PATTERN AVOIDANCE

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The following theorem of Erdős and Szekeres [1] (see the section "Second proof") has the answer of REU Exercise 19 as a corollary.

Theorem 1. *Every permutation of length $N > (m-1)(n-1)$ contains either $1 \cdots n$ or $m \cdots 1$ as a pattern.*

One can also prove a converse.

Proposition 2. *If w and u are two patterns other than those that appear in the Erdős–Szekeres theorem, then there exist permutations of every length avoiding w and u.*

In 2003, Marcus and Tardos [2] proved the following theorem about the growth rates of permutation classes.

Theorem 3. *For any permutation w, there is a constant c_w such that the number of permutations in S_n avoiding w is bounded above by $(c_w)^n$.*

Corollary 4. *Suppose that w and u are any two patterns other than those that appear in the Erdős–Szekeres theorem. Then for sufficiently large n,*

$$\Pr(v \in S_n \text{ avoids } w) < \Pr(v \in S_n \text{ avoids } w \mid v \text{ avoids } u).$$

Proof. By Marcus–Tardos, we have

$$\Pr(v \text{ avoids } w) < \frac{(c_w)^n}{n!}.$$

By Marcus–Tardos and the converse to Erdős–Szekeres, we have

$$\Pr(v \text{ avoids } w \mid v \text{ avoids } u) \geq \frac{1}{(c_u)^n}.$$

For any constant C and sufficiently large n, $n! \gg C^n$, so in particular $n! \gg (c_u c_w)^n$ and thus

$$\Pr(v \text{ avoids } w) < \frac{(c_w)^n}{n!} \ll \frac{1}{(c_u)^n} \leq \Pr(v \text{ avoids } w \mid v \text{ avoids } u). \qquad \square$$

At a high level, the problem is that "avoiding a pattern" is so rare that pattern-avoidance classes are not large enough to support events with smaller probability.

One possible fix is to consider everything inside a smaller (say, exponential-sized) universe. For example, one could introduce a third permutation and take everything relative to that. (Obviously this loses a certain amount of elegance; but perhaps the case could be made that it is nice enough when the third pattern is length 3, or is a monotone pattern.)

References

[1] P. Erdös and G. Szekeres, A combinatorial problem in geometry. *Compositio Math.* **2** (1935), pp. 463–470.

[2] Adam Marcus and Gábor Tardos, Excluded permutation matrices and the Stanley-Wilf conjecture. *J. Combin. Theory Ser. A* **107** (2004), pp. 153–160.