6/13/17

Dihedral Actions

1. q-counts
2. cyclic actions
3. q-Catalan
4. dihedral action
5. representation theory
6. REU problem 2

1. q-counts:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n!]_q}{[k!]_q [(n-k)!]_q} \quad \text{is the } q\text{-binomial coefficient}$$

where $[n!]_q = [n]_q [n-1]_q \cdots [2]_q [1]_q$

and $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$

$$= \frac{1-q^n}{1-q}$$

EX!!
$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[4]_q [3]_q \cancel{[2]_q} \cancel{[1]_q}}{[2]_q [1]_q \cancel{[2]_q} \cancel{[1]_q}}$$

$$= \frac{[4]_q [3]_q}{[2]_q}$$

$$= \frac{(1+q+q^2+q^3)(1+q+q^2)}{(1+q)}$$

$$= (1+q^2)(1+q+q^2)$$

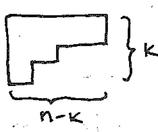
$$= 1 + q + 2q^2 + q^3 + q^4 \quad (\in \mathbb{N}[q]!)$$

Exercise 3

(a) Show that $\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \quad \forall n \geq 1$ (q-Pascal relation)

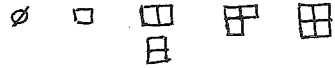
(note $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ if $k \notin [0,n]$)

(b) Show $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\substack{\text{partition } \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots) \\ \lambda_1 \leq n-k \\ \#\text{parts} \leq k}} q^{|\lambda|}$

(i.e., $\lambda \subset$ 

)

Ex!! $\begin{bmatrix}4\\2\end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$



(c) Show that if $q = p^d$ (a prime power), $\begin{bmatrix}n\\k\end{bmatrix}_{q=p^d}$ = # of k-dimensional $\mathbb{F}_q$-linear subspaces of $(\mathbb{F}_q)^n$

(note: $\mathbb{F}_q$-linear ⇒ non-affine)

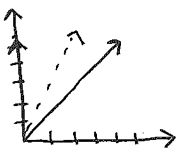
Ex!! $n = 2$

$\begin{bmatrix}2\\0\end{bmatrix}_q = 1$ counts $\{\vec{0}\} \subseteq \mathbb{F}_q^2$

$\begin{bmatrix}2\\1\end{bmatrix}_q = \frac{[2]_q}{[1]_q} = 1 + q = \#\{\underbrace{\mathbb{F}_q\begin{bmatrix}1\\0\end{bmatrix}, \mathbb{F}_q\begin{bmatrix}1\\1\end{bmatrix}, \mathbb{F}_q\begin{bmatrix}1\\2\end{bmatrix}, \ldots,}_{\text{finite slopes}} \underbrace{\mathbb{F}_q\begin{bmatrix}0\\1\end{bmatrix}}_{\text{slope } \infty}\}$

$\begin{bmatrix}2\\2\end{bmatrix}_q = 1$ counts $\mathbb{F}_q^2$ itself

$q = 5$

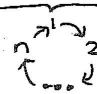


$(\mathbb{F}_5)^2$

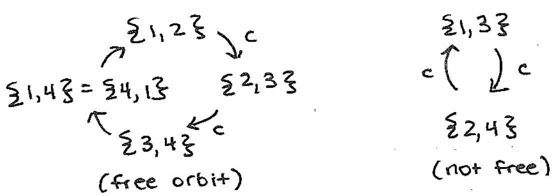
2. Cyclic Actions:

Evaluating $\begin{bmatrix}n\\k\end{bmatrix}_q$ at $q = n^{th}$ roots of unity (i.e., $q = \zeta_n^d$) turns out to count things related to an n-cycle $c = (1, 2, \ldots, n)$, permuting k-element subsets of $\{1, 2, \ldots, n\}$

$\zeta_n = e^{\frac{2\pi i}{n}}$



Ex!! $n = 4$, $k = 2$ $\quad c = (1, 2, 3, 4)$ permutes 2-subsets of $\{1, 2, 3, 4\}$:



there are two orbits.

Theorem: (R - Stanton - White)
When $c = (1, 2, \ldots, n)$ permutes k-subsets of $\{1, 2, \ldots, n\}$, $\begin{bmatrix}n\\k\end{bmatrix}_{q = \zeta_n^d}$ counts k-subsets fixed by c^d.

Ex!! $\zeta_4 = e^{\frac{\pi i}{2}}$ $\quad \begin{bmatrix}4\\2\end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$

If you plug in [redacted] $\zeta_4^1, \zeta_4^3 = \pm i$, you get 0

——— " ——— $q = \zeta_4^2 = -1$, you get 2

Exercise 4 Prove this theorem via

(a) Show if ζ is a primitive m^{th}-root of unity, then

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q=\zeta} = \binom{n'}{k'} \begin{bmatrix} n'' \\ k'' \end{bmatrix}_{q=\zeta}$$

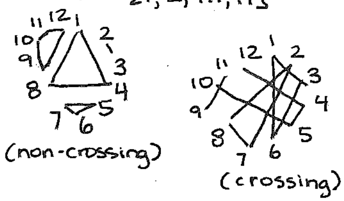
where $m \overset{n'}{\overline{)n}}$ and $m \overset{k'}{\overline{)k}}$, remainders n'' and k'':

$n = mn' + n''$, $0 \le n'' \le m-1$ $\qquad$ $k = k'm + k''$, $0 \le k'' \le m-1$

(b) Apply this where $\zeta = \zeta_n^d$ and compare to brute force count of k-subsets fixed by c^d.

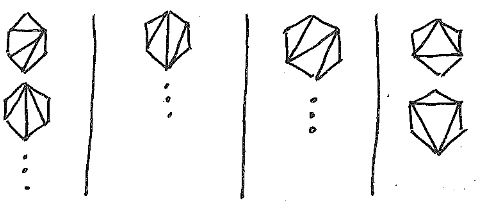
3. q-Catalan

The Catalan number $C_n := \frac{1}{n+1}\binom{2n}{n}$ counts
- triangulations of an n-gon
- non-crossing partitions of $\{1, 2, \ldots, n\}$



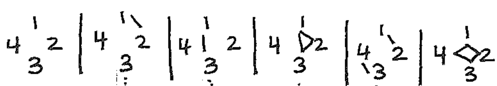
Ex // $n=4$, $C_4 = \frac{1}{5}\binom{8}{4} = 14$ counts...

6-gon triangulations



$6 + 3 + 3 + 2 = 14$

non-crossing partitions



$1 + 4 + 2 + 4 + 2 + 1 = 14$

Theorem: (R.-Stanton-White)

MacMahon's q-Catalan number $C_n(q) := \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n \end{bmatrix}_q$ has

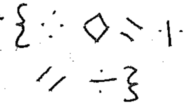
(a) $C_n(q)\big|_{q=(\zeta_{n+2})^d}$ counts (n+2)-gon triangulations fixed by c_{n+2}^d $(c_{n+2} = (1, 2, \ldots, n+2))$

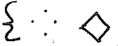
(b) $C_n(q)\big|_{q=(\zeta_n)^d}$ counts non-crossing partitions of $\{1,2,\ldots,n\}$ fixed by c_n^d

<u>EX</u>|| $n=4$, $C_4(q) = \frac{1}{[5]_q}\begin{bmatrix}8\\4\end{bmatrix}_q = 1+q^2+q^3+2q^4+q^5+2q^6+q^7+2q^8$
$+\,q^9+q^{10}+q^{12}$

($\in \mathbb{N}[q]$!, a theorem by MacMahon)

$q=1$ gives $C_4(1)=14$

$q=-1=\zeta_6^3$ —"— $C_4(-1=\zeta_6^3)=6$ $\#\{$  $\}$ or $\#\{$  $\}$

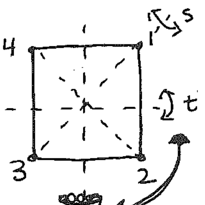
$q=e^{\frac{2\pi i}{3}}=\zeta_6^2$ —"— $C_4(\zeta_6^2)=2$ $\#\{$  $\}$ or $\{$  $\}$

$q=e^{\frac{2\pi i}{6}}$ —"— $C_4(e^{\frac{\pi i}{3}})=0$

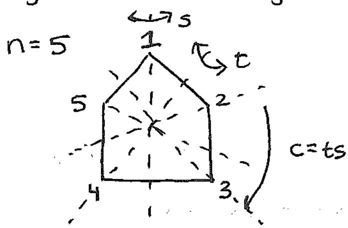
4. Dihedral actions

In fact, $\{1,\ldots,n\}$ aren't just cycled by $c_n=(1,2,\ldots,n)$ and its powers c^d; it carries an action of the <u>dihedral group $I_2(n)$ of order $2n$</u>.
:= symmetries of a regular n-gon

<u>EX</u>|| $n=4$



$c=ts$
(s acts, then t)



<u>Abstractly</u>, $I_2(n) = \langle \underbrace{s,t}_{\text{generators}} \mid \underbrace{s^2=e=t^2=(ts)^n}_{\text{relations}} \rangle$

$= \langle s,c \mid s^2=c^n,\ scs^{-1}=c^{-1}\rangle$

$= \{\underbrace{e,c,c^2,\ldots,c^{n-1}}_{\text{rotations}},\ \underbrace{s,sc,sc^2,\ldots,sc^{n-1}}_{\text{reflections}}\}$

The symmetry structure gives us a <u>representation</u> ← group homomorphism $G \xrightarrow{\rho} GL_n(\mathbb{F})$ (↑ field)

$I_2(n) \xrightarrow{\rho_{def}} GL_2(\mathbb{R})$

$c \longmapsto \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$, $\theta=\frac{2\pi}{n}$

$t \longmapsto \begin{bmatrix}1&0\\0&-1\end{bmatrix}$

The permutation actions on vertices $\{1,2,\ldots,n\}$

or k-subsets of $\{1,2,\ldots,n\}$

or triangulations

or non-crossing partitions

⋮

give various representations.

$$I_2(n) \xrightarrow{\rho} GL_N(\mathbb{R}) \subset GL_N(\mathbb{C})$$

$$g \longmapsto \text{some } N \times N \text{ permutation matrix}$$

general question: How do you decompose these permutation reps into direct sums of irreducible reps, that is, there are no non-zero proper subspaces stabilized. (even working over $\mathbb{C}$, not just $\mathbb{R}$)

$V = V_1 \oplus V_2$

$g \mapsto \begin{bmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{bmatrix}$

Theorem: (Maschke) This can always be done.

Reference: Steinberg, Ch. 2-4

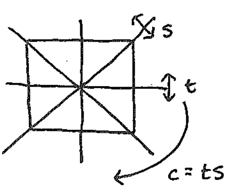
Theorem: (Frobenius, Schur, ...) One can do it by computing the character

$$\chi_\rho : G \to \mathbb{C}$$

$$g \mapsto \mathrm{Trace}(\rho(g))$$

There are tricks like orthogonality relations that help with computation.

EX11 $G = I_2(4) = \{e, c, c^2, c^3, s, sc, sc^2, sc^3\}$



there are five conjugacy classes:

$e \mid c, c^{-1} \mid c^2 \mid s, sc^2 \mid sc, sc^3$ (where $c^{-1} = c^3$ and $sc^3 = t$)

and five irreducible reps / chars

Character table:

irreducible reps \ conjugacy classes	e	c, c^{-1}	c^2	s, sc^2	sc, sc^3
$\mathbb{1}$	1	1	1	1	1
χ_s	1	-1	$+1$	-1	$+1$
χ_t	1	-1	$+1$	$+1$	-1
$\chi_s \chi_t$	1	$+1$	$+1$	-1	-1
χ_{def}	2	0	-2	0	0

$\chi_{\{1,2,3,4\}}$

e	c	c^2	s	sc
4	0	0	2	0

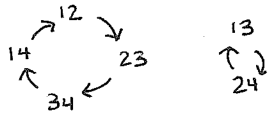
(imagine this on the bottom of the char

note these count vertices fixed by the reflection/rotation

$\chi_{\{2\text{-subsets}\}}$

e	c	c^2	s	sc
6	0	2	2	2

recall,



notice $\chi_{\{2\text{-subsets}\}} = \mathbb{1} + \mathbb{1} + \chi_t + \chi_s + \chi_{def}$

$$\chi_{\{1,2,3,4\}} = \mathbb{1} + \chi_t + \chi_{def}$$

REU Question 2. Expand into irreducibles

(a) $I_2(n)$ permuting k-subsets of $\{1, 2, \ldots, n\}$ (warm-up)

(b) $I_2(n)$ permuting non-crossing partitions of n

(c) $I_2(n+2)$ ⟵ " ⟶ triangulations of $(n+2)$-gon

- What binomial identity ensues?
- Does $C_n(q)$ help describe the reps/chars?
- Or does the (q,t)-Catalan number of Garsia-Haiman, $C_n(q,t)$, help?