Def: a minor of a matrix M is the determinant of a matrix obtained from M by selecting entries only in certain rows and columns.

Ex: $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

$\left| M_{\{2,3\},\{1,3\}} \right| = \left| \begin{bmatrix} 4 & 1 \\ 2 & 4 \end{bmatrix} \right| = 16 - 2 = 14$ size 2

$\left| M_{\{1\},\{2\}} \right| = \left| [2] \right| = 2$ size 1

Def: a <u>totally positive</u> (<u>totally nonnegative</u>) matrix is a matrix where all minors (of all sizes) are positive (nonneg.)

- These are related to networks / plabic graphs / wiring diagrams / cluster algs.
- They have nice eigenvalues.

<u>Notation</u>: $[n] = \{1, 2, \dots, n\}$.

<u>Cauchy-Binet Formula</u>: Let A, B be matrices of size $n \times m$, $m \times n$, $n \le m$.

$$\det(AB) = \sum_{\substack{S \subseteq [m] \\ |S| = n}} \det(A[S]) \det(B[S])$$ where $A[S]$, $B[S]$ are $n \times n$ submatrices corresponding to S.

Ex: $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ $n = 2$, $m = 3$

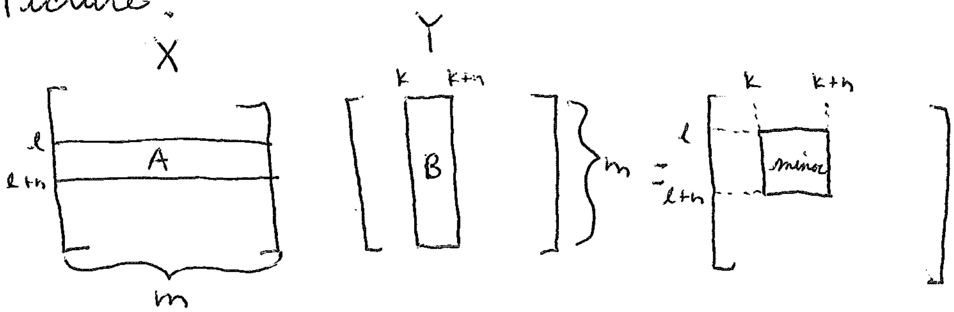
$\det(AB) = \det(A[1,2]) \det(B[1,2]) + \det(A[1,3]) \det(B[1,3]) + \det(A[2,3]) \det(B[2,3])$

$= \left| \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \right| \left| \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right| + \left| \begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix} \right| \left| \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \right| + \left| \begin{bmatrix} 3 & 5 \\ 4 & 6 \end{bmatrix} \right| \left| \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \right|$

$= (-2)(-2) + (-4)(-4) + (-2)(-2) = 4 + 16 + 4 = 24$

If we want to know a minor of XY, we can think of A, B from above as submatrices of X, Y.

Picture:



<u>Consequence</u>: Totally pos matrices, totally nonneg matrices are semigps (sets with associative binary relation).

<u>LDU Factorization</u>: Let X be a TNN nonsingular matrix. Then one can write $X = LDU$ where L is a lower triangular matrix with 1's on the diagonal, D is a (nonsingular) diagonal matrix, and U is an upper triangular matrix with 1's on the diagonal. These matrices are given by:

$$l_{ij} = \frac{|X_{[j-1]\cup i,[j]}|}{|X_{[j],[j]}|}, \quad u_{ij} = \frac{|X_{[i],[i-1]\cup j}|}{|X_{[i],[i]}|}, \quad d_{i,i} = \frac{|X_{[i],[i]}|}{|X_{[i-1],[i-1]}|}$$

Exercise: The leading principal minors $|X_{[k],[k]}|$ are positive for invertible TNN matrix X.

Exercise: The matrices in the LDU factorization are TNN.

<u>Def</u>: Let the <u>Chevalley generators</u> be $\{e_i(a) =$ matrix with 1's on the diagonal and a in entry $i,i+1\}$ and $\{f_i(a) =$ matrix with 1's on the diagonal and a in entry $i+1,i\}$.

<u>Thm</u>: A TNN upper triangular matrix with 1's on the diagonal can be factored into $e_i(a)$'s with $a \geq 0$. A TNN lower triangular matrix with 1's on the diagonal can be factored into $f_i(a)$'s with $a \geq 0$.

Thm: The following identities hold:

(1) $e_i(a)\, e_{i+1}(b)\, e_i(c) = e_{i+1}\left(\frac{bc}{a+c}\right) e_i(a+c)\, e_{i+1}\left(\frac{ab}{a+c}\right)$

(2) $f_i(a)\, f_{i+1}(b)\, f_i(c) = f_{i+1}\left(\frac{bc}{a+c}\right) f_i(a+c)\, f_{i+1}\left(\frac{ab}{a+c}\right)$

Def: A k-nonnegative matrix is a matrix where all minors of size $\leq k$ are nonnegative.

k-nonnegative matrices are a semigroup for the same reasons as before.

Problem: What are the generators/relations for the semigroup of nonsingular k-nonnegative matrices?

Might be easiest to start with $k = n-1$ or $k = 1$.

Exercise: For $n=2, k=1$, show the generators are $e_1(a), f_1(a)$, diagonal matrices, and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Another variation: Restrict to upper triangular matrices with 1's on the diagonal.