecting this cone with certain affine hyperplanes.

DEFINITION 1.8. For any point $\mathbf{x} \in X_n$, we define the *highest weight* $\mathbf{hwt}(\mathbf{x})$ of $\mathbf{x}$ to be the top row $(x_{1n}, \ldots, x_{nn})$ of $\mathbf{x}$. We define the *weight* $\mathbf{wt}(\mathbf{x}) = (\beta_1, \ldots, \beta_n)$ of $\mathbf{x}$ by $\beta_1 = x_{11}$ and $\beta_j = \sum_{i=1}^{j} x_{ij} - \sum_{i=1}^{j-1} x_{i,j-1}$ for $2 \le j \le n$. Thus, the entries of $\mathbf{wt}(\mathbf{x})$ are the successive row-sum differences of $\mathbf{x}$.

For each $\lambda \in \mathbb{R}^n$, let GT_λ be the polyhedron of GT-patterns with highest weight λ:

$$GT_\lambda = \{\mathbf{x} \in X_n : \mathbf{x} \text{ is a GT-pattern and } \mathbf{hwt}(\mathbf{x}) = \lambda\}.$$

Observe that GT_λ is in fact a bounded polyhedron, *i.e.*, a polytope, because $\lambda_1 \ge x_{ij} \ge \lambda_n$ for each entry x_{ij} of a point in GT_λ. Note also that the GT inequalities (1.3) force the top row of a GT-pattern to be weakly decreasing, so that $GT_\lambda = \varnothing$ if λ is not weakly decreasing. (The converse is also true, since if λ is weakly decreasing, then $(x_{ij})_{1 \le i \le j \le n}$ with $x_{ij} = \lambda_i$ is a GT-pattern in GT_λ.) For each $\beta \in \mathbb{R}^n$, let $W_\beta \subset X_n$ be the affine subspace of points in X_n with weight β:

$$W_\beta = \{\mathbf{x} \in X_n : \mathbf{wt}(\mathbf{x}) = \beta\}.$$

DEFINITION 1.9. Given $\lambda, \beta \in \mathbb{Z}^n$, the *Gelfand–Tsetlin polytope* (or *GT-polytope*) $GT_{\lambda\beta}$ is the convex polytope of GT-patterns with highest weight λ and weight β:

$$GT_{\lambda\beta} = GT_\lambda \cap W_\beta.$$

Given a GT-pattern $\mathbf{x}$, let $GT(\mathbf{x})$ denote the unique GT-polytope $GT_{\mathbf{hwt}(\mathbf{x}),\mathbf{wt}(\mathbf{x})}$ containing $\mathbf{x}$.

The polytope $GT_{\lambda\beta}$ is the set of all GT-patterns in X_n in which the top row is λ and the sum of the entries in the jth row is $\sum_{i=1}^{j} \beta_i$ for $1 \le j \le n$. Note that when we speak of a GT-polytope $GT_{\lambda\beta}$, we assume that λ and β are integral, but GT_λ is defined for arbitrary $\lambda \in \mathbb{R}^n$.

The importance of GT-polytopes stems from a classic result of I. M. Gelfand and M. L. Tsetlin in [**24**], which states that the number of integral lattice points in the GT-polytope $GT_{\lambda\beta}$ equals the dimension of the weight-β subspace of the irreducible representation of $\mathfrak{gl}_n\mathbb{C}$ with highest weight λ. These subspaces are indexed by the set $SSYT(\lambda, \beta)$ of semi-standard Young tableaux with shape λ and content β [**64**]. It is well-known that the elements of $SSYT(\lambda, \beta)$ are in one-to-one correspondence with the integral GT-patterns in $GT_{\lambda\beta}$ under the bijection exemplified in Figure 1.3: Given an integral GT-pattern in X_n, let $\lambda^{(j)}$ be the jth row (so that $\lambda^{(n)} = \lambda$). For $1 \leq j \leq n$, place j's in each of the boxes in the skew shape $\lambda^{(j)}/\lambda^{(j-1)}$ in the Young diagram of shape λ. (Here we put $\lambda^{(0)} = \varnothing$ to deal with the $j = 1$ case.) See [**64**] for details and [**39, 37**] for additional interesting applications of GT-polytopes.

In 1995, Berenstein and Kirillov conjectured that GT-polytopes are always integral [**39**]. This conjecture seems to have been motivated by the fact that, for an integer parameter m, the Kostka number $K_{m\lambda,m\beta}$ is a univariate polynomial function of m when λ and β are fixed. This was proved by Kirillov and Reshetikhin using fermionic formulas in [**40**]. For completeness, we give another proof here.

PROPOSITION 1.10. *Given a highest weight λ and a weight β for a Lie algebra of type A, the value of the Kostka coefficient $K_{m\lambda,m\beta}$ is given by a polynomial in m.*

PROOF. We write $f(m) = K_{m\lambda,m\beta}$. It follows from the polyhedral encoding of Kostka numbers as GT-polytopes that $f(m)$ is a quasi-polynomial function of m. This means that there exist an integer M and polynomials $g_0, g_1, \ldots, g_{M-1}$ such that $f(m) = g_i(m)$ if $m \equiv i \bmod M$ (see details in Section 1.2.2). So it is then enough to prove that, for some large enough value of m, a single polynomial interpolates all values from then on, because then the g_i's are forced to coincide infinitely many times, which proves that they are the same polynomial.

We use the algebraic meaning of $f(m)$ as the multiplicity of the weight $n\beta$ in the irreducible representation $V_{n\lambda}$ of $\mathfrak{gl}_n\mathbb{C}$. The well-known Kostant's multiplicity formula (see page 421 of [**23**]) gives that

$$f(m) = \sum_{\sigma \in S_n} (-1)^{\varepsilon(\sigma)} K(\sigma(m\lambda + \delta) - m\beta - \delta), \tag{1.4}$$

where K is Kostant's partition function for the root system A_n, $\varepsilon(\sigma)$ denotes the number of inversions of σ, and δ is one-half of the sum of the positive roots in A_n.

Kostant's partition function is a vector partition function in the sense of [**67**]. More precisely, $K(b)$ is equal to the number of nonnegative integral solutions x of a linear system $Ax = b$. The columns of A in this case are exactly the positive roots of the system A_n. Because the matrix A is unimodular [**60**], $K(b)$ is a multivariate piecewise polynomial function of the coordinates of b. The regions where K is a polynomial are convex polyhedral cones called *chambers* [**67**]. The chamber that contains b determines the polynomial value of $K(b)$; in fact, it is the vector direction of b, not its norm, that determines the polynomial formula to be used.

In equation (1.4) the right-hand-side vector for Kostant's partition function is $b = \sigma(m\lambda + \delta) - (m\beta + \delta)$. Initially, as m grows, we might be moving from one chamber to another. Our claim is that, from some value of m on, the vectors $\sigma(m\lambda+\delta)-(m\beta+\delta)$ are all contained in the same chamber. To see this, note that, in equation (1.4), β, λ, and δ are constant vectors. For a given permutation σ, the direction of the vector $\sigma(m\lambda + \delta)$ approaches that of $\sigma(\lambda)$ as m grows. Similarly, the direction of the vector $m\beta + \delta$ approaches that of β as m grows. Thus, the

$$\begin{array}{ccccccccc}
6 & & 3 & & 2 & & 2 & & 0 \\
& 4 & & 2 & & 2 & & 0 & \\
& & 4 & & 2 & & 1 & & \\
& & & 3 & & 1 & & & \\
& & & & 3 & & & &
\end{array}$$

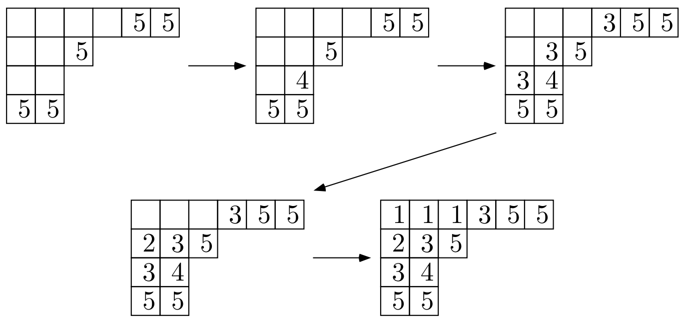

FIGURE 1.3. A bijection mapping $GT_{\lambda\beta} \cap Z^{(n(n+1)/2)} \to SSYT(\lambda, \beta)$

direction of $b = \sigma(m\lambda + \delta) - (m\beta + \delta)$ approaches the direction of $b' = \sigma(\lambda) + \beta$ along a straight line. For sufficiently large m, the vectors b and b' are contained in the same chamber, where a single polynomial gives the value of $K(b)$.

We have shown that, for all sufficiently large m, equation (1.4) represents an alternating sum of polynomials in the variable m. Therefore, $f(m)$ is a polynomial, exactly as we wished to prove. $\square$

Billey, Guillemin, and Rassart have shown that, more strongly, $K_{\lambda\beta}$ is a piecewise multivariate polynomial in λ and β [**9**]. It is natural to ask whether the above polynomial properties of the Kostka numbers extend to the Littlewood–Richardson coefficients $c^{\nu}_{\lambda\mu}$. Indeed, Derksen and Weyman established that the one-parameter dilations of these numbers (*i.e.*, $c^{m\nu}_{m\lambda,m\mu}$ with λ, μ, ν fixed) are again univariate polynomials in m [**18**]. Rassart has now extended the piecewise multivariate polynomiality of Kostka numbers to Littlewood-Richardson coefficients [**59**].

1.5. Summary of results

In Chapter 2, we combine the lattice point enumeration algorithm of Barvinok [**3**] with the results of Berenstein and Zelevinsky [**8**] on the polyhedral realization of Clebsch–Gordan coefficients to produce a new algorithm for computing these coefficients. The results in this chapter have been published in a joint paper with J. De Loera [**17**]. Our main theoretical result in this chapter is the following.

THEOREM 1.11 (Proved on p. 20). *Given a fixed finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$, one can compute a Clebsch–Gordan coefficients $C^{\nu}_{\lambda\mu}$ of $\mathfrak{g}$ in time polynomial in the input size of the defining weights.*

Moreover, in the type-A case, deciding whether the Littlewood–Richardson coefficient $c^{\nu}_{\lambda\mu}$ is nonzero can be done in polynomial time even when the rank is not fixed.

We have implemented this algorithm for types A_r, B_r, C_r, and D_r (the so-called "classical" Lie algebras) using the software packages `LattE` [**14**] and `Maple 9` [**50**]. We present computations demonstrating that, in many instances, our implementation performs faster than standard methods, such as those implemented in the software LiE [**69**]. In particular, our software permits the computation of Clebsch–Gordan coefficients parameterized by long highest weights, which had not previously been possible. Our software is also freely available at `http://math.ucdavis.edu/~deloera`.

In Chapter 3, we put the polyhedral algorithm discussed in Chapter 2 to work. The results in this chapter also appeared in [**17**]. Using our software, we explore general properties satisfied by the Clebsch–Gordan coefficients for the classical Lie algebras under the operation of *stretching of multiplicities* in the sense of [**35**]. We prove the following proposition motivated by our computational experiments using the polyhedral algorithm from Theorem 1.11.

Proposition 1.12 (Proved on p. 32). *The minimum quasi-period of a stretched Clebsch–Gordan coefficient for a classical Lie algebra is at most 2.*

Using our implementation of the polyhedral algorithm, we provide abundant experimental evidence for two conjectured generalizations of the saturation property of Littlewood–Richardson coefficients. The first of our conjectures states that there is a unimodular triangulation of the cone generated by triples of dominant weights corresponding to nonempty hive polytopes (Conjecture 3.5). The second of our conjectures seems to be valid for all of the classical root systems: the stretched Clebsch–Gordan coefficient $C_{n\lambda,n\mu}^{n\nu}$ is a quasi-polynomial with only nonnegative coefficients (Conjecture 3.8).

In Chapter 4, in pursuit of a proof of these conjectures, we turn to a theoretical study of stretched Clebsch–Gordan coefficients in the special case of stretched Kostka coefficients for type-A Lie algebras. We approach this problem via the geometry and combinatorics of GT-polytopes, which encode the Kostka coefficients of $\mathfrak{gl}_n(\mathbb{C})$. The results in this chapter have been published in collaboration with J. De Loera in [**16**]. We present a combinatorial structure (Definition 4.1) on GT-patterns, which constitute the polyhedral cone within which GT-polytopes exist. This combinatorial structure, which we call a *tiling*, encodes both the combinatorics of the polytope and the geometry of its embedding with respect to the integer lattice. Each tiling $\mathscr{T}$ has associated to it a certain matrix $A_{\mathscr{T}}$. The motivation for introducing tilings, and one of the main results of this chapter, is the following.

Theorem 1.13 (Proved on p. 37). *Suppose that $\mathscr{T}$ is the tiling of a GT-pattern $\mathbf{x}$. Then the dimension of the kernel of $A_{\mathscr{T}}$ is equal to the dimension of the minimal (dimensional) face of the GT-polytope containing $\mathbf{x}$.*

As a corollary to this result, we get an easy-to-check criterion for a GT-pattern being a vertex of the GT-polytope containing it (Corollary 4.5). In addition, we use the machinery of tilings to describe the precise conditions under which a denominator $q > 1$ appears in a vertex of a GT-polytope. Combining these results, we present a negative solution to the conjecture by Berenstein and Kirillov [**39**, Conjecture 2.1] that GT-polytopes are always integral.

Theorem 1.14 (Proved on p. 41). *The Berenstein–Kirillov conjecture is true for $n \leq 4$. However, counterexamples to this conjecture exist for all $n \geq 5$. More* *strongly, by choosing n sufficiently large, we can find GT-polytopes in which the denominators of the vertices are arbitrarily large:*

In particular, for any positive integer k, let $\lambda = (k^k, k-1, 0^k)$ and $\beta = ((k-1)^{k+1}, 1^k)$. Then one of the vertices of $GT_{\lambda\beta} \subset X_{2k+1}$ contains entries with denominator k.

We repeat that it is quite natural to conjecture integrality of the vertices of GT-polytopes, because the Ehrhart counting functions are known to be polynomials when the vertices of a polytope are integral. Indeed, in our disproof of the Berenstein–Kirillov conjecture, we are presenting the first known infinite family of non-integral polyhedra whose Ehrhart counting functions are still polynomials. Other low-dimensional families have been found recently by the author in collaboration with K. Woods [**51**] (see Appendix A). Finally, we must remark that R. P. Stanley communicated to us that a student of his, Peter Clifford, noticed non-integrality for GT-polytopes (unpublished). King et al. had also independently noticed non-integrality for K-hive polytopes with $n = 5$ (which are isomorphic to Gelfand-Tsetlin polytopes under a lattice-preserving linear map) and integrality of vertices for $n \leq 4$ [**35**]. However, our work appears to be the first demonstrating that GT-polytopes may take on arbitrarily large denominators in the coordinates of their vertices.

Having applied the tiling machinery to understanding the embedding of GT-polytopes with respect to the integer lattice, we conclude Chapter 4 with a discussion of the combinatorics of tilings and their connection to the face lattices of GT-polytopes. We give a characterization of the tilings of GT-patterns considered as partitions of $\mathcal{I}_n$, and we prove that tilings naturally form posets that are isomorphic to the face lattices of GT-polytopes. Hence, tilings also determine the combinatorics of GT-polytopes.

In Chapter 5, continuing our application of the Gelfand–Tsetlin tilings, we study the stretched Kostka coefficient $\mathcal{K}_{\lambda\beta}$, which is the map $n \mapsto K_{n\lambda, n\beta}$ sending each positive integer n to the Kostka coefficient indexed by $n\lambda$ and $n\beta$. Kirillov and Reshetikhin [**40**] have shown that stretched Kostka coefficients are polynomial functions of n. King, Tollu, and Toumazet have conjectured that these polynomials always have nonnegative coefficients [**35**] (a special case of our Conjecture 3.8), and they have given a conjectural expression for their degrees [**36**]. We prove the values conjectured by King, Tollu, and Toumazet for the degrees of stretched Kostka coefficients.

THEOREM 1.15 (Proved on p. 54). *If $\lambda = (\kappa_1^{v_1}, \dots, \kappa_m^{v_m}) \in \mathbb{Z}^r_{\geq 0}$ is a partition with $m \geq 2$ and $\beta \lhd \lambda$, then the degree of the stretched Kostka coefficient $\mathcal{K}_{\lambda\beta}(n)$ is given by*

$$\deg \mathcal{K}_{\lambda\beta}(n) = \binom{r-1}{2} - \sum_{p=1}^{m} \binom{v_p}{2} \tag{1.5}$$

(where we evaluate $\binom{1}{2} = 0$).

As stated, this theorem gives the degree of a stretched Kostka coefficient only when $\lambda \lhd \beta$—that is, when λ *strictly* dominates β. In the language of Chapter 5, this means that λ and β must form a primitive pair. However, Berenstein and Zelevinsky have shown that all Kostka coefficients factor into a product of Kostka coefficients indexed by primitive pairs [**7**]. It follows from this factorization that

Theorem 1.15 suffices to describe the degrees of stretched Kostka coefficients in all cases.

In Appendix A, we address a peculiar phenomenon exhibited by GT-polytopes and, more generally, hive polytopes. The members of these families of polytopes all have polynomial Ehrhart counting functions. This property motivated Berenstein and Kirillov to conjecture that GT-polytopes are integral. That they are not integral makes it seem a surprising coincidence that their Ehrhart counting functions are nonetheless polynomials. Appendix A discusses this phenomenon of *quasi-period collapse* in the context of dimension 2. We exhibit a family of triangles in which the denominators take on arbitrarily large values while the Ehrhart counting function remains a polynomial (Theorem A.2). Motivated by the proof of this result, we make a conjecture regarding precisely when quasi-period collapse occurs (Conjecture A.4), and we conjecture when two polytopes have the same Ehrhart quasi-polynomial (Conjecture A.6). Some of the material in this Appendix, including the example used in Theorem A.2, has previously been published in collaboration with K. Woods [**51**].

CHAPTER 2

Computing Clebsch–Gordan coefficients with Berenstein–Zelevinsky polytopes

2.1. Clebsch–Gordan coefficients: Polyhedral algorithms

As stated in the introduction, we are interested in the problem of efficiently computing $C_{\lambda\mu}^{\nu}$ in the tensor product expansion $V_\lambda \otimes V_\mu = \bigoplus_\nu C_{\lambda\mu}^{\nu} V_\nu$. It appears that the most common method used to compute Clebsch–Gordan coefficients is based on Klimyk's formula (see Lemma 2.1 below). For example, it is used in LiE [**69**], the `Maple` packages `coxeter/weyl` [**65, 66**], and in the computer algebra system `Magma` [**13**].

LEMMA 2.1. [**27**, Exercise 24.9] *Fix a complex semisimple Lie algebra* $\mathfrak{g}$*, and let* $\mathfrak{W}$ *be the associated Weyl group. For each weight* β *of* $\mathfrak{g}$*, let* $\operatorname{sgn}(\beta)$ *denote the parity of the minimum length of an element* $w \in \mathfrak{W}$ *such that* $w(\beta)$ *is a highest weight, and let* $\{\beta\}$ *denote that highest weight. Let* δ *be one-half the sum of the positive simple roots of* $\mathfrak{g}$*. Finally, for each highest weight* λ *of* $\mathfrak{g}$*, let* $K_{\lambda\beta}$ *be the multiplicity of* β *in* V_λ*.*

Then, given highest weights λ *and* μ *of* $\mathfrak{g}$*, we have that*

$$V_\lambda \otimes V_\mu = \bigoplus_{\varepsilon} K_{\lambda\varepsilon} \operatorname{sgn}(\varepsilon + \mu + \delta) V_{\{\varepsilon+\mu+\delta\}-\delta},$$

where the sum is over weights ε *of* $\mathfrak{g}$ *with trivial stabilizer subgroup in* $\mathfrak{W}$*.*

Implementations of Klimyk's formula begin by computing the weight spaces appearing with nonzero multiplicity in the representation V_λ. Then, for each such weight ε with trivial stabilizer, one computes the Weyl group orbit of $\varepsilon+\mu+\delta$. One then finds the dominant member of the orbit and notes the number ℓ of reflections needed to reach it. Finally, one adds $(-1)^\ell K_{\lambda\varepsilon}$ to the multiplicity of $V_{\{\varepsilon+\mu+\delta\}-\delta}$.

Observe that as we perform the algorithm just described, we compute the coefficient of each V_ν in $V_\lambda \otimes V_\mu$ "in parallel". In other words, we do not know the value of any particular Clebsch–Gordan coefficient until we have carried out the entire computation and produced the complete decomposition $V_\lambda \otimes V_\mu = \bigoplus_\nu C_{\lambda\mu}^{\nu} V_\nu$. Since the number of terms in this decomposition grows exponentially as the sizes of λ and μ grow, these sizes need to be small in practice. This is the main disadvantage of Klimyk's formula from the point of view of computational complexity. One can then ask for an algorithm that behaves well as the sizes of the input weights increase, at least if some other parameter is fixed.

2.1.1. Computational Consequences. The specific properties of the BZ-polytopes that we need to prove our theorem are (1) for fixed rank r, the dimensions of the BZ-polytopes are bounded above by a constant, (2) for fixed rank, the input size of a BZ-polytope grows linearly with the input sizes of the weights λ, μ, and ν, and (3) the following result describing the relationship between BZ-polytopes and Clebsch–Gordan coefficients:

LEMMA 2.2. [**8**, Theorems 2.3 and 2.4] *Fix a finite-dimensional complex semisimple Lie algebra* $\mathfrak{g}$ *and a triple of highest weights* (λ, μ, ν) *for* $\mathfrak{g}$. *Then the Clebsch–Gordan coefficient* $C^{\nu}_{\lambda\mu}$ *equals the number of integer lattice points in the corresponding BZ-polytope.*

The final necessary ingredient is Barvinok's algorithm for counting lattice points in polytopes in polynomial time for fixed dimension. Several detailed descriptions of the algorithm in Lemma 2.3 are now available in the literature (see, for example, [**15**] and references therein).

LEMMA 2.3. [**3**] *Fix* $d \in \mathbb{Z}_{\geq 0}$. *Then, given a system of equalities and inequalities defining a rational convex polytope* $P \subset \mathbb{R}^d$, *we can compute* $\#(P \cap Z^d)$ *in time polynomial in the input size of the polytope.*

Having stated these prior results, we are now ready to prove the following theorem.

THEOREM 2.4. *Given a fixed finite-dimensional complex semisimple Lie algebra* $\mathfrak{g}$, *one can compute a Clebsch–Gordan coefficients* $C^{\nu}_{\lambda\mu}$ *of* $\mathfrak{g}$ *in time polynomial in the input size of the defining weights.*

Moreover, in the type-A case, deciding whether $c^{\nu}_{\lambda\mu} \neq 0$ *can be done in polynomial time even when the rank is not fixed.*

PROOF. First, if we fix the rank of the Lie algebra, then we fix an upper bound on the dimension of the hive or BZ polytope. Moreover, the input sizes of these polytopes grow linearly with the input sizes of the weights. Thus, by Barvinok's theorem (Lemma 2.3 stated above), their lattice points can be computed in time polynomial in the input sizes of the weights. Therefore, the first part of the theorem follows by Lemma 2.2.

For the second part of the theorem, regarding type A, the hive polytopes provide a very fast method for determining whether $c^{\nu}_{\lambda\mu} \neq 0$. According to the saturation theorem (see Chapter 3), $c^{\nu}_{\lambda\mu} \neq 0$ if and only if the corresponding hive polytope is nonempty. Hence, it suffices to check whether the system of inequalities defining the hive polytope is (real) feasible, which can be done in polynomial time for *arbitrary* dimension as a corollary of the polynomiality of linear programming via Khachiyan's ellipsoid algorithm (see [**60**]). □

An analog of Theorem 2.4 also applies to Kostka numbers $K_{\lambda\beta}$, which are another significant family of invariants in the representation theory of type-A Lie algebras.

PROPOSITION 2.5. *For fixed rank, the Kostka number* $K_{\lambda\beta}$ *can be computed in polynomial time in the size of the highest weight* λ *and the weight* β. *For arbitrary rank, one can decide in polynomial time whether* $K_{\lambda\beta} \neq 0$.

The polynomiality of computing Kostka numbers for fixed rank follows because these numbers can be expressed as the number of lattice points in Gelfand–Tsetlin polytopes [**24**]. The polynomiality of determining whether $K_{\lambda\beta} \neq 0$ for arbitrary rank follows from the well-known criterion that $K_{\lambda\beta} \neq 0$ if and only if λ dominates β, which may be checked in polynomial time.

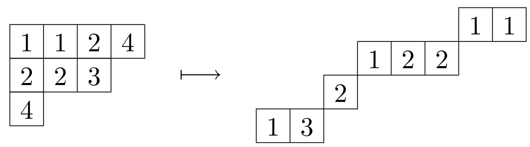

FIGURE 2.1. Corresponding semi-standard Young and Littlewood–Richardson tableaux

It is also worth noticing that Proposition 2.5 follows directly from Theorem 2.4. This is because each Kostka number $K_{\lambda\beta}$ is a Littlewood–Richardson coefficient for some choice of highest weights. For example, if $\lambda, \beta \in \mathbb{Z}^r$, then $K_{\lambda\beta} = c^{\nu}_{\mu\lambda}$, where

$$\begin{cases} \nu_i = \beta_i + \beta_{i+1} + \cdots \\ \mu_i = \beta_{i+1} + \beta_{i+2} + \cdots \end{cases} \qquad \text{for } i = 1, 2, \ldots, r.$$

For those familiar with the enumeration of semi-standard Young tableaux and Littlewood–Richardson tableaux by Kostka numbers and Littlewood–Richardson coefficients respectively (see, *e.g.*, [**64**]), the bijection establishing this relation is straightforward: Given a semi-standard Young tableau Y with shape λ and content β, construct a Littlewood–Richardson tableau L with shape ν/μ and content λ by filling the boxes as follows. Start with a skew Young diagram D with shape ν/μ. For $1 \le i, j \le r$, place a number of j's in the ith row of D equal to the number of i's in the jth row of Y, ordering the entries in each row so that they are weakly increasing. Let L be the tableau produced by filling the boxes of D in this fashion. (See Figure 2.1 for an example.)

It is not hard to see that, under this map, the column-strictness condition on Y is equivalent to the lattice permutation condition on L. It follows that the map just described is a bijection between semi-standard Young tableaux with shape λ and content β and Littlewood–Richardson tableaux with shape ν/μ and content λ. Thus, computing Kostka numbers reduces to computing Littlewood–Richardson coefficients.

2.2. Using the algorithm in practice

Definitions 1.3, 1.5, and 1.6 give the hive polytopes and the BZ-polytopes as the sets of solutions to explicit systems of linear inequalities and equalities. Using these definitions, we wrote `Maple` notebooks which, when given a triple of highest weights for one of the classical root systems, produce the corresponding hive or BZ-polytope in a `LattE`-readable format. These notebooks may be downloaded from `http://math.ucdavis.edu/~deloera`.

We compared the running time necessary to compute Clebsch–Gordan coefficients with `LattE` to the time required by LiE. All of our computations were performed on a Linux PC with a 2 GHZ CPU and 4 Gigabytes of memory. From our experiments, we conclude that (1) The polyhedral method of computing tensor product multiplicities complements the method employed in LiE. LiE is effective for slightly larger ranks (up to $r = 10$, say), but the sizes of the weights must be kept small. This is because LiE uses the Klimyk formula to generate the entire direct sum decomposition of the tensor product, after which it dispenses with all but the single desired term. However, computing all of the terms in the direct sum decomposition is not feasible when the sizes of the entries in the weights grow into the 100s. On the other hand, (2) lattice point enumeration is often effective for very large weights (in particular, the algorithm is suitable for investigating the stretching properties of Chapter 3). However, the rank must be relatively low (roughly $r < 6$) because lattice point enumeration complexity grows exponentially in the dimension of the polytope, and the dimensions of these polytopes grow quadratically with the rank of the Lie algebra. Together, the two algorithms cover a larger range of problems.

We would also like to mention that Charles Cochet [**11**] also uses lattice points in polytopes to compute Clebsch–Gordan coefficients. Using the Steinberg formula (see equation (3.3)), together with techniques developed in [**1**], he has written software that, like ours, can compute with large sizes of weight entries. Indeed, in the comparison of running times reported in [**11**], his software seems to compute Clebsch–Gordan coefficients approximately five to ten times more quickly than ours. It would be interesting to determine whether these computation times differ by a constant factor in general and whether this factor is due to the theoretical complexity of the computations or to implementation issues.

Applying the Steinberg formula consists of computing an alternating sum of vector partition functions over $\mathfrak{W} \times \mathfrak{W}$, the Cartesian square of the Weyl group. Since this is a fixed set for fixed rank, and since evaluating each vector partition function in the sum amounts to enumerating the lattice points in a polytope, Cochet's techniques also yield a polynomial time algorithm for computing Clebsch–Gordan coefficients in fixed rank. However, because applying the Steinberg formula involves computing an alternating sum, the techniques in [**11**] cannot be used to yield the second theoretical result in Theorem 2.4.

2.2.1. Experiments for type A_r. In the tables below, we index Littlewood–Richardson coefficients for type A_r with triples of partitions with $r + 1$ parts. Experiments indicate that lattice point enumeration is very efficient for computing Littlewood–Richardson numbers when $r \leq 4$. First, we computed over 30 instances with randomly generated weights with leading entries larger than 40 with our approach and with LiE. In all cases our algorithm was faster. After that, we did a "worst case" sampling for Table 2.1 comparing the computation times of `LattE` and LiE. To produce the ith row of that table, we selected uniformly at random 1000 triples of weights (λ, μ, ν) in which the largest parts of λ and μ were bounded above by $10i$ and $|\nu| = |\lambda| + |\mu|$ (this is a necessary condition for $c_{\lambda\mu}^{\nu} \neq 0$). Then we evaluated the corresponding hive polytopes with `LattE`. The `LattE` input files are created with our `Maple` program. The weight triple in the ith row is the one that `LattE` took the longest time to compute. We then computed the same tensor product multiplicity with LiE. Table 2.2 shows the running time needed when using `LattE` to compute weight triples with entries in the thousands or millions.

When $r \geq 5$, the running time under `LattE` begins to blow up. Still, for $r = 5$, all examples we attempted could be computed in under 30 minutes using `LattE`, and most could be computed in under 5 minutes. For example, among 54 nonempty hive polytopes chosen uniformly at random among those in which the weights had entries less than 100, all but seven could be computed in under 5 minutes with `LattE`, and the remaining seven could all be computed in under 30 minutes. None of these Littlewood–Richardson coefficients could be computed with LiE. At $r = 6$, lattice point enumeration becomes less effective, with examples typically taking several hours or more to evaluate.

λ, μ, ν	$c^{\nu}_{\lambda\mu}$	`LattE` runtime	`LiE` runtime
(9,7,3,0,0), (9,9,3,2,0), (10,9,9,8,6)	2	0m00.74s	0m00.01s
(18,11,9,4,2), (20,17,9,4,0), (26,25,19,16,8)	453	0m03.86s	0m00.12s
(30,24,17,10,2), (27,23,13,8,2), (47,36,33,29,11)	5231	0m05.21s	0m02.71s
(38,27,14,4,2), (35,26,16,11,2), (58,49,29,26,13)	16784	0m06.33s	0m25.31s
(47,44,25,12,10),(40,34,25,15,8),(77,68,55,31,29)	5449	0m04.35s	1m55.83s
(60,35,19,12,10),(60,54,27,25,3),(96,83,61,42,23)	13637	0m04.32s	23m32.10s
(64,30,27,17,9), (55,48,32,12,4), (84,75,66,49,24)	49307	0m04.63s	45m52.61s
(73,58,41,21,4), (77,61,46,27,1), (124,117,71,52,45)	557744	0m07.02s	$>$ 24 hours

TABLE 2.1. A "worst case" comparison of running times between `LattE` and `LiE`for A_4

λ, μ, ν	$c^{\nu}_{\lambda\mu}$	`LattE` runtime
(935,639,283,75,48) (921,683,386,136,21) (1529,1142,743,488,225)	1303088213330	0m07.84s
(6797,5843,4136,2770,707) (6071,5175,4035,1169,135) (10527,9398,8040,5803,3070)	459072901240524338	0m09.63s
(859647,444276,283294,33686,24714) (482907,437967,280801,79229,26997) (1120207,699019,624861,351784,157647)	11711220003870071391294871475	0m08.15s

TABLE 2.2. Computing large weights with `LattE` for case A_4

2.2.2. Experiments for types B_r, C_r, and D_r. To compute Clebsch–Gordan coefficients in types B_r, C_r, and D_r, we used the BZ-polytopes. In the tables that follow, all weights are given in the basis of fundamental weights for the corresponding Lie algebra.

Our experiments followed the same process we used for A_r: First, for each root system, we computed over 30 instances with randomly generated weights with entries larger than 40 with our approach and with LiE. In all cases our algorithm was faster. After that, we did a "worst case" sampling to produce Table 2.3 comparing the computation times of LattE and LiE. As in Section 2.2.1, these weight triples were the ones which LattE took the longest to evaluate among thousands of instances generated with the following procedure: First, to produce line i of a table, we set an upper bound U_i for the entries of each weight. Then, we generated 1000 random weight triples with entry sizes no larger than U_i. Here are the specific values of U_i used in each of the three subtables in Table 2.3. For type B_r, the bounds U_i were 50, 60, 70, and 10,000, respectively. For type C_r, the bounds U_i were 50, 60, 80, and 10,000, respectively. Finally, for type D_r, the bounds U_i were 20, 30, 40, and 10,000, respectively. For each generated triple of weights, we produced the associated BZ-polytopes (using our Maple notebook) and counted their lattice points with LattE. Table 2.3 includes those instances that were slowest in LattE. We also recorded in the table the time taken by LiE for the same instances. One can see the running time needed by LattE is hardly affected by growth in the size of the input weights, while the time needed by LiE grows rapidly.

We found that for types B_r and C_r, lattice point enumeration with the BZ-polytopes is very effective when $r \leq 3$. Each of the many thousands of examples we generated could be evaluated by LattE in under 10 seconds (the examples in Table 2.3 were the worst cases). When $r = 4$, the computation time begins to blow up, with examples typically taking half an hour or more to compute. The polyhedral method is also reasonably efficient for type-D Lie algebras with rank 4, the lowest rank in which they are defined. All of the examples we generated could be evaluated by LattE in under 5 minutes.

	λ, μ, ν	$C^{\mu}_{\lambda\nu}$	`LattE` runtime	LiE runtime
B_3	(46,42,38), (38,36,42), (41,36,44)	354440672	0m09.58s	1m45.27s
	(46,42,41), (14,58,17), (50,54,38)	88429965	0m06.38s	3m16.01s
	(15,60,67), (58,70,52), (57,38,63)	626863031	0m07.14s	6m01.43s
	(5567,2146,6241), (6932,1819,8227), (3538,4733,3648)	215676881876569849679	0m7.07s	n/a
C_3	(25,42,22), (36,38,50), (31,33,48)	87348857	0m07.48s	0m17.21s
	(34,56,36), (44,51,49), (37,51,54)	606746767	0m08.42s	2m57.27s
	(39,64,58), (65,15,72), (70,41,44)	519379044	0m07.63s	8m00.35s
	(5046,5267,7266), (7091,3228,9528), (9655,7698,2728)	1578943284716032240384	0m07.66s	n/a
D_4	(13,20,10,14), (10,20,13,20), (5,11,15,18)	41336415	2m46.88s	0m12.29s
	(12,22,9,30),(28,14,15,26),(10,24,10,26)	322610723	3m04.31s	7m03.44s
	(37,16,31,29),(40,18,35,41),(36,27,19,37)	18538329184	4m29.63s	>60m
	(2883,8198,3874,5423),(1901,9609,889,4288), (5284,9031,2959,5527)	189129325670457435656514934 4	2m06.42s	n/a

TABLE 2.3. A "worst case" sample comparison of running times between `LattE` and LiE

λ, μ, ν	$C_{n\lambda, n\nu}^{n\mu}$	`LattE` runtime
(0, 15, 5) (12, 15, 3) (6, 15, 6)	$\begin{cases} \frac{68339}{64} n^5 + \frac{407513}{384} n^4 + \frac{13405}{32} n^3 + \frac{9499}{96} n^2 + \frac{107}{8} n + 1, n \text{ even} \\ \frac{68339}{64} n^5 + \frac{407513}{384} n^4 + \frac{13405}{32} n^3 + \frac{16355}{192} n^2 + \frac{659}{64} n + \frac{75}{128},\ n \text{ odd} \end{cases}$	7m08.57s
(4, 8, 11) (3, 15, 10) (10, 1, 3)	$\begin{cases} \frac{13}{4} n^2 + 3n + 1, n \text{ even} \\ \frac{13}{4} n^2 + 3n + 3/4, n \text{ odd} \end{cases}$	0m01.64s
(8, 1, 3) (11, 13, 3) (8, 6, 14)	$\begin{cases} \frac{121}{576} n^6 + \frac{1129}{640} n^5 + \frac{6809}{1152} n^4 + \frac{163}{16} n^3 + \frac{2771}{288} n^2 + \frac{191}{40} n + 1, n \text{ even} \\ \frac{121}{576} n^6 + \frac{1129}{640} n^5 + \frac{6809}{1152} n^4 + \frac{1933}{192} n^3 + \frac{659}{72} n^2 + \frac{8003}{1920} n + \frac{93}{128}, n \text{ odd} \end{cases}$	0m10.26s
(8, 9, 14) (8, 4, 5) (1, 5, 15)	$\begin{cases} \frac{4117}{192} n^6 + \frac{50369}{640} n^5 + \frac{14829}{128} n^4 + \frac{703}{8} n^3 + \frac{3541}{96} n^2 + \frac{341}{40} n + 1, n \text{ even} \\ \frac{4117}{192} n^6 + \frac{50369}{640} n^5 + \frac{14829}{128} n^4 + \frac{5599}{64} n^3 + \frac{3451}{96} n^2 + \frac{5001}{640} n + \frac{97}{128}, n \text{ odd} \end{cases}$	0m13.29s
(10, 5, 6) (5, 4, 10) (0, 7, 12)	$\begin{cases} \frac{669989}{960} n^5 + \frac{286355}{384} n^4 + \frac{10803}{32} n^3 + \frac{7993}{96} n^2 + \frac{1427}{120} n + 1, n \text{ even} \\ \frac{669989}{960} n^5 + \frac{286355}{384} n^4 + \frac{10803}{32} n^3 + \frac{15509}{192} n^2 + \frac{10081}{960} n + \frac{65}{128}, n \text{ odd} \end{cases}$	2m52.39s

TABLE 2.4. Ehrhart quasi-polynomials for BZ-polytopes of type B_3.

λ, μ, ν	$C_{n\lambda,n\nu}^{n\mu}$	`LattE` runtime
(1,13,6) (14,15,5) (5,11,7)	$\begin{cases} \frac{5937739}{5760}\,n^6 + \frac{87023}{40}\,n^5 + \frac{936097}{576}\,n^4 + \frac{27961}{48}\,n^3 + \frac{85397}{720}\,n^2 + \frac{883}{60}\,n + 1, n \text{ even} \\ \frac{5937739}{5760}\,n^6 + \frac{87023}{40}\,n^5 + \frac{936097}{576}\,n^4 + \frac{27961}{48}\,n^3 + \frac{657931}{5760}\,n^2 + \frac{3097}{240}\,n + 3/4, n \text{ odd} \end{cases}$	21m20.59s
(4,15,14) (12,12,10) (4,9,8)	$\begin{cases} \frac{22199219}{2880}\,n^6 + \frac{8154617}{960}\,n^5 + \frac{4500665}{1152}\,n^4 + \frac{31297}{32}\,n^3 + \frac{226903}{1440}\,n^2 + \frac{2021}{120}\,n + 1, n \text{ even} \\ \frac{22199219}{2880}\,n^6 + \frac{8154617}{960}\,n^5 + \frac{4500665}{1152}\,n^4 + \frac{31297}{32}\,n^3 + \frac{217363}{1440}\,n^2 + \frac{13513}{960}\,n + \frac{85}{128}, n \text{ odd} \end{cases}$	17m05.74s
(9,0,8) (8,12,9) (7,7,3)	$1/30\,n^5 + 3/8\,n^4 + \frac{19}{12}\,n^3 + \frac{25}{8}\,n^2 + \frac{173}{60}\,n + 1$	0m00.61s
(10,2,7) (8,10,1) (7,5,5)	$\begin{cases} \frac{596153}{1152}\,n^6 + \frac{53425}{48}\,n^5 + \frac{502621}{576}\,n^4 + \frac{5577}{16}\,n^3 + \frac{11941}{144}\,n^2 + \frac{149}{12}\,n + 1, n \text{ even} \\ \frac{596153}{1152}\,n^6 + \frac{53425}{48}\,n^5 + \frac{502621}{576}\,n^4 + \frac{5577}{16}\,n^3 + \frac{94097}{1152}\,n^2 + \frac{131}{12}\,n + \frac{23}{32}, n \text{ odd} \end{cases}$	19m24.55s
(10,10,15) (11,3,15) (10,7,15)	$\begin{cases} \frac{6084163}{320}\,n^6 + \frac{507527}{30}\,n^5 + \frac{1185853}{192}\,n^4 + \frac{59995}{48}\,n^3 + \frac{43039}{240}\,n^2 + \frac{357}{20}\,n + 1, n \text{ even} \\ \frac{6084163}{320}\,n^6 + \frac{507527}{30}\,n^5 + \frac{1185853}{192}\,n^4 + \frac{59995}{48}\,n^3 + \frac{144751}{960}\,n^2 + \frac{883}{80}\,n + \frac{25}{64}, n \text{ odd} \end{cases}$	16m05.08s

TABLE 2.5. Ehrhart quasi-polynomials for BZ-polytopes of type C_3.

λ, μ, ν	$C_{n\lambda,n\nu}^{n\mu}$	LattE runtime
(0, 2, 10, 5) (4, 11, 9, 11) (5, 8, 6, 9)	$\begin{cases} \frac{625007}{10080}\,n^7 + \frac{729157}{2880}\,n^6 + \frac{77197}{180}\,n^5 + \frac{449539}{1152}\,n^4 + \frac{298979}{1440}\,n^3 + \frac{95189}{1440}\,n^2 \\ \quad + \frac{10079}{840}\,n + 1, n \text{ even} \\ \frac{625007}{10080}\,n^7 + \frac{729157}{2880}\,n^6 + \frac{77197}{180}\,n^5 + \frac{449539}{1152}\,n^4 + \frac{298979}{1440}\,n^3 + \frac{95189}{1440}\,n^2 \\ \quad + \frac{10079}{840}\,n + \frac{127}{128}, n \text{ odd} \end{cases}$	20m24.79s
(2, 7, 12, 2) (11, 10, 5, 9) (13, 11, 1, 1)	$\begin{cases} \frac{34675903}{80640}\,n^8 + \frac{3037051}{1680}\,n^7 + \frac{9121453}{2880}\,n^6 + \frac{241181}{80}\,n^5 + \frac{615083}{360}\,n^4 + \frac{8947}{15}\,n^3 \\ \quad + \frac{107791}{840}\,n^2 + \frac{6721}{420}\,n + 1, n \text{ even} \\ \frac{34675903}{80640}\,n^8 + \frac{3037051}{1680}\,n^7 + \frac{9121453}{2880}\,n^6 + \frac{241181}{80}\,n^5 + \frac{615083}{360}\,n^4 + \frac{8947}{15}\,n^3 \\ \quad + \frac{107791}{840}\,n^2 + \frac{6721}{420}\,n + \frac{239}{256}, n \text{ odd} \end{cases}$	123m59.76s
(3, 11, 0, 10) (2, 15, 10, 15) (10, 12, 11, 0)	$\begin{cases} \frac{53609}{60}\,n^6 + \frac{25631}{15}\,n^5 + \frac{63779}{48}\,n^4 + \frac{1627}{3}\,n^3 \\ \quad + \frac{2497}{20}\,n^2 + \frac{239}{15}\,n + 1, n \text{ even} \\ \frac{53609}{60}\,n^6 + \frac{25631}{15}\,n^5 + \frac{63779}{48}\,n^4 + \frac{1627}{3}\,n^3 \\ \quad + \frac{2497}{20}\,n^2 + \frac{239}{15}\,n + \frac{15}{16}, n \text{ odd} \end{cases}$	2m37.73s
(10, 1, 12, 4) (1, 12, 0, 3) (0, 5, 3, 4)	$5\,n^2 + 4\,n + 1$	0m01.63s
(12, 2, 5, 13) (15, 6, 10, 11) (2, 0, 12, 13)	$\begin{cases} \frac{455263}{2016}\,n^7 + \frac{447281}{576}\,n^6 + \frac{198433}{180}\,n^5 + \frac{971011}{1152}\,n^4 + \frac{108787}{288}\,n^3 \\ \quad + \frac{28969}{288}\,n^2 + \frac{12631}{840}\,n + 1, n \text{ even} \\ \frac{455263}{2016}\,n^7 + \frac{447281}{576}\,n^6 + \frac{198433}{180}\,n^5 + \frac{971011}{1152}\,n^4 + \frac{108787}{288}\,n^3 \\ \quad + \frac{28969}{288}\,n^2 + \frac{12631}{840}\,n + \frac{127}{128}, n \text{ odd} \end{cases}$	4m25.90s

TABLE 2.6. Ehrhart quasi-polynomials for BZ-polytopes of type D_3.

CHAPTER 3

Conjectures generalizing the saturation theorem

In 1999, Knutson and Tao used the hive polytopes to prove the saturation theorem.

THEOREM 3.1 (Saturation [**41**]). *Given highest weights λ, μ, and ν for $\mathfrak{sl}_r(\mathbb{C})$, and given an integer $n > 0$, the Littlewood–Richardson coefficient $c^{\nu}_{\lambda\mu}$ satisfies*

$$c^{\nu}_{\lambda\mu} \neq 0 \quad \Longleftrightarrow \quad c^{n\nu}_{n\lambda,n\mu} \neq 0.$$

Buch [**10**] has written a very readable exposition of Knutson and Tao's proof of this theorem. Several conjectured generalizations of Knutson and Tao's result have appeared in the literature since then. For example, Littlewood–Richardson coefficients can be expressed as coefficients of certain polynomials arising in the study of parabolic Kostka polynomials. Kirillov conjectures an extension of the saturation theorem to other coefficients of these polynomials [**38**]. Another conjecture, by Kapovich and Millson, concerns the degree to which the other classical Lie algebras fail to be saturated [**30**]. This conjecture implies saturation in the type-A case.

In this chapter, we state two additional conjectures that, if true, generalize Theorem 3.1. Our conjectures concern the polyhedral geometry arising in the interpretation provided by Berenstein and Zelevinsky (see Section 1.3). First, we translate Theorem 3.1 into the language of hive polytopes, where it may be restated as

$$\#\left(H^{\nu}_{\lambda\mu} \cap \mathbb{Z}^d\right) \neq \varnothing \quad \Longleftrightarrow \quad \#\left(H^{n\nu}_{n\lambda,n\mu} \cap \mathbb{Z}^d\right) \neq \varnothing,$$

where $d = \binom{r+2}{2}$. The definition of hive polytopes (see Definition 1.3 above) implies that $H^{n\nu}_{n\lambda,n\mu} = n\, H^{\nu}_{\lambda\mu}$. Since $n\, H^{\nu}_{\lambda\mu}$ clearly contains an integer lattice point for some n (*e.g.*, the least common multiple of the denominators appearing in its vertices), it follows that the saturation theorem is equivalent to the statement that every nonempty hive polytope contains an integral lattice point.

3.1. First conjecture

To show that every hive polytope contains an integral point, Knutson and Tao actually proved that every hive polytope contains an integral *vertex*. Our idea is to take a different approach to show a generalization of this last result using the theory of triangulations of semigroups. To develop this idea, observe that the boundary equalities and rhombus inequalities that define a hive polytope may be expressed as the set of solutions to a system of matrix equalities and inequalities:

$$H^{\nu}_{\lambda\mu} = \left\{ h \in \mathbb{R}^{(r+1)(r+2)/2} \;:\; \begin{matrix} Bh = b(\lambda,\mu,\nu), \\ Rh \leq 0 \end{matrix} \right\}, \tag{3.1}$$

where B and R are integral matrices (depending on r), and $b(\lambda,\mu,\nu)$ is an integral vector depending linearly on λ, μ, and ν. Here we think of a hive pattern h as a column vector of dimension $(r+1)(r+2)/2$. There is some degree of choice in how the boundary equalities and rhombus inequalities are encoded as matrices B and R, respectively, but all such encodings are equivalent for our purposes.

Recall that in our definition of hive polytopes (Definition 1.2), we required that $\lambda, \mu, \nu \geq 0$. This has the convenient consequence that the hive polytope $H^{\nu}_{\lambda\mu}$ is contained in the nonnegative orthant. Such a polytope defined by a system of equalities and inequalities may be *homogenized* by adding "slack variables" to produce an equivalent polytope defined as the set of nonnegative solutions to a system of linear equations. Following this procedure, we define the *homogenized hive polytope* $\tilde{H}^{\nu}_{\lambda\mu}$ by

$$\tilde{H}^{\nu}_{\lambda\mu} = \left\{ \tilde{h} \,:\, \begin{bmatrix} B & 0 \\ R & I \end{bmatrix} \tilde{h} = \begin{bmatrix} b(\lambda,\mu,\nu) \\ 0 \end{bmatrix},\ \tilde{h} \geq 0 \right\}$$

(where I is an identity matrix). The equivalence between $H^{\nu}_{\lambda\mu}$ and $\tilde{H}^{\nu}_{\lambda\mu}$ is given by the linear map

$$h \mapsto \begin{bmatrix} h \\ -Rh \end{bmatrix}.$$

Note that this linear map preserves vertices and integrality. Therefore, to prove the saturation theorem, it suffices to show that every homogenized hive polytope contains an integral vertex. Proceeding with this idea, we make the following definitions.

Definition 3.2. Fix $r \in \mathbb{Z}$. Define the *homogenized hive matrix* to be

$$M = \begin{bmatrix} B & 0 \\ R & I \end{bmatrix}$$

(where B and R are as in equation (3.1)). Given an integral vector b with dimension equal to the number of rows in M, define the *generalized hive polytope* or *g-hive polytope* H_b by

$$H_b = \left\{ \tilde{h} : M\tilde{h} = b,\ \tilde{h} \geq 0 \right\}. \tag{3.2}$$

Note that the homogenized hive polytopes are g-hive polytopes that satisfy certain additional conditions on the right-hand-side vector b, such as its final entries being 0.

We now state some very basic facts about vertices of polyhedra expressed in the form $\{x : Ax = b,\ x \geq 0\}$. Let a finite collection of integral points $\{a_1, \dots, a_d\} \subset \mathbb{Z}^m$ be given, and let A be the matrix with columns $a_1, \dots, a_d$. Define $\operatorname{cone} A$ to be the cone in $\mathbb{R}^m$ generated by the point-set $\{a_1, \dots, a_d\}$:

$$\operatorname{cone} A = \{x_1 a_1 + \cdots + x_d a_d \,:\, x_1, \dots, x_d \geq 0\}.$$

Then, for each vector $b \in \mathbb{Z}^m$, we have a polytope

$$P_b = \{x : Ax = b,\ x \geq 0\} \subset \mathbb{R}^d,$$

and $P_b \neq \varnothing$ if and only if $b \in \operatorname{cone} A$. In other words, there is a correspondence between nonempty polytopes P_b, $b \in \mathbb{Z}^m$, and the elements of the semigroup of integral lattice points contained in the cone generated by the columns of A. The crucial property for our purposes is the following.

Lemma 3.3. *If $b \in (\operatorname{cone} A') \cap \mathbb{Z}^m$ for some $m \times m$ submatrix A' of A with $\det A' = \pm 1$, then P_b has an integral vertex.*

PROOF. Suppose that $b \in (\operatorname{cone} A') \cap \mathbb{Z}^m$ for some $m \times m$ submatrix A' of A with $\det A' = \pm 1$. Let the columns of A' be $a_{i_1}, \ldots, a_{i_m}$, and let $J = \{i_1, \ldots, i_m\}$ be the indices of these columns. Then there is a vector $x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d_{\geq 0}$ such that $Ax = b$ and $x_i = 0$ for each $i \notin J$. Letting $x' = (x_{i_1}, \ldots, x_{i_m})^T$ and using Cramer's rule to solve for x' in $A'x' = b$, we find that x is an integral vector. Thus, x is an integral lattice point in the polytope P_b.

To see that x is in fact a vertex of P_b, recall that the codimension (with respect to the ambient space $\mathbb{R}^d$) of the face containing a solution to the system $Ax = b$, $x \geq 0$, of linear equalities and inequalities is the number of linearly independent equalities or inequalities satisfied with *equality*. Observe that x is a solution to the system of d equalities

$$\begin{cases} Ax = b \\ x_i = 0, \quad i \notin J. \end{cases}$$

We claim that this is a linearly independent system of equalities. For suppose otherwise. Then the zero row-vector is a nontrivial linear combination of the rows of A and the row-vectors $(0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the ith position, $i \notin J$. But this implies that the zero row-vector is a nontrivial linear combination of the rows of A', which is impossible because $\det A' \neq 0$.

Thus, having shown that x satisfies the d linearly independent equalities above, we have shown that x lies in a codimension-d face of P_b, *i.e.*, x is a vertex. $\square$

We say that $a_{i_1}, \ldots, a_{i_m}$ is a *unimodular* subset if the submatrix A' of A with columns $a_{i_1}, \ldots, a_{i_m}$ satisfies $\det A' = \pm 1$. We say that the matrix A has a *unimodular cover* (resp. *unimodular triangulation*) if the point set $\{a_1, \ldots, a_d\}$ has a unimodular cover (resp. unimodular triangulation).

COROLLARY 3.4. *If A has a unimodular cover, then P_b has an integral vertex for every integral $b \in \operatorname{cone}(A)$.*

Our conjecture is that this corollary applies to the homogenized hive matrix. More precisely, we conjecture the following.

CONJECTURE 3.5. *The homogenized hive matrix has a unimodular triangulation. Consequently, every g-hive polytope has an integral vertex.*

Since the hive polytopes are special cases of the g-hive polytopes, Conjecture 3.5 generalizes the saturation theorem.

THEOREM 3.6. *Conjecture (3.5) is true for $r \leq 6$.*

To compute the unimodular triangulations that provide a proof of Theorem 3.6 we used the software `topcom` [**58**]. It may be worth noting that the triangulations used to prove Theorem 3.6 were all placing triangulations.

3.2. Second conjecture

Our second conjecture is motivated by observed general properties satisfied by Clebsch–Gordan coefficients for semisimple Lie algebras of types B_r, C_r, and D_r under the operation of *stretching of multiplicities*. By *stretching of multiplicities*, we refer to the function $f\colon \mathbb{Z}_{>0} \to Z_{\geq 0}$ defined by $f(n) = C^{n\nu}_{n\lambda, n\mu}$.

The BZ-polytopes are defined (Definitions 1.5 and 1.6) as the set of solutions to a system of linear equalities and inequalities $Ax \leq b$, $Cx = d$, where b and d are linear functions of λ, μ, and ν, with rational coefficients. It follows that, given any highest weights λ, μ, and ν of a semisimple Lie algebra, $f(n)$ is a quasi-polynomial in n. Indeed, $f(n)$ is, in polyhedral language, the Ehrhart quasi-polynomial of the corresponding BZ-polytope (see Section 1.2.2).

If we put $P = H^{\nu}_{\lambda\mu}$, then the Ehrhart quasi-polynomial of P is just the stretched Littlewood–Richardson coefficient $c^{n\nu}_{n\lambda,n\mu}$. The Ehrhart quasi-polynomials of hive polytopes have been studied by several authors. Since lattice point enumeration can compute with large weights, it is possible to produce the Ehrhart quasi-polynomials for the stretched Clebsch–Gordan coefficients in the other types. See Tables 2.4–2.6 for some sample examples out of the many hundreds generated.

The reader will observe that each of the quasi-polynomials in Tables 2.4–2.6 have quasi-period 2. We now show that this property holds in general for each of the classical Lie algebras. Derksen and Weyman have already shown in [**18**] that the stretched Littlewood–Richardson coefficients are polynomials, so it remains only to consider the root systems B_r, C_r, and D_r.

Proposition 3.7. *The minimum quasi-period of a stretched Clebsch–Gordan coefficient for a classical Lie algebra is at most 2.*

Proof. Let $\mathfrak{g}$ be a classical Lie algebra of type B_r, C_r, or D_r. Since we already know that $C^{n\nu}_{n\lambda,n\mu}$ is a quasi-polynomial in n, it will suffice to show that, for all sufficiently large n, $C^{n\nu}_{n\lambda,n\mu}$ has quasi-period 2. Once this is established, we can interpolate to show that the claim holds for all values of n.

Let Λ be the weight lattice of $\mathfrak{g}$ and let $N_{\mathfrak{g}}(b)$ denote the number of ways to write a vector $b \in \Lambda$ as an integral linear combination of the positive roots of $\mathfrak{g}$. The function $N_{\mathfrak{g}}$ is a *vector partition function*, which means that its support is contained in a union of polyhedral cones in $\Lambda \otimes \mathbb{R}$, called *chambers*, such that the restriction of $N_{\mathfrak{g}}(b)$ to the lattice points in any chamber is a multivariate quasi-polynomial function of the coordinates of b [**67**].

The Clebsch–Gordan coefficients can be expressed in terms of the vector partition function $N_{\mathfrak{g}}$ with the Steinberg multiplicity formula (see, *e.g.*, [**23**]), according to which,

$$C^{n\nu}_{n\lambda,n\mu} = \sum_{(w,w')\in\mathfrak{W}\times\mathfrak{W}} (-1)^{\sigma(ww')} N_{\mathfrak{g}}\big(w(n\lambda+\rho) + w'(n\mu+\rho) - (n\nu+2\rho)\big), \tag{3.3}$$

where $\mathfrak{W}$ is the Weyl group of $\mathfrak{g}$ and $\sigma(w)$ is the sign of w in $\mathfrak{W}$.

To prove that $C^{n\nu}_{n\lambda,n\mu}$ is a quasi-period 2 quasi-polynomial of n, it will suffice to show that each term in the sum on the right-hand side of (3.3) is a quasi-period 2 quasi-polynomial of n. The key fact from which this will follow is Corollary 3.6 in [**1**], which states that, in each of its chambers, $N_{\mathfrak{g}}(b)$ is a multivariate quasi-polynomial function of b with quasi-period 2. Thus, we need only show that, for all sufficiently large n, the vectors

$$b_n = w(n\lambda+\rho) + w'(n\mu+\rho) - (n\nu+2\rho)$$

remain in a single chamber.

To see this, note that as n increases, the direction of b_n approaches that of $b' = w(\lambda) + w'(\mu) - \nu$ along a straight line. Hence, for all n sufficiently large, b_n and b' share a chamber, so that the value of $N_{\mathfrak{g}}(b_n)$ is given by a single quasi-polynomial function of the coordinates of b_n. Because we are looking at a particular term in the right-hand side of (3.3), w, w', λ, μ, and ν are all fixed, so $N_{\mathfrak{g}}(b_n)$

reduces to a quasi-polynomial of n. This reduction does not increase the quasi-period, so we have shown that the right-hand side of (3.3) is a sum of quasi-period 2 quasi-polynomials when n is sufficiently large. Thus it follows that $C_{n\lambda,n\mu}^{n\nu}$ is also a quasi-polynomial of quasi-period 2. $\square$

Our experiments also motivate the following "stretching conjecture," which generalizes the saturation theorem.

CONJECTURE 3.8 (Stretching Conjecture). *Given highest weights λ, μ, ν of a Lie algebra of type A_r, B_r, C_r, or D_r, let*

$$C_{n\lambda,n\mu}^{n\nu} = \begin{cases} f_1(n) & \textit{if } n \textit{ is odd,} \\ f_2(n) & \textit{if } n \textit{ is even} \end{cases}$$

be the quasi-polynomial representation of the stretched Clebsch–Gordan coefficient $C_{n\lambda,n\mu}^{n\nu}$. Then the coefficients of f_1 and f_2 are all nonnegative.

This conjecture was made by King, Tollu, and Toumazet in [**35**] for the type-A case of Littlewood–Richardson coefficients. That Conjecture 3.8 implies the saturation theorem follows from a result of Derksen and Weyman [**18**] showing that the Ehrhart quasi-polynomials of hive polytopes are in fact just polynomials.

This conjecture does not appear to be any easier to prove when we restrict our attention to stretched Kostka coefficients, which are special cases of stretched Littlewood–Richardson coefficients (see the remarks following Proposition 2.5.) This is remarkable because the saturation property

$$K_{n\lambda,n\beta} \neq 0 \Longleftrightarrow K_{\lambda\beta} \neq 0$$

for Kostka coefficients is comparatively very easy to prove, as has been observed, for example, in [**36**]. One way to see this is to recall the well-known fact that $K_{\lambda\beta} \neq 0$ if and only if $\beta \trianglelefteq \lambda$—that is, if and only if λ exceeds β in the dominance order (see, *e.g.*, [**23**, Exercise A.11]). Since the relative order of two weights is not changed by scaling both by n, we have that $K_{n\lambda,n\beta} \neq 0$ if and only if $K_{\lambda\beta} \neq 0$.

We should remark that the saturation property is known not to hold for Clebsch–Gordan coefficients in the root systems B_r, C_r, and it's status is unknown for D_r. A simple counterexample to saturation in B_2, due to Kapovich, Leeb, and Millson [**29**], is given by setting $\lambda = \mu = \nu = (1,0)$ (with respect to the basis of fundamental weights). In this case we have

$$C_{n\lambda,n\mu}^{n\nu} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

This example also demonstrates why the stretching conjecture is not contradicted by the failure of the saturation property in the root systems B_r or C_r. Since the stretched multiplicities are not necessarily polynomials in these cases, it is possible for them to evaluate to zero for some nonnegative integer while still having all nonnegative coefficients.

CHAPTER 4

Faces of Gelfand–Tsetlin polytopes

As discussed in the introduction, the geometry of Gelfand–Tsetlin polytopes, or GT-polytopes, determines the behavior of the Kostka coefficients $K_{\lambda\beta}$ as the parameters λ and β are varied. In this chapter, we introduce and study a class of combinatorial objects that encodes the structure of GT-polytopes. We will use these objects, which we call *Gelfand–Tsetlin tilings*, or simply *tilings*, to examine both the combinatorics of GT-polytopes (that is, the properties of their face posets), and the manner in which GT-polytopes are embedded with respect to the integer lattice. Both lines of inquiry will need to be pursued to illuminate the behavior of $K_{\lambda\beta}$. We will continue to apply tilings to study Kostka coefficients in Chapter 5.

The outline of this chapter is as follows. We define tilings of Gelfand–Tsetlin patterns in Section 4.1. We also define certain matrices, called tiling matrices, which are associated to these tilings. We prove that the kernel of the tiling matrix of a given Gelfand–Tsetlin pattern is the dimension of the minimal face of the GT-polytope containing that pattern. In Section 4.2 we study the special case of 0-dimensional faces, that is, the vertices. We conclude with section 4.3, in which we prove additional properties of tilings and establish their direct connection to the face lattices of GT-polytopes.

4.1. Tilings of GT-Patterns

4.1.1. Tilings Defined.

Definition 4.1. Given a GT-pattern $\mathbf{x} \in X_n$ (see Definition 1.7), the *tiling* $\mathscr{T}$ of $\mathbf{x}$ is the partition of the set of entries of $\mathbf{x}$ into subsets, called *tiles*, that results from grouping together those entries in $\mathbf{x}$ that are equal and adjacent. More precisely, $\mathscr{T}$ is that partition of $\mathcal{I}_n$ such that two pairs $(i,j), (\tilde{i},\tilde{j})$ are in the same tile if and only if there are sequences

$$i = i_1, i_2, \dots, i_r = \tilde{i},$$
$$j = j_1, j_2, \dots, j_r = \tilde{j}$$

such that, for each $k \in \{1, \dots, r-1\}$, we have that

$$(i_{k+1}, j_{k+1}) \in \{(i_k+1, j_k+1), (i_k, j_k+1), (i_k-1, j_k-1), (i_k, j_k-1)\}$$

and $x_{i_{k+1}j_{k+1}} = x_{i_kj_k}$.

Another way to view tiles is as the connected components of the graph of a GT-pattern. Here we view a GT-pattern as a map assigning a real number to each square in a triangle-shaped grid. See Figure 4.1 for examples of GT-patterns and their tilings. The shading of some of the tiles in that figure is explained in Example 4.2 below.

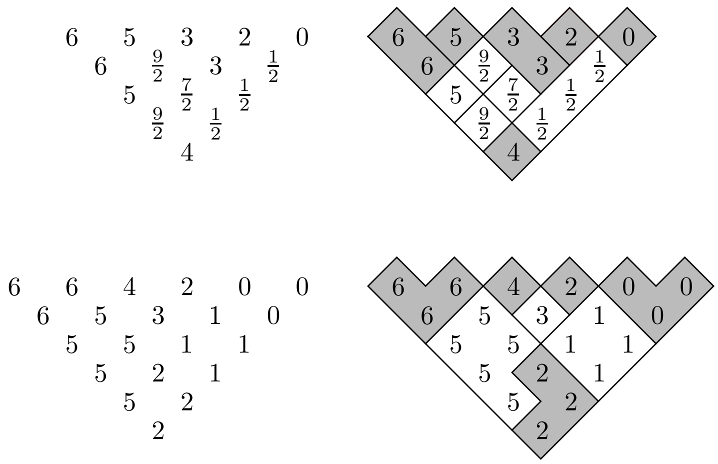

FIGURE 4.1. Tilings of GT-patterns

4.1.2. Tiling Matrices of GT-patterns. Given a GT-pattern $\mathbf{x} \in X_r$ with tiling $\mathscr{T}$, we associate to $\mathscr{T}$ (or, equivalently, to $\mathbf{x}$) a matrix $A_{\mathscr{T}}$ (or $A_{\mathbf{x}}$) as follows. Define the *free rows* of $\mathbf{x}$ to be those rows that are neither the top row nor the bottom row of $\mathbf{x}$. The *free tiles* $T_1, \ldots, T_s$ of $\mathscr{T}$ are those tiles in $\mathscr{T}$ that contain entries only from the free rows of $\mathbf{x}$, *i.e.*, those tiles that contain neither $(1,1)$ nor (i,r) for $1 \leq i \leq r$. The remaining tiles are the *non-free tiles*. The order in which the free tiles are indexed will not matter for our purposes, but, for concreteness, we adopt the convention of indexing the free tiles in the order in which they are initially encountered as the entries of $\mathbf{x}$ are read from left to right and bottom to top. The terminology *free tile* represents the fact that the entries in the non-free tiles of $\mathbf{x}$ are fixed once the weight $\mathbf{wt}(\mathbf{x})$ and highest weight $\mathbf{hwt}(\mathbf{x})$ are fixed.

Define the *tiling matrix* $A_{\mathscr{T}} = A_{\mathbf{x}} = (a_{jk})_{1 \leq j \leq r-2,\, 1 \leq k \leq s}$ by

$$a_{jk} = \#\left\{i : (i, j+1) \in T_k\right\}.$$

That is, a_{jk} counts the number of entries in the jth free row of $\mathbf{x}$ that are contained in the free tile T_k. (Note that the jth *free* row is the $(j+1)$st row of the GT-pattern.) While a different choice of order for the free tiles would result in a tiling matrix with permuted columns, this will be immaterial for our purposes because we will ultimately be interested only in the kernel of $A_{\mathscr{T}}$ up to isomorphism.

EXAMPLE 4.2. Two GT-patterns and their tilings are given in Figure 4.1. The unshaded tiles are the free tiles. The associated tiling matrices are respectively

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

4.1.3. Tilings and the Faces of GT-polytopes. The motivation for introducing tilings, and the main result of this section, is the following theorem.

THEOREM 4.3. *Suppose that $\mathscr{T}$ is the tiling of a GT-pattern $\mathbf{x}$. Then the dimension of the kernel of $A_{\mathscr{T}}$ is equal to the dimension of the minimal (dimensional) face of the GT-polytope containing $\mathbf{x}$.*

PROOF. Suppose that $\mathscr{T}$ is the tiling of a GT-pattern $\mathbf{x}$ in the GT-polytope $GT_{\lambda\beta} \subset X_n$. Let s be the number of free tiles in $\mathscr{T}$. Let $(\varepsilon^{(1)}, \dots, \varepsilon^{(d)})$ be a basis for $\ker A_{\mathscr{T}}$ whose elements are sufficiently short so that

$$|\varepsilon_k^{(m)}| < 1/2 \min\{|x_{i_1j_1} - x_{i_2j_2}| : x_{i_1j_1} \neq x_{i_2j_2}\}, \qquad \text{for } 1 \le m \le d,\ 1 \le k \le s,$$

where $\varepsilon_k^{(m)}$ is the kth coordinate of $\varepsilon^{(m)}$.

Let $H \subset X_n$ be the linear subspace of X_n such that $H + \mathbf{x}$ is the affine span of the minimal face of $GT(\mathbf{x})$ containing $\mathbf{x}$. Define a linear map $\varphi\colon \ker A_{\mathscr{T}} \to X_n$ by $\varphi(\varepsilon^{(m)}) = \mathbf{y}^{(m)}$, where

$$y_{ij}^{(m)} = \begin{cases} \varepsilon_k^{(m)} & \text{if } (i,j) \text{ is in the free tile } T_k \text{ of } \mathscr{T}, \\ 0 & \text{if } (i,j) \text{ is not in a free tile of } \mathscr{T}. \end{cases}$$

and $\mathbf{y}^{(m)} = (y_{ij}^{(m)})_{1 \le i \le j \le n}$. (See Example 4.4 following this proof.) Thus, $\mathbf{x} + \mathbf{y}^{(m)}$ is the result of adding $\varepsilon_k^{(m)}$ to each entry in the kth free tile of $\mathbf{x}$ for $1 \le k \le s$.

To prove the theorem, we show that $(\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(d)})$ is a basis for H. First, since the $\varepsilon_k^{(m)}$'s are sufficiently small, $\mathbf{x} \pm \mathbf{y}^{(m)}$ is a GT-pattern. Moreover, $y_{11}^{(m)} = 0$, $y_{in}^{(m)} = 0$ for $1 \le i \le n$, and each row-sum of $\mathbf{y}^{(m)}$ is 0. This last fact is true because $\varepsilon^{(m)} \in \ker A_{\mathscr{T}}$ and the row-sum vector of $\mathbf{y}^{(m)}$ is, by construction, the same as the result of multiplying $\varepsilon^{(m)}$ with the matrix $A_{\mathscr{T}}$ on the left. Taken together, these properties yield that $\mathbf{x} \pm \mathbf{y}^{(m)} \in GT(\mathbf{x})$. That is, $\mathbf{x} + \mathbf{y}^{(m)}$ and $\mathbf{x} - \mathbf{y}^{(m)}$ are the endpoints of a line segment contained in $GT(\mathbf{x})$ that contains $\mathbf{x}$ in its relative interior. This establishes that $\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(d)} \in H$.

That $\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(d)}$ are linearly independent clearly follows from the fact that $\varepsilon^{(1)}, \dots, \varepsilon^{(d)}$ are linearly independent. Thus, it remains only to prove that $\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(d)}$ span H. Suppose that $\mathbf{y} \in H$, and assume that $\mathbf{y}$ is scaled by a nonzero amount so that $\mathbf{x} \pm \mathbf{y} \in GT(\mathbf{x})$. We construct an element ε of $\ker A_{\mathscr{T}}$ such that $\varphi(\varepsilon) = \mathbf{y}$. Note that

- $y_{ij} = 0$ when (i,j) is not in a free tile of $\mathscr{T}$,
- each row-sum of $\mathbf{y}$ is 0, and
- if (i_1, j_1) and (i_2, j_2) are in the same free tile of $\mathscr{T}$, then $y_{i_1j_1} = y_{i_2j_2}$.

To see that this last property holds, it suffices (see Definition (4.1)) to examine the case where $y_{i_1j_1}$ and $y_{i_2j_2}$ are adjacent entries, *i.e.* where

$$(i_2, j_2) \in \{(i_1+1, j_1+1), (i_1, j_1+1), (i_1-1, j_1-1), (i_1, j_1-1)\}.$$

Since $\mathbf{x} \pm \mathbf{y}$ is a GT-pattern (see Definition (1.7)), we must have either

$$x_{i_1j_1} + y_{i_1j_1} \le x_{i_2j_2} + y_{i_2j_2} \qquad \text{and} \qquad x_{i_1j_1} - y_{i_1j_1} \le x_{i_2j_2} - y_{i_2j_2}$$

or

$$x_{i_1j_1} + y_{i_1j_1} \ge x_{i_2j_2} + y_{i_2j_2} \qquad \text{and} \qquad x_{i_1j_1} - y_{i_1j_1} \ge x_{i_2j_2} - y_{i_2j_2}.$$

But since (i_1, j_1) and (i_2, j_2) are in the same tile of $\mathscr{T}$, we have $x_{i_1j_1} = x_{i_2j_2}$. Thus, in either case, we can subtract the $\mathbf{x}$ entries from both sides, yielding $y_{i_1j_1} = y_{i_2j_2}$, as claimed.

For $1 \le k \le s$ and for each (i,j) in the free tile T_k, put $\varepsilon_k = y_{ij}$. Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_s)$. Then, from the conditions on $\mathbf{y}$ given above, $\varepsilon \in \ker A_{\mathscr{T}}$ and

$\varphi(\varepsilon) = \mathbf{y}$. Hence, the coordinates of ε with respect to the basis $(\varepsilon^{(1)}, \dots, \varepsilon^{(d)})$ of $\ker A_{\mathscr{T}}$ will also be the coordinates of $\mathbf{y}$ with respect to $(\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(d)})$. In particular, $\mathbf{y}$ is in the span of $(\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(d)})$. Since $\mathbf{y}$ was an arbitrary element of H, this establishes the claim. $\square$

EXAMPLE 4.4. Let $\mathbf{x}$ be the GT-pattern from Figure 4.1:

$$\mathbf{x} = \begin{array}{ccccccccc} 6 & & 5 & & 3 & & 2 & & 0 \\ & 6 & & \frac{9}{2} & & 3 & & \frac{1}{2} & \\ & & 5 & & \frac{7}{2} & & \frac{1}{2} & & \\ & & & \frac{9}{2} & & \frac{1}{2} & & & \\ & & & & 4 & & & & \end{array}.$$

We apply the constructions in the proof of Theorem 4.3 to $\mathbf{x}$ to generate explicitly an affine basis for the face of $GT(\mathbf{x})$ containing $\mathbf{x}$. This GT-pattern has tiling matrix

$$A_{\mathscr{T}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

A "sufficiently short" basis for $\ker A_{\mathscr{T}}$ is

$$(\varepsilon^{(1)}, \varepsilon^{(2)}) = \left(1/3 \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, 1/3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right).$$

(Here, "sufficiently short" refers to the fact that $\mathbf{x} + \mathbf{y}^{(1)}$ and $\mathbf{x} + \mathbf{y}^{(2)}$, which are constructed presently, will lie within the minimal face of $GT(\mathbf{x})$ containing $\mathbf{x}$.) Therefore, $\mathbf{x}$ lies in the relative interior of a 2-dimensional face of

$$GT(\mathbf{x}) = GT_{(6,5,3,2,0),\,(4,1,4,5,2)}.$$

Applying the map φ from the proof to $(\varepsilon^{(1)}, \varepsilon^{(2)})$ yields

$$\mathbf{y}^{(1)} = \begin{array}{ccccccccc} 0 & & 0 & & 0 & & 0 & & 0 \\ & 0 & & 0 & & 0 & & 0 & \\ & & -\frac{1}{3} & & \frac{1}{3} & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 0 & & & & \end{array}$$

and

$$\mathbf{y}^{(2)} = \begin{array}{ccccccccc} 0 & & 0 & & 0 & & 0 & & 0 \\ & 0 & & \frac{1}{3} & & 0 & & -\frac{1}{3} & \\ & & \frac{1}{3} & & 0 & & -\frac{1}{3} & & \\ & & & \frac{1}{3} & & -\frac{1}{3} & & & \\ & & & & 0 & & & & \end{array}.$$

From the proof just given, the affine subspace affinely spanned by the minimal face containing $\mathbf{x}$ is affinely spanned by $\{\mathbf{x}, \mathbf{x} + \mathbf{y}^{(1)}, \mathbf{x} + \mathbf{y}^{(2)}\}$.

4.2. Vertices and Denominators of GT-polytopes

As a corollary to Theorem 4.3, we get an easy-to-check criterion for a GT-pattern being a vertex of the GT-polytope containing it.

COROLLARY 4.5. *If* $\mathbf{x} \in GT_{\lambda\beta}$ *has tiling* $\mathscr{T}$ *containing* s *free tiles, then the following conditions are equivalent:*

- $\mathbf{x}$ *is a vertex of* $GT_{\lambda\beta}$*; and*
- $A_{\mathscr{T}}$ *has trivial kernel; i.e, for some* $s \times s$ *submatrix* $\tilde{A}$ *of* $A_{\mathscr{T}}$, $\det \tilde{A} \neq 0$.

PROOF. The only nontrivial assertion in this corollary is the claim that $A_{\mathscr{T}}$ has a trivial kernel if and only if, for some $s \times s$ submatrix $\tilde{A}$ of $A_{\mathscr{T}}$, $\det \tilde{A} \neq 0$. This is an application of the well-known general result stating that an $m \times n$ matrix A has trivial kernel if and only if, for some $n \times n$ submatrix $\tilde{A}$ of A, $\det \tilde{A} \neq 0$. We include a proof of this result for completeness.

The "if" direction is straightforward: any solution $x \in \mathbb{R}^n$ to $Ax = 0$ is in particular a solution to $\tilde{A}x = 0$. Hence, if the latter has only the trivial solution, then so must the former. To prove the "only if" direction, let $a_i \in (\mathbb{R}^n)^*$ be the ith row of A, and let $H_i = \{x \in \mathbb{R}^n : \langle a_i, x\rangle = 0\}$ be the corresponding normal hyperplane. We show that there exist $H_{i_1}, \dots, H_{i_n}$ such that $\dim \cap_{j=1}^{n} H_{i_j} = 0$. The corresponding rows $a_{i_1}, \dots, a_{i_n}$ will then constitute an $n \times n$ submatrix $\tilde{A}$ of A with $\det \tilde{A} \neq 0$.

Begin by choosing H_{i_1} arbitrarily. Since $\dim \ker A = 0$, there must exist an H_{i_2} such that $\dim H_{i_1} \cap H_{i_2} = \dim H_{i_1} - 2 = n - 1$, for otherwise every H_i would coincide with H_{i_1}, implying that $\dim \ker A = n-1$. Now suppose that $H_{i_1}, \dots, H_{i_j}$, $1 \leq j \leq n-1$, have been selected so that $\dim \cap_{k=1}^{j} H_{i_k} = n - j$. Then there exists an $H_{i_{j+1}}$ such that

$$\dim \cap_{k=1}^{j+1} H_{i_k} = \dim \cap_{k=1}^{j} H_{i_k} - 1 = n - j - 1,$$

since otherwise every H_i would contain $\dim \cap_{k=1}^{j} H_{i_k}$, implying that $\dim \ker A = n - j > 0$. Proceeding in this fashion, we construct hyperplanes $H_{i_1}, \dots, H_{i_n}$ with trivial intersection, as desired. $\square$

The corollary just proved shows that tiling matrices determine which GT-tilings are the tilings of vertices. The next result shows that, moreover, tiling matrices encode the denominators that may appear in GT-patterns with the corresponding tilings. This result allows us to construct nonintegral vertices of GT-polytopes by looking for a tiling with a tiling matrix satisfying certain properties given below. Then the tiling can be "filled" in a systematic way with the entries of a GT-pattern that is a counter-example to the Berenstein–Kirillov conjecture [**39**]. Indeed, we will be able to construct infinite families of counter-examples in which the denominators take on arbitrarily large values.

LEMMA 4.6. *Suppose that* $\mathscr{T}$ *is a tiling with* s *free tiles such that* $A_{\mathscr{T}}$ *has trivial kernel. Then the following conditions are equivalent:*

1. $\mathscr{T}$ *is the tiling of a nonintegral vertex* $\mathbf{x}$ *of a GT-polytope in which* $q \in \mathbb{N}$ *is the least common multiple of the denominators of the entries in* $\mathbf{x}$ *(written in reduced form); and*
2. *there is an integral vector* $\xi = (\xi_1, \dots, \xi_s)$ *such that* $A_{\mathscr{T}}\xi \equiv 0$ *mod* q *and such that, for some* $k \in \{1, \dots, s\}$, $\gcd(\xi_k, q) = 1$.

PROOF. [(1) $\Rightarrow$ (2)] Suppose that $\mathbf{x}$ is a nonintegral vertex in which q is the least common multiple of the denominators of the entries. For each entry x_{ij}, $1 \leq i \leq j \leq n$, let $p_{ij} = qx_{ij}$. Let $T_1, \dots, T_s$ be the free tiles of $\mathscr{T}$, and define $\xi = (\xi_1, \dots, \xi_s)$ by $\xi_k = p_{ij}$ for some $(i,j) \in T_k$ (all values of p_{ij} being equal within a tile). Since $\mathbf{x}$ has entries with denominator q, we have that, for some $k \in \{1, \dots, s\}$, $\gcd(\xi_k, q) = 1$. Moreover, since each row-sum of $\mathbf{x}$ is an integer, we have that, for *fixed* $j \in \{1, \dots, n\}$,

$$q \quad \text{divides} \quad \sum_{\substack{1 \leq k \leq s \\ (i,j) \in T_k}} p_{ij} = \sum_{1 \leq k \leq s} a_{jk} \xi_k.$$

Therefore, $A_{\mathscr{T}} \xi \equiv 0 \bmod q$.

[(2) $\Rightarrow$ (1)] $\mathscr{T}$ is given to be a tiling, so some GT-pattern $\tilde{\tilde{\mathbf{x}}}$ with rational entries has tiling $\mathscr{T}$. If necessary, multiply $\tilde{\tilde{\mathbf{x}}}$ by some integer to produce an integral GT-pattern $\tilde{\mathbf{x}}$ with tiling $\mathscr{T}$. Choose $\xi = (\xi_1, \dots, \xi_s)$ satisfying condition (2) such that $0 \leq \xi_1, \dots, \xi_s < q$. Define $\mathbf{y} \in X_n$ by

$$y_{ij} = \begin{cases} \xi_k / q & \text{if } (i,j) \text{ is in the free cell } P_k \text{ of } \mathscr{T}, \\ 0 & \text{if } (i,j) \text{ is not in a free cell of } \mathscr{T}. \end{cases}$$

Then $\mathbf{x} = \tilde{\mathbf{x}} + \mathbf{y}$ satisfies condition (1). $\square$

Now we are ready to give the details of the proof of Theorem 1.14. In particular, Propositions 4.7 and 4.8 settle the Berenstein-Kirillov conjecture. Proposition 4.7 has also been stated by King et al. [**35**] with respect to K-hive polytopes, which are isomorphic to GT-polytopes under a lattice-preserving linear map. We give here a "tiling" proof.

PROPOSITION 4.7. *When $n \leq 4$, every GT-polytope in X_n is integral.*

PROOF. Note that it suffices to prove the $n = 4$ case since there is a natural embedding $X_n \hookrightarrow X_{n+1}$ defined by $\mathbf{x} \mapsto \tilde{\mathbf{x}}$, where

$$\tilde{x}_{ij} = \begin{cases} 0 & \text{if } 1 \leq i = j \leq n+1, \\ x_{i,j-1} & \text{if } 1 \leq i < j \leq n+1. \end{cases}$$

Hence, $\tilde{\mathbf{x}}$ is the GT-pattern that arises from $\mathbf{x}$ by appending a zero to the right end of each row.

Suppose that $\mathbf{x} \in X_4$ is a vertex. Then, by Corollary 4.5, the associated tiling matrix $A_{\mathscr{T}}$ has trivial kernel. Therefore, $A_{\mathscr{T}}$ is either a 2×1 or a 2×2 matrix. Note also that the first and last nonzero entries of each column of a tiling matrix associated with a GT-pattern must be 1. Therefore, $A_{\mathscr{T}}$ is a 0/1-matrix.

If $A_{\mathscr{T}}$ is 2×1, then the only possibilities are

$$A_{\mathscr{T}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; A_{\mathscr{T}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ or } A_{\mathscr{T}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In each case, there exists no vector $\xi \not\equiv 0 \bmod q$ such that $A_{\mathscr{T}} \xi \equiv 0 \bmod q$ for $q > 1$, so Lemma 4.6 implies that the entries of $\mathbf{x}$ are integral. On the other hand, if $A_{\mathscr{T}}$ is 2×2, then $\det A_{\mathscr{T}} \in \{-1, 1\}$; that is, $\gcd(\det A_{\mathscr{T}}, q) = 1$ for $q > 1$. Therefore, $A_{\mathscr{T}}$, considered as a module homomorphism on $\mathbb{Z}/q\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z}$, is invertible for $q > 1$, so, by Lemma 4.6, $\mathbf{x}$ is integral. $\square$

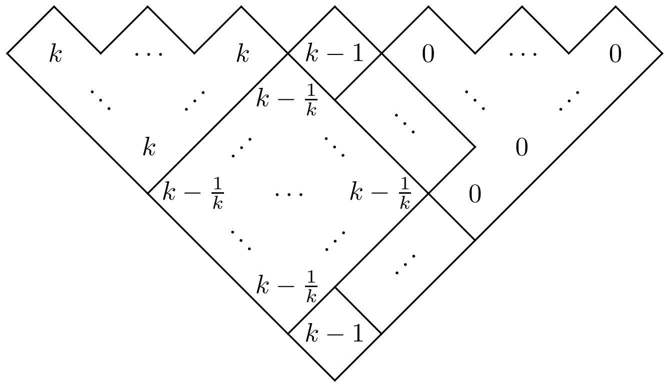

FIGURE 4.2. An infinite family of counterexamples to the Berenstein–Kirillov conjecture

Now we show that nonintegral GT-polytopes exist in X_n for each $n \geq 5$. Moreover, by choosing n sufficiently large, we can find GT-polytopes in which the denominators of the vertices are arbitrarily large.

PROPOSITION 4.8. *For positive integer k, let $\lambda = (k^k, k-1, 0^k)$ and $\beta = ((k-1)^{k+1}, 1^k)$. Then a vertex of $GT_{\lambda\beta} \subset X_{2k+1}$ contains entries with denominator k.*

PROOF. Define $\mathbf{x}^{(k)} \in X_{2k+1}$ by

$$x_{ij}^{(k)} = \begin{cases} \frac{(k-j+1)(k+1)}{k} & \text{if } 1 \leq i = j \leq k+1, \\ k - \frac{1}{k} & \text{if } 1 \leq i < j \leq k+1, \\ k & \text{if } k+1 < j \leq 2k+1 \text{ and } 1 \leq i < j-k, \\ k - \frac{1}{k} & \text{if } k+1 < j \leq 2k+1 \text{ and } j-k \leq i \leq k, \\ \frac{(j-k-1)(k-1)}{k} & \text{if } k+1 < j \leq 2k+1 \text{ and } i = k+1, \\ 0 & \text{if } k+1 < j \leq 2k+1 \text{ and } k+1 < i \leq 2k+1. \end{cases}$$

(See Figure 4.2.) Then $\mathbf{x}^{(k)} \in GT_{\lambda\beta}$. The tiling matrix associated with $\mathbf{x}^{(k)}$ is

$$A_{\mathscr{T}} = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 2 & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k-1 & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 \\ k & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ k-1 & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

It is easy to see that $\det A_{\mathscr{T}} = \pm k$ (for example, by cofactor expansion along the kth row). In particular, $\det A_{\mathscr{T}} \neq 0$. Hence, by Corollary 4.5, $\mathbf{x}^{(k)}$ is a vertex of $GT_{\lambda\beta}$. □

Proposition 4.8 explicitly constructs counterexamples to the Berenstein–Kirillov conjecture in X_n where $n \geq 5$ is odd. Counterexamples with even $n \geq 6$ may be constructed from these using the embedding $X_n \hookrightarrow X_{n+1}$ given in the proof of Theorem 4.7. Less trivial examples with even n may be constructed using other tilings. For example, if $\lambda = (3, 3, 1, 0, 0, 0)$ and $\beta = (1, 1, 1, 2, 1, 1)$, then the GT-pattern

$$\mathbf{x} = \begin{array}{ccccccccccc}
3 & & 3 & & 1 & & 0 & & 0 & & 0 \\
& 3 & & \frac{7}{3} & & \frac{2}{3} & & 0 & & 0 & \\
& & \frac{7}{3} & & \frac{7}{3} & & \frac{1}{3} & & 0 & & \\
& & & \frac{7}{3} & & \frac{1}{3} & & \frac{1}{3} & & & \\
& & & & \frac{5}{3} & & \frac{1}{3} & & & & \\
& & & & & 1 & & & & &
\end{array}$$

in $GT_{\lambda\beta} \subset X_6$ has the tiling matrix

$$A_{\mathbf{x}} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Since $\det A_{\mathbf{x}} = 3 \neq 0$, $\mathbf{x}$ is a vertex of $GT_{\lambda\beta}$.

We now prove a necessary condition for a GT-tiling $\mathscr{T}$ to be the tiling of a vertex of a GT-polytope containing a denominator $q > 1$. Observe that Lemma 4.6 says that if $\mathbf{x}$ is a nonintegral vertex in which q appears as a denominator, then the tiling matrix $A_{\mathscr{T}}$ has trivial kernel as a linear operator $\mathbb{R}^s \to \mathbb{R}^{n-2}$ (since $\mathbf{x}$ is a vertex), but $A_{\mathscr{T}}$ has *nontrivial* kernel when considered as an operator $(\mathbb{Z}/q\mathbb{Z})^s \to (\mathbb{Z}/q\mathbb{Z})^{n-2}$. Moreover, this nontrivial kernel contains a vector in which one of the coordinates is a unit in $\mathbb{Z}/q\mathbb{Z}$. The next result shows that these properties imply that each $s \times s$ submatrix of $A_{\mathscr{T}}$ has determinant equal to 0 modulo q.

LEMMA 4.9. *Let A be an integral matrix with s columns. Suppose that there is an integral vector $\xi = (\xi_1, \ldots, \xi_s)$ such that $A\xi \equiv 0 \bmod q$ and such that some ξ_k is a unit in $\mathbb{Z}/q\mathbb{Z}$. Then every square row submatrix of A has determinant equal to 0 modulo q.*

We remind the reader that a *row submatrix* of a matrix A is a matrix whose rows are all rows of A.

PROOF. Let a row submatrix A' of A be given. Let $\tilde{\xi}$ be a vector that results from permuting the entries of ξ so that ξ_k is the last coordinate. Let $\tilde{A}$ be the matrix that results from applying the same permutation to the columns of A'. Thus we have that $\tilde{A}\tilde{\xi} = A'\xi \equiv 0 \bmod q$. If $\det \tilde{A} = 0$, then we are done, so suppose that $\tilde{A}$ is nonsingular. Recall that every rational square nonsingular matrix has a *Hermite normal form* (see [**60**], Theorem 4.1). In particular, there is a unimodular matrix U such that $H = U\tilde{A}$ is integral and upper triangular. (Here, H^t is the Hermite normal form of $\tilde{A}^t$.) Consequently, we have that

$$H\xi = U\tilde{A}\tilde{\xi} \equiv 0 \bmod q \qquad \text{and} \qquad |\det H| = |\det \tilde{A}|.$$

Let h_{ss} be the last entry in the last row of H. Since all other entries in the last row of H are equal to 0, we must have that $h_{ss}\xi_k \equiv 0 \bmod q$. But since ξ_k is a unit in $\mathbb{Z}/q\mathbb{Z}$, it is not a zero divisor, which implies that $h_{ss} \equiv 0 \bmod q$. Since the determinant of H is the product of the entries on its diagonal, and since $|\det H| = |\det \tilde{A}| = |\det A'|$, the claim is proved. $\square$

As an application of this Lemma, we derive a bound on the size of the denominators in the vertices of GT-polytopes in fixed dimension.

PROPOSITION 4.10. *For fixed n, the numbers that may appear as denominators of entries in vertices of GT-polytopes in X_n are no larger than $\frac{(n-1)!}{2^{n-1}}(n-1)^{n-3}$.*

PROOF. Fix $n \in \mathbb{N}$. Since only finitely many partitions of $\mathcal{I}_n$ exist, there is an upper bound on the set

$$\left\{ |m| : \begin{array}{l} m \text{ is the determinant of a square row submatrix of} \\ \text{the tiling matrix of some GT-pattern } \mathbf{x} \in X_n \end{array} \right\}.$$

By a "row submatrix", we mean a submatrix whose rows are a subset of the rows of the tiling matrix. Let N be an upper bound on this set. The claim is that no GT-polytope in X_n has a vertex with denominators greater than N.

Let $q > N$ be given. Suppose that $\mathbf{x} \in X_n$ is a vertex. Let s be the number of free tiles in $\mathbf{x}$, and let $A_{\mathscr{T}}$ be the tiling matrix of $\mathbf{x}$. Then no $s \times s$ submatrix of $A_{\mathscr{T}}$ has determinant greater than or equal to q. Moreover, by Corollary 4.5, some $s \times s$ submatrix of $A_{\mathscr{T}}$ has nonzero determinant. Therefore, this $s \times s$ submatrix has determinant not equal to 0 modulo q. However, by Lemmas 4.6 and 4.9, if q were a denominator of one of the entries in $\mathbf{x}$, then *every* $s \times s$ submatrix would have determinant equal to 0 modulo q. This proves that N is a bound as claimed.

Our second claim is that N is no more than $\frac{(n-1)!}{2^{n-1}}(n-1)^{n-3}$. To prove this, observe that every tiling matrix of a GT-pattern in X_n has $n-2$ rows. Moreover, if the GT-pattern is a vertex, then the number of rows must be greater than or equal to the number of columns (since the kernel is trivial), so we have that $s \leq n-2$. In addition, the tiling matrix contains only nonnegative entries, and, since the first and last entry in each column must be a 1, and since each entry can differ by at most ± 1 from the entry above it, the largest possible entry in a tiling matrix is $\frac{n-1}{2}$. Therefore, if $A = (a_{ij})$ is an $s \times s$ submatrix of a tiling matrix, we have that

$$\begin{aligned} \det A &= \sum_{\sigma \in \mathfrak{S}_s} a_{1\sigma(1)} \cdots a_{s\sigma(s)} \\ &\leq \sum_{\sigma \in \mathfrak{A}_s} \left(\frac{n-1}{2} \right)^s \\ &\leq \frac{(n-2)!}{2} \left(\frac{n-1}{2} \right)^{n-2} \\ &= \frac{(n-1)!}{2^{n-1}} (n-1)^{n-3}, \end{aligned}$$

where $\mathfrak{A}_s$ denotes the alternating group in $\mathfrak{S}_s$. □

The bound in Proposition 4.10 is not tight. For example, it is easy to show that, when n=5, the largest possible denominator is 2, while the proof just given bounds the largest possible denominator by $\frac{(5-1)!}{2^{5-1}}(5-1)^{5-3} = 24$.

4.3. Combinatorics of Gelfand–Tsetlin Tilings

In this section, we discuss the combinatorics of Gelfand–Tsetlin tilings. In the previous sections, tilings were defined constructively in terms of GT-patterns (Definition 4.1). That is, tilings were defined to be those partitions of $\mathcal{I}_n$ that arise when one begins with a GT-pattern and then groups together entries that

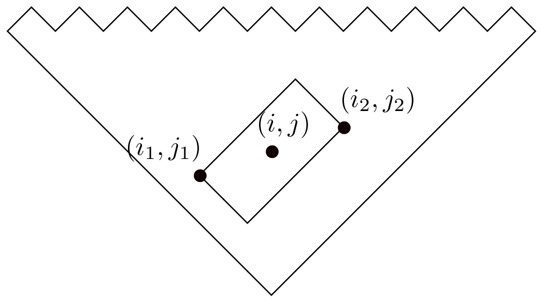

FIGURE 4.3. The GT-tiling condition

are equal and adjacent. We now give a definition of tilings that does not depend upon having a GT-pattern in hand. More precisely, we define a class of partitions of $\mathcal{I}_n$, which we call GT-tilings, without reference to GT-patterns. We then prove that the GT-tilings are precisely the partitions of $\mathcal{I}_n$ that arise as the tilings of GT-patterns.

We say that a subset $T \subset \mathcal{I}_n$ is *connected* if, for every pair $(i,j),(i',j') \in T$, there are sequences

$$i = i_1, i_2, \ldots, i_r = i',$$
$$j = j_1, j_2, \ldots, j_r = j'$$

such that, for each $k \in \{1, \ldots, r-1\}$, we have that

$$(i_{k+1}, j_{k+1}) \in T \cap \{(i_k+1, j_k+1), (i_k, j_k+1), (i_k-1, j_k-1), (i_k, j_k-1)\}$$

That is, for every pair of points in T, it must be possible to travel from one point to the other while remaining in T and only moving one step at a time in the North-East, North-West, South-West, or South-East direction.

DEFINITION 4.11. A *GT-tiling* is a partition $\mathscr{T}$ of the set $\mathcal{I}_n$ into disjoint nonempty connected subsets, called *tiles*, such that, for each tile $T \in \mathscr{T}$, if (i_1, j_1), $(i_2, j_2) \in T$, then $(i,j) \in T$ whenever

$$i_1 \leq i \leq i_2 \quad \text{and} \quad i_1 - j_1 \leq i - j \leq i_2 - j_2.$$

If we think of a GT-tiling as a partition of the entries of a triangular array, then the inequalities in Definition 4.11 state that if a tile T contains two entries that are respectively the left-most and the right-most elements in the region bounded by the diagonals passing through those two entries, then T contains all of the elements within that region (see Figure 4.3).

Before stating the next theorem, we remind the reader that the word "partition" is used in two distinct, but conventional, senses. On the one hand, we speak of partitions of a set into disjoint subsets. On the other hand, a partition of n is a weakly decreasing sequence of nonnegative integers that sum to n. These meanings are related, since there is a many-to-one map sending partitions of n in the second sense to partitions of $[n]$ in the first sense. Nonetheless, we warn the reader that both of these meanings are employed in close proximity to each other in the following theorem and its proof.

THEOREM 4.12. *A partition $\mathscr{T}$ of $\mathcal{I}_n$ is a GT-tiling if and only if there is a GT-pattern of which $\mathscr{T}$ is the tiling.*

Indeed, given a partition $\lambda = (\kappa_1^{v_1}, \ldots, \kappa_m^{v_m})$, if $\mathscr{T}$ is a GT-tiling that partitions the top row of $\mathcal{I}_n$ into subsets with cardinalities $v_1, \ldots, v_m$ as read from left to right, then there exists a GT-pattern with tiling $\mathscr{T}$ and highest weight λ.

PROOF. Suppose that $\mathscr{T}$ is the tiling of a GT-pattern $\mathbf{x} = (x_{ij})_{1 \le i \le j \le n}$. Let (i_1, j_1) and (i_2, j_2) be two entries in a tile $T \in \mathscr{T}$, and suppose that $(i,j) \in \mathcal{I}_n$ is such that

$$i_1 \le i \le i_2 \quad \text{and} \quad i_1 - j_1 \le i - j \le i_2 - j_2. \tag{4.1}$$

To prove that $\mathscr{T}$ is a GT-tiling, we show that $(i,j) \in T$.

Observe that the inequalities (4.1) imply that $j_1 \ge j-(i-i_1)$ and $j \ge j_2-(i_2-i)$. Thus, since $\mathbf{x}$ is a GT-pattern, we have the sequence of inequalities

$$\begin{aligned} x_{i_1,j_1} &\ge x_{i_1,j_1-1} \ge \cdots \ge x_{i_1,j-(i-i_1)} \\ &\ge x_{i_1+1,j-(i-i_1)+1} \ge \cdots \ge x_{ij} \\ &\ge x_{i,j-1} \ge \cdots \ge x_{i,j_2-(i_2-i)} \\ &\ge x_{i+1,j_2-(i_2-i)+1} \ge \cdots \ge x_{i_2,j_2}. \end{aligned}$$

But since (i_1, j_1) and (i_2, j_2) are in the same tile, we have that the first and last terms in this sequence are equal: $x_{i_1,j_1} = x_{i_2,j_2}$. Hence, all of the entries in this sequence are equal. As a result, they belong to the same tile by Definition 4.1. In particular, $(i,j) \in T$, as desired.

To prove the converse claim, we must show that, for an arbitrary GT-tiling $\mathscr{T}$, there is a GT-pattern with tiling $\mathscr{T}$. Indeed, we show that, more strongly, if $\lambda = (\kappa_1^{v_1}, \ldots, \kappa_m^{v_m})$ is a partition, and if $\mathscr{T}$ is a GT-tiling that partitions the top row of $\mathcal{I}_n$ into subsets with cardinalities $v_1, \ldots, v_m$ as read from left to right, then there exists a GT-pattern with tiling $\mathscr{T}$ and highest weight λ.

We construct a GT-pattern with tiling $\mathscr{T}$ as follows. Our first step is to assign the value κ_i to the ith of the tiles intersecting the top row of $\mathcal{I}_n$ as it is read from left to right. Observe that, by Definition 4.11, each remaining tile T has a unique highest element (i,j), that is, an element (i,j) such that $(i',j') \in T$ implies that $j' < j$. For suppose otherwise; that is, suppose that (i_1, j) and (i_2, j) are distinct elements of T with $i_1 < i_2$, and that no element of T is higher than either of these. Then, since T does not intersect the top row, we have that $(i_1+1, j+1) \in \mathcal{I}_n$, and so, by Definition 4.11, $(i_1+1, j+1) \in T$. Hence, there is an element of T higher than (i_1, j), a contradiction. We call the unique highest element $(i,j) \in T$ the *peak* of T.

We associate, to each tile T not intersecting the top tow of $\mathcal{I}_n$, two disjoint nonempty subsets of $\mathcal{I}_n$ defined as follows:

$$\begin{aligned} R_{T<} &= \{(i,j) \in I_n \backslash T : \exists (k,\ell) \in T \text{ such that } i \le k \text{ and } i-j \le k-\ell\} \\ R_{T>} &= \{(i,j) \in I_n \backslash T : \exists (k,\ell) \in T \text{ such that } i \ge k \text{ and } i-j \ge k-\ell\} \end{aligned}$$

Note that $R_{T<}$ and $R_{T>}$ are both disjoint from T. The definitions of these regions are motivated by the fact that the entries in $R_{T<}$ (resp. $R_{T>}$) must be assigned values larger than (resp. smaller than) the value assigned to T for the resulting array to be a GT-pattern with tiling $\mathscr{T}$.

It follows from the definition of a GT-tiling (Definition 4.11) that the regions $R_{T<}$ and $R_{T>}$ are disjoint. For suppose otherwise; that is, suppose that there exists an $(i, j) \in R_{T<} \cap R_{T>}$. Then there exist $(k_1, \ell_1), (k_2, \ell_2) \in T$ such that

$$\begin{aligned} i \le k_1 \quad &\text{and} \quad i - j \le k_1 - \ell_1 \\ i \ge k_2 \quad &\text{and} \quad i - j \ge k_2 - \ell_2. \end{aligned}$$

That is,

$$k_1 \le i \le k_2 \qquad \text{and} \qquad k_2 - \ell_2 \le i - j \le k_1 - \ell_1.$$

By Definition 4.11, this implies that $(i, j) \in T$, a contradiction.

We are now ready to fill the tiles not intersecting the top row of $\mathcal{I}_n$ by induction on the heights of their peaks. For those tiles whose peaks are the same height, we proceed by induction from left to right. Fix a tile T, and assume that all tiles with peaks higher than, or equally high and to the left of, T have been assigned distinct values consistent with the GT-inequalities in Definition 1.7.

Let ξ_1 be the least value assigned so far to a tile intersecting $R_{T<}$, and let T_1 be a tile assigned that value. Similarly, let ξ_2 be the greatest value assigned so far to a tile intersecting $R_{T>}$, and let T_2 be a tile assigned that value. Our goal is to show that $\xi_1 > \xi_2$, from which it will follow that we may assign a value ξ to T such that $\xi_1 > \xi > \xi_2$, thereby maintaining conformity to the GT-inequalities.

The crucial observation is that since T_1 and T_2 have peaks higher than that of T, we have that $R_{T_1>} \cap T_2 \neq \varnothing$. Consequently, since the entries assigned so far are consistent with the GT-inequalities, we have that $\xi_1 > \xi_2$, and the claim is proved. $\square$

Observe that if two points lie in the relative interior of the same face of GT_λ, then they have the same tiling. To see this, recall that two points are in the relative interior of the same face if and only if they satisfy the same set of facet-defining inequalities with equality. The facet-defining inequalities of GT_λ are a subset of the GT-inequalities in Definition 1.7, and if two points have the same tiling, then they satisfy exactly the same set of GT-inequalities with equality.

Therefore, we may associate to each face F of GT_λ a certain tiling: the tiling $\mathscr{T}(F)$ shared by all points in that face's relative interior. In particular, given a partition λ, there is a tiling shared by all points in the relative interior of GT_λ. We call this tiling the *interior tiling* of GT_λ, and we denote it by $\mathscr{T}(\lambda)$.

DEFINITION 4.13. Given two GT-tilings $\mathscr{T}$ and $\mathscr{T}'$ of $\mathcal{I}_n$, we say that $\mathscr{T} \le \mathscr{T}'$ if and only if (1) $\mathscr{T}$ is refined by $\mathscr{T}'$ as a partition of $\mathcal{I}_n$, and (2) the top row of $\mathcal{I}_n$ is partitioned identically by $\mathscr{T}$ as by $\mathscr{T}'$. Let $P(\mathscr{T})$ be the order ideal generated by $\mathscr{T}$ within this poset structure on GT-tilings. That is, let

$$P(\mathscr{T}) = \{\mathscr{T}' : \mathscr{T}' \le \mathscr{T}\}.$$

THEOREM 4.14. *Let $F(\lambda)$ be the face poset of GT_λ. Then $F(\lambda)$ and $P(\mathscr{T}(\lambda))$ are isomorphic as posets.*

PROOF. We proceed as follows. We first show that if F and F' are faces of GT_λ and $F \subset F'$, then $\mathscr{T}(F) < \mathscr{T}(F')$. This establishes that $P(\mathscr{T}(\lambda))$ contains an isomorphic image of $F(\lambda)$. We then show that every GT-tiling in $P(\mathscr{T}(\lambda))$ is the tiling of some point in GT_λ, completing the proof.

Suppose that $F \subset F'$ are faces of GT_λ. Let $\mathscr{T} = \mathscr{T}(F)$ and $\mathscr{T}' = \mathscr{T}(F')$. The facet-defining inequalities satisfied with equality by a point in the relative interior of F' form a subset of those satisfied with equality by a point in the relative interior of F. Therefore, $\mathscr{T}$ is refined by $\mathscr{T}'$ as a partition of $\mathcal{I}_n$.

To complete the proof, let $\mathscr{T}$ be a GT-tiling such that $\mathscr{T} \leq \mathscr{T}(\lambda)$. Put $\lambda = (\kappa_1^{v_1}, \ldots, \kappa_m^{v_m})$. Then $\mathscr{T}(\lambda)$ partitions the top row of $\mathcal{I}_n$ into subsets of cardinalities $v_1, \ldots, v_m$ as read from left to right. Since $\mathscr{T}$ partitions the top row of $\mathcal{I}_n$ identically, it follows from Theorem 4.12 that there is a GT-pattern $\mathbf{x}$ with top row λ and tiling $\mathscr{T}$. Hence, $\mathbf{x} \in GT_\lambda$ and if F is the face containing $\mathbf{x}$ in its relative interior, then $\mathscr{T} = \mathscr{T}(F)$, completing the proof. $\square$

The idea of partially ordering GT-tilings can also be used to give a "face poset" version of Theorem 4.3. Just as we did with GT_λ above, we associate, to each face F of a GT-polytope $GT_{\lambda\beta}$, a certain tiling: the tiling $\mathscr{T}(F)$ shared by all points in that face's relative interior. In particular, given $\lambda, \beta \in \mathbb{Z}^n$ such that $GT_{\lambda\beta} \neq \varnothing$, there is a tiling shared by all points in the relative interior of $GT_{\lambda\beta}$. We call this tiling the *interior tiling* of $GT_{\lambda\beta}$, and we denote it by $\mathscr{T}(\lambda, \beta)$.

Recall the construction in Theorem 4.3 of an affine linear map embedding $\ker A_{\mathscr{T}}$ into the space X_n of triangular arrays. We now carry out this construction for every GT-tiling $\mathscr{T}$ partitioning $\mathcal{I}_n$. Let $(\varepsilon^{(1)}, \ldots, \varepsilon^{(d)})$ be a basis for $\ker A_{\mathscr{T}}$ and define a linear map $\varphi_{\mathscr{T}}\colon \ker A_{\mathscr{T}} \to X_n$ by $\varphi_{\mathscr{T}}(\varepsilon^{(m)}) = \mathbf{y}^{(m)}$, where

$$y_{ij}^{(m)} = \begin{cases} \varepsilon_k^{(m)} & \text{if } (i,j) \text{ is in the free tile } T_k \text{ of } \mathscr{T}, \\ 0 & \text{if } (i,j) \text{ is not in a free tile of } \mathscr{T}. \end{cases}$$

and $\mathbf{y}^{(m)} = (y_{ij}^{(m)})_{1 \leq i \leq j \leq n}$ (see Example 4.4). Recall that in Theorem 4.3 we showed that if $\mathbf{x}$ is a point in a GT-polytope with tiling $\mathscr{T}$, then $\mathbf{x} + \varphi_{\mathscr{T}}(\ker A_{\mathscr{T}})$ is the affine span of the face containing $\mathbf{x}$ in its relative interior. Therefore, we have an injective affine linear map

$$\Phi_{\lambda,\beta,\mathscr{T}}\colon \ker A_{\mathscr{T}} \to \mathbf{hwt}^{-1}(\lambda) \cap \mathbf{wt}^{-1}(\beta)$$

defined by

$$\Phi_{\lambda,\beta,\mathscr{T}}(\varepsilon) = \mathbf{x} + \varphi_{\mathscr{T}}(\varepsilon).$$

Moreover, since the affine span of any face is contained in the affine span of the entire polytope, we have, for each tiling $\mathscr{T}$ of a face of $GT_{\lambda\beta}$, an injection

$$\Phi_{\lambda,\beta,\mathscr{T}}\colon \ker A_{\mathscr{T}} \to \operatorname{Im} \Phi_{\lambda,\beta,\mathscr{T}(\lambda\beta)}.$$

Using this embedding, we define for each tiling $\mathscr{T}$ a poset $\mathcal{P}(\mathscr{T})$ as follows: we say that $\mathscr{T}' \prec \mathscr{T}$ if and only if the composition $\Phi^{-1}_{\lambda,\beta,\mathscr{T}} \circ \Phi_{\lambda,\beta,\mathscr{T}'}$ is an embedding $\ker A_{\mathscr{T}'} \hookrightarrow \ker A_{\mathscr{T}}$, and we write

$$\mathcal{P}(\mathscr{T}) = \{\mathscr{T}' : \mathscr{T}' \preceq \mathscr{T}\}.$$

As a consequence of the preceding discussion, we can reformulate Theorem 4.3 as the following result.

THEOREM 4.15. *Let $F(\lambda, \beta)$ be the face poset of $GT_{\lambda\beta}$. Then $F(\lambda, \beta)$ and $\mathcal{P}(\mathscr{T}(\lambda, \beta))$ are isomorphic as posets.*

CHAPTER 5

Degrees of stretched Kostka coefficients

5.1. Introduction

Kostka coefficients are important numbers appearing in many branches of mathematics, including representation theory, the theory of symmetric functions, and algebraic geometry (see, *e.g.*, [**21, 62, 64**] and references therein). Given a highest weight λ and a weight β of the Lie algebra $\mathfrak{gl}_r(\mathbb{C})$, the Kostka coefficient $K_{\lambda\beta}$ is the dimension of the weight subspace with weight β of the irreducible representation V_λ of $\mathfrak{gl}_r(\mathbb{C})$ [**21**]. In the theory of symmetric functions, Kostka coefficients are defined by the expansion of Schur functions s_λ into monomials. That is, given a partition λ of $n \in \mathbb{N}$, we have

$$s_\lambda = \sum_{\substack{\text{compositions} \\ \beta \text{ of } n}} K_{\lambda\beta} x^\beta, \tag{5.1}$$

where $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_r^{\beta_r}$.

For our purposes, it will suffice to assume that λ is a partition and β is a composition of the same length and size as λ. That is, we take $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{Z}^r$ such that $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$, and $\beta = (\beta_1, \dots, \beta_r) \in \mathbb{Z}^r$ such that $\beta_i \geq 0$ and $\sum_i \beta_i = \sum_i \lambda_i$. The Kostka coefficients $K_{\lambda\beta}$ are indexed by such pairs (λ, β).

We sometimes write $\lambda = (\kappa_1^{v_1}, \dots, \kappa_m^{v_m})$ to indicate that λ has v_p parts equal to κ_p for $1 \leq p \leq m$. The use of this notation always presumes that $v_p \geq 1$ for $1 \leq p \leq m$. Let $|\lambda| = \sum_i \lambda_i$ denote the sum of the parts in λ. Given an arbitrary sequence of nonnegative integers $\beta = (\beta_1, \dots, \beta_r) \in \mathbb{Z}^r_{\geq 0}$, let $\bar{\beta} = (\bar{\beta}_1, \dots, \bar{\beta}_r)$ be the unique partition that may be produced by permuting the terms of β. We say that λ *dominates* β, denoted $\beta \trianglelefteq \lambda$, if $|\lambda| = |\beta|$ and $\sum_{k=1}^i \lambda_k \geq \sum_{k=1}^i \bar{\beta}_k$ for $1 \leq i < r$. If, more strongly, $\sum_{k=1}^i \lambda_k > \sum_{k=1}^i \bar{\beta}_k$ for $1 \leq i < r$, we write $\beta \triangleleft \lambda$, and we say that λ and β form a *primitive pair*.

Since the parameters defining a Kostka coefficient are themselves vectors, they may be "stretched" by a scaling factor n. This procedure defines a function $n \mapsto K_{n\lambda,n\beta}$, which, following [**35**], we call a *stretched Kostka coefficient*. We denote this function by $\mathcal{K}_{\lambda\beta}(n) = K_{n\lambda,n\beta}$. Kirillov and Reshetikhin [**40**] have shown that $\mathcal{K}_{\lambda\beta}(n)$ is a polynomial function of the scaling parameter n. King, Tollu, and Toumazet have conjectured that these polynomials have only positive coefficients [**35**], and they have given a conjectural expression for the degree of $\mathcal{K}_{\lambda\beta}(n)$ [**36**]. In this chapter, we prove that the stretched Kostka coefficients indeed have the degrees conjectured in [**36**]:

Theorem 5.1 (Proved on p. 54). *If $\lambda = (\kappa_1^{v_1}, \dots, \kappa_m^{v_m}) \in \mathbb{Z}^r_{\geq 0}$ is a partition with $m \geq 2$ and $\beta \triangleleft \lambda$, then the degree of the stretched Kostka coefficient $\mathcal{K}_{\lambda\beta}(n)$*

is given by

$$\deg \mathcal{K}_{\lambda\beta}(n) = \binom{r-1}{2} - \sum_{p=1}^{m} \binom{v_p}{2} \tag{5.2}$$

(*where we evaluate* $\binom{1}{2} = 0$).

As stated, this theorem gives the degree of a stretched Kostka coefficient only when λ and β are a primitive pair. However, Berenstein and Zelevinsky have shown that all Kostka coefficients factor into a product of Kostka coefficients indexed by primitive pairs [**7**]. It follows from this factorization that Theorem 5.1 suffices to describe the degrees of stretched Kostka coefficients in all cases.

The factorization into primitive pairs works as follows. It is well-known that $K_{\lambda\beta}$ is invariant under permutations of the coordinates of β. For example, this follows from equation (5.1) and the fact that Schur functions are symmetric. Consequently, $\mathcal{K}_{\lambda\beta}(n) = \mathcal{K}_{\lambda\bar{\beta}}(n)$. In particular, to compute the degree of stretched Kostka coefficients, we need only consider the case where β is a partition.

Suppose that $\lambda, \beta \in \mathbb{Z}^r$ are both partitions with $|\lambda| = |\beta|$. If λ and β do not form a primitive pair, then we may write both λ and β as a concatenation of partitions such that each of the partitions contained in λ forms a primitive pair with the corresponding partition contained in β. More precisely, there exists a unique sequence of integers

$$1 = i_1 < i_2 < \cdots < i_s < i_{s+1} = r + 1$$

such that each pair

$$\lambda^{(t)} = (\lambda_{i_t}, \ldots, \lambda_{i_{t+1}-1}),\ \beta^{(t)} = (\beta_{i_t}, \ldots, \beta_{i_{t+1}-1}), \quad 1 \le t \le s$$

is primitive. We then have

$$K_{\lambda\beta} = \prod_{t=1}^{s} K_{\lambda^{(t)}\beta^{(t)}}.$$

This observation of Berenstein and Zelevinsky [**7**] is the justification for the terminology *primitive pair*. Since the set of indices $i_1, \ldots, i_{s+1}$ decomposing (λ, β) into primitive pairs does not change when we scale λ and β by a parameter n, this decomposition carries over to the stretched Kostka coefficients:

$$\mathcal{K}_{\lambda\beta}(n) = \prod_{t=1}^{s} \mathcal{K}_{\lambda^{(t)}\beta^{(t)}}(n).$$

Several combinatorial interpretations of Kostka coefficients have been constructed. Most classically, $K_{\lambda\beta}$ is the number of semi-standard Young tableaux with shape λ and content β (see, *e.g.*, [**63**]). Of particular interest for our study is the representation of Kostka coefficients as the number of lattice points in particular families of rational polytopes (*i.e.*, polytopes whose vertices have rational coordinates). Gelfand and Tsetlin provided the first such model in 1950 [**24**], which we employ in our study below. More recently, King, Tollu, and Toumazet have introduced K-hive polytopes [**35**], which they used to motivate their conjectures described above. Moreover, they deduce from their model additional information about the structure of the polynomials $\mathcal{K}_{\lambda\beta}(n)$. Among other results, they provide an interpretation for the roots of $\mathcal{K}_{\lambda\beta}(n)$.

The polyhedral models provide a natural geometric interpretation of the polynomial $\mathcal{K}_{\lambda\beta}(n)$ and its degree. For example, to each Kostka coefficient $K_{\lambda\beta}$, there corresponds a Gelfand–Tsetlin polytope $GT_{\lambda\beta} \subset \mathbb{R}^D$ such that $K_{\lambda\beta} = \left|GT_{\lambda\beta} \cap \mathbb{Z}^D\right|$. As a consequence of the definition of Gelfand–Tsetlin polytopes (Definition 1.9), scaling λ and β by a factor of n corresponds to dilating the polytope $GT_{\lambda\beta}$ by a factor of n:

$$\mathcal{K}_{\lambda\beta}(n) = \left|nGT_{\lambda\beta} \cap \mathbb{Z}^D\right|,$$

where $D = \dim X_r = \binom{r+1}{2}$. In other words, $\mathcal{K}_{\lambda\beta}(n)$ is the Ehrhart polynomial of a rational polytope. Ehrhart proved that if P is a d-dimensional rational polytope, then the function $i_P(n)$ counting the number of integer lattice points in nP is a quasi-polynomial function of n with degree d [**19**]. Our proof of Theorem 5.1 depends on this interpretation of $\deg \mathcal{K}_{\lambda\beta}(t)$ as the dimension of the Gelfand–Tsetlin polytope $GT_{\lambda\beta}$.

Tilings of GT-patterns were used in Chapter 4 to study the properties of vertices of GT-polytopes, establishing in particular that they can have arbitrarily large denominators. In the present chapter, we move to the opposite end of the face lattice and apply the tiling machinery to points in the interior of GT-polytopes. We use the notation $\operatorname{int}(P)$ to denote the *relative interior* of a polytope P, that is, the interior of P with respect to the affine space that it spans. As an immediate corollary to Theorem 4.3, we get:

Corollary 5.2. *If* $\mathbf{x} \in \operatorname{int}(GT_{\lambda\beta})$*, then* $\dim GT_{\lambda\beta} = \dim\ker A_{\mathbf{x}}$.

If $GT_\lambda \neq \varnothing$, the tilings of GT-patterns in the relative interior of GT_λ have an easy characterization. Observe that if a block of entries $x_{kr}, x_{k+1,r}, \ldots, x_{\ell r}$ in the top row of a GT-pattern all have the same value κ_p, then the Gelfand–Tsetlin inequalities (1.3) require that the entries x_{ij} with $k \le i \le \ell-(r-j)$ all assume that same value κ_p. These are the entries that lie within the triangular region whose horizontal edge consists of the terms $x_{kr}, x_{k+1,r}, \ldots, x_{\ell r}$ and whose diagonal edges run parallel to the diagonals of the GT-pattern. Let T_p be the tile of entries in this region. That is, if $\lambda = (\kappa_1^{v_1}, \ldots, \kappa_m^{v_m}) \in \mathbb{R}^r$, put

$$T_p = \left\{ (i,j) \in \mathbb{Z}^2 \,:\, \sum_{q=1}^{p-1} v_q < i \le \sum_{q=1}^{p} v_q - (r-j) \right\}, \quad 1 \le p \le m.$$

Define the *generic interior tiling* $\mathscr{T}_\lambda$ associated with λ to be the tiling consisting of the T_p's together with a distinct tile for each entry not contained in one of the T_p's. See Figure 4.1 for an example of a generic interior tiling when $(v_1, \ldots, v_5) = (3,1,2,1,4)$. The shaded tile at the bottom of the pattern, and each of the unshaded free tiles, contains only a single entry.

The next theorem characterizes when a GT-pattern $\mathbf{x}$ has the generic interior tiling and gives the dimension of the GT-polytope containing $\mathbf{x}$. This is the main result from which Theorem 5.1 will follow.

Theorem 5.3. *Let* $\lambda = (\kappa_1^{v_1}, \ldots, \kappa_m^{v_m}) \in \mathbb{R}^r$.

(1) *A GT-pattern* $\mathbf{x} \in GT_\lambda$ *is in the relative interior of* GT_λ *if and only if the tiling of* $\mathbf{x}$ *is the generic interior tiling* $\mathscr{T}_\lambda$.
(2) *If* $\mathbf{x}$ *has the generic interior tiling* $\mathscr{T}_\lambda$ *and* $m \ge 2$*, then*

$$\dim GT(\mathbf{x}) = \binom{r-1}{2} - \sum_{p=1}^{m} \binom{v_p}{2},$$

(*where we evaluate* $\binom{1}{2} = 0$).

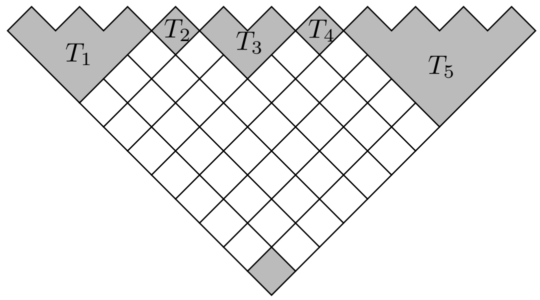

FIGURE 5.1. The tiling of a point in the interior of GT_λ.

PROOF. To prove part (1), note that the GT-inequalities given in (1.3) include the facet-defining inequalities of GT_λ in X_r. A point $\mathbf{x}$ lies in the relative interior of GT_λ if and only if $\mathbf{x}$ satisfies with equality *only* those GT-inequalities that are satisfied with equality by every point in GT_λ.

We claim that the GT-inequalities satisfied with equality by every point in GT_λ are precisely the ones implied by forcing each entry in T_p to be equal to κ_p for $1 \le p \le m$. Since every point in GT_λ must satisfy those equalities, it remains only to exhibit a point $\mathbf{x} \in GT_\lambda$ satisfying *only* those equalities. That is, we need to construct an $\mathbf{x} \in GT_\lambda$ with the generic interior tiling.

Such a point exists as a direct consequence of Theorem 4.12. However, we give here a simpler construction that suffices for interior tilings. For $1 \le p \le m-1$, consider the upper-left-to-lower-right diagonals that abut the tile T_p. Let S_p be the set of entries in these diagonals. Apply a total order to S_p by reading each diagonal from left to right, and then reading the diagonals themselves from left to right. Finally, construct $\mathbf{x}$ by assigning values to the entries in S_p according to a strictly monotonically decreasing function mapping S_p into the open interval $]\kappa_{p+1}, \kappa_p[$. For example, in the portion depicted in Figure 5.2, we fill the tiles with strictly decreasing values in the order indicated by the arrows. It is easy to see that $\mathbf{x}$ is a GT-pattern and that its tiling is $\mathscr{T}_\lambda$, so part (1) is proved.

To prove part (2), suppose that $\mathbf{x}$ has the interior tiling. Then, by part (1), we have that $\mathbf{x} \in \operatorname{int}(GT(\mathbf{x}))$. Thus, we can apply Corollary 5.2 to compute $\dim GT(\mathbf{x})$. The hypothesis that $m \ge 2$ implies that every free row of $\mathbf{x}$ contains a free tile. Hence, each of the $r-2$ rows of the tiling matrix $A_\mathbf{x}$ contains a nonzero entry. Moreover, since every free tile of the interior tiling $\mathscr{T}_\lambda$ contains only a single entry, every column of $A_\mathbf{x}$ contains only a single 1. Thus, $A_\mathbf{x}$ is in reduced row echelon form (perhaps after a suitable permutation of its columns, which amounts to re-indexing the free tiles). This means that

$$\begin{aligned}\dim GT(\mathbf{x}) &= \dim \ker A_\mathbf{x} \\ &= \#\text{ of columns of } A_\mathbf{x} \ - \ \text{dimension of row span of } A_\mathbf{x} \\ &= \#\text{ of free tiles in } \mathscr{T}_\lambda \ - (r-2).\end{aligned}$$

Thus the computation reduces to finding the number of free tiles in $\mathscr{T}_\lambda$, which is easily done:

$$\binom{r}{2} - \sum_{p=1}^{m} \left(|T_p| - v_p\right) - 1 = \binom{r}{2} - \sum_{p=1}^{m} \binom{v_p}{2} - 1.$$

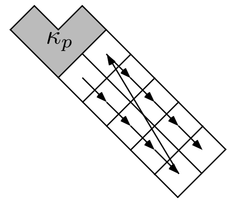

FIGURE 5.2. Filling of a portion of the generic interior tiling.

Hence,

$$\dim GT(\mathbf{x}) = \binom{r}{2} - \sum_{p=1}^{m} \binom{v_p}{2} - (r-1) = \binom{r-1}{2} - \sum_{p=1}^{m} \binom{v_p}{2},$$

as claimed. □

To complete the proof of Theorem 5.1, we call upon a well-known fact from the theory of convex polytopes.

LEMMA 5.4. *Given a polytope $P \subset \mathbb{R}^m$ and a linear map $\pi \colon \mathbb{R}^m \to \mathbb{R}^n$, we have that* $\operatorname{int}(\pi(P)) \subseteq \pi(\operatorname{int} P)$.

The converse containment is also true and easy to prove, but it is unnecessary for our purposes. It is also worth mentioning that this is the point at which the geometry of *convex* polyhedra is crucial to the argument. Lemma 5.4 does not necessarily hold for all polyhedral sets, even if they are connected and full-dimensional.

PROOF OF LEMMA 5.4. Without loss of generality, we assume that $\dim P = m$ and $\dim \pi(P) = n$. Suppose that $y \notin \pi(\operatorname{int} P)$. We show that $y \notin \operatorname{int}(\pi(P))$ by exhibiting an "exit vector" $\bar{y}$ such that, for every $\varepsilon > 0$, $y + \varepsilon\bar{y} \notin \pi(P)$. Since we assume that $\pi(P)$ is full-dimensional, this will prove the claim.

Since $y \notin \pi(\operatorname{int}(P))$, we have that $\pi^{-1}(y) \cap \operatorname{int}(P) = \varnothing$. This means that there is a hyperplane H separating $\pi^{-1}(y)$ from $\operatorname{int}(P)$. Let $\bar{x} \in \mathbb{R}^m$ be a normal to H pointing away from P. Then, for every $x' \in \pi^{-1}(y)$ and $\varepsilon > 0$, we have that $x' + \varepsilon\bar{x} \notin P$. Let $\bar{y} = \pi(\bar{x})$. Suppose that $\varepsilon > 0$ and that $x \in \mathbb{R}^m$ is such that $\pi(x) = y + \varepsilon\bar{y}$. Then $x - \varepsilon\bar{x} \in \pi^{-1}(y)$, so $x = x - \varepsilon\bar{x} + \varepsilon\bar{x} \notin P$. In other words, $y + \varepsilon\bar{y} \notin \pi(P)$, proving the claim. □

Let $D(\lambda) \subset \mathbb{R}^r$ be the image of GT_λ under the map $\mathbf{wt} \colon X_r \to \mathbb{R}^r$. Note that for $\beta \in D(\lambda)$, we have $GT_{\lambda\beta} = \mathbf{wt}^{-1}(\beta) \cap GT_\lambda$. It is well-known that $GT_{\lambda\beta} \neq \varnothing$ if and only if $\beta \trianglelefteq \lambda$ [**39**]. For, if $\beta \trianglelefteq \lambda$, then $K_{\lambda\beta} > 0$, so $GT_{\lambda\beta}$ contains an integral point. Conversely, if $GT_{\lambda\beta} \neq \varnothing$, then some integral multiple $nGT_{\lambda\beta} = GT_{n\lambda,n\beta}$ contains an integral point. Hence, $K_{n\lambda,n\beta} \neq 0$, which, since Kostka coefficients satisfy the saturation property (see the remarks following Conjecture 3.8), implies that $\beta \trianglelefteq \lambda$. This establishes the following lemma:

LEMMA 5.5. *Suppose that $\lambda \in \mathbb{Z}^r$ is a partition. Then $D(\lambda) = \{\beta \in \mathbb{R}^r : \beta \trianglelefteq \lambda\}$. Consequently, if $\beta \triangleleft \lambda$, then $\beta \in \operatorname{int}(D(\lambda))$.*

Putting together the preceding results, we are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. From Lemmas 5.5 and 5.4, we have that

$$\beta \in \mathrm{int}(D(\lambda)) \subset \mathbf{wt}(\mathrm{int}(GT_\lambda)).$$

Hence,

$$GT_{\lambda\beta} \cap \mathrm{int}(GT_\lambda) = \mathbf{wt}^{-1}(\beta) \cap \mathrm{int}(GT_\lambda) \neq \varnothing.$$

Choose $\mathbf{x} \in GT_{\lambda\beta} \cap \mathrm{int}(GT_\lambda)$. By part (1) of Theorem 5.3, $\mathbf{x}$ has the interior tiling $\mathscr{T}_\lambda$, so, by part (2) of Theorem 5.3,

$$\begin{aligned}\deg \mathcal{K}_{\lambda\beta}(n) &= \dim GT_{\lambda\beta} = \dim GT(\mathbf{x}) \\ &= \binom{r-1}{2} - \sum_{p=1}^{m} \binom{v_p}{2},\end{aligned}$$

as claimed. $\square$

APPENDIX A

The minimum quasi-period of the Ehrhart quasi-polynomial of a rational polytope

A.1. Introduction

Recall that an *integral* (respectively, *rational*) *polytope* is a polytope whose vertices have integral (respectively, rational) coordinates. Given a rational polytope $P \subset \mathbb{R}^d$, the *denominator* of P is

$$\mathcal{D}(P) = \min\{n \in \mathbb{Z}_{>0} : nP \text{ is an integral polytope}\}.$$

Ehrhart proved [**19**] that, if $P \subset \mathbb{R}^d$ is a rational polytope, then there is a quasi-polynomial function $i_P : \mathbb{Z} \mapsto \mathbb{Z}$ with *quasi-period* $\mathcal{D}(P)$ such that, for $n \geq 0$,

$$i_P(n) = |nP \cap \mathbb{Z}^d|.$$

This means that there exist polynomial functions $f_1, \ldots, f_{\mathcal{D}(P)}$ such that $i_P(n) = f_j(n)$ for $n \equiv j \bmod \mathcal{D}(P)$. In particular, if P is integral, then $\mathcal{D}(P) = 1$, so i_P is a polynomial function. We call i_P the *Ehrhart quasi-polynomial of* P.

We know that $\mathcal{D}(P)$ is *a* quasi-period of the Ehrhart quasi-polynomial of P, but what is the *minimum* quasi-period? Of course, it must divide $\mathcal{D}(P)$, and it very often equals $\mathcal{D}(P)$. Though this is not always the case, very few counterexamples were previously known. R. P. Stanley provided an example of a polytope P with denominator $\mathcal{D}(P) = 2$ where the minimum period is 1, that is, where the Ehrhart quasi-polynomial is actually a polynomial [**63**, Example 4.6.27]. Stanley's example is the 3-dimensional pyramid P with vertices $(0,0,0)^t$, $(1,0,0)^t$, $(0,1,0)^t$, $(1,1,0)^t$, and $(1/2, 0, 1/2)^t$. In this case, $i_P(n) = \binom{n+3}{3}$.

We say that *quasi-period collapse* occurs when the minimum quasi-period is strictly less than the denominator of the polytope. We say that P has *full quasi-period* if the minimum quasi-period equals the denominator of the polytope. Stanley's example raises some natural questions. In what dimensions can quasi-period collapse occur? Can quasi-period collapse occur for P such that $\mathcal{D}(P) > 2$? What values may the minimum quasi-period be when it is not $\mathcal{D}(P)$? All of these questions are answered in the next section.

In Section A.2, we provide (Theorem A.2) an infinite class of 2-dimensional triangles such that, for any $\mathcal{D}$, there is a triangle P in this class with denominator $\mathcal{D}$, but such that $i_P(n)$ is actually a polynomial. In fact, for any $d \geq 2$ and for any $\mathcal{D}$ and s with $s|\mathcal{D}$, there is a d-dimensional polytope with denominator $\mathcal{D}$ but with minimum period s. Such quasi-period collapse cannot occur in dimension 1, however: rational 1-dimensional polytopes always have full quasi-period (Theorem A.1). We use the method of proof in Theorem A.2 to motivate two conjectures on when the Ehrhart quasi-polynomial of a polytope is a polynomial (Conjecture A.4) when two polytopes have the same Ehrhart quasi-polynomial (Conjecture A.6).

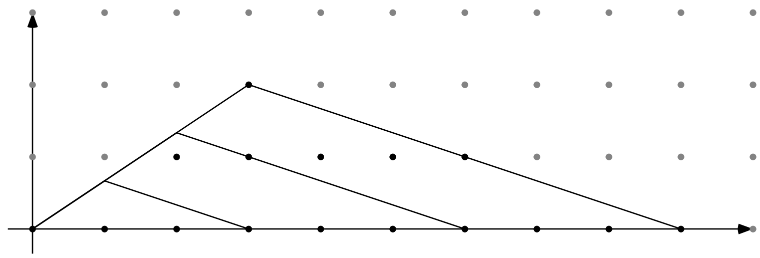

FIGURE A.1. The first three dilations of P when $\mathcal{D} = 3$

A.2. Example of quasi-period collapse

First, we prove that period collapse cannot happen in dimension 1.

THEOREM A.1. *The quasi-polynomials of rational 1-dimensional polytopes always have full period.*

PROOF. In this case, P is simply a segment $[\frac{p}{q}, \frac{r}{s}]$ (where the integers p, q, r, and s are chosen so that the fractions are fully reduced). Write $\mathcal{D} = \mathcal{D}(P) = \operatorname{lcm}(s, q)$.

On the one hand, we clearly have that

$$i_P(n) = \left\lfloor n\frac{r}{s} \right\rfloor - \left\lceil n\frac{p}{q} \right\rceil + 1. \tag{A.1}$$

On the other hand, there exist $\mathcal{D}$ polynomials $f_1(n), \ldots, f_{\mathcal{D}}(n)$ such that $i_P(n) = f_j(n)$, for $n \equiv j \bmod \mathcal{D}$. The claim is that i_P has period $\mathcal{D}$. To show this, it suffices to show that the constant term of $f_j(n)$ is 1 if and only if $j = \mathcal{D}$.

Since P is one-dimensional, we have that, for each $j \in \{1, 2, \ldots, \mathcal{D}\}$, the polynomial $f_j(n)$ is linear, and therefore it is determined by its values at $n = j$ and $n = j + \mathcal{D}$. Interpolating using (A.1) yields

$$f_j(n) = \left(\frac{r}{s} - \frac{p}{q}\right) n + 1 - \left(\left\lceil j\frac{p}{q} \right\rceil - j\frac{p}{q}\right) - \left(j\frac{r}{s} - \left\lfloor j\frac{r}{s} \right\rfloor\right).$$

The constant term is 1 if and only if q and s both divide j, which happens if and only if $j = \mathcal{D}$. □

While, in dimension 1, nothing (with respect to quasi-period collapse) is possible, in dimension 2 and higher, anything is possible, as the following theorem demonstrates.

THEOREM A.2. *Given $d \geq 2$, and given $\mathcal{D}$ and s such that $s|\mathcal{D}$, there exists a d-dimensional polytope with denominator $\mathcal{D}$ whose Ehrhart quasi-polynomial has minimum period s.*

PROOF. We first prove the theorem in the case where $d = 2$ and $s = 1$; that is, we exhibit a polygon with denominator $\mathcal{D}$ for which $i_P(n)$ is actually a polynomial in n. Given $\mathcal{D} \geq 2$, let P be the triangle with vertices $(0,0)^t$, $(1, \frac{\mathcal{D}-1}{\mathcal{D}})^t$, and $(\mathcal{D}, 0)^t$ (see Figure A.1). We will prove that $i_P(n)$ is a polynomial. Our method is to divide the triangle into two pieces and then to glue a unimodular image of one of the pieces to the other so as to form an integral triangle.

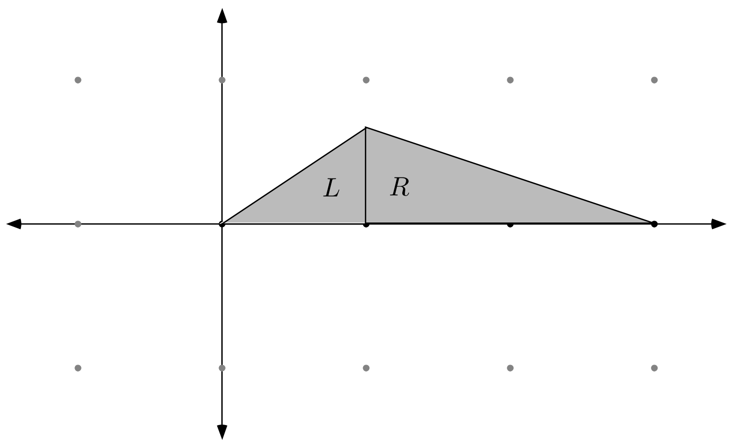

FIGURE A.2. Dissecting P into two right triangles.

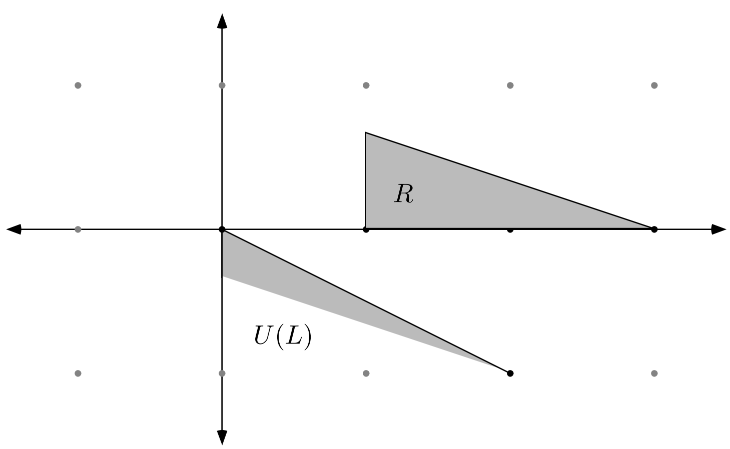

FIGURE A.3. Acting on L by the unimodular transformation U.

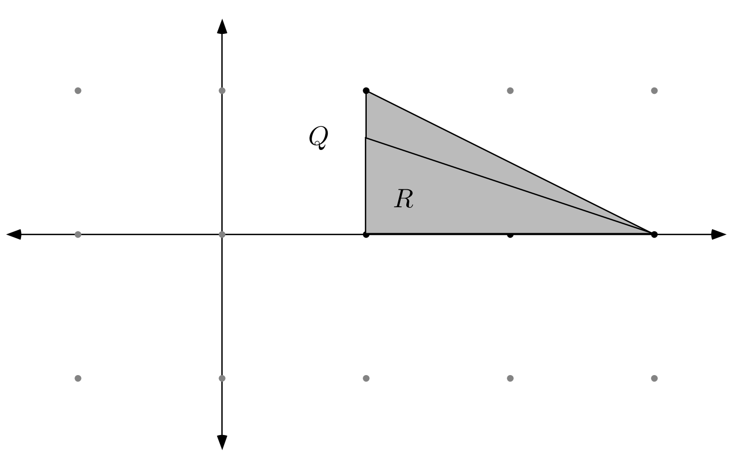

FIGURE A.4. Translating $U(L)$ to form the integral triangle Q.

Divide P into two right triangles by the line $x = 1$ (see Figure A.2). Let L be the "one-third-open" right triangle strictly to the left of the line, and let R be the closed right triangle to the right. Thus we have

$$L = \operatorname{conv}\{(0,0)^t, (1,0)^t, (1, \tfrac{\mathcal{D}-1}{\mathcal{D}})^t\} \setminus [(1,0)^t, (1, \tfrac{\mathcal{D}-1}{\mathcal{D}})^t]$$
$$R = \operatorname{conv}\{(1,0)^t, (\mathcal{D},0)^t, (1, \tfrac{\mathcal{D}-1}{\mathcal{D}})^t\}$$

Recall that an *affine unimodular map* $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is a map of the form $\varphi(x) = Ax + b$, where $A \in \mathbb{Z}^{2\times 2}$ has determinant ± 1 and $b \in \mathbb{Z}^2$. The number of lattice points in a convex set and its dilations is invariant under the action of a unimodular map. Let T be the image of L under the linear map $U : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$U(x) = \begin{bmatrix} \mathcal{D}-1 & -\mathcal{D} \\ -1 & 1 \end{bmatrix} x.$$

Then

$$T = \operatorname{conv}\{(0,0)^t, (\mathcal{D}-1,-1)^t, (0,-1/\mathcal{D})^t\} \setminus [(\mathcal{D}-1,-1)^t, (0,-1/\mathcal{D})^t].$$

(See Figure A.3.) Now translate T by the vector $(1,1)^t$. Then the triangle $T+(1,1)^t$ is the result of applying an affine unimodular map to L. Gluing this triangle to R produces a partition of the *integral* triangle

$$Q = \operatorname{conv}\{(1,0)^t, (1,1)^t, (\mathcal{D},0)^t\}.$$

(See Figure A.4). Since this triangle is integral, it has a polynomial Ehrhart quasi-polynomial. Indeed, it is easy to compute that

$$i_Q(n) = \frac{\mathcal{D}-1}{2}n^2 + \frac{\mathcal{D}+1}{2}n + 1.$$

Moreover, since only unimodular actions were performed, this final triangle has the same Ehrhart quasi-polynomial as the original; that is,

$$i_P(n) = \frac{\mathcal{D}-1}{2}n^2 + \frac{\mathcal{D}+1}{2}n + 1.$$

Now suppose $d = 2$ and s is any divisor of $\mathcal{D}$. Let P' be the pentagon with vertices $(0,0)^t, (1, \frac{\mathcal{D}-1}{\mathcal{D}})^t, (\mathcal{D},0), (\mathcal{D}, -\frac{1}{s})^t$, and $(0, -\frac{1}{s})^t$. If P is the triangle defined as before, then $nP' \setminus nP$ contains $\lfloor \frac{n}{s} \rfloor \cdot (\mathcal{D}n+1)$ lattice points, and so

$$i_{P'}(n) = i_P(n) + \left\lfloor \frac{n}{s} \right\rfloor \cdot (\mathcal{D}n+1),$$

which has minimum period s.

Now suppose d is greater than 2. Let P' be the pentagon defined as before, and let $P'' = P' \times [0,1]^{d-2}$, a polytope of dimension d. Then

$$i_{P''}(n) = (n+1)^{d-2} i_{P'}(n),$$

which also has minimum period s. □

A.3. Conjectures

In the preceding proof, we showed that a nonintegral polygon had a polynomial Ehrhart function because it was, in some sense, a disguised integral polygon—it was an integral polygon up to rearrangement and unimodular transformation of its pieces. We conjecture that all polytopes with polynomial Ehrhart quasi-polynomials are "disguised" integral polytopes in this sense. To make this precise, let $\mathcal{G} = \mathbb{Z}^d \rtimes \operatorname{Aut}(\mathbb{Z}^d)$ be the group of affine unimodular transformation on $\mathbb{R}^d$.

DEFINITION A.3. We say that two polytopes P and Q are *$\mathcal{G}$-equidecomposable* if there exist simplices $T_1, \ldots, T_r$ and affine unimodular transformations $U_1, \ldots, U_r \in \mathcal{G}$ such that

$$\coprod_{j=1}^{r} \operatorname{int} T_j = P \qquad \text{and} \qquad \coprod_{j=1}^{r} \operatorname{int} U_j(T_j) = Q.$$

(Here, $\coprod$ indicates disjoint union and $\operatorname{int} T_j$ is the relative interior of T_j.)

CONJECTURE A.4. *Suppose that P is a rational polytope and that $i_P(n)$ is a polynomial function of n. Then there exists an integral polytope Q such that P and Q are $\mathcal{G}$-equidecomposable.*

One way to disprove this conjecture would be to exhibit an Ehrhart polynomial that is not the Ehrhart polynomial of any *integral* polytope. Haase and Schicho have characterized all Ehrhart polynomials of integral polygons [**26**]. Therefore, a partial result towards confirming Conjecture A.4 would be to show that the characterization of Haase and Schicho includes the Ehrhart polynomials of all rational polygons.

Another phenomenon that appeared in the proof of Theorem A.2 was the equality of the Ehrhart quasi-polynomials of two distinct polytopes.

DEFINITION A.5. We say that two rational polytopes P and Q are *Ehrhart equivalent* if and only if their Ehrhart quasi-polynomials are equal: $i_P(n) = i_Q(n)$.

We also conjecture that this equality only happens when, as in the proof of Theorem A.2, the two polytopes are $\mathcal{G}$-equidecomposable. If all Ehrhart polynomials are the Ehrhart polynomials of integral polytopes, then this conjecture implies Conjecture A.4:

CONJECTURE A.6. *Two rational polytopes are Ehrhart equivalent if and only if they are $\mathcal{G}$-equidecomposable.*

I. Bárány and J. Kantor have studied integral polytopes with equal *universal counting functions*, a stronger condition that the Ehrhart equivalence defined above [**2**]. The universal counting function $\mathcal{U}_P$ of an integral polytope maps each lattice $\mathbb{L}$ containing $\mathbb{Z}^d$ to the number of points of $\mathbb{L}$ contained in P. Thus we have that $i_P(n) = \mathcal{U}_P(\frac{1}{n}\mathbb{Z}^d)$. Bárány and Kantor give necessary conditions for two integral polytopes having the same universal counting function, and they give necessary and sufficient conditions in the case of $d = 2$. They ask whether an analogue of Conjecture A.6 holds for integral polytopes with the same universal counting functions. In this setting, the rearrangement of pieces equivalent up to unimodular transformation is replaced by the rearrangement of pieces equivalent up to translation and reflection through the origin.

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