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RTU Day 4 A. Garver
2016

Noncrossing tree partitions and shard intersection orders

1. Lattices
2. Shellability
3. Noncrossing tree partitions
4. Shard intersection order of biclosed sets

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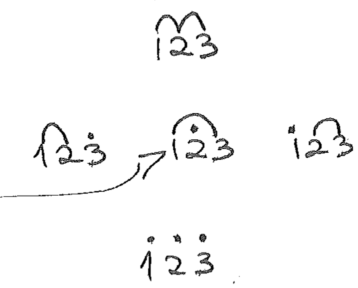
0 Noncrossing partitions

A set partition $\underline{B} = (B_1, \ldots, B_r)$ is a family of subsets of $[n] := \{1,2,\ldots,n\}$ such that $\bigcup_{i=1}^{r} B_i = [n]$ and $B_i \cap B_j = \emptyset \ \forall i \neq j$

$\underline{B}$ is noncrossing if no two of its blocks B_s, B_t $s \neq t$ have $i,k \in B_s$, $j,l \in B_t$ with $i<j<k<l$

$NC(n) :=$ all noncrossing set partitions

e.g. $NC(3) =$



$=$ all set partitions of $[3]$

means $\underline{B} = \{B_1, B_2\}$, $B_1 = \{1,3\}$, $B_2 = \{2\}$

Well-known FACT: $\#NC(n) = \frac{1}{n+1}\binom{2n}{n} =$ Catalan number

$NC(4) \neq$ all set partitions of $[4]$ since

$\leftrightarrow \underline{B} = \{B_1, B_2\}$, $B_1 = \{1,3\}$, $B_2 = \{2,4\}$

is not noncrossing

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1 Lattices

A poset is a set P with a partial order $\leq$ (or $\leq_P$) such that

- $x \leq x$
- $x \leq y, y \leq x \Rightarrow x = y$
- $x \leq y, y \leq z \Rightarrow x \leq z$

RMK: One can think of $(P, \leq)$ as an acyclic directed graph with at most one arrow $x \to y$ for any $x, y \in P$ and no sub-digraph

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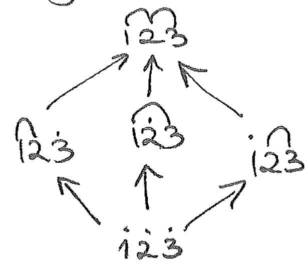
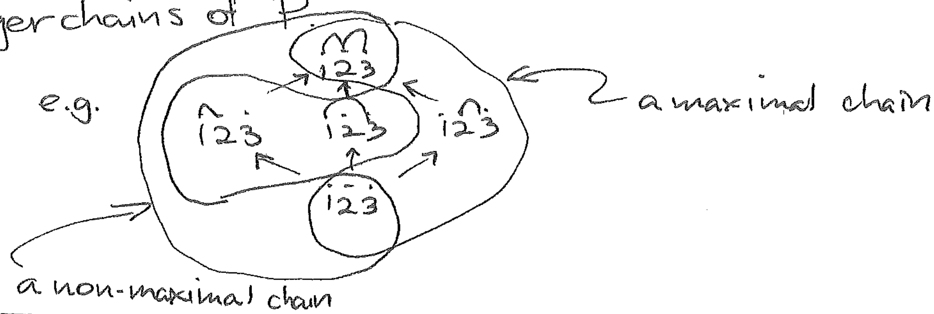
NC(n) is partially ordered by <u>refinement</u>, i.e. $\underline{B} \le \underline{B}'$ if $\forall B_i \in \underline{B}\ \exists B_j \in \underline{B}'$ with $B_i \subseteq B_j$

e.g. NC(3)



A <u>chain</u> in P is a subposet where any two elements are comparable. A chain is <u>maximal</u> if it is not contained in any larger chains of P.

e.g.

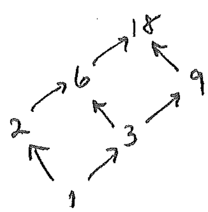


A <u>lattice</u> L is a poset where for any $x, y \in L$ there exist

$x \vee y \in L$ "x join y"
$x \wedge y \in L$ "x meet y"

such that $x \le x \vee y$ and any $z \in L$ with $x \le z$ has $x \vee y \le z$
$x \ge x \wedge y$ — " — $x \ge z$ has $x \wedge y \ge z$

<u>EXAMPLE</u>: L = {all divisors of 18}, ordered by <u>divisibility</u>

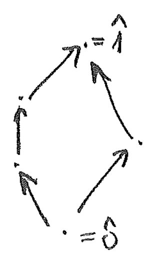


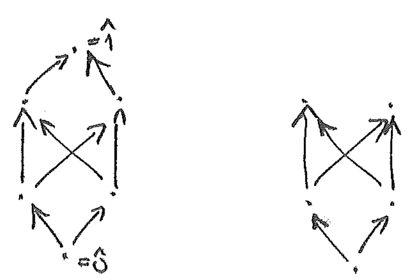
$x \vee y = \mathrm{l.c.m}\{x,y\}$ least common multiple
$x \wedge y = \mathrm{g.c.d}\{x,y\}$ greatest common divisor

We will focus on <u>finite</u> posets and lattices.
In this situation, a lattice L always has a unique minimal element $\hat{0}$ and maximal element $\hat{1}$

<u>EXAMPLE</u>:

a lattice





<u>not</u> lattices

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② Shellability

Assume all maximal chains of P have the same <u>length</u> r and $\exists!$ minimal, maximal elements $\hat{0}, \hat{1}$

— # of <u>arrows</u> in the ~~maximal~~ chain

- call such a P a <u>graded poset</u>.

$\mathrm{Cov}(P) := \{ (x,y) : x \to y \text{ in } P \}$ = <u>covering</u> re<u>lat</u>ions of P

(but $\nexists\, z \neq x, y$ with $x \to z \to y$)

Call a set map $\lambda : \mathrm{Cov}(P) \longrightarrow \underset{\text{a poset}}{Q}$ a <u>(edge-) labeling</u>

A maximal chain $\underset{c_1 < c_2 < \cdots < c_{r+1}}{\overset{\shortparallel}{C}}$ in P is <u>increasing</u> if $\lambda(c_1,c_2) \leq_Q \lambda(c_2,c_3) \leq_Q \cdots \leq_Q \lambda(c_r, c_{r+1})$

Given C, C' max. chains, say C is lex-smaller than C' if $(\lambda(c_1,c_2), \ldots, \lambda(c_r,c_{r+1}))$ lexicographically precedes $(\lambda(c_1',c_2'), \ldots, \lambda(c_r',c_{r+1}'))$ in Q^r

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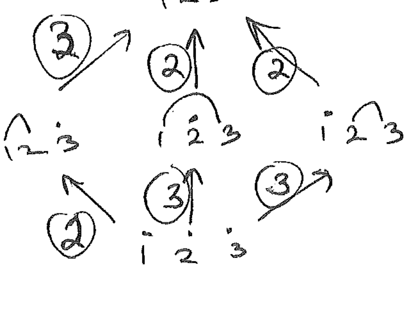
DEF'N: A labeling $\lambda : \mathrm{Cov}(P) \to Q$ is an <u>EL-labeling</u> (and P is <u>EL-shellable</u>) if

- every interval $[x,y] := \{ z : x \leq_P z \leq_P y \}$ has a <u>unique</u> <u>increasing</u> maximal chain C_0
- C_0 is <u>lex-smaller</u> than all other maximal chains C in $[x,y]$

EXAMPLE: Björner (1980) proved that $NC(n)$ has an EL-labeling:

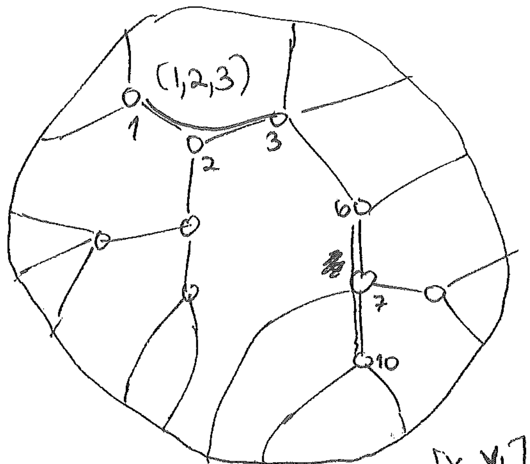
Let $(\underline{B}, \underline{B}') \in \mathrm{Cov}(NC(n))$, so one merges two blocks B_1, B_2 of $\underline{B}$ to produce $\underline{B}'$. ~~Assume $\min B_1 < \min B_2$, and~~ Label

$$\lambda(\underline{B}, \underline{B}') = \max\{ \min \underline{B}_1, \min B_2 \}$$



③ Noncrossing tree partitions

Start with a tree T embedded in the disk D^2, having interior vertices (shown as circles $\circ$ here) of degree at least three.



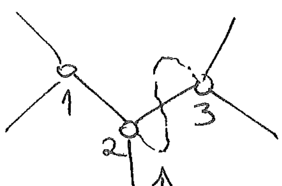
$(6,7,10)$ is not a segment, but $(7,10)$ is a segment

A segment $s = (v_0, \ldots, v_t) =: [v_0, v_t]$ is a sequence of internal vertices v_i such that

- (v_i, v_{i+1}) is an edge of T $\forall i$
- s turns sharply at each vertex
  - the hardest right or the hardest left at each vertex

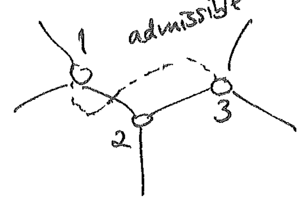
An admissible curve $\gamma : [0,1] \to D^2$ for a segment $s = [v_0, v_t]$ is a simple curve where

- v_0, v_t are its endpoints
- γ may only intersect edges in s
- γ must leave its endpoints to the right.

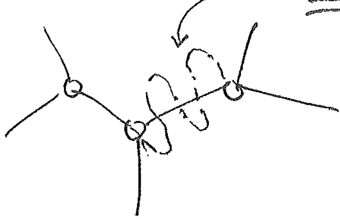


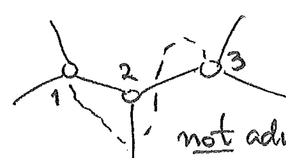
an admissible curve for [2,3]

admissible for [1,3]



the same admissible curve for [2,3]





not admissible for [1,3]

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A <u>noncrossing tree partition</u> of T

$\underline{B} = (B_1, B_2, \ldots, B_k)$ is a set partition of the interior vertices of T where

- there is a unique set $seg(B_i) \subset seg(T)$ connecting the vertices of B_i connecting the vertices of B_i where any two segments may agree only at their endpoints

- any segments $s_1 \in Seg(B_i)$, $s_2 \in Seg(B_j)$ are <u>noncrossing</u> (i.e. they admit non-intersecting admissible curves)

$NCP(T) := \{\text{noncrossing tree partitions of } T\}$ ordered by refinement

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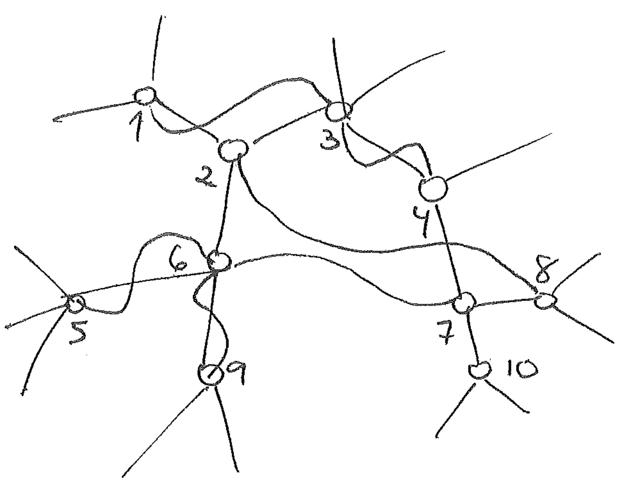
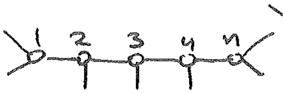
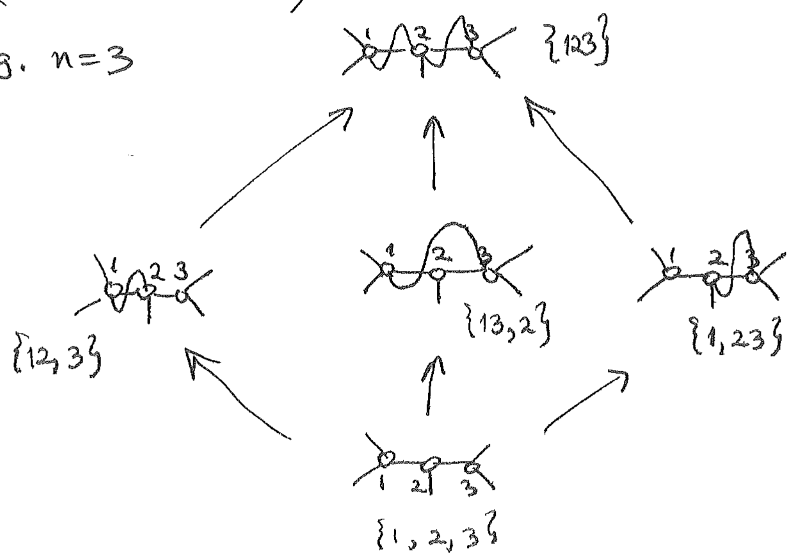
e.g. $\underline{B} = \{134, 28, 5679, 10\} \in NCP(T)$ for this T



NCP() = NC(n)

e.g. n=3



THM [G.-McConville] NCP(T) is a lattice, and it is graded by $n - \#\text{blocks}(B)$.
arXiv 1604.09....

REU EXERCISE 10

Find a tree T where

a) $\#NCP(T) \neq C_n$ for any n

b) $\#NCP(T) = C_n$ for some n, but $NCP(T) \not\cong NC(n)$ for any n

REU PROBLEM 4(a):

1) Show that NCP(T) is EL-shellable, that is, it has an EL-labeling

2) Find a formula for $\#\{\text{max chains in } NCP(T)\}$

e.g. $\#\{\text{maximal chains in } NC(n)\} = n^{n-2}$ by an old result of Kreweras

A mysterious diagram related to REU Problem 4(b), to be further explained...

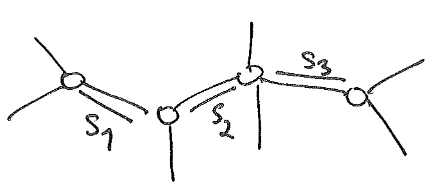
$$\begin{array}{ccc} Bic(T) & \xrightarrow{\Psi} & \Psi(Bic(T)) \\ \cup & & \cup \, ? \\ \vec{FG}(T) & \xrightarrow{\Psi} & \Psi(\vec{FG}(T)) \\ & & \| \wr \\ & & NCP(T) \end{array}$$

oriented flip-graph

shard intersection order

④ Shard intersection order of biclosed sets

Two segments $s_1, s_2 \in Seg(T)$ are composable if $s_1 \circ s_2 \in Seg(T)$



s_1, s_2 are composable, but s_2, s_3 are not composable.

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Say $B \subset Seg(T)$ is <u>closed</u> if any $s_1, s_2 \in B$ and s_1, s_2 are composable then $s_1 \circ s_2 \in B$ also.

Say B is <u>biclosed</u> if, in addition, its complement $Seg(T) - B$ is closed.

$Bic(T) := \{$biclosed subsets of $Seg(T)\}$, partially ordered by inclusion.

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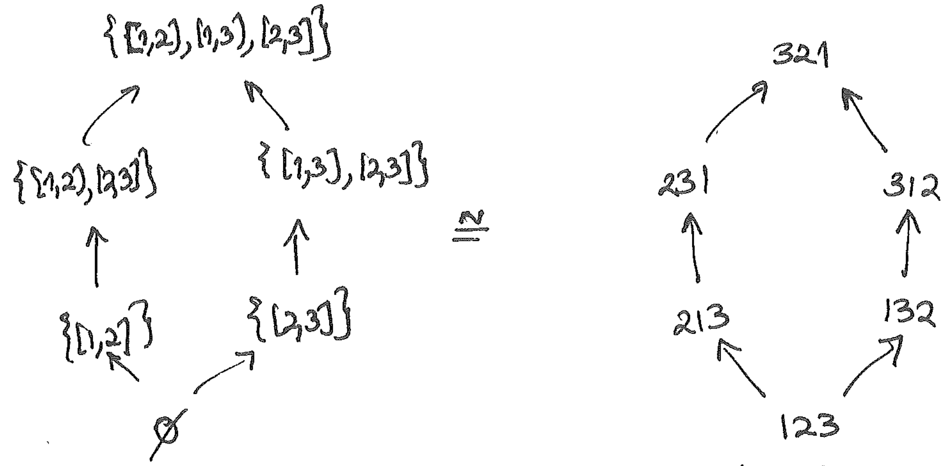
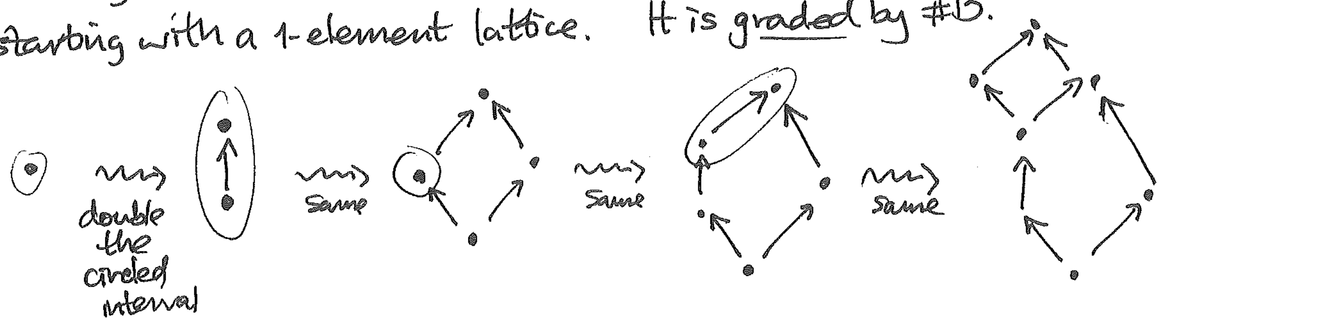
e.g. $T =$ 

has $Bic(T) =$ 

$Weak(S_n) :=$ inclusion order on the <u>inversion sets</u> of permutations in S_n

((i,j) is an <u>inversion</u> of $\pi = \pi_1 \pi_2 \cdots \pi_n$ if $i<j$ but $\pi_i^{-1} > \pi_j^{-1}$)

THM [B.-M.] $Bic(T)$ is a <u>congruence-uniform</u> lattice, meaning that it can be constructed by a finite sequence of <u>interval doublings</u> starting with a 1-element lattice. It is <u>graded</u> by $\#B$.



PROPOSITION (Reading) A lattice L is congruence uniform

$\iff$ L admits a <u>CU-labeling</u>.

defined on next page

DEF'N: A labeling $\lambda : Cov(L) \to Q$ is a CU-labeling if

defines a CN-labeling $\left\{ \begin{array}{l} \text{(CN1)} \leftarrow \text{(omitted here)} \\ \text{(CN2)} \\ \text{(CN3)} \end{array} \right.$

(CU1) for $j, j' \in L$ that cover unique elements j_*, j'_* respectively, one has $\lambda(j_*, j) \neq \lambda(j'_*, j')$

(CU2) the dual condition to (CU1), where you reverse the order

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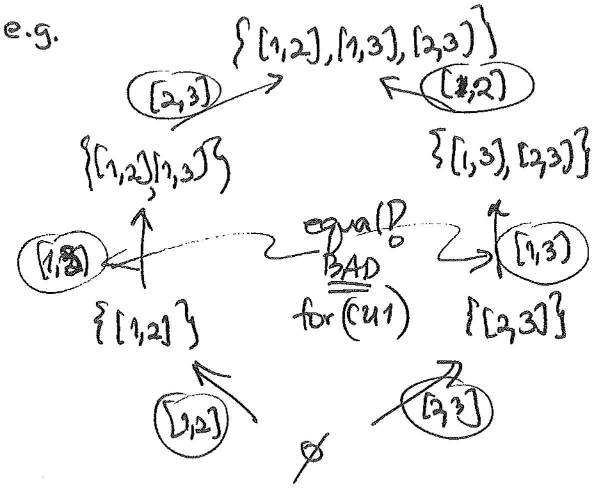
THM [G.-M.] $\lambda : Cov(Bic(T)) \longrightarrow Seg(T)$ ← partially ordered via $s_1 \leq s_2$ if s_1 is a subsequence of s_2

$(B, B \sqcup \{s\}) \longmapsto \lambda(B, B \sqcup \{s\}) = s$

is a CN-labeling, but not a CN-labeling.

e.g.



DEF'N: Let L be CU with $\lambda : Cov(L) \to P$ a CU-labeling.
Take $x \in L$ and $y_1, \ldots, y_k \in L$ such that $(y_i, x) \in Cov(L)$.
Define $\Psi(L)$, the shard intersection order of L, as the sets

$$\psi(x) := \{\text{labels appearing between } \bigwedge_{i=1,\ldots,k} y_i \text{ and } x\}, \text{ ordered by inclusion.}$$

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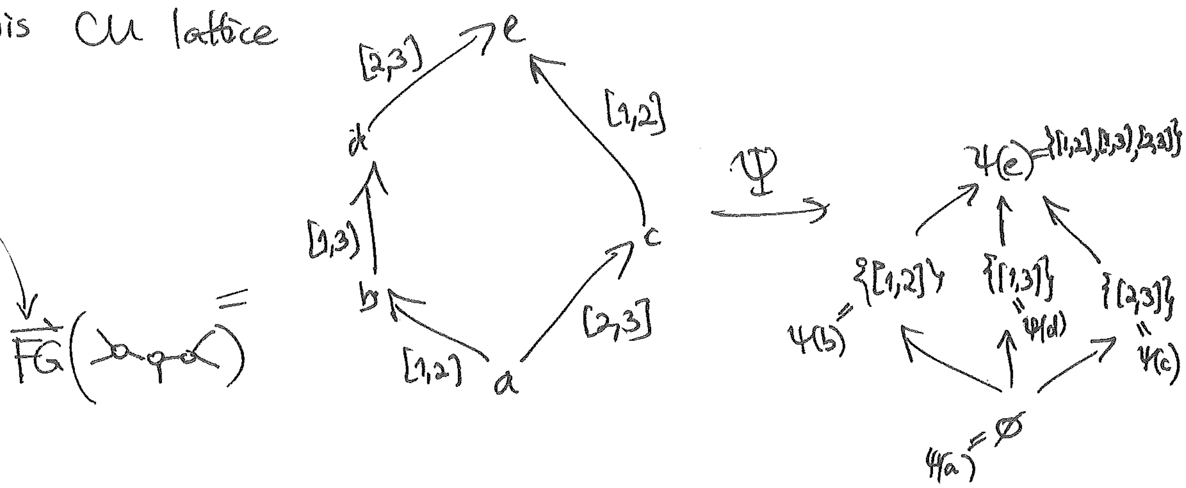
EXAMPLE: This CU lattice

the oriented flip graph construction $\overrightarrow{FG}(-)$ has not been defined.



REU PROBLEM 4(b):

Describe $\boxed{\Psi(\mathrm{Bic}(T))}$:= the image of Ψ, ordered by set inclusion

i) Construct a CU-labeling $\lambda : \mathrm{Cov}(\mathrm{Bic}(T)) \longrightarrow S$ where S involves, in some way, $\mathrm{Seg}(T)$

ii) Is $\boxed{\text{it}}$ a lattice?

iii) Is it EL-shellable?

REU EXERCISE 11

Given $B_1, B_2 \in \mathrm{Bic}(T)$

(a) Describe $B_1 \vee B_2$

(b) Use (a) to show $\mathrm{Bic}(T)$ is a lattice.