Coincidences Among Skew Grothendieck Polynomials

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University of Minnesota REU, July 2016

Partitions and Young Diagrams

- A partition λ of a positive integer n is a weakly decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ whose sum is n.

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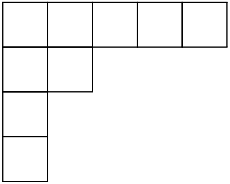
- A partition λ of a positive integer n is a weakly decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ whose sum is n.
- The Young diagram of a partition λ is a collection of left-justified boxes where the i-th row from the top has λ_i boxes. For example, the Young diagram of $\lambda = (5, 2, 1, 1)$ is



Skew Shapes

- Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots, \mu_k)$ be two partitions with $k \leq m$ and $\mu_i < \lambda_i$. We define the skew shape λ/μ by $\lambda/\mu = (\lambda_1 - \mu_1, \ldots, \lambda_k - \mu_k, \lambda_{k+1}, \ldots, \lambda_m)$.

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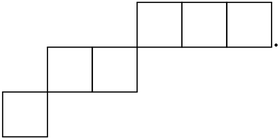
- We form the Young diagram of a skew shape λ/μ by superimposing the Young diagrams of λ and μ and removing the boxes which are contained in both. For example, the Young diagram of the skew shape where $(6, 3, 1)/(3, 1)$ is



Semistandard Young Tableaux

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		1	1
	1	2	3
1	3	4	
2	5		

Schur Function

- Given a SSYT T, we associate a monomial x^T given by

$$x^T = \prod_{i \in \mathbb{N}} x_i^{m_i},$$

where m_i is the number of times the integer i appears as an entry in T.

		1	1
	1	2	3
1	3	4	
2	5		

$$x_1^4 x_2^2 x_3^2 x_4 x_5$$

Schur Function

- We define the Schur function $s_{\lambda/\mu}$ by

$$s_{\lambda/\mu} = \sum_{T} x^{T},$$

where the sum is across all semistandard Young tableau of shape λ/μ.

Stable Grothendieck Polynomials

- We can also create a set valued tableuax by filling the boxes of the shape λ/μ with nonempty sets of positive integers such that the entries weakly increase from left to right across rows and strictly increase from top to bottom down columns.
- For two sets of positive integers A and B, we say that $A \leq B$ if $\max A \leq \min B$. We define the size $|T|$ of T to be the sum of the sizes of the sets appearing as entries in T.
- For example,

	$2,3$	$3,4$	9
	5	$7,8$	
3	$6,7$		

is a set-valued tableau of shape $\lambda/\mu = (4,3,2)/(1,1)$ and size 11 with associated monomial $x_2 x_3^3 x_4 x_5 x_6 x_7^2 x_8 x_9$.

Stable Grothendieck Polynomials

- We define the stable Grothendieck polynomial $G_{\lambda/\mu}$ by

$$G_{\lambda/\mu} = \sum_T (-1)^{|T|-|\lambda|} x^T,$$

where the sum is across all set-valued tableau of shape λ/μ.

- Notice that $G_{\lambda/\mu} = s_{\lambda/\mu} +$ higher order terms.

Dual Stable Grothendieck Polynomials

- A reverse plane partition of shape λ/μ is a filling of the boxes of the Young diagram of λ/μ with positive integers such that the entries weakly increase from left to right across rows and weakly increase from bottom to top down columns. For example,

	2	3	4
	2	4	
2	2		

is a reverse plane partition of shape $\lambda/\mu = (4, 3, 2)/(1, 1)$.

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	2	3	4
	2	4	
2	2		

  is a reverse plane partition of shape $\lambda/\mu = (4,3,2)/(1,1)$.
- Given a reverse plane partition T, the associated monomial x^T is given by

$$x^T = \prod_{i \in \mathbb{N}} x_i^{m_i},$$

  where m_i is the number of columns of T which contain the integer i as an entry.
- The above RPP has associated monomial $x_2^2 x_3 x_4^2$.

Dual Stable Grothendieck Polynomial

- We define the dual-stable Grothendieck polynomial $g_{\lambda/\mu}$ by

$$g_{\lambda/\mu} = \sum_T x^T,$$

where the sum is across all reverse plane partitions of shape λ/μ.
- Notice that $g_{\lambda/\mu} = s_{\lambda/\mu} +$ lower order terms.

Problem

Question: For what shapes is it true that

$$G_{\lambda/\mu} = G_{\gamma/\nu}$$

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Necessary Condition for $g_A = g_B$

Let λ/μ have *m* rows and *n* columns.

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Let λ/μ have m rows and n columns.

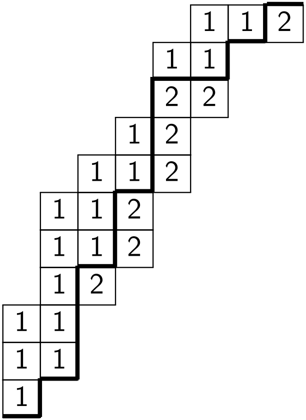
Idea: compute terms in $g_{\lambda/\mu}$ of the form $x_1^i x_2^j$ of degree $n+1$.

These terms correspond to fillings of λ/μ that have $i-1$ columns containing only 1, $j-1$ columns containing only 2, and 1 column containing both 1 and 2.

		2	2
1	1	2	
1	1	2	
1	2		

Lattice Paths

Fillings with only 1's and 2's correspond to lattice paths from the top right corner of λ/μ to the bottom left corner.



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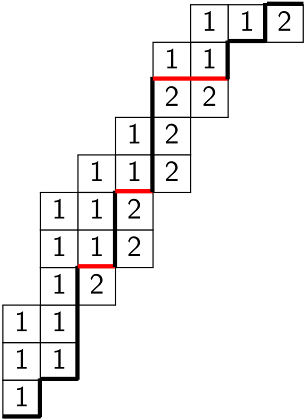
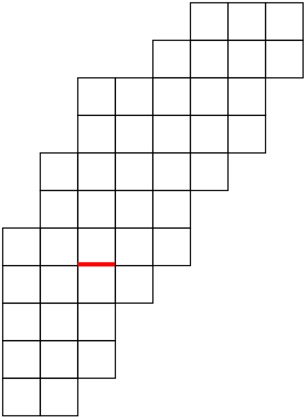
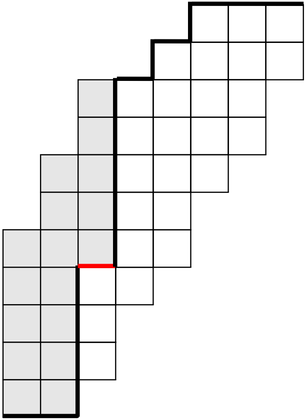


Interior horizontal edges correspond to rows containing both 1's and 2's.

$x_1^i x_2^{n-i+1}$

Example: $n = 8$, $x_1^4 x_2^5$.





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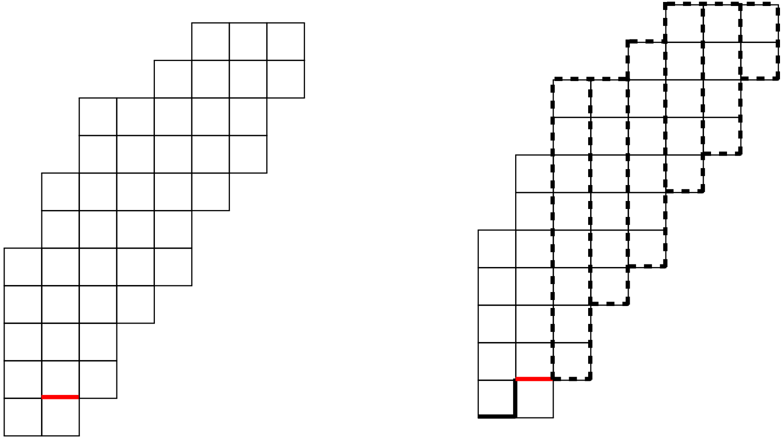

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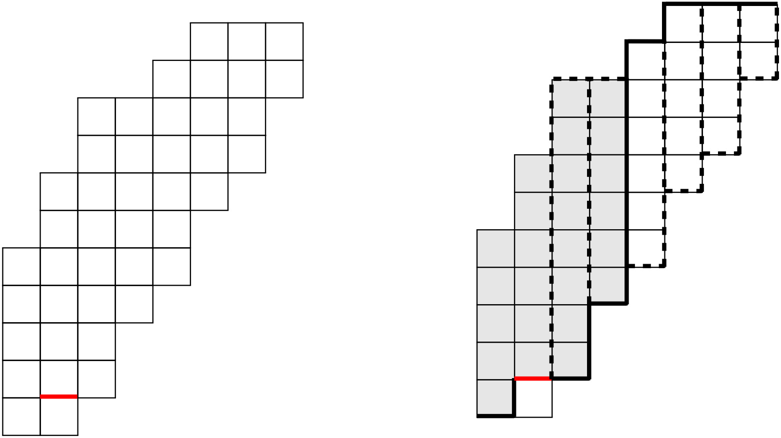

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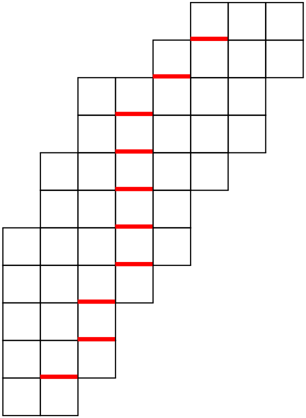


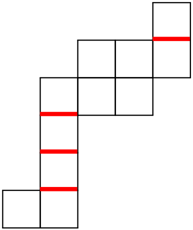
Each lattice path giving the monomial $x_1^4 x_2^5$ uses one of the red interior horizontal edges. There are $m - 1$ such edges, where m is the number of rows. Each red edge is used by exactly one lattice path, unless it touches both boundaries.

Bottleneck Edges

Definition

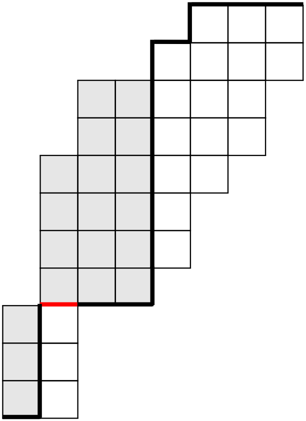
Bottleneck edges are interior horizontal edges touching both boundaries. The number of bottleneck edges in column i is

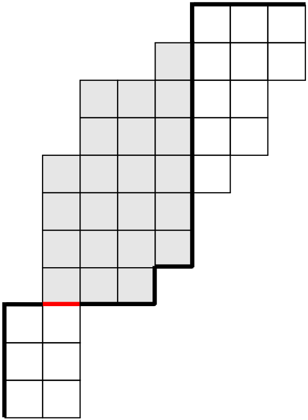
$$b_i := |\{1 \leq j \leq m-1 \mid \mu_j = i-1, \lambda_{j+1} = i\}|.$$



$b_2 = 3, b_5 = 1$

Example: $n = 8$, $x_1^4 x_2^5$.





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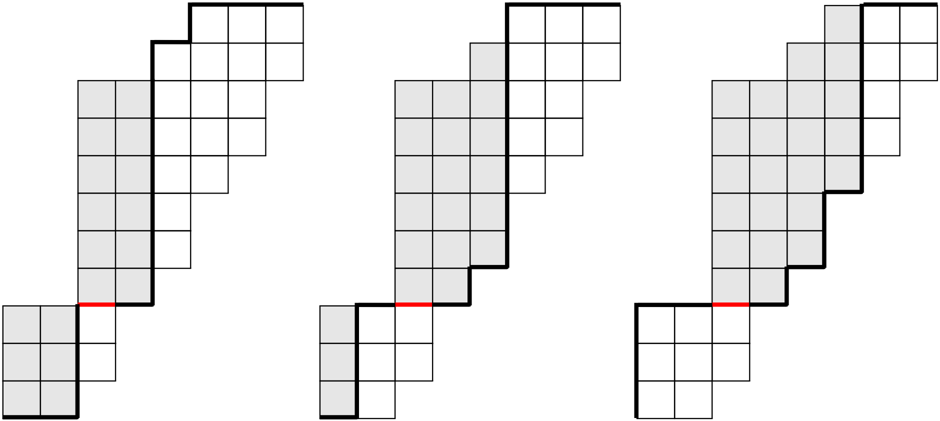


Proposition

The coefficient of $x_1^i x_2^{n-i+1}$ is

$$\begin{aligned}(m-1) &+ (b_2 + b_{n-1}) \\ &+ 2(b_3 + b_{n-2}) \\ &+ 3(b_4 + b_{n-3}) \\ &+ \cdots \\ &+ (i-1)(b_i + b_{n-i+1}) \\ &+ \cdots \\ &+ (i-1)(b_k + b_{n-k+1}).\end{aligned}$$

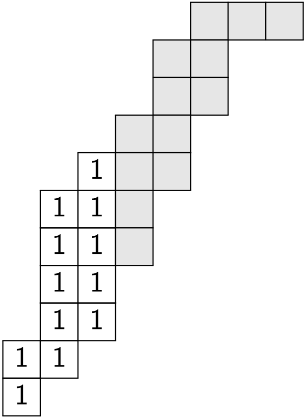
Theorem

Suppose $g_{\lambda/\mu} = g_{\gamma/\nu}$ for skew shapes λ/μ and γ/ν with m rows and n columns. Then for $i = 1, \ldots, n$ the sums $b_i + b_{n-i+1}$ are the same for the two shapes.

Higher Terms

Theorem

Terms of degree $n+1$ *are determined by* m *and the sums* $b_2 + b_{n-1}, \ldots, b_k + b_{n-k+1}$.



Higher Terms

Proposition

The coefficient of $x_1^2 x_2^n$ is

$$\binom{m}{2} - \sum_{i=1}^{n} \binom{b_i + 1}{2}.$$

Proposition

The coefficient of $x_1 x_2 x_3^n$ is

$$(m-1)^2 - \sum_{i=1}^{n} \binom{b_i + 1}{2}.$$

Corollary

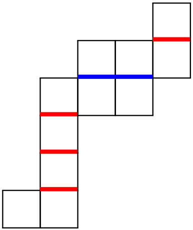
Suppose $g_{\lambda/\mu} = g_{\gamma/\nu}$. Then $b_1^2 + \cdots + b_n^2$ is the same for the two shapes.

Higher Terms

Definition

A **bottleneck of width** w is a segment of w adjacent interior horizontal edges touching both boundaries. The number of bottlenecks of width w at column i is

$$b_i^{(w)} := |\{1 \leq j \leq m-1 \mid \mu_j = i-1, \lambda_{j+1} = i+w-1\}|.$$



Higher Terms

The coefficient of $x_1^3x_2^{n-1}$ in $g_{\lambda/\mu}$ is

$$\left(\binom{m}{2}-\sum_{i=1}^{n}\binom{b_i^{(1)}+1}{2}\right)+\sum_{i=2}^{n-2}\binom{b_i^{(2)}+1}{2}+(m-2)\sum_{i=2}^{n-1}b_i^{(1)}$$

$$-\left(b_2^{(1)}(m-\mu_1'-1)+b_{n-1}^{(1)}(\lambda_n'-1)+\sum_{i=2}^{n-2}b_i^{(1)}b_{i+1}^{(1)}\right).$$

The coefficient of $x_1^3x_2^{n}$ in $g_{\lambda/\mu}$ is

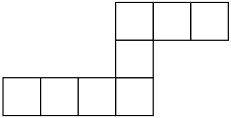
$$\binom{m+1}{3}-\sum_{i=1}^{n}\left((m-1)\binom{b_i^{(1)}+1}{2}-2\binom{b_i^{(1)}}{3}-b_i^{(1)}(b_i^{(1)}-1)\right)$$

$$-\sum_{i=1}^{n-1}\left(\binom{b_i^{(2)}+2}{3}+(b_i^{(1)}+b_{i+1}^{(1)})\binom{b_i^{(2)}+1}{2}+b_i^{(1)}b_i^{(2)}b_{i+1}^{(1)}\right).$$

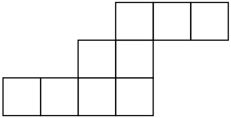
Ribbons

Ribbons

- A ribbon is a connected skew shape containing no 2x2 rectangles.
- Ribbons are in bijection with compositions by letting the number of boxes in the ith row from the bottom be the ith summand in the composition.



is a ribbon with corresponding composition (4,1,3).

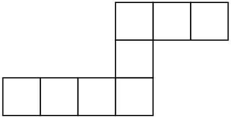
- 

is not a ribbon.

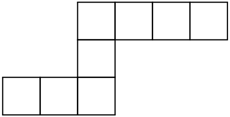
Ribbons

- If $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k)$, then we define $\alpha^* = (\alpha_k, ..., \alpha_1)$. This is a 180 degree rotation of α.

$\alpha = (4, 1, 3)$



$\alpha^* = (3, 1, 4)$



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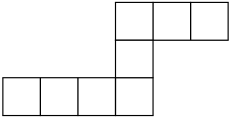

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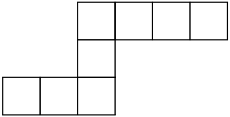


- We will also use column notation $[\alpha_1, \alpha_2, ..., \alpha_k]$ where α_i is the number of boxes in column i of the Young diagram.

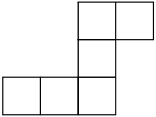
Operations on Ribbons

- Concatenation:

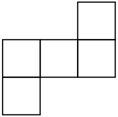
$$\alpha \cdot \beta = (\alpha_1, \ldots, \alpha_k, \beta_1 \ldots, \beta_m)$$

. Visually this attaches β on top of α.

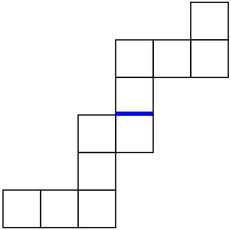
$\alpha = (3, 1, 2)$



$\beta = (1, 3, 1)$



$\alpha \cdot \beta = (3, 1, 2, 1, 3, 1)$



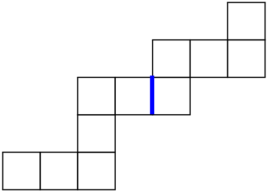
Operations on Ribbons

- Near Concatenation:

$$\alpha \odot \beta = (\alpha_1, \ldots, \alpha_{k-1}, \alpha_k + \beta_1, \beta_2, \ldots, \beta_m).$$

Visually this attaches β to the right of α.

$\alpha \cdot \beta = (3, 1, 2, 1, 3, 1)$



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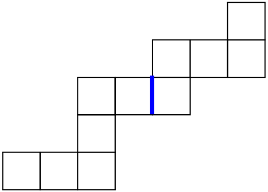
Visually this attaches β to the right of α.

$$\alpha \cdot \beta = (3, 1, 2, 1, 3, 1)$$



- We define

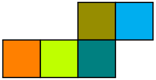
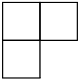
$$\alpha^{\odot n} = \underbrace{\alpha \odot \cdots \odot \alpha}_{n}.$$

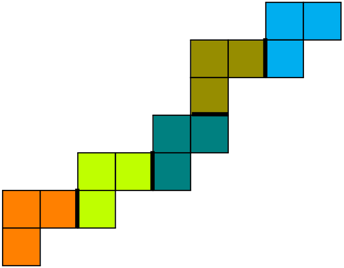
Operations on Ribbons

We can combine the two concatenation operations to define a third operation $\circ$, defined by

$$\alpha \circ \beta = \beta^{\odot \alpha_1} \cdots \beta^{\odot \alpha_k}.$$

Visually, the operation $\circ$ replaces each square of α with a copy of β.

$\alpha = (3, 2)$  $\beta = (1, 2)$ 

$\alpha \circ \beta =$ 

Irreducible Factorizations of Ribbons

Billera, Thomas, and vanWilligenburg proved the following:

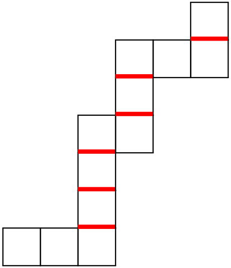
1. Every ribbon α has a unique irreducible factorization $\alpha = \alpha_m \circ \cdots \circ \alpha_1$.
2. Two ribbons α and β are Schur equivalent if and only if α and β have irreducible factorizations

$$\alpha = \alpha_m \circ \cdots \circ \alpha_1 \quad \text{and} \quad \beta = \beta_m \circ \cdots \circ \beta_1,$$

where each β_i is equal to either α_i or α_i^*.

Ribbon Bottlenecks

In the case of ribbons, every interior horizontal edge is a bottleneck. Thus the bottleneck number b_i is the size of column i minus 1.



Then by the bottleneck condition, if $\alpha = [\alpha_1, \ldots, \alpha_k]$ and $\beta = [\beta_1, \ldots, \beta_k]$ are ribbons such that $g_\alpha = g_\beta$, we have

$$\alpha_i + \alpha_{k-i+1} = \beta_i + \beta_{k-i+1}.$$

A Necessary and Sufficient Condition for g of Ribbons

We will prove the following theorem:

Theorem

Let α, β be ribbons. Then $g_\alpha = g_\beta$. if and only if α equals β or β^*.

We will require the following lemma:

Lemma

Suppose α and β are distinct ribbons such that $g_\alpha = g_\beta$, and there exist ribbons σ, τ, μ such that $\alpha = \sigma \circ \mu$ and $\beta = \tau \circ \mu$. Then $\mu = \mu^*$.

Proof of Lemma

- Let $\mu = [\mu_1, \dots, \mu_t], \alpha = [\alpha_1, \dots, \alpha_k], \beta = [\beta_1, \dots, \beta_k]$. Let m and M be the minimal and maximal indices, respectively, such that $\alpha_m \neq \beta_m$ and $\alpha_M \neq \beta_M$.
- We have

$$\alpha_m + \alpha_{k-m+1} = \beta_m + \beta_{k-m+1}$$
$$\alpha_M + \alpha_{k-M+1} = \beta_M + \beta_{k-M+1}.$$

  If $k - m + 1 \neq M$, then $\alpha_m = \beta_m$ or $\alpha_M = \beta_M$, a contradiction. Therefore $k - m + 1 = M$, hence

$$\alpha_m + \alpha_M = \beta_m + \beta_M.$$

Proof of Lemma (cont.)

- We examine columns 1 through m and M through k of α and β:

$$\alpha = (*, \mu_2, \ldots, \mu_{t-1}, \mu_t \diamond \mu_1, \ldots\ldots, \mu_t \diamond' \mu_1, \mu_2, \ldots, \mu_{t-1}, *')$$
$$\beta = (*, \mu_2, \ldots, \mu_{t-1}, \mu_t \star \mu_1, \ldots\ldots, \mu_t \star' \mu_1, \mu_2, \ldots, \mu_{t-1}, *').$$

- We use the equation

$$\alpha_m + \alpha_M = \beta_m + \beta_M$$

to reduce to the case where $\alpha_m = \mu_t$ and $\alpha_M = \mu_1 + \mu_t$. Then the above equation is

$$\mu_1 + 2\mu_t = 2\mu_1 + \mu_t,$$

hence $\mu_1 = \mu_t$. We examine columns $m+1$ through $M-1$ to see that

$$\mu_i + \mu_{t-i} = \mu_{i+1} + \mu_{t-i+1},$$

thus $\mu_{i+1} = \mu_{t-i}$ by induction.

We have a bijection of reverse plane partitions of a ribbon α with reverse plane partitions of α^*:

		3	5	5
1	2	4		
1				

$\longleftrightarrow$

				5
		2	4	5
1	1	3		

$$x_1 x_2 x_3 x_4 x_5^2 \qquad \longleftrightarrow \qquad x_1^2 x_2 x_3 x_4 x_5.$$

Since g is symmetric it follows that $g_\alpha = g_{\alpha^*}$.

Proof of Theorem (only if direction)

Proof.

Since $g_\alpha = g_\beta$ we have $s_\alpha = s_\beta$. Then we have irreducible factorizations

$$\alpha = \alpha_m \circ \cdots \circ \alpha_1$$
$$\beta = \beta_m \circ \cdots \circ \beta_1,$$

where β_i equals α_i or α_i^*. Assume by induction that $\beta_{r-1} \circ \cdots \circ \beta_1$ equals $\alpha_{r-1} \circ \cdots \circ \alpha_1$ or $(\alpha_{r-1} \circ \cdots \circ \alpha_1)^*$. By letting $\mu = \alpha_{r-1} \circ \cdots \circ \alpha_1$, and applying the lemma to α and β or β^*, we have

$$\alpha_{r-1} \circ \cdots \circ \alpha_1 = (\alpha_{r-1} \circ \cdots \circ \alpha_1)^*$$

by the lemma. Since α_r equals β_r or β_r^* we are done. □

Further Explorations

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Conjecture

Suppose $g_A = g_B$. Then $g_{A^t} = g_{B^t}$.

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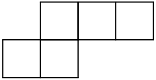
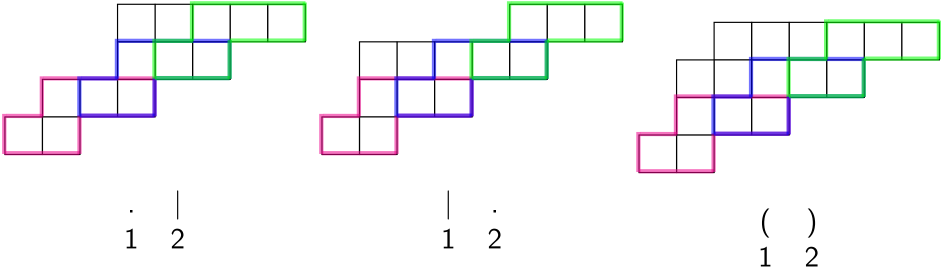
Ribbon Staircases

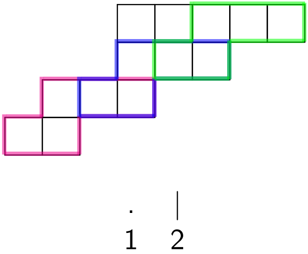
$\alpha =$ 

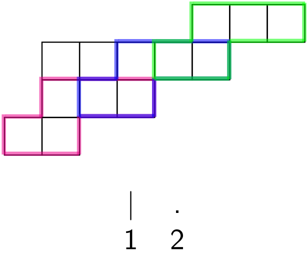


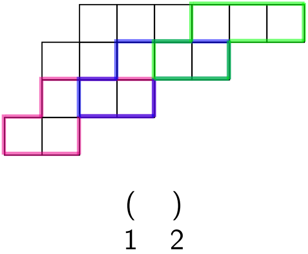
Ribbon Staircases

Theorem (RSvW)

Skew shapes that can be decomposed into the same α that have opposite nestings are Schur equivalent.







Ribbon Staircases

Question

For which ribbons α and nestings $\mathcal{N}$ will the shape with decomposition into α with nesting $\mathcal{N}$ match the shape with decomosition into α and nesting $\mathcal{N}^*$?

Conjecture: $\alpha = (1,2)$

For any μ contained in the staircase partition $\delta_n = (n-1, \ldots, 1)$ we have

$$g_{\delta_n/\mu} = g_{\delta_n/\mu^t}$$

$$G_{\delta_n/\mu} = G_{\delta_n/\mu^t}$$

Conjecture: $\alpha = (2,3)$

Let A be the shape with nesting $\mathcal{N}$ and B the shape with nesting $\mathcal{N}^*$. Then $G_A = G_B$ iff $\mathcal{N}$ contains only vertical slashes "|" and dots "."

G-Positivity

Conjecture

$G_\alpha = G_\beta$ for ribbons α and β iff $\alpha = \beta$ or $\alpha = \beta^*$.

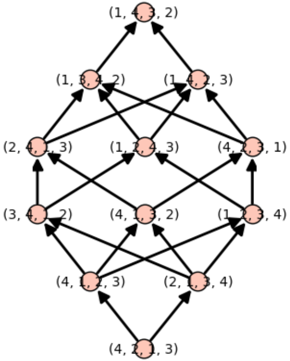
Littlewood-Richardson Coefficients

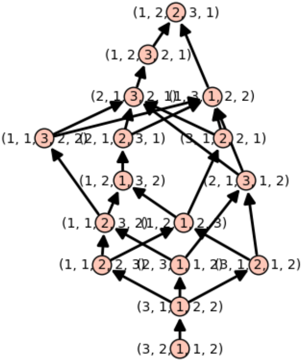
$G_{\lambda/\mu} = \sum_\nu a_{\lambda/\mu,\nu} G_\nu$

Definition

$A \leq B$ if $a_{A,\nu} \leq a_{B,\nu}$ for all ν.

G-Positivity





G-Positivity

Conjecture

For fixed λ, the set of ribbons which are permutations of λ has both a least and a greatest element.

Conjecture

Conjugation acts as an isomorphism.

Question

Permutations of a fixed λ follow the general pattern that ribbons where larger rows are in the middle are larger. In what way can this be made formal?

Question

Are there ribbons α and β such that $s_\alpha = s_\beta$ and $G_\alpha \neq G_\beta$ but α and β are incomparable?

Acknowledgements

This research was carried out as part of the 2016 summer REU program at the School of Mathematics, University of Minnesota, Twin Cities, and was supported by NSF RTG grant DMS-1148634.

We would also like to thank Rebecca Patrias and Sunita Chepuri for their mentorship and many helpful comments.