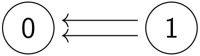
Stable Cluster Variables

Grace Zhang

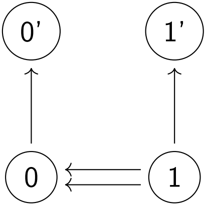
August 1, 2016

Outline

Background



A **quiver** is a directed graph. Multiple edges are allowed. Self-loops are not allowed.



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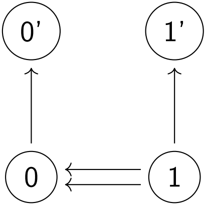
Frame a quiver by adding a new "frozen vertex" i' for each vertex i and drawing an arrow $i \to i'$.



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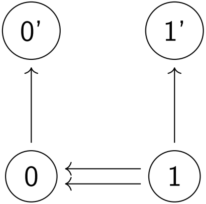
Set the initial cluster variable for each non-frozen vertex as 1, and for each frozen vertex i' as y_i.



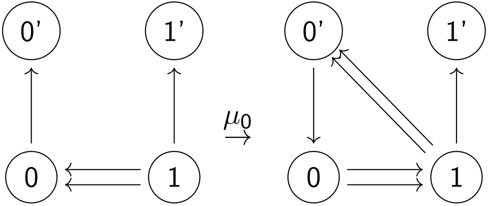
$$\{1, 1, y_0, y_1\}$$

Mutation at a vertex i:

1. Update the cluster variable for vertex i:

$$\frac{\prod_{v \to i} \text{cluster var for } v + \prod_{i \to v} \text{cluster var for } v}{\text{old cluster var for } i}$$

2. For every 2-path $u \to i \to v$, draw an arrow $u \to v$.
3. If any self-loops or 2-cycles were newly created, delete them.
4. Reverse all arrows incident to i.



$\{1, 1, y_0, y_1\}$ $\{\mathbf{1 + y_0}, 1, y_0, y_1\}$

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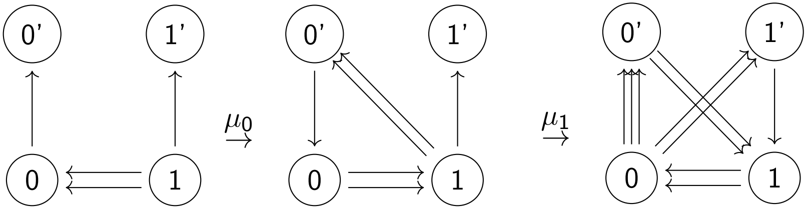

$\{1, 1, y_0, y_1\}$ $\quad$ $\{\mathbf{1 + y_0}, 1, y_0, y_1\}$ $\quad$ $\{1 + y_0, \mathbf{y_0^2 y_1 + (1 + y_0)^2}, y_0, y_1\}$

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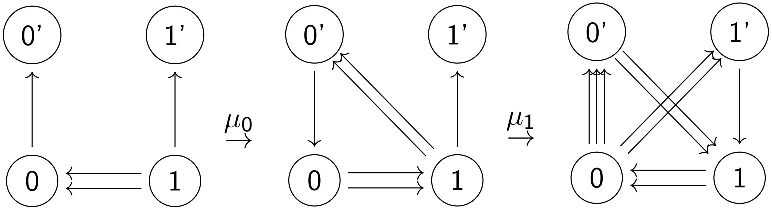
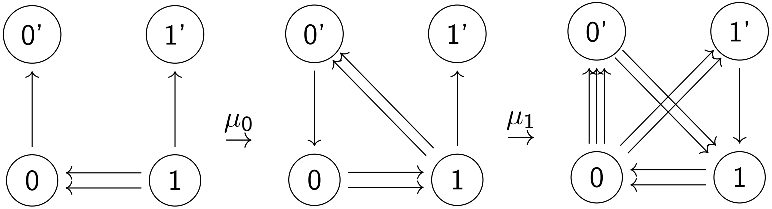


$F_0 = 1$ $\qquad$ $F_1 = 1 + y_0$ $\qquad$ $F_2 = y_0^2 y_1 + (1 + y_0)^2$

For framed quivers we mutate only at non-frozen vertices. The resulting cluster variables are known as **F-polynomials**.



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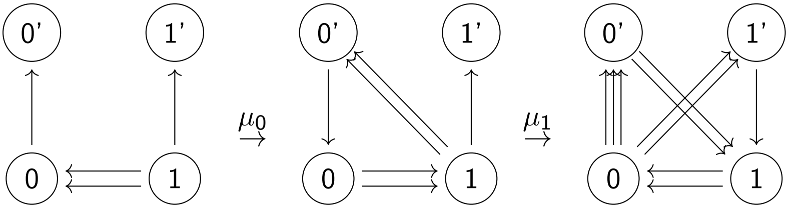
We will keep this running example and fix the mutation sequence

$$\mu = (0, 1, 0, 1, \ldots)$$

Stable Cluster Variables

Eager and Franco defined a transformation on F-polynomials that seems to stabilize them, or make them converge to a limit as a formal power series.



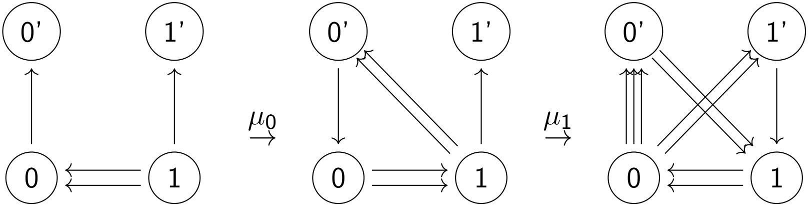
$F_0 = 1$ $\quad$ $F_1 = y_0 + 1$ $\quad$ $F_2 = y_0^2 y_1 + y_0^2 + 2y_0 + 1$

$$C_0 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad C_1 = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix} \qquad C_2 = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$$

At any step in the mutation sequence, define the C-**matrix**:

$$C_{ij} = \# \text{ arrows } i' \to j$$

(negative value if the arrows point from j to i')



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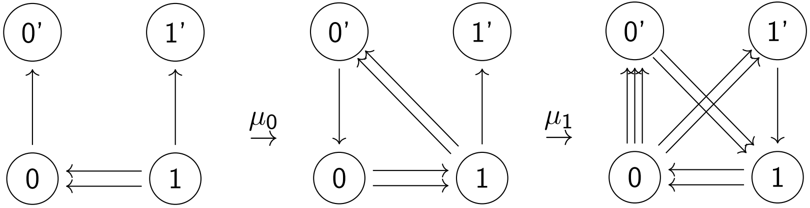
$$C_0^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad C_1^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix} \qquad C_2^{-1} = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$

Given a C-matrix and a monomial $m = y_0^{a_0} y_1^{a_1}$, its **C-matrix transform** is

$$\tilde{m} = y_0^{b_0} y_1^{b_1}$$

$$\text{where } C^{-1} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$



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$\tilde{F}_0 = 1$ | $\tilde{F}_1 = y_0 + 1$ | $\tilde{F}_2 = y_0^2 y_1^4 + 2y_0 y_1^2 + y_1 + 1$

For each F_n, get the **C-matrix transformation** $\tilde{F}_n$ by transforming each monomial individually, using C_n.

Table of the first few transformed cluster variables, illustrating the stabilization property. The low order terms match, up to a fluctuation between y_0 and y_1.

n	$\tilde{F}_n$
1	$y_0 + 1$
2	$y_0^2 y_1^4 + 2y_0 y_1^2 + y_1 + 1$
3	$y_0^9 y_1^6 + 3y_0^6 y_1^4 + 2y_0^5 y_1^3 + 3y_0^3 y_1^2 + 2y_0^2 y_1 + y_0 + 1$
4	$\ldots + 3y_0^4 y_1^6 + 4y_0^3 y_1^4 + 3y_0^2 y_1^3 + 2y_0 y_1^2 + y_1 + 1$

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It appears that

$$\lim_{n \to \infty} \tilde{F}_n = 1 + y_0 + 2y_0^2 y_1 + 3y_0^3 y_1^2 + 4y_0^4 y_1^3 + \ldots$$

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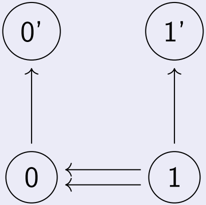
In the remainder of the talk, I prove this convergence and present two more examples of quivers where stabilization happens. I also give a combinatorial interpretation of the limit in each case.

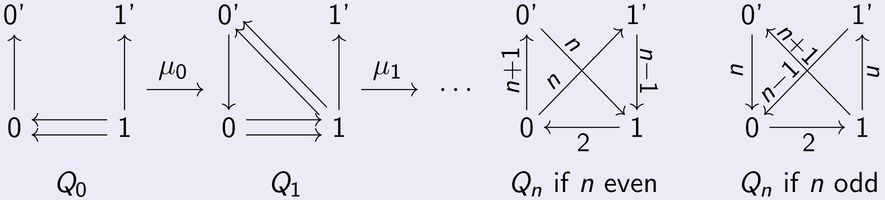
Kronecker Quiver

Framed Kronecker Quiver



Fix the mutation sequence $\mu = (0, 1, 0, 1, \ldots)$.

The Kronecker quiver mutates with a predictable structure.



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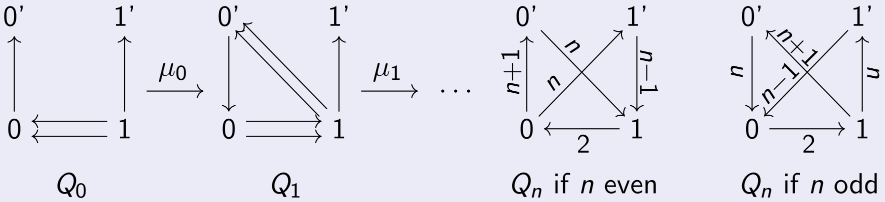

Hence, the C-matrix has a predictable structure.

$$C_n = \begin{cases} \begin{bmatrix} -(n+1) & n \\ -n & n-1 \end{bmatrix} & \text{if } n \text{ even} \\ \\ \begin{bmatrix} n & -(n+1) \\ n-1 & -n \end{bmatrix} & \text{if } n \text{ odd} \end{cases} \qquad C_n^{-1} = \begin{cases} \begin{bmatrix} n-1 & -n \\ n & -(n+1) \end{bmatrix} & \text{if } n \text{ even} \\ \\ \begin{bmatrix} n & -(n+1) \\ n-1 & -n \end{bmatrix} & \text{if } n \text{ odd} \end{cases}$$

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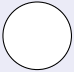
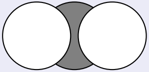
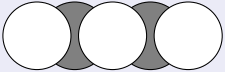
Then for any monomial $m = y_0^a, y_1^b$, C_n transforms it to

$$\tilde{m} = y_0^{n(a-b)-b} y_1^{n(a-b)-a}$$

Definition (**Row pyramid** of length n)

$R_n :=$ two-layer arrangement of stones with n white stones on the top and $n-1$ black stones on the bottom, as shown.

R_1  R_2  R_3 

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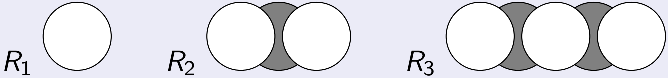

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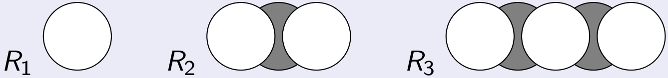

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$$y_0^{\#\text{ white stones removed}}\, y_1^{\#\text{ black stones removed}}$$

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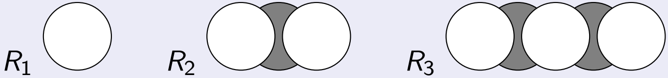

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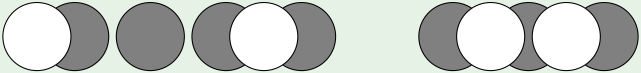
Example (A partition of R_9 with weight $y_0^5 y_1$)



Lemma

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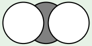
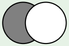
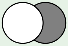
Example

$F_2 = 1 + 2y_0 + y_0^2 + y_0^2 y_1$

1:  y_0:  y_0:  y_0^2:  $y_0^2 y_1$:

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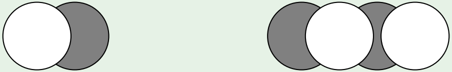

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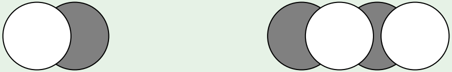

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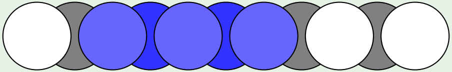

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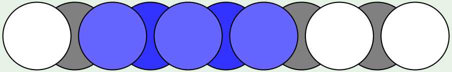

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$$= {y_0}^{\#\text{ non-removed white stones } + 1} {y_1}^{\#\text{ non-removed black stones}}$$

Theorem

For the Kronecker quiver with $\mu = (0, 1, 0, 1, \dots)$

$$\lim_{n\to\infty} \tilde{F}_n = 1 + y_0 + 2y_0^2 y_1 + 3y_0^3 y_1^2 + 4y_0^4 y_1^3 + \dots$$

Proof sketch:

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It can be shown that for any monomial $y_0^a y_1^b \neq 1$ in $\tilde{F}_n$, $a > b$.

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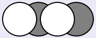
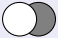
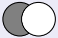
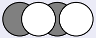
Proof by example: $a = 3$.

$y_0^{n-2} y_1^{n-3}$ transforms to $y_0^3 y_1^2$.
Always 3 simple partitions leaving 2 white and 2 black stones (for $n \geq 3$):

1. 

2.  

3. 

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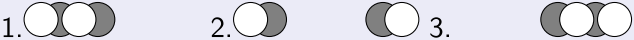

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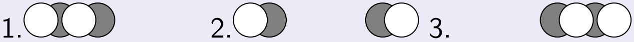

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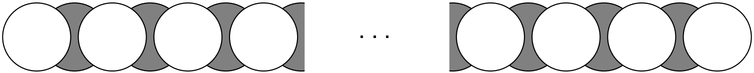
Partitions of F_z and F_{z+1} mapping to the same $\tilde{m}$ differ by k stones of each color. (i.e. bump up each exponent by k). But only 2 stones are added to R_z. So eventually exponents grow too large for any partition.

$R_\infty :=$ infinite row pyramid as shown.



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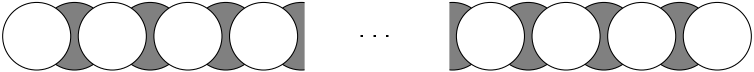

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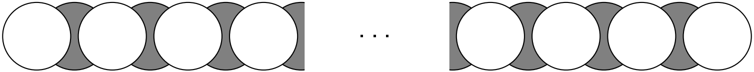

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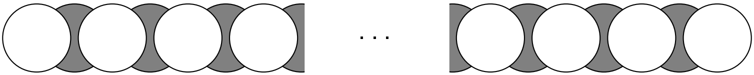

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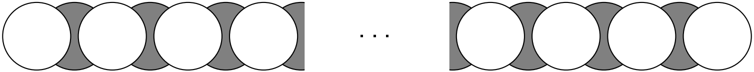

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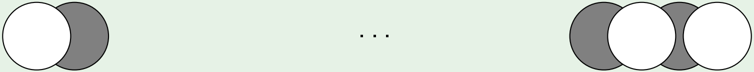
Example (A simple partition of R_∞ with weight $y_0^4 y_1^3$)



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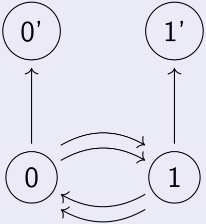
Theorem

$$\lim_{n\to\infty} \tilde{F}_n = 1 + R$$

Conifold Quiver

Framed Conifold quiver



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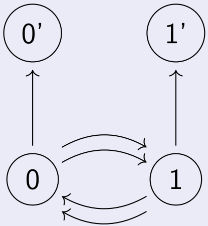

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A table again suggests that the C-matrix transformation stabilizes the cluster variables.

n	$\tilde{F}_n$
1	$\underline{y_0 + 1}$
2	$y_0^2 y_1^5 + y_0^2 y_1^4 + \underline{2y_0 y_1^3 + 2y_0 y_1^2 + y_1 + 1}$
3	$\ldots + \underline{4y_0^4 y_1^2 + 3y_0^3 y_1^2 + 2y_0^3 y_1 + 2y_0^2 y_1 + y_0 + 1}$
4	$\ldots + \underline{4y_0^2 y_1^4 + 3y_0^2 y_1^3 + 2y_0 y_1^3 + 2y_0 y_1^2 + y_1 + 1}$

The stable cluster variables do converge, and the limit can be combinatorially interpreted in an analogous way as in the previous section.

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Here is a larger number of stable terms:

$$\ldots + 33y_0^{10}y_1^6 + 60y_0^9y_1^7 + 63y_0^9y_1^6 + 8y_0^8y_1^7 + 10y_0^9y_1^5 + 40y_0^8y_1^6 + 32y_0^8y_1^5 + 7y_0^7y_1^6 + 3y_0^8y_1^4 + 28y_0^7y_1^5 + 14y_0^7y_1^4 + 6y_0^6y_1^5 + 16y_0^6y_1^4 + 6y_0^6y_1^3 + 5y_0^5y_1^4 + 10y_0^5y_1^3 + y_0^5y_1^2 + 4y_0^4y_1^3 + 4y_0^4y_1^2 + 3y_0^3y_1^2 + 2y_0^3y_1 + 2y_0^2y_1 + y_0 + 1$$

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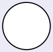
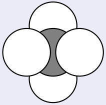
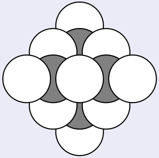
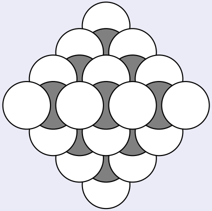
The conifold mutates with a predictable structure, and the C-matrix has the same form as in the previous section.

$$C_n = C_n^{-1} = \begin{bmatrix} n & -(n+1) \\ n-1 & -n \end{bmatrix}$$

$AD_n^{(2)}$:= the 2-color Aztec diamond pyramid shown below.

$AD_1^{(2)}$  $AD_2^{(2)}$  $AD_3^{(2)}$  $AD_4^{(2)}$ 

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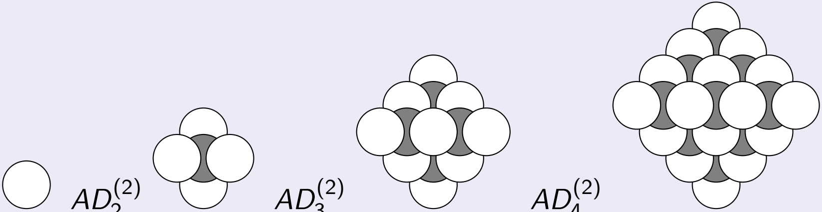

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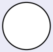
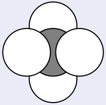
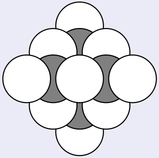
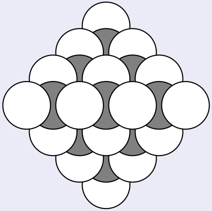
$AD_1^{(2)}$  $AD_2^{(2)}$  $AD_3^{(2)}$  $AD_4^{(2)}$ 

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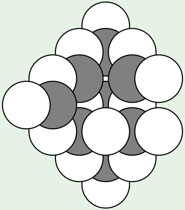
Example (A partition of $AD_4^{(2)}$ with weight $y_0^4 y_1^2$)



Theorem

$$F_n = \sum_{Partitions\ P\ of\ AD_n^{(2)}} weight(P)$$

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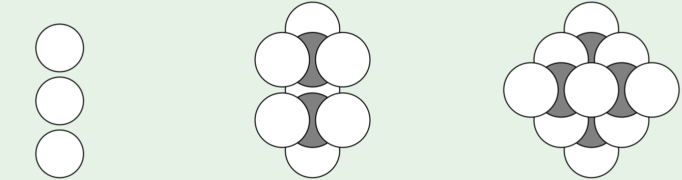
Example (Row pyramid decomposition of $AD_3^{(2)}$, shown layer by layer)



3 rows of length 1 — 2 rows of length 2 — 1 row of length 3

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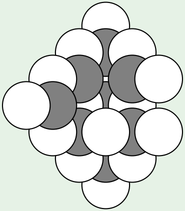

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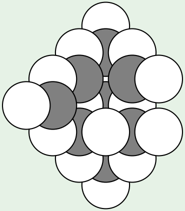

Analogous to the situation before, the idea of the proof that $\tilde{F}_n$ stabilizes is that the stable terms are contributed by the simple partitions.

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For the conifold, $\lim_{n\to\infty} \tilde{F}_n$ *converges as a formal power series.*

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For the same reason as before, every monomial $y_0^a y_1^b \neq 1$ appearing in $\tilde{F}_n$ for any n has $a > b$.

Claim:

Let $\tilde{m} = y_0^a y_1^{a-k}$, with $k \geq 1$. For sufficiently large n, the terms in F_n transforming to $\tilde{m}$ come only from simple partitions (possibly none).

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The only possible partitions left are those altering exactly k rows.

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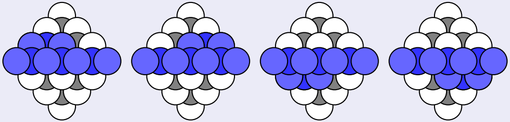

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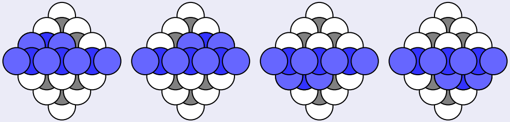
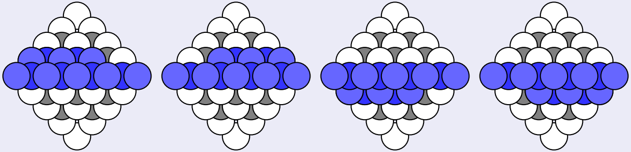


$n = 5$



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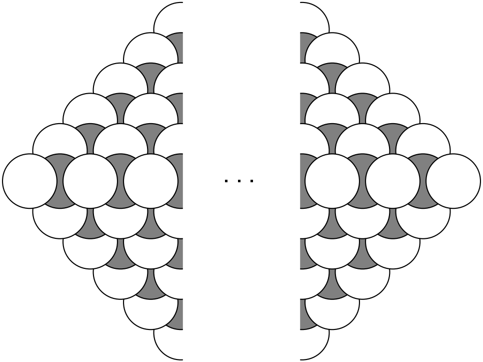

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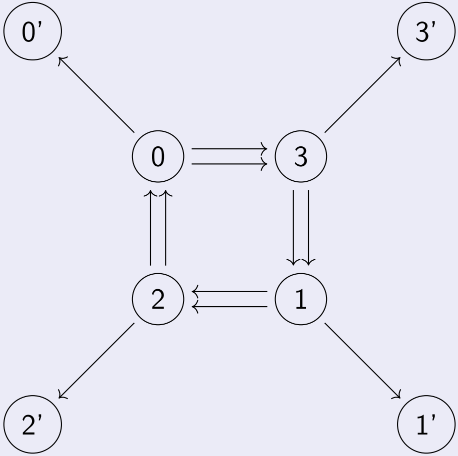
Compare to:

$$\sum_{P \text{ a simple partition of } R_\infty} y_0^{\#\text{ non-removed white stones } + 1} y_1^{\#\text{ non-removed black stones}}$$

F_0 Quiver

Framed F_0 Quiver



Fix $\mu = 01230123\ldots$.

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A table of the odd-indexed cluster variables.

n	F_n	$\tilde{F}_n$
1	$y_0 + 1$	$y_0 + 1$
3	$y_0^2 y_1^2 y_2 + 2y_0^2 y_1 y_2 + y_0^2 y_2 + y_0^2 + 2y_0 + 1$	$y_0^2 y_2^4 + y_1^2 y_2 + 2y_0 y_2^2 + 2y_1 y_2 + y_2 + 1$
5	$\ldots + 4y_0^2 y_1 y_2 + y_0^3 + 2y_0^2 y_2 + 3y_0^2 + 3y_0 + 1$	$\ldots + 4y_0 y_1 y_3^2 + y_0 y_3^2 + 2y_0^2 y_2 + 2y_0 y_3 + y_0 + 1$
7	$\ldots + 6y_0^2 y_1 y_2 + 4y_0^3 + 3y_0^2 y_2 + 6y_0^2 + 4y_0 + 1$	$\ldots + 4y_1^2 y_2 y_3 + y_1^2 y_2 + 2y_0 y_2^2 + 2y_1 y_2 + y_2 + 1$

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A table of the even-indexed cluster variables.

n	F_n	$\tilde{F}_n$
2	$y_1 + 1$	$y_1 + 1$
4	$y_0^2 y_1^2 y_3 + 2y_0 y_1^2 y_3 + y_1^2 y_3 + y_1^2 + 2y_1 + 1$	$y_0^2 y_2^4 y_3 + y_1^2 y_3^4 + 2y_0 y_2^2 y_3 + 2y_1 y_3^2 + y_3 + 1$
6	$\ldots + 4y_0 y_1^2 y_3 + y_1^3 + 2y_1^2 y_3 + 3y_1^2 + 3y_1 + 1$	$\ldots + 4y_0^3 y_1 y_2^2 + 3y_1^3 y_3^2 + 2y_0^2 y_1 y_2 + 2y_1^2 y_3 + y_1 + 1$
8	$\ldots + 6y_0 y_1^2 y_3 + 4y_1^3 + 3y_1^2 y_3 + 6y_1^2 + 4y_1 + 1$	$\ldots + 4y_0^2 y_2^3 y_3 + 3y_1^2 y_3^3 + 2y_0 y_2^2 y_3 + 2y_1 y_3^2 + y_3 + 1$

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Here is a larger number of stable terms:

$$\ldots + 6y_0^6y_1^5 + 4y_0^5y_1^3y_2y_3^2 + 10y_0y_2^4y_3^6 + 8y_0^2y_1y_2^3y_3^4 + 8y_0y_2^4y_3^5 + 5y_0^5y_1^4 + 2y_0^4y_1^2y_2y_3^2 + 4y_0y_2^3y_3^5 + 4y_0^2y_1y_2^2y_3^3 + 6y_0y_2^3y_3^4 + 4y_0^4y_1^3 + y_0y_2^2y_3^4 + 4y_0y_2^2y_3^3 + 3y_0^3y_1^2 + 2y_0y_2y_3^2 + 2y_0^2y_1 + y_0 + 1$$

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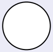
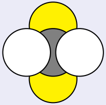
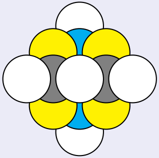
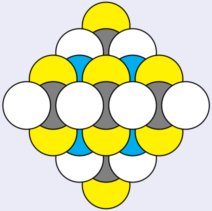
If you identify pairs of y_i's, this collapses down to the conifold case.

$AD_n^{(4)}$:= the 4-color Aztec diamond pyramid shown.

$AD_1^{(4)}$  $AD_2^{(4)}$  $AD_3^{(4)}$  $AD_4^{(4)}$ 

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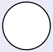
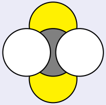
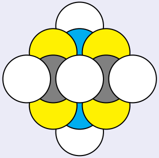
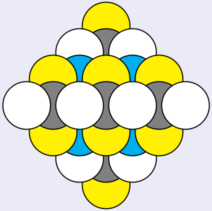
$AD_1^{(4)}$  $AD_2^{(4)}$  $AD_3^{(4)}$  $AD_4^{(4)}$ 

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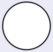
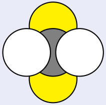
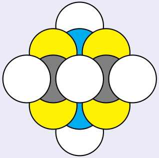
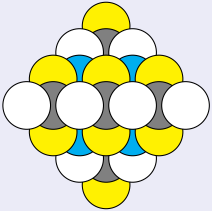
$AD_1^{(4)}$ 

$AD_2^{(4)}$ 

$AD_3^{(4)}$ 

$AD_4^{(4)}$ 

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$$weight(P) = y_0^{\#\text{ yellow removed}} y_1^{\#\text{ white removed}} y_2^{\#\text{ blue removed}} y_3^{\#\text{ black removed}}$$

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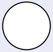
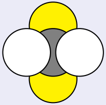
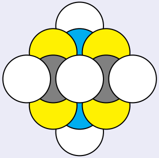
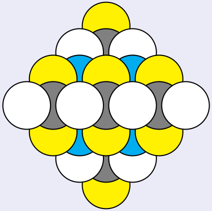
$AD_1^{(4)}$  $AD_2^{(4)}$  $AD_3^{(4)}$  $AD_4^{(4)}$ 

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$$F_n = \sum_{\text{Partitions } P \text{ of } AD_n^{(4)}} weight(P)$$

It can be shown by the same method as before that the $\tilde{F}$'s converge.

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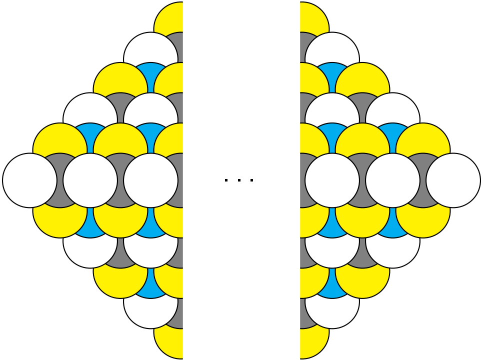

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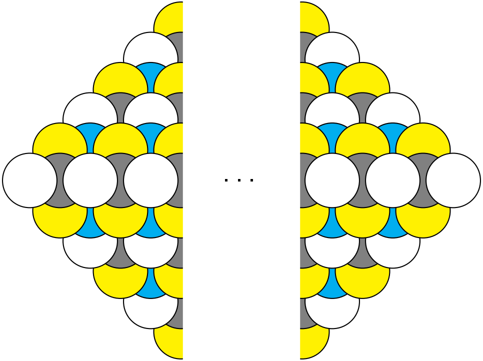

Analogous to before, the limit can be interpreted as a partition function for $AD_\infty^{(4)}$. This function generalizes that of the previous case.

Conclusion

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- Explain what it is about the C-matrix that causes stabilization.
- Characterize for which quivers and mutation sequences stabilization occurs.

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