REU 2016 Day 2

G. Musiker

Toric cluster variables for the dP_2 quiver

1. Three-term recurrences
2. Cluster algebras and quiver mutation
3. Aztec diamonds, Somos 4 and Somos 5
4. Brane tiling for dP_2 and REU Problem 2

1. $\{x_n : n \geq 1\}$ defined by

$$x_1 = x_2 = 1, \quad x_n = \frac{x_{n-1}+1}{x_{n-2}}$$

$$x_3 = \frac{1+1}{1} = 2, \quad x_4 = \frac{2+1}{1} = 3$$

$$x_5 = \frac{3+1}{2} = 2, \quad x_6 = \frac{2+1}{3} = 1,$$

$$x_7 = \frac{1+1}{2} = 1$$

5-periodic!

Instead, consider

$\{x_n : n \geq 1\}$ defined by

$$x_1 = x_2 = 1, \quad x_n = \frac{x_{n-1}^2 + 1}{x_{n-2}}$$

$$x_3 = \frac{1^2+1}{1} = 2, \quad x_4 = \frac{2^2+1}{1} = 5,$$

$$x_5 = \frac{5^2+1}{2} = 13, \quad x_6 = \frac{13^2+1}{5} = 34,$$

$$x_7 = \frac{34^2+1}{13} = \frac{1157}{13} = 89$$

CLAIM: These are

even-indexed Fibonacci numbers

Let's do the 1st one with variables x_1, x_2

$$x_n = \frac{x_{n-1}+1}{x_{n-2}}$$

$$x_3 = \frac{x_2+1}{x_1}, \quad x_4 = \frac{x_3+1}{x_2} = \frac{\left(\frac{x_2+1}{x_1}\right)+1}{x_2}$$

$$= \frac{x_2+1+x_1}{x_1 x_2}$$

$$x_5 = \frac{x_4+1}{x_3} = \frac{\left[\frac{x_2+1+x_1}{x_1 x_2}\right]+1}{\left[\frac{x_2+1}{x_1}\right]}$$

$$= \frac{(x_2+1+x_1+x_1x_2)}{x_1x_2} \cdot \frac{x_1}{x_2+1} \overset{!}{=} \frac{x_1+1}{x_2}$$

$x_6 = x_1, \quad x_7 = x_2$ Still periodic!

DEF'N: A Laurent polynomial is a rational function whose denominator is a single monomial.

Let's do the 2^{nd} one with variables...

$$\{x_n : n \geq 1\}, \quad x_n = \frac{x_{n-1}^2+1}{x_{n-2}}$$

$$x_3 = \frac{x_2^2+1}{x_1}, \quad x_4 = \frac{\left(\frac{x_2^2+1}{x_1}\right)^2+1}{x_2} = \frac{(x_2^2+1)^2+x_1^2}{x_1^2 x_2}$$

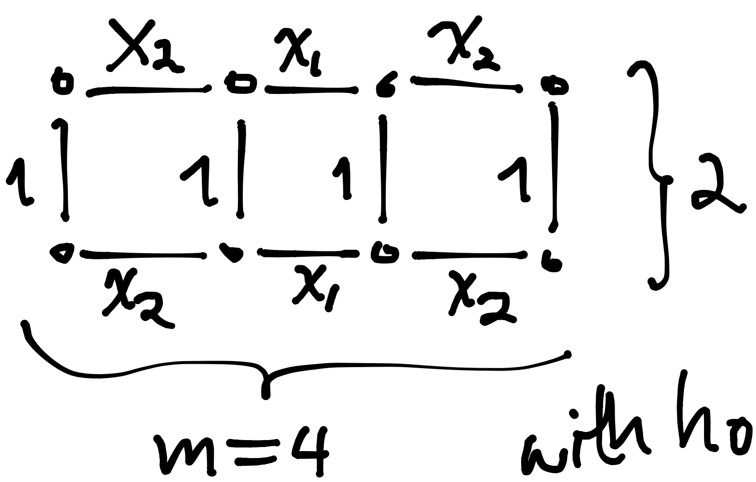
$$x_5 = \frac{\left[\left(\frac{(x_2^2+1)^2+x_1^2}{x_1^2 x_2}\right)^2+1\right]}{\left[\frac{x_2^2+1}{x_1}\right]} = \frac{(x_2^2+1)^3+x_1^4+2x_1^2+2x_1^2x_2^2}{x_1^3x_2^2}$$

Laurent!

CLAIM: These are Laurent polynomials with positive integer coefficients whose numerators have (even-indexed) Fibonacci numbers of terms.

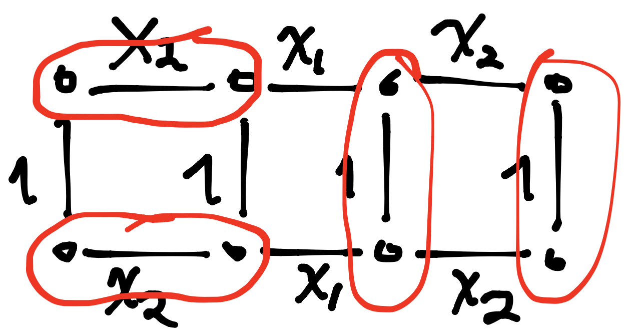
REU EXERCISE 4

i) Let $G_m = 2 \times m$ grid graph



with horizontal edges weighted $x_2, x_1, x_2, x_1, \ldots$

A perfect matching M of a graph G is a subset of edges so that every vertex is covered exactly once.



Prove that $x_n x_{n-2} = x_{n-1}^2 + 1$ satisfies

$$x_n = \frac{1}{x_1^{n-2} x_2^{n-3}} \sum_{\substack{\text{perfect} \\ \text{matchings } M \\ \text{of } G_{2n-4}}} x(M)$$

where $x(M) = \prod_{e \in M} x_e$

ii) (Easy) Corollary: if $x_1 = x_2 = 1$,

then $x_n = F_{2n-4}$ ← a typo was fixed

where $F_0 = F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$

for $n \geq 2$.

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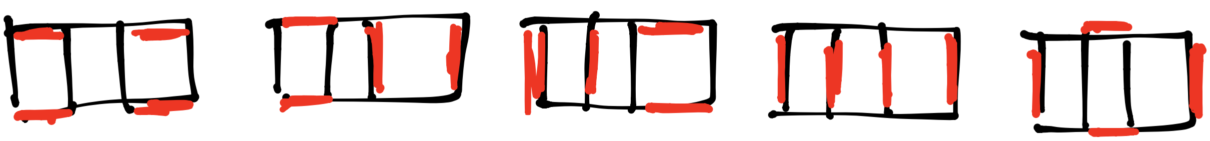
e.g. $n=4$ M in G_4



$$x_2^4 + x_2^2 + x_2^2 + 1 + x_1^2$$

cf. $$\frac{(x_2^2+1)^2 + x_1^2}{x_1^2 x_2}$$

Somos-4 sequence

$\{x_n\}$ $x_1 = x_2 = x_3 = x_4 = 1$

$$x_n x_{n-4} = x_{n-1} x_{n-3} + x_{n-2}^2$$

$$x_5, x_6, x_7, x_8, x_9, \ldots = 2, 3, 7, 23, 59, 314, \ldots$$

$$314 = \frac{59 \cdot 7 + 23^2}{3}$$

Somos-5 sequence

$$x_n x_{n-5} = x_{n-1} x_{n-4} + x_{n-2} x_{n-3}$$

$$1, 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, 274, \ldots$$

$$274 = \frac{83 \cdot 5 + 11 \cdot 37}{3}$$

② DEF'N (Fomin-Zelevinsky 2001)

A cluster algebra is defined to be a subalgebra of $K(x_1,\dots,x_n)$, the field of rational functions, constructed cluster-by-cluster via certain exchange relations.

Generators: specify a finite set $\{x_1,\dots,x_n\}$ called the initial cluster and get n new clusters

$$\{x_1,\dots,x_n\}$$

μ_1 $\quad$ μ_2 $\quad$ .... $\quad$ μ_n

via binomial exchange relations

of the form

$$x_\alpha x_{\alpha'} = \prod_i x_{\gamma_i}^{\alpha_i^+} + \prod_i x_{\gamma_i}^{\alpha_i^-}$$

The set of all such generators are called the cluster variables.

Relations: Induced by these binomial exchange relations

To be more precise, let's introduce quivers...

DEF'N: A *quiver* is a directed graph (but considered within the context of representation theory)

[For the moment, no loops $\bullet \circlearrowleft$ and no 2-cycles $\bullet \rightleftarrows \bullet$, but we'll relax this later]

EXAMPLES: $\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet}$

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, bend left] \arrow[r, bend right] & \underset{2}{\bullet} \end{tikzcd} = \underset{1}{\bullet} \rightrightarrows \underset{2}{\bullet} = \underset{1}{\bullet} \xrightarrow{2} \underset{2}{\bullet}$$

$$= \underset{1}{\bullet} \twoheadrightarrow \underset{2}{\bullet}$$

$$\underset{1}{\bullet} \longleftarrow \underset{2}{\bullet} \longleftarrow \underset{3}{\bullet}$$

For each vertex j of the quiver

$$x_j x_j' = \prod_{i \to j} x_i + \prod_{j \to i} x_i$$

e.g. $\quad 1 \leftarrow 2 \leftarrow 3$

$$x_1 x_1' = 1 + x_2$$

$$x_2 x_2' = x_1 + x_3$$

$$x_3 x_3' = x_2 + 1$$

Quiver mutation (at vertex j)

1) For every 2-path $i \to j \to k$ in Q, add a new arrow $i \to k$.
2) Reverse all arrows incident to j.
3) (Unless otherwise stated) erase all 2-cycles created by the above.

e.g.

$$\mu_1\left(\underset{1}{\bullet} \leftarrow \underset{2}{\bullet} \leftarrow \underset{3}{\bullet}\right) = \underset{1}{\bullet} \rightarrow \underset{2}{\bullet} \leftarrow \underset{3}{\bullet}$$

$$\mu_2\left(\underset{1}{\bullet} \leftarrow \underset{2}{\bullet} \leftarrow \underset{3}{\bullet}\right) = \begin{tikzcd} \underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} \arrow[r] & \underset{3}{\bullet} \arrow[ll, bend right] \end{tikzcd}$$

$$\mu_3\left(\underset{1}{\bullet} \leftarrow \underset{2}{\bullet} \leftarrow \underset{3}{\bullet}\right) = \underset{1}{\bullet} \leftarrow \underset{2}{\bullet} \rightarrow \underset{3}{\bullet}$$

$$\mu_2^2\left(1 \leftarrow 2 \leftarrow 3\right) = \mu_2\left(\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[ll, bend right] \end{tikzcd}\right)$$

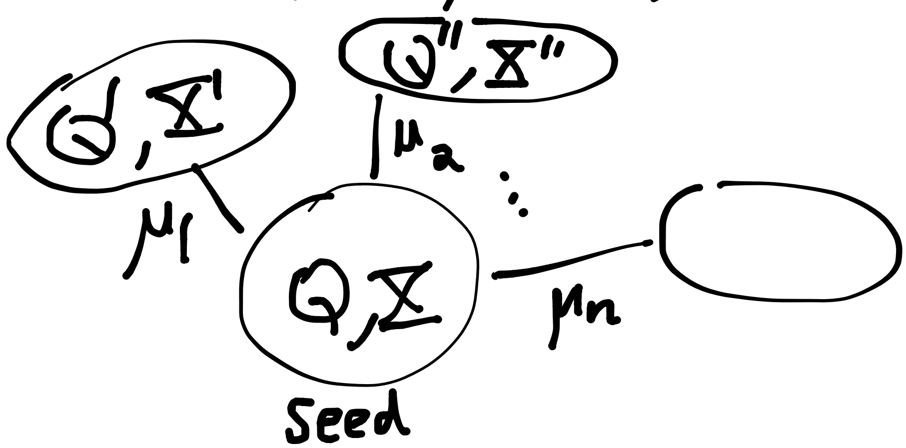
$$= \begin{tikzcd} 1 \arrow[rr, bend left] & 2 \arrow[l] \arrow[l, bend right] & 3 \arrow[l] \end{tikzcd} = 1 \leftarrow 2 \leftarrow 3$$

Cluster mutation

$$\mathbb{X} = \{x_1, x_2, \ldots, x_n\}$$

$$\mu_i \downarrow$$

$$\mathbb{X}' = \{x_1, x_2, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n\}$$



EXAMPLE:

$$\left(\underset{1}{\bullet} \to \underset{2}{\bullet},\ \{x_1, x_2\}\right)$$

$$\Big\downarrow \mu_1$$

$$\left(\underset{1}{\bullet} \leftarrow \underset{2}{\bullet},\ \left\{\frac{x_2+1}{x_1}, x_2\right\}\right)$$

$$\Big\downarrow \mu_2$$

$$\left(\underset{1}{\bullet} \to \underset{2}{\bullet},\ \left\{\frac{x_2+1}{x_1}, \frac{x_2+1+x_1}{x_1 x_2}\right\}\right)$$

$$\Big\downarrow \mu_1$$

$$\left(\underset{1}{\bullet} \leftarrow \underset{2}{\bullet},\ \left\{\frac{x_1+1}{x_2}, \frac{x_2+1+x_1}{x_1 x_2}\right\}\right)$$

$$\Big\downarrow \mu_2$$

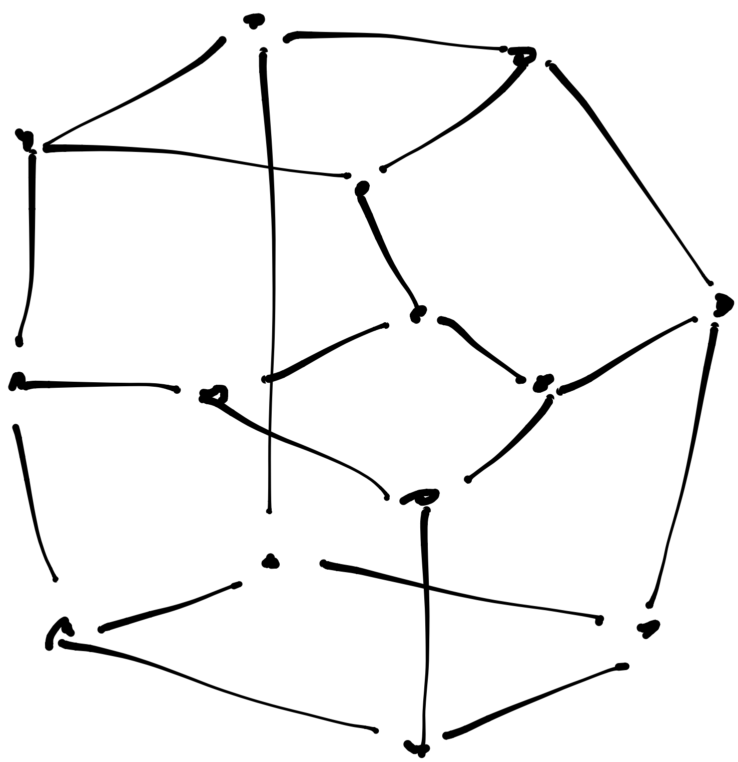
$$\left(\underset{1}{\bullet} \to \underset{2}{\bullet},\ \left\{\frac{x_1+1}{x_2}, x_1\right\}\right)$$

$$\Big\downarrow \mu_1$$

$$\left(\underset{1}{\bullet} \leftarrow \underset{2}{\bullet},\ \{x_2, x_1\}\right)$$

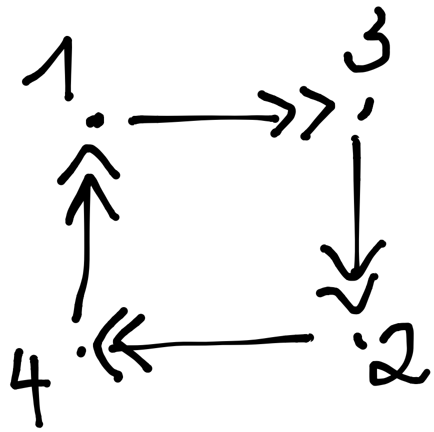
same up to permutation

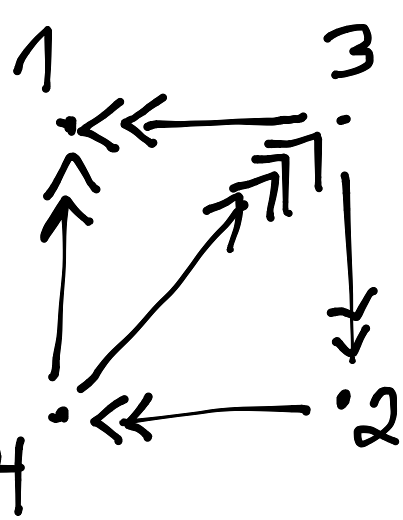
If you do $(1 \leftarrow 2 \leftarrow 3, \{x_1, x_2, x_3\})$
the mutation graph will look like
the associahedron

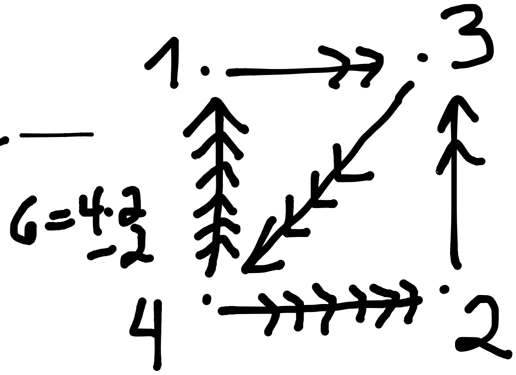


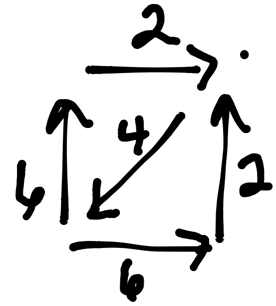
See Example 4.18 of
Fomin-Reading "Root systems and
generalized associahedra"

3. Consider

$Q =$ 

$\mu_1(Q) =$ 

$\mu_3\mu_1(Q) =$ 

$=$ 

$$\mu_2\mu_3\mu_1(Q) = \begin{tikzcd} \bullet \arrow[r, "2"] & \bullet \arrow[d, "2"] \\ \bullet \arrow[u, "6"] \arrow[ur, "8"] & \bullet \arrow[l, "6"] \end{tikzcd}$$

$$\mu_1\mu_2\mu_3\mu_1(Q) = \begin{tikzcd} \bullet \arrow[d, "6"'] & \bullet \arrow[l, "2"'] \arrow[d, "2"] \\ \bullet \arrow[ur, "20"] & \bullet \arrow[l, "6"] \end{tikzcd}$$

$\mu_3\mu_1\mu_2\mu_3\mu_1, \ldots$

THM (Fomin-Zelevinsky)
All cluster variables are Laurent polynomials in the initial seed $\{x_1, \ldots, x_n\}$, with integer coefficients.
"The Laurent phenomenon"

Try in SAGE:

```
Az = ClusterSeed([[1,3,2],
      [3,2,2],[2,0,2],[0,1,2]])

Az.mutate([1,3,2]);

Az.cluster()
```

has 2 monomials 26 5

$$\left[x_0, \frac{*}{x_1}, \frac{*}{x_1^4 x_2 x_3^2}, \frac{*}{x_1^2 x_3}\right]$$

Repeat and they get big quickly...

13 20079877 22849

$$\left[x_0, \frac{*}{x_1^3 x_3^2}, \frac{*}{x_1^{20} x_2^3 x_3^{12}}, \frac{*}{x_1^{12} x_2^2 x_3}\right]$$

Instead let's do this:

$$\cdots\; \mu_2 \mu_1 \mu_4 \mu_3 \mu_2 \mu_1 Q$$

periodically

$$\mu_2\mu_1 Q = \begin{tikzcd} \overset{1}{\bullet} \arrow[d] & \overset{3}{\bullet} \arrow[l, shift left=0.5ex] \arrow[l, shift right=0.5ex] \\ \underset{4}{\bullet} \arrow[r, shift left=0.5ex] \arrow[r, shift right=0.5ex] & \underset{2}{\bullet} \arrow[u] \end{tikzcd} = Q^{opp}$$

(all arrows reversed)

Call $x_5, x_6, x_7, \ldots$ the cluster variables obtained at each stage by this period sequence, i.e.,

$$\{x_1, x_2, x_3, x_4\} \xrightarrow{\mu_1} \{x_5, x_2, x_3, x_4\} \xrightarrow{\mu_2} \{x_5, x_6, x_3, x_4\}$$

$$\longrightarrow \ldots$$

REU EXERCISE 5

i) Argue why (easy)

$$x_n x_{n-4} = \begin{cases} x_{n-1}^2 + x_{n-2}^2 & n \text{ odd} \\ x_{n-2}^2 + x_{n-3}^2 & n \text{ even} \end{cases}$$

a typo was fixed

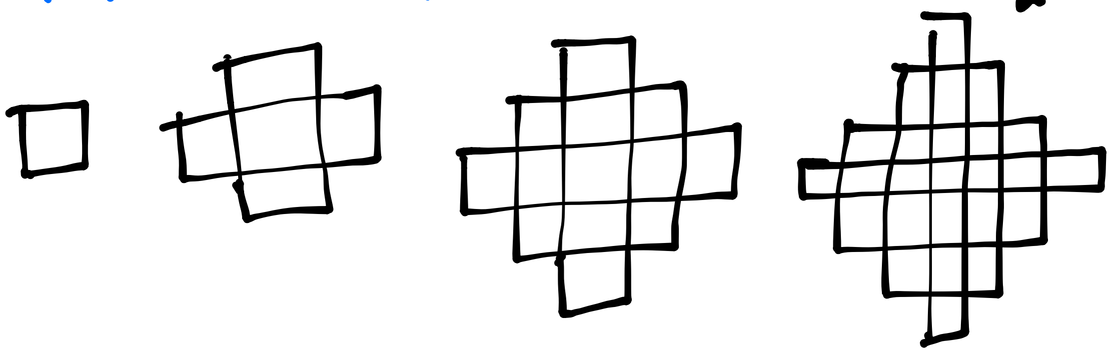
ii) let $x_{2n-1} = x_{2n} = T_n \quad \forall n \geq 1$

Show $T_n T_{n-2} = 2T_{n-1}^2$

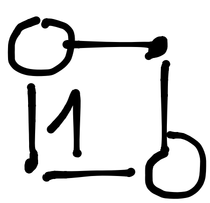
(easy) Show $T_n = 2^{\frac{(n-1)(n-2)}{2}}$ if $T_1 = T_2 = 1$

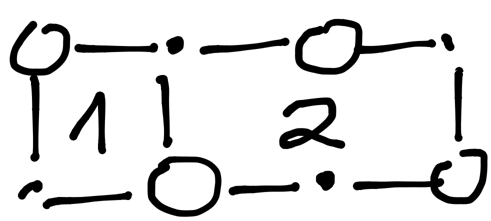
$\{1, 1, 2, 8, 64, 1024, 32768, \ldots\}$

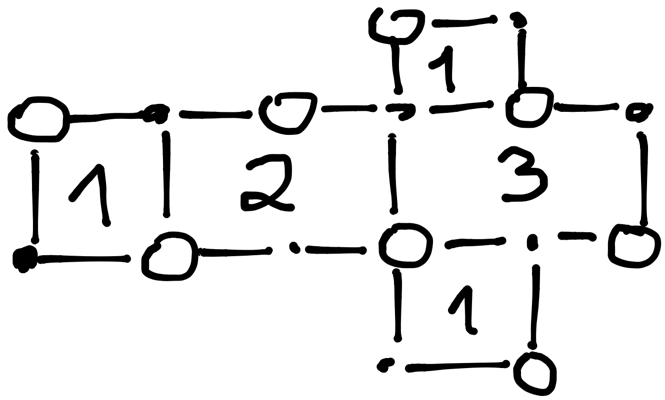
(iii) Show that T_n counts the perfect matchings in the n^{th} Aztec diamond (if $T_1 = T_2 = 1$)

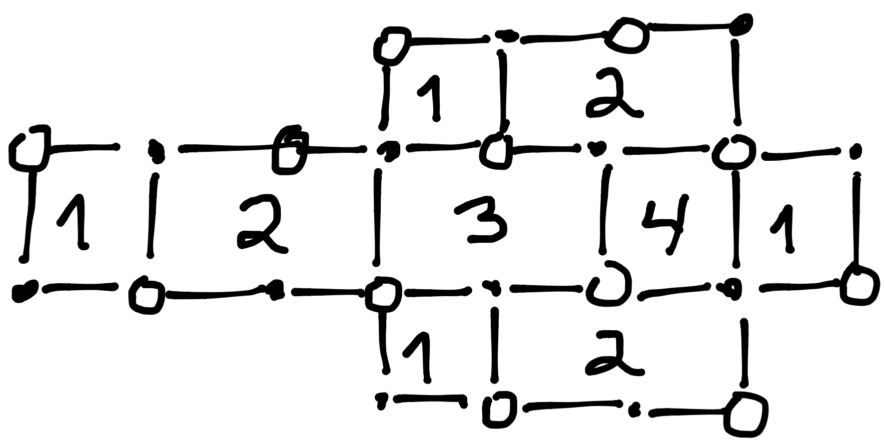


CLAIM: There is a family of graphs

$G_5 =$ 

$G_6 =$ 

$G_7 =$ 

$G_8 =$ 

⋮

(see Bousquet-Melou, Propp, West and Speyer)

whose numbers of perfect matchings is x_n in the Somos-4 sequence.

REU EXERCISE 6

i) Explain why mutating periodically

$$\cdots \mu_2 \mu_1 \mu_4 \mu_3 \mu_2 \mu_1 (Q_{S_4})$$

$$\{x_1, x_2, x_3, x_4\} \xrightarrow{\mu_1} \{x_5, x_2, x_3, x_4\} \rightarrow \cdots$$

gives the Somos-4 sequence

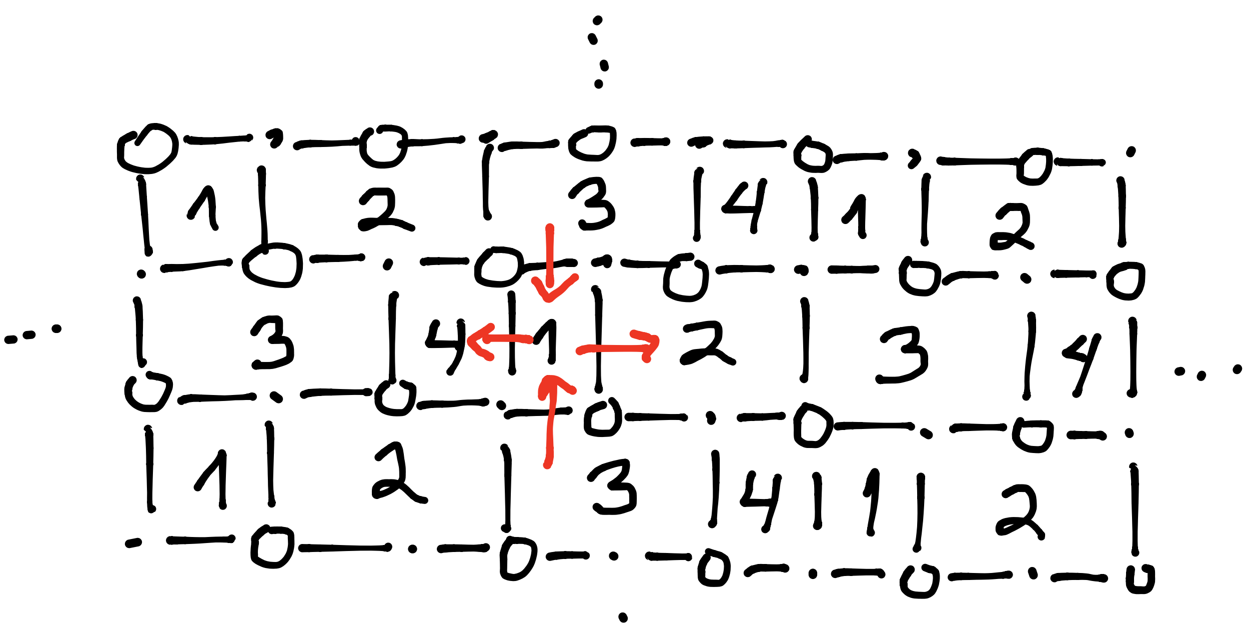
where

$$Q_{S_4} = \begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[d, "3"] \\ 4 \arrow[ur, "2"] & 3 \arrow[l] \arrow[ul, "2"'] \end{tikzcd}$$

ii) Use this recurrence to express x_7 as a Laurent polynomial, and illustrate the perfect matchings of G_7, and spot-check that the weights agree.

to be explained...

The dP_1 brane tiling

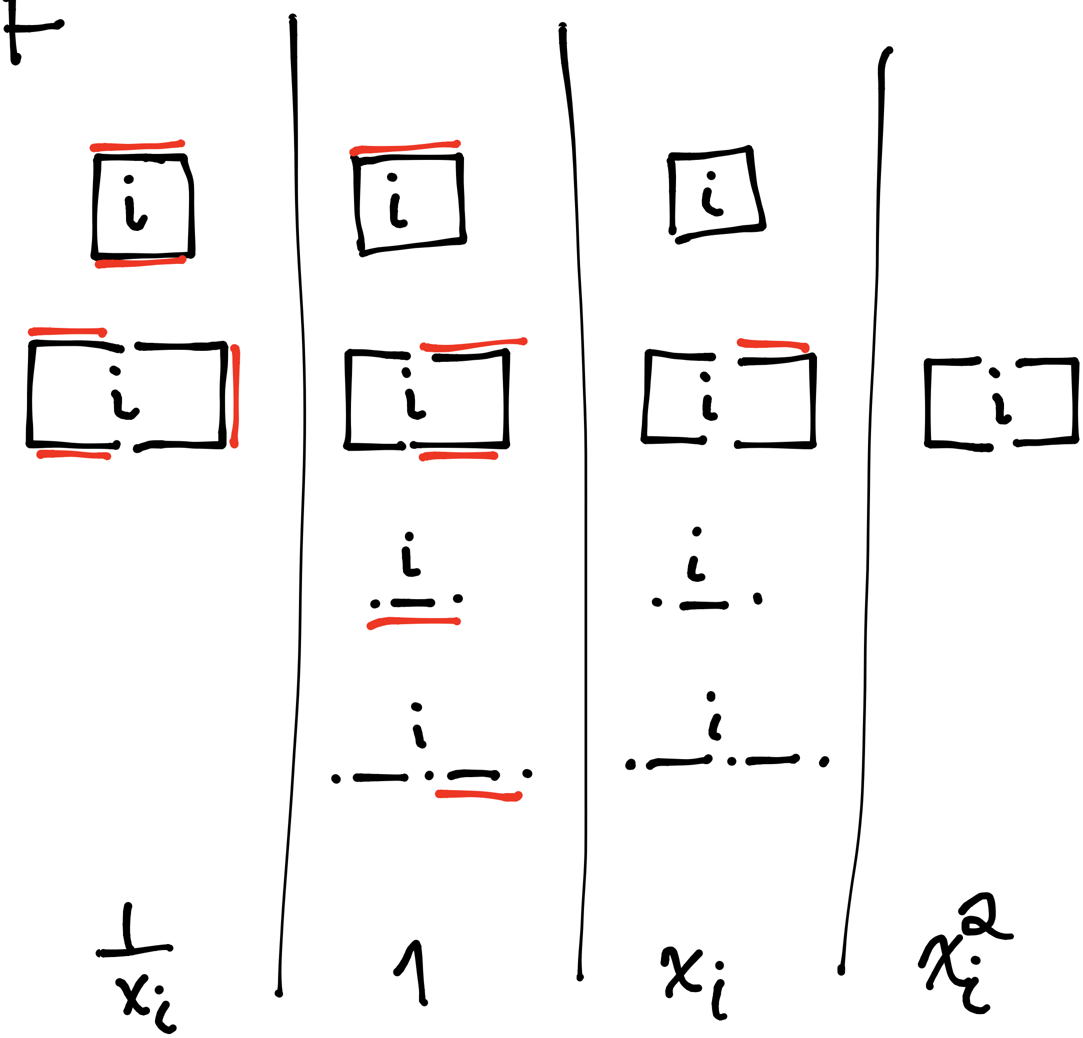


These → correspond to the quiver

Weighting

$$x(M) = \prod_{\substack{\text{faces of } G_n \\ \text{and faces on boundary} \\ \text{of } G_n}} x_F$$

$x_F =$

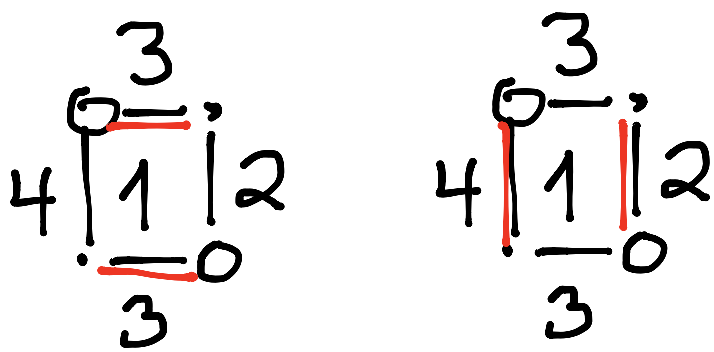


THM: (Special case of Speyer's "octahedron recurrence")

If x_n is given by the Somos-4 sequence, then $$x_n = \sum_{\substack{\text{perfect matchings of} \\ G_n}} x(M)$$

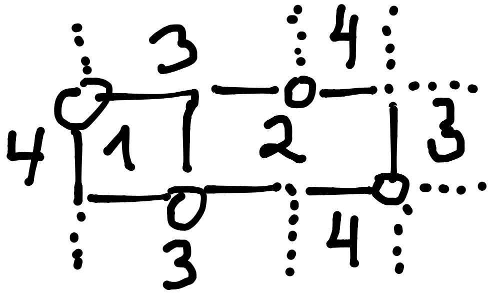
e.g.

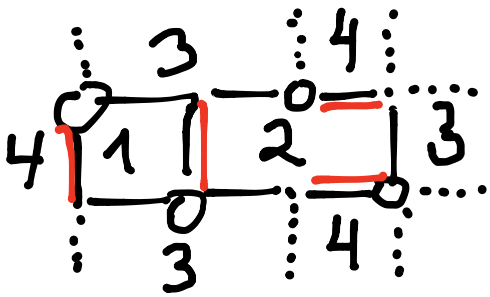
$G_5 =$

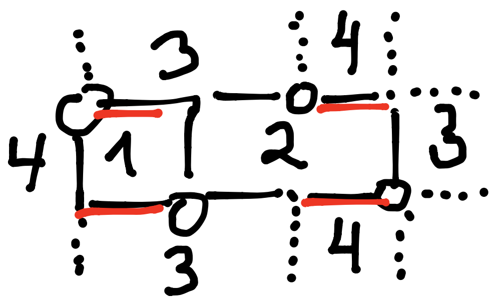


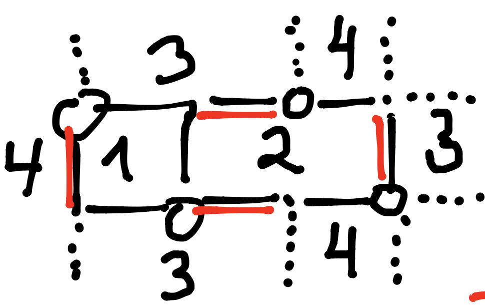
$$\frac{1}{x_1} x_2 \cdot 1 \cdot x_4 \quad + \quad \frac{1}{x_1} \cdot x_3 \cdot x_3$$

$$x_5 = \frac{x_4 x_2 + x_3^2}{x_1}$$

$G_6 =$ 

 $\frac{1}{x_1}\frac{1}{x_2}x_3x_3x_3$

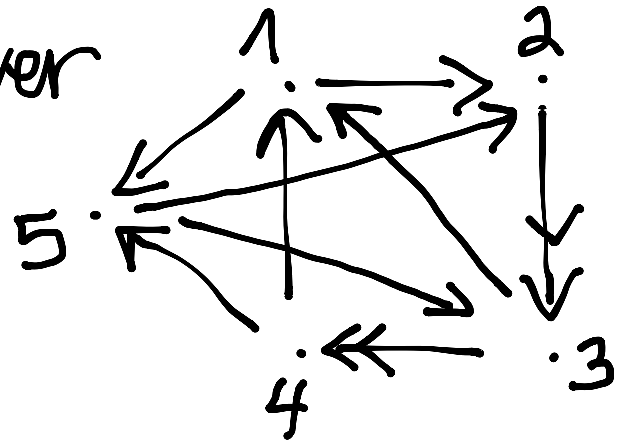
 $+\frac{1}{x_1}x_3x_4$

 $+\frac{1}{x_2}x_4^2$

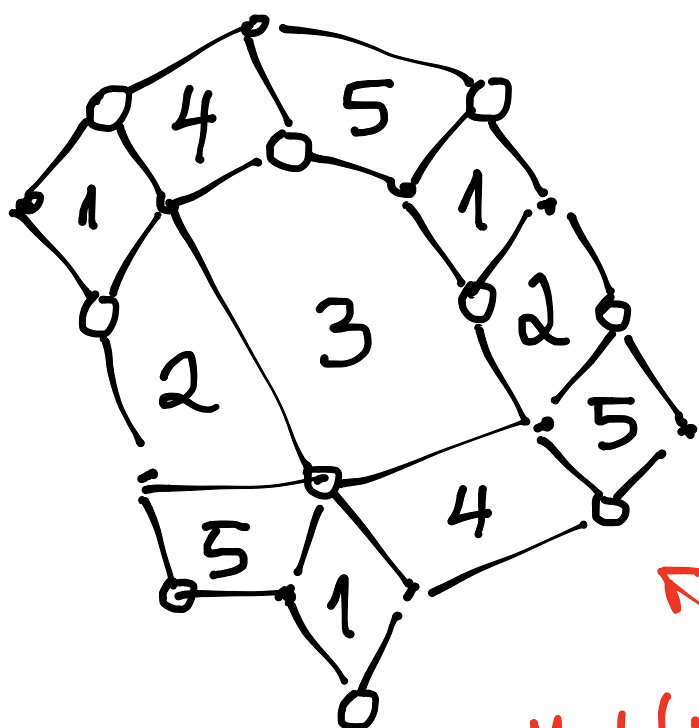
$$= x_6$$

REU PROBLEM 2

dP_2 quiver



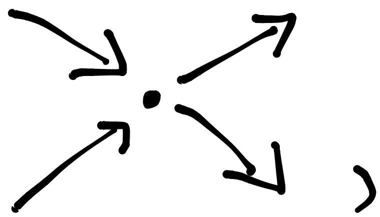
dP_2 brane tiling



Model 12

(see Hanany-Seong "Brane tilings of reflexive polygons")

Consider cluster variables that come from mutating only at vertices



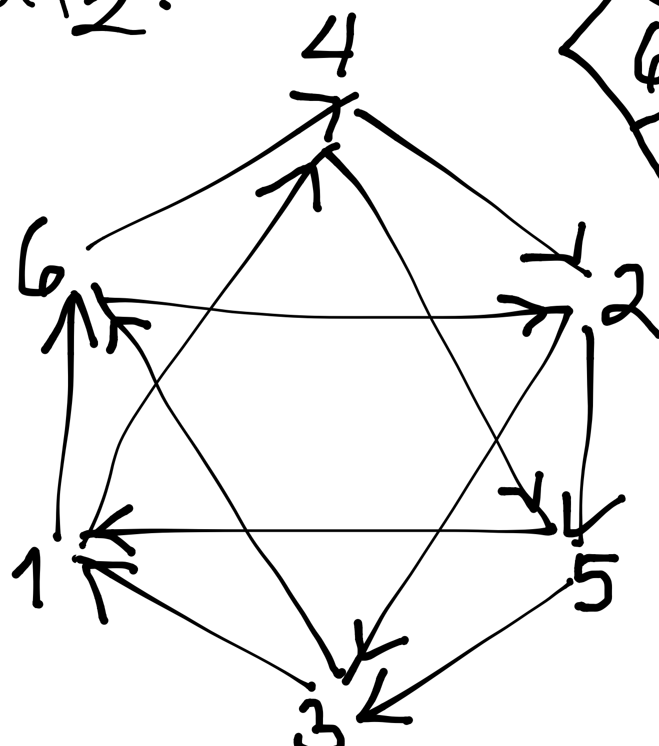
called *toric cluster variables*

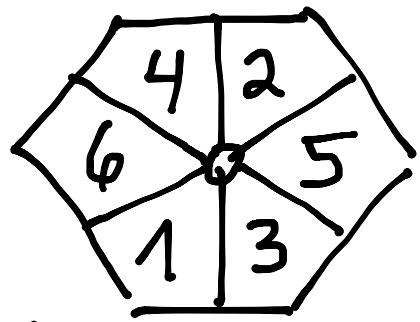
i) Give a $\mathbb{Z}^2$-parametrization for the toric cluster variables for the dP_2 quiver.

ii) Find a family $\{G_{i,j}\}_{(i,j)\in\mathbb{Z}^2}$ of subgraphs of the dP_2 brane tiling so that toric cluster variables x_{ij} satisfy $x_{ij} = \sum_{M \in G_{ij}} x(M)$.

Suggestion: $G_{i,j}$ should be cut out by (nonconvex?) pentagons?

Why dP_2?

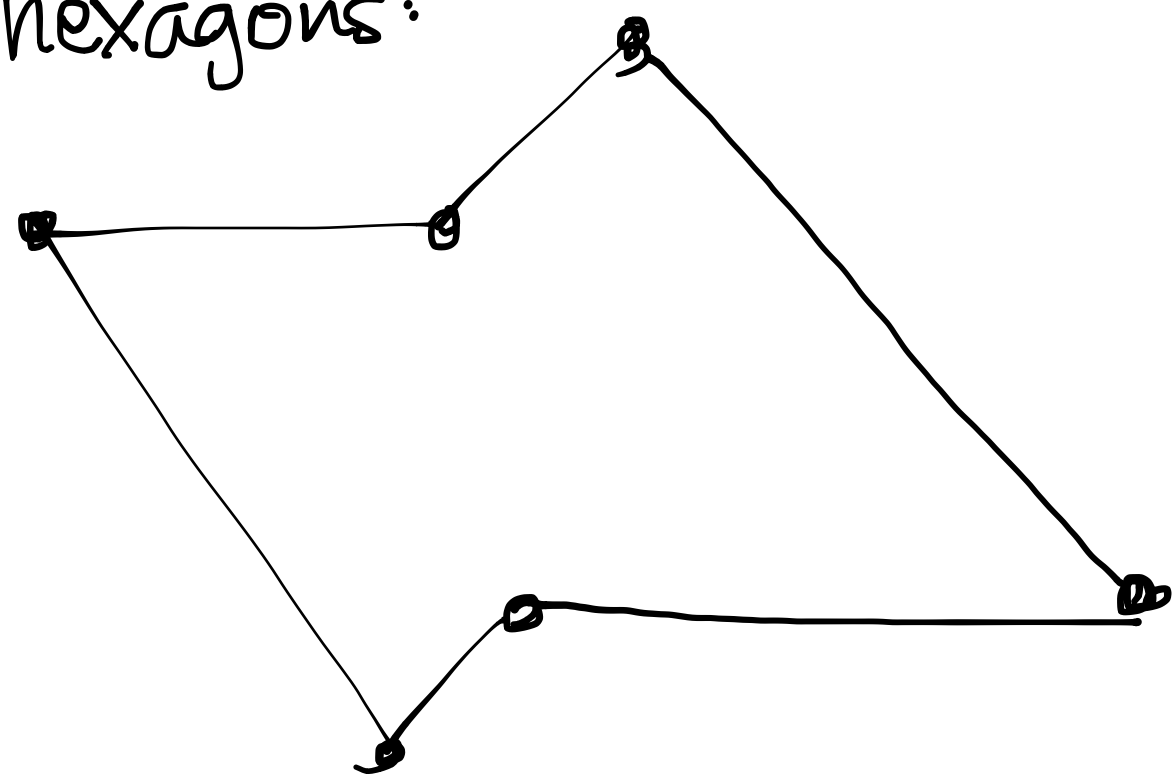
$dP_3 =$ 



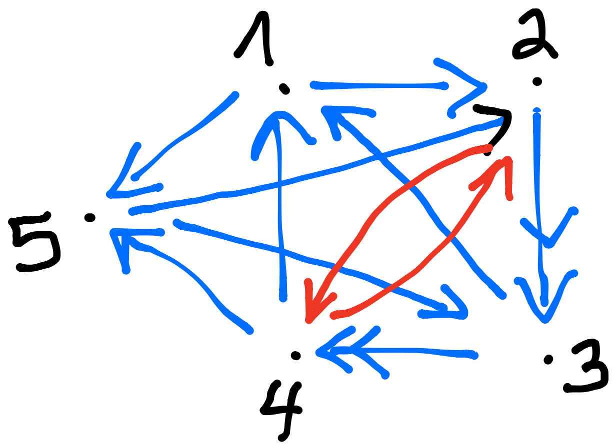
fund-amental domain for the brane tiling

had a story like this, where the subgraphs G inside...

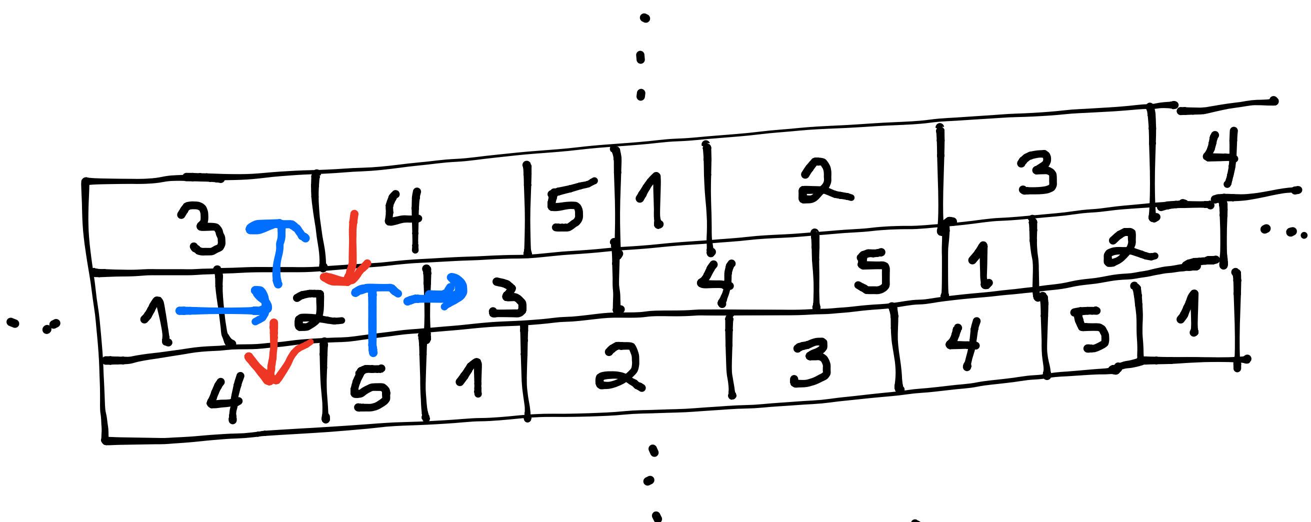
...the brane tiling for dP_3 were cut out by (non-convex) hexagons:



iii) Explain Somos-5 via a $\mathbb{Z}$-parametrized subfamily, adding a 2-cycle to dP_2 as follows...



which eliminates some cluster variables from being toric.



Somos-5 (pseudo-dP_2) brane tiling

iv) Match up the subgraphs of the pseudo-dP_2 brane tiling to the $\mathbb{Z}^1$-subfamily for Somos-5 in the dP_2 brane tiling.