REU 2016 Day 3
K. Patrias

1. Symmetric functions and bases
2. Young tableaux and Schur functions
3. Jacobi-Trudi
4. Problem + Exercises

(1) DEF'N: A (polynomial) function $f(x_1,\dots,x_n)$ is symmetric if for any $\sigma \in S_n$ one has

$$f(x_1,\dots,x_n) = f(x_{\sigma(1)},\dots,x_{\sigma(n)})$$

EXAMPLE:

$$f(x_1,x_2,x_3) = x_1x_2x_3 + 2x_1 + 2x_2 + 2x_3$$

NON EXAMPLE:

$$\cancel{h(x_1,x_2,x_3) = 2x_1x_2 + x_2x_3}$$

Some bases for the algebra of symmetric functions

DEF'N: A partition is a tuple of weakly decreasing positive integers

$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k)$

If $n = \lambda_1 + \dots + \lambda_k$, call it a partition of n.

e.g.

$(4,2,2)$ is a partition of 8

- Monomial symmetric functions (a linear basis)

$$m_\lambda(x_1,\dots,x_n)=\sum_{\{\alpha_1,\dots,\alpha_k\}\subset[n]} x_{\alpha_1}^{\lambda_1}x_{\alpha_2}^{\lambda_2}\cdots x_{\alpha_k}^{\lambda_k}$$

e.g.

$$m_3(x_1,x_2,x_3)=x_1^3+x_2^3+x_3^3$$

$$m_{21}(x_1,x_2,x_3)=x_1^2x_2+x_1^2x_3+x_2^2x_1+x_2^2x_3+x_2x_3^2$$

- Elementary symmetric functions

$$e_\ell(x_1,\dots,x_n)=\sum_{i_1<i_2<\dots<i_\ell} x_{i_1}x_{i_2}\cdots x_{i_\ell}$$

e.g. $e_2(x_1,x_2,x_3) = x_1x_2 + x_1x_3 + x_2x_3$

$$e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$$

e.g.
$$e_{21}(x_1,x_2,x_3) = e_2 \cdot e_1$$
$$= (x_1x_2 + x_1x_3 + x_2x_3)(x_1 + x_2 + x_3)$$

- Complete homogeneous symmetric functions

$$h_\ell(x_1,\dots,x_n) = \sum_{i_1 \le i_2 \le \dots \le i_\ell} x_{i_1} x_{i_2} \cdots x_{i_\ell}$$

e.g. $h_2(x_1,x_2,x_3) = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$

Similarly to e_λ,

$$h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}$$

e.g. $h_{431} = h_4 \cdot h_3 \cdot h_1$

In-class exercise:

Decompose $x_1x_2x_3 + 2x_1 + 2x_2 + 2x_3 =: f$ in each basis

$$f = m_{111}(x_1,x_2,x_3) + 2m_1(x_1,x_2,x_3)$$

$$f = e_3(x_1,x_2,x_3) + 2e_1(x_1,x_2,x_3)$$

$$f = 2h_1(x_1,x_2,x_3) + h_3(x_1,x_2,x_3) + h_{111}(x_1,x_2,x_3) - 2h_{21}(x_1,x_2,x_3)$$

REMARK: $f(x_1, x_2, \ldots)$ is symmetric if $\forall m \geq 1$ and $\forall \sigma \in S_m$ one has

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(m)}, x_{m+1}, x_{m+2}, \ldots)$$
$$= f(x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots)$$

Write $e_\lambda, h_\lambda, m_\lambda$ for these symmetric functions in $x_1, x_2, \ldots$

e.g.

$$m_1 = e_1 = h_1 = x_1 + x_2 + x_3 + \ldots$$

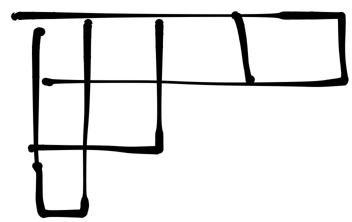
$$h_4 = x_1^4 + x_1^3 x_2 + \ldots + x_{100}^2 x_{10000}^2 + \ldots$$

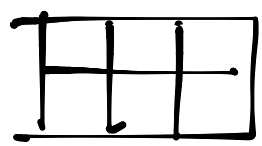
If we set $x_i = 0$ for $i > n$, we recover

$$f(x_1, x_2, \ldots, x_n, 0, 0, 0, 0, \ldots) = f(x_1, \ldots, x_n).$$

② Young tableaux and Schur functions

We can view partitions as a shape

$\lambda = (4,2,1) \longleftrightarrow$ 

$(3,3) \longleftrightarrow$ 

To get a semistandard Young tableau of shape λ, fill each box with a positive integer so that

- rows weakly increase left-to-right
- columns strictly increase top-to-bottom

$$T_1 = \begin{array}{c|c|c|c|c|}
 & \leq & \leq & \leq & \\
\cline{2-5}
 & 1 & 1 & 1 & 2 \\
\cline{2-5}
\wedge & 2 & 5 & \multicolumn{2}{c}{} \\
\cline{2-3}
\wedge & 3 & \multicolumn{3}{c}{} \\
\cline{2-2}
\end{array} \rightsquigarrow x^{T_1} = x_1^3 x_2^2 x_3 x_5$$

$$T_2 = \begin{array}{|c|c|c|c|}
\hline
1 & 3 & 3 & 7 \\
\hline
2 & 9 & \multicolumn{2}{c}{} \\
\cline{1-2}
8 & \multicolumn{3}{c}{} \\
\cline{1-1}
\end{array} \rightsquigarrow x^{T_2} = x_1 x_2 x_3^2 x_7 x_8 x_9$$

$$\cancel{\begin{array}{|c|c|c|}
\hline
1 & 1 & 1 \\
\hline
1 & 2 & 2 \\
\hline
\end{array}}$$

not a tableau

DEF'N: The Schur function

$$s_\lambda = \sum_{\substack{\text{semistandard}\\ \text{Young tableaux } T\\ \text{of shape } \lambda}} x^T$$

e.g.

$$s_{\ydiagram{2,1}} = x_1^2x_2 + x_1^2x_3 + 2x_1x_2x_3 + \dots$$

$$\ytableaushort{11,2} \qquad \ytableaushort{11,3} \qquad \ytableaushort{12,3} \qquad \ytableaushort{13,2}$$

$$s_{\ydiagram{1,1,1}} = x_1x_2x_3 + x_1x_2x_4 + x_{20}x_{30}x_{50} + \dots$$

$$\ytableaushort{1,2,3} \qquad \ytableaushort{1,2,4} \qquad \begin{ytableau} 20 \\ 30 \\ 50 \end{ytableau} \quad = e_3$$

$$s_{\square\square\square} = \underset{\boxed{1|1|1}}{x_1^3} + \underset{\boxed{1|1|2}}{x_1^2 x_2} + \underset{\boxed{2|4|7}}{x_2 x_4 x_7} + \dots = h_3$$

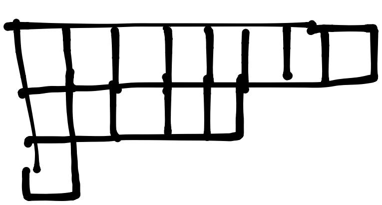
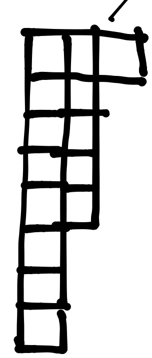
FACT (not obvious)

s_λ is a symmetric function.

---

(3) Jacobi-Trudi

DEF'N: λ' is transpose/conjugate partition to λ

$\lambda =$  $\lambda' =$ 

THM (The Jacobi-Trudi identity):

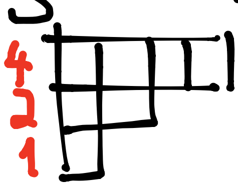
$$s_\lambda = \det\left(h_{\lambda_i - i + j}\right)$$

$$s_{\lambda'} = \det\left(e_{\lambda_i - i + j}\right)$$

proof: Student paper presentation!

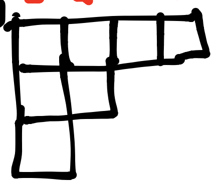
---

e.g.

s  $= \begin{vmatrix} h_4 & h_5 & h_6 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix}$ row sizes go in the diagonal subscripts

$h_0 := 1$

$h_r := 0$ if r is negative

$s_{3211} =$  $\begin{vmatrix} e_3 & e_4 & e_5 & e_6 \\ e_1 & e_2 & e_3 & e_4 \\ 0 & 1 & e_1 & e_2 \\ 0 & 0 & 1 & e_1 \end{vmatrix}$ column sizes give the diagonal subscripts

---

4 REU PROBLEM 3

If you take a random homomorphism

$$\left\{ \begin{array}{c} \text{Symmetric functions} \\ \text{over } \mathbb{Z} \end{array} \right\} \longrightarrow \mathbb{F}_q$$

chosen by picking the image of each e_i uniformly, what is the probability that $s_\lambda \longmapsto 0$?

e.g. $\Pr\left(s_{(1,1,1)} \to 0\right) = \frac{1}{q}$ since $s_{(1,1,1)} = e_3$

$\Pr\left(s_{(2,1)} \to 0\right) = \frac{q^2}{q^3} = \frac{1}{q}$

since $s_{(2,1)} = \begin{vmatrix} e_2 & e_3 \\ 1 & e_1 \end{vmatrix} = e_1 e_2 - e_3$

q choices q choices

needs to equal their product

Not all of them are $\frac{1}{q}$!

e.g. $\Pr\left(s_{(2,2,1)} \to 0\right) = \frac{q^3+q^2-q}{q^4}$

(... I think)

EXTENDED REU PROBLEM 3

- $\Pr(s_\lambda \to 0)$? (the original problem)

- What is the distribution of the $\mathbb{F}_q$-values of s_λ?

- What about the distribution of the ranks of the Jacobi-Trudi matrices?

- When are probabilities independent? i.e. $\Pr(s_\lambda \to 0 \mid s_\mu \to 0) = \Pr(s_\lambda \to 0)$?

REU EXERCISE 7

Compute $\#GL(n,q)$ where

$GL(n,q) = \{n\times n$ invertible matrices with entries in $\mathbb{F}_q\}$

REU EXERCISE 8

(Stanley, "Enumerative Combinatorics, Vol. 1", Chap. 1 Exercise 179)

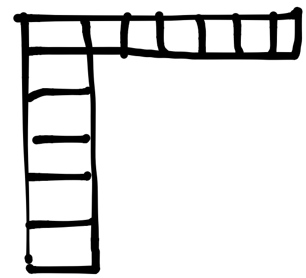
How many pairs (A,B) in

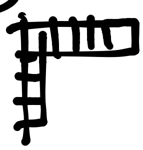
$Mat(n,q) = \{n\times n$ matrices over $\mathbb{F}_q\}$

satisfy $A+B=AB$?

REU EXERCISE 9

A hook shape is a partition $(a, 1^b)$ $= (a, 1, 1, \ldots, 1)$



Show $\Pr(S$$\longrightarrow 0) = \frac{1}{8}$

for any hook