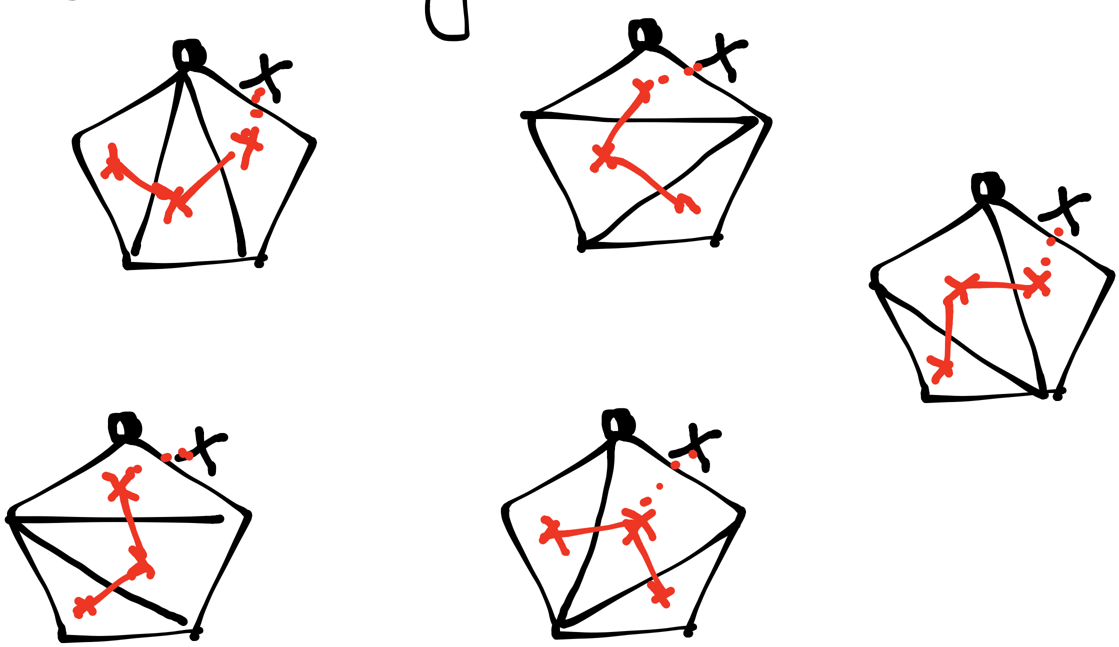
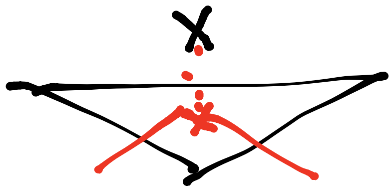
REU 2016 Day 5

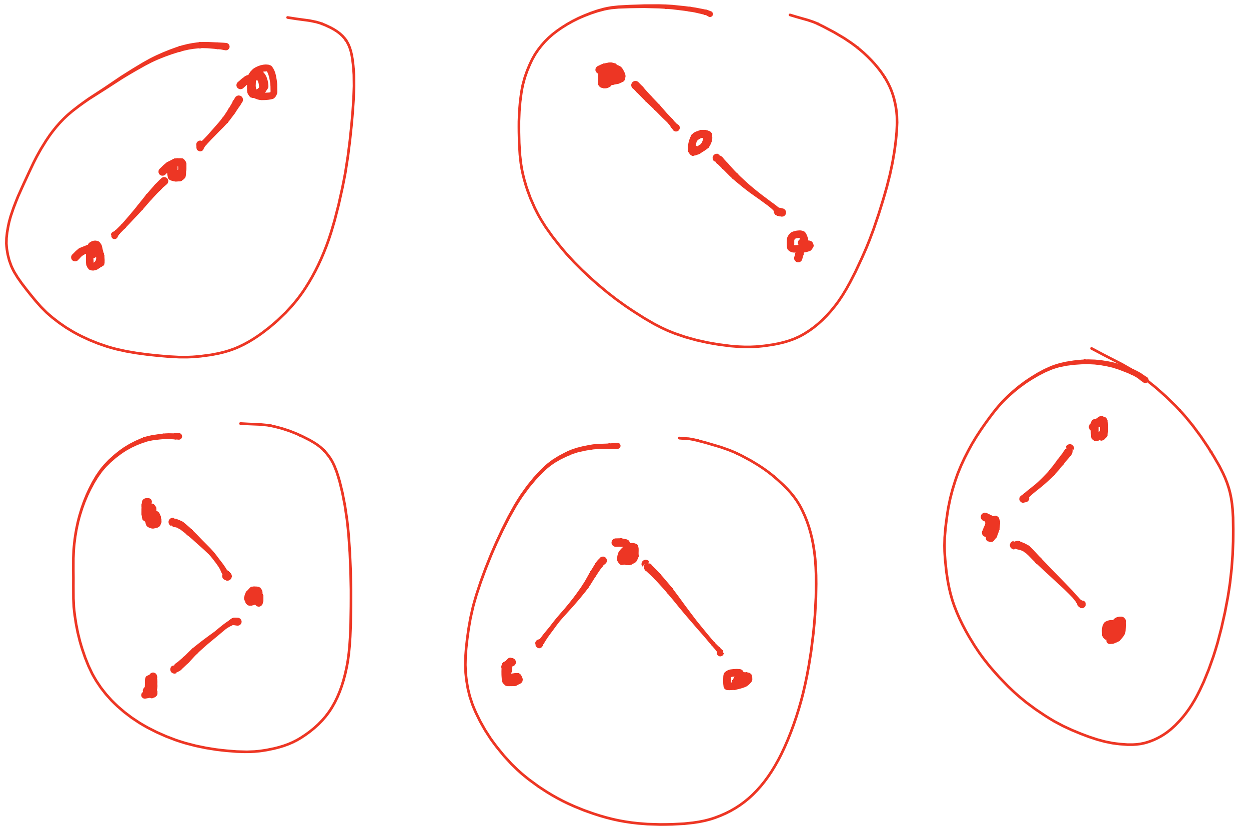
T. Scrimshaw

Let's start with triangulations of an n-gon



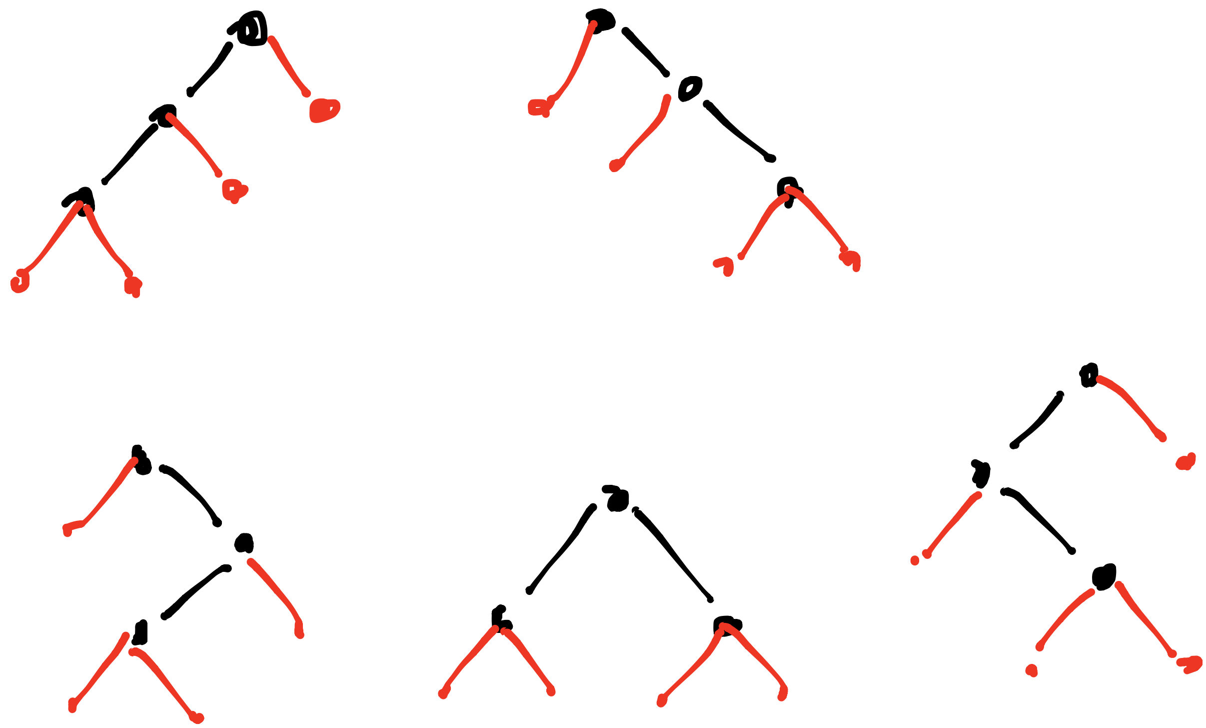
dual tree in red





This gives a bijection between triangulations of $(n+2)$-gons and binary trees with n nodes.

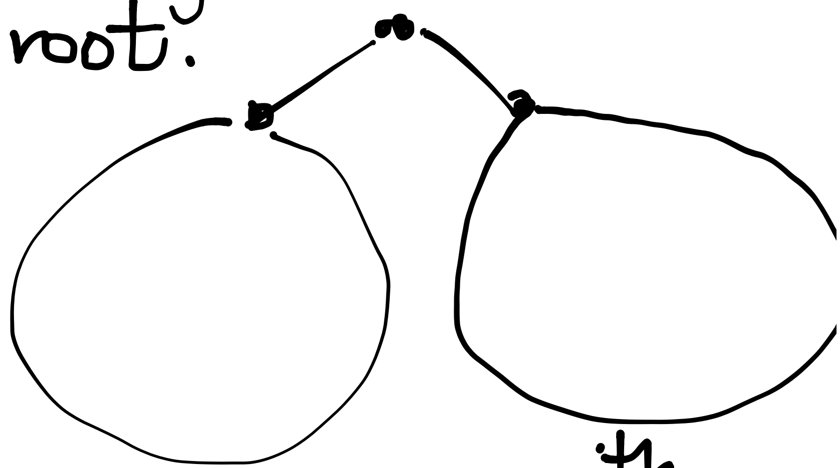
Now add all possible children ...



... so they are also in bijection with complete binary trees with $n+1$ leaves.

How to count them?

Partition the leaves in the complete binary tree according to the left- and right-subtrees below the root.



complete binary tree with k leaves

same with $n+1-k$ leaves

for some $k = 1, 2, \ldots, n$

Hence c_n = # complete binary trees with $n+1$ leaves

satisfies $c_n = \sum_{k=0}^{n-1} c_k c_{n-k-1}$

(note re-indexing)

Base case : $C_0 = 1$

n	0	1	2	3	4	5
C_n	1	1	2	5	14	42

$C_n =$ n^{th} Catalan number

$$= \frac{1}{n+1}\binom{2n}{n}$$

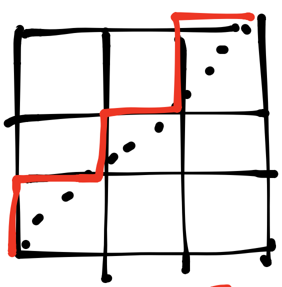
not obvious; well-known; see e.g. Wikipedia

There are over 200 objects counted by Catalan numbers; see Stanley's book "Catalan numbers"

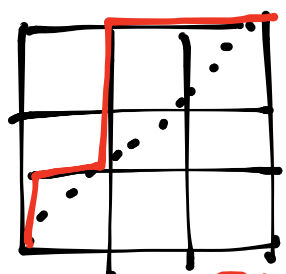
e.g. Dyck paths

<u>DEF'N</u>: A **Dyck path** is a walk from $(0,0)$ to (n,n) in $\mathbb{Z}^2$ taking unit steps up (U) and right (R), staying weakly above $y=x$.

e.g. $n=3$



URURUR



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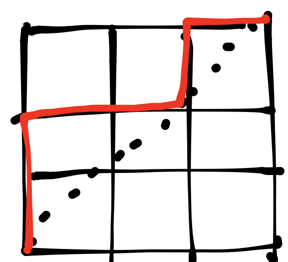

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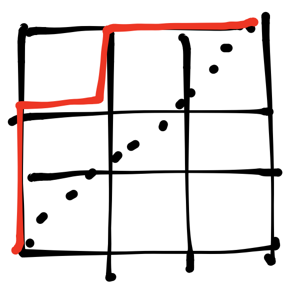

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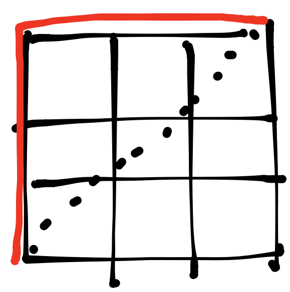
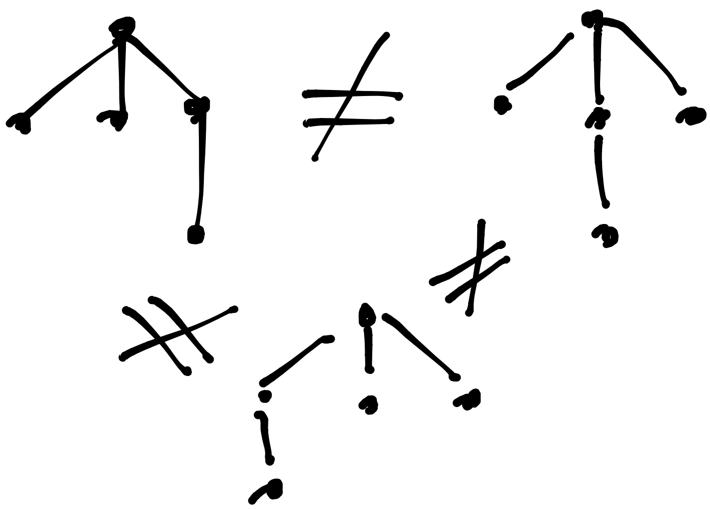


UUURRR

DEF'N: A rooted planar tree is a rooted tree in which each vertex has a linear ordering (left-to-right) of its children.

e.g.

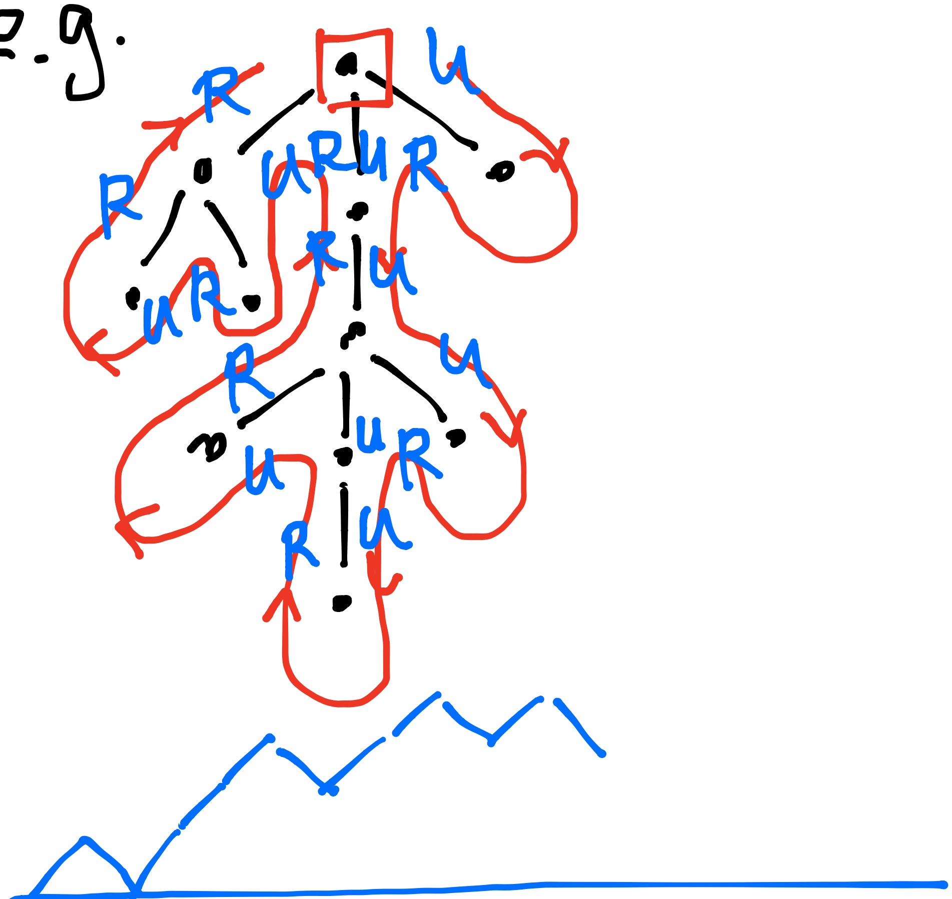


They biject with Dyck ~~paths~~ words, via the planar code of the RPT

You get the planar code by starting at the root and walking around the tree recording

- U when moving away from root,
- R when moving toward the root.

e.g.



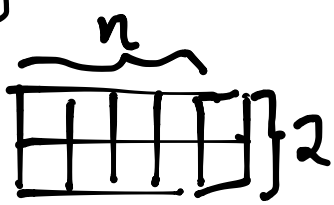
URUUURUURURRR URURR

$\exists$ a bijection from binary trees to Dyck paths (omitted here)

REU EXERCISE 12

Describe a bijection (KEYWORD HINT: R.S.K.)

$$\left\{ \begin{array}{c} 2\times n \text{ standard} \\ \text{Young tableaux} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} 321\text{-avoiding} \\ \text{permutations} \\ \text{in } S_n \end{array} \right\}$$

a filling of  with $1,2,\ldots,2n$ each appearing exactly once, increasing in rows and columns

$\pi = \pi_1\pi_2\cdots\pi_n$ having no $i<j<k$ with $\pi_k > \pi_j > \pi_i$

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e.g. $n=5$

1	2	4	5	8
3	6	7	9	10

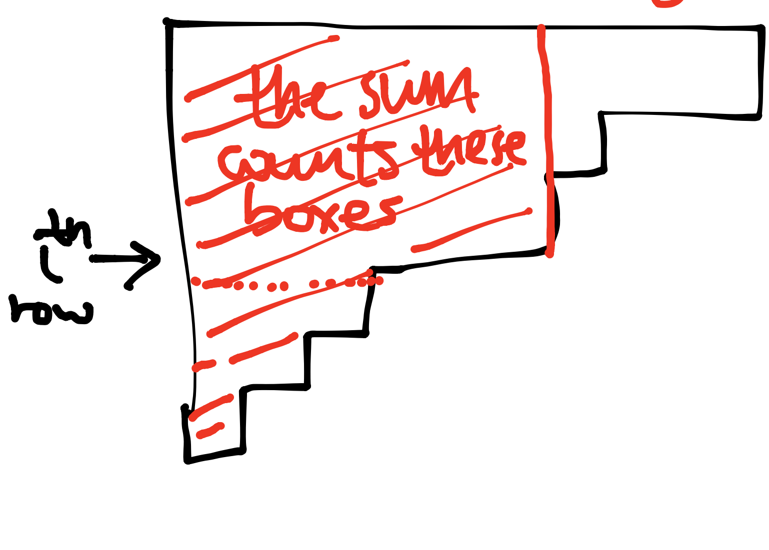
$\pi = 24351$ fails

$\left.\begin{array}{l} 12345 \\ 13254 \end{array}\right\}$ OK

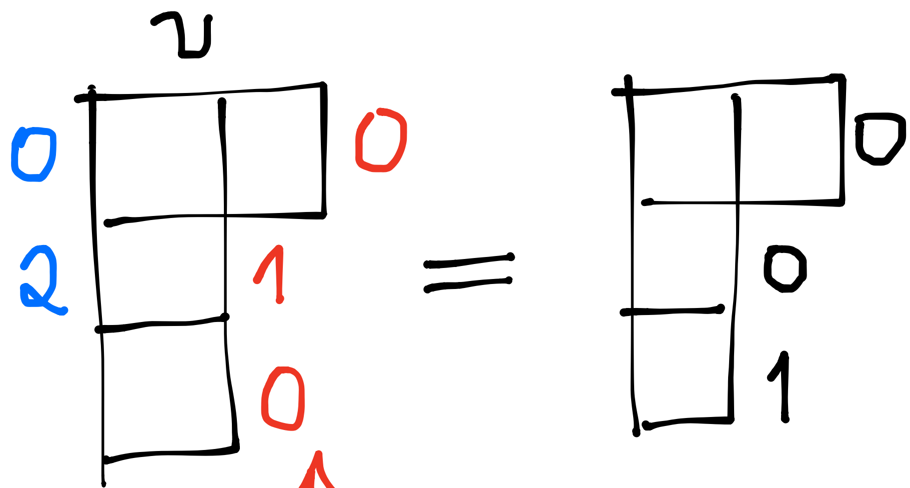
One more set of objects...

DEF'N: A rigged configuration is a partition ν together with integers J_i for each part ν_i of ν, satisfying $0 \leq J_i \leq P_i$ where P_i = the i^{th} vacancy number

$$= 2n - 2\sum_j \min(\nu_j, \nu_i)$$



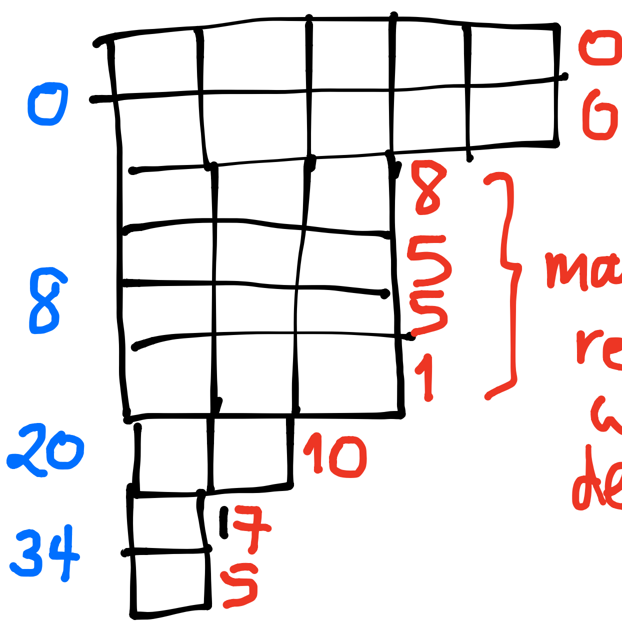
e.g.



the vacancy numbers

the rigging - it associates a multiset of values to each part size

e.g.



may as well reorder weakly decreasing

Multiplicity notation:

$$\nu = 1^{m_1} 2^{m_2} 3^{m_3} \cdots$$

means ν has m_3 parts of size 3

$$P_\ell = 2n - 2 \sum_i{}' m_i \min(i, \ell)$$

(= vacancy number for all parts $\nu_i = \ell$)

There is a fermionic formula

$$C_n = \sum_{\nu \vdash n}{}' \prod_{\ell=1}^{\infty} \binom{m_\ell + p_\ell}{m_\ell}$$

"ν is a partition of n"

which, using the fact that

$$\binom{a+b}{a} = \#\left\{ \begin{array}{c} \text{multisets of size } a \\ \text{from } \{0, 1, \ldots, b\} \end{array} \right\} \ldots$$

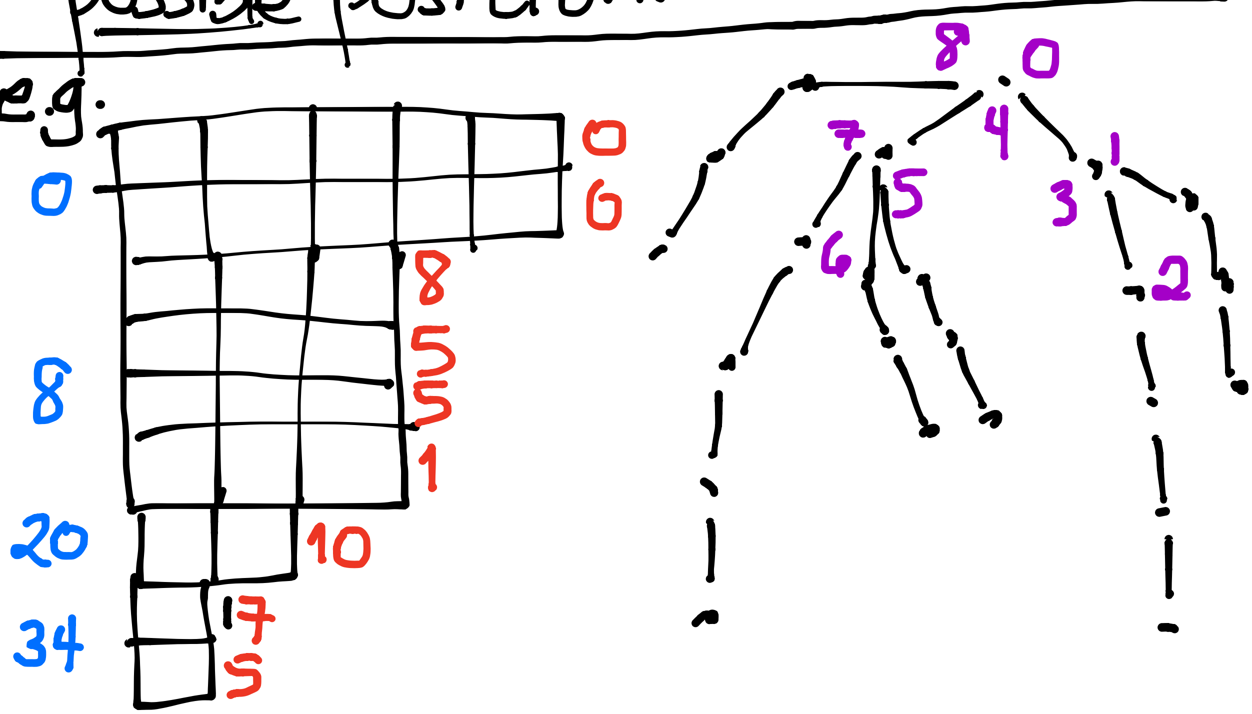
... suggests the following.

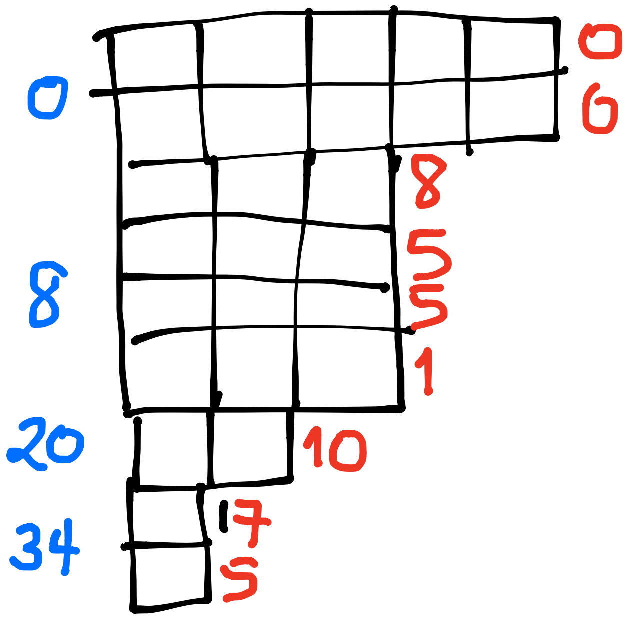
<u>CLAIM</u>: There is a bijection

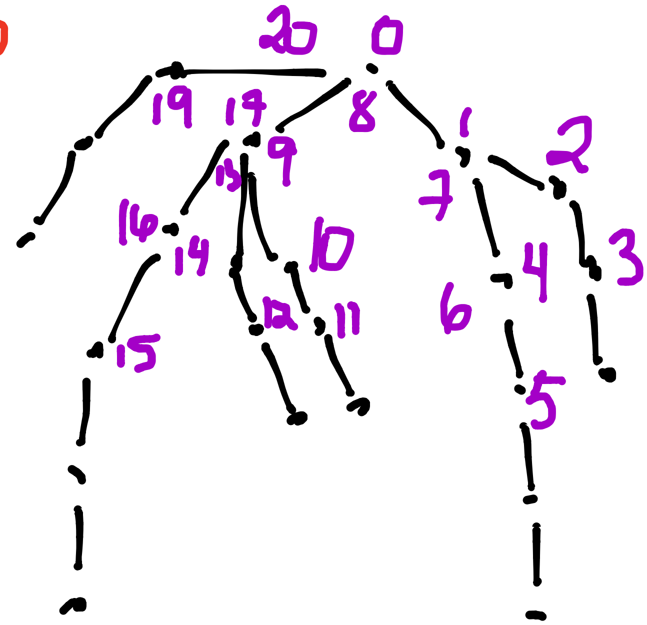
$$RC \xrightarrow{\Psi} RPT$$

given by reading the RC from top to bottom, adding a path of length ν_i at the J_i-th <u>possible</u> position.

e.g.







(Sorry, note-taker got a little lost at this next stage...)

The key issue is in the numbering of the possible positions (purple) when adding the group of next smallest parts.

$$\underbrace{\Phi : D_n}_{\text{Dyck words of length } 2n} \longrightarrow \underbrace{RC_n}_{\text{rigged configs with } \nu \vdash n}$$

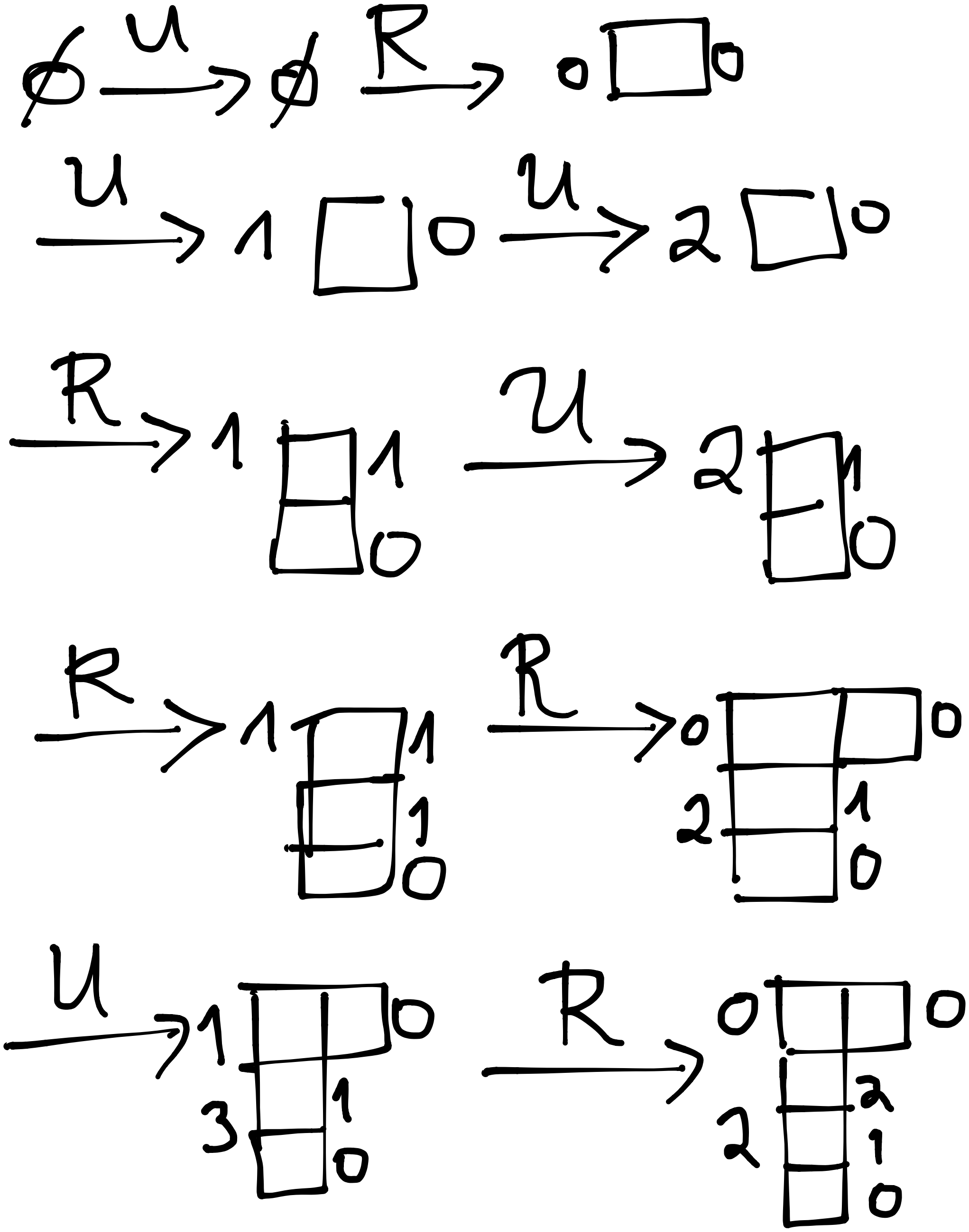
Given the Dyck word, define at the k^{th} stage of the bijection Φ

$$P_l = k - 2\sum_i \min(\nu_i, l)$$

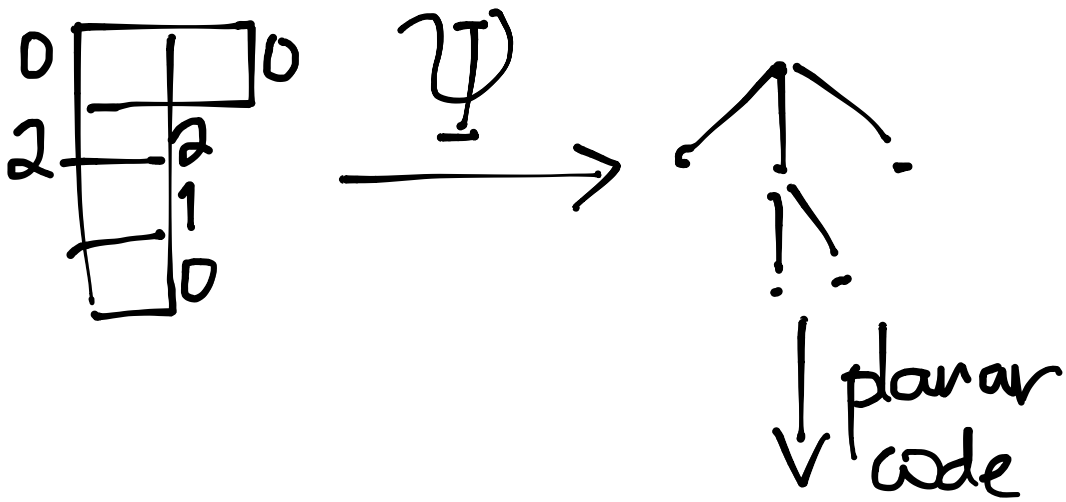
and define Φ recursively by adding a box to the longest row of ν such that $J_i = P_i$ (which we call singular) and keep the row singular if we add R, and do nothing for U.

e.g.

URUURURRUR



Can check



URUURURRUR

<u>THM</u> (Reynolds '15)

This diagram commutes

$$\begin{tikzcd} RC \arrow[r, "\Psi"] & RPT \arrow[d, "\text{planar code}"] \\ & D \arrow[ul, "\Phi"] \end{tikzcd}$$

REU EXERCISE 13

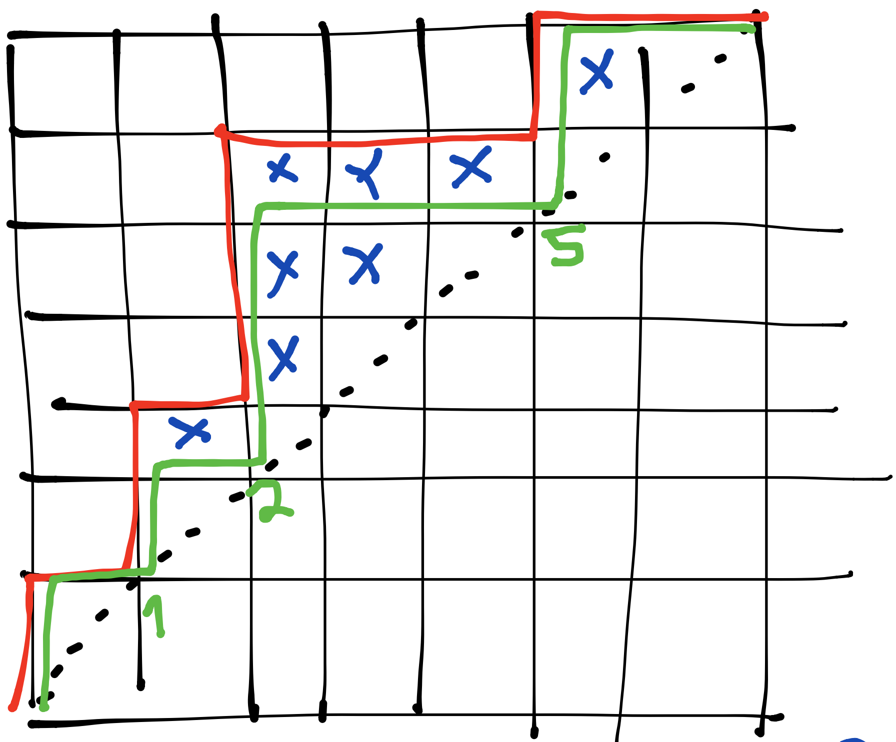
Show Ψ, Φ are bijections (directly, without using Reynolds's Thm)

Two statistics on Dyck paths

Area = # full boxes under the Dyck path

Bounce = the sum of the positions of the bounce path

e.g.



Area $= 8$ ($=$ # of boxes marked x)

Bounce $= 1+2+5 = 8$

The green bounce path bounces off the red Dyck path and off the diagonal $y=x$.

DEF'N: The (q,t)-Catalan number

$$C_n(q,t) := \sum_{d \in D_n} q^{\mathrm{area}(d)} t^{\mathrm{bounce}(d)}$$

THM (Garsia, Haglund et al)

$$C_n(q,t) = C_n(t,q)$$

OPEN PROBLEM (Not REU!)
Prove this combinatorially.
(Their proof is algebraic.)

REU PROBLEM 5

Determine area and bounce on RC's under Φ, i.e. find statistics α, β such that

$$\text{area} = \alpha \circ \Phi$$

$$\text{bounce} = \beta \circ \Phi$$

REU EXERCISE 14

(i) Find a bijection
{complete binary trees} $\to$ RPT
or
$\to$ Dyck paths

(ii) Find the definition of area on RPT under planar code.