Chow rings of matroids

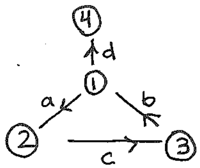
6/22/17

1. What's a matroid?
2. Characteristic polynomial
3. Chow ring & Adiprasito-Huh-Katz's 2015 work
4. REU problem 9

1. What's a matroid?

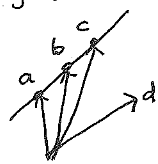
It abstracts a graph $G=(V,E)$ and a matrix $M=M_G$ (thought of as their column vectors)

$E = \{a,b,c,d\}$

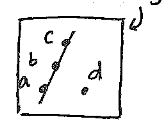


To get M_G, arbitrarily orient edges.

$$\text{Here, } M_G = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

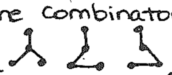


"affine drawing"



The column vectors live in a 3-dimensional space.

Matroid only specifies the combinatorial data of (in)dependence, in various equivalent forms:



- the <u>bases</u> $\mathcal{B}(M) = \{abd, acd, bcd\}$
  $\leftrightarrow$ {spanning trees in G}

- the <u>independent sets</u> $\mathcal{I}(M) = \{$subsets of bases$\}$
  $= \{abd, acd, bcd, ab, ac, ad, bc, bd, cd, a, b, c, d, \emptyset\}$
  $\leftrightarrow$ {forests in G}

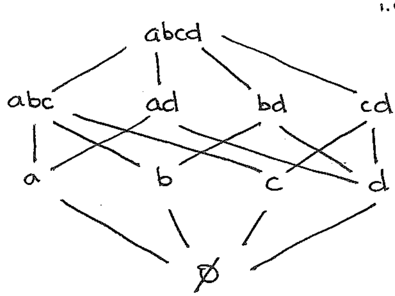
- the <u>circuits</u> $\mathcal{C}(M) = \{$minimal dependent sets under inclusion$\}$
  $= \{abc\}$
  $\leftrightarrow$ {minimal cycles in G}

If M has no zero vectors (loops in G) and no (anti-)parallel vectors (multiple edges), then an equivalent way to specify the matroid is via

$L(M) :=$ lattice of flats $F \subset E$ for M, ordered by inclusion

subsets F of vectors closed under linear span i.e., closed under adding in vectors that don't increase $r(F) := \max\{|I| : I \subset F, I \in \mathcal{I}(M)\}$

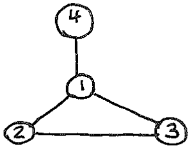
<u>EX1</u>



is $L(M)$ for our running example

EX!! $L(M)$ helps abstract $p_G(t) :=$ chromatic polynomial of G

$:= \#\{\text{proper vertex } t\text{-colorings of } G\}$



Choose color for ② first, t choices

then ③, $t-1$ choices

then ①, $t-2$ choices

then ④, $t-1$ choices

So $p_G(t) = t(t-1)^2(t-2)$

$= t^4 - 4t^3 + 5t^2 - 2t$ ← coefficients always alternate in sign.

$p_G(t)$ has unsigned coefficient sequence $(1, 4, 5, 2)$ } ← conjectured to be unimodal by Read 1968

conjectured log-concave, which ($a_k^2 \geq a_{k-1} a_{k+1}$) implies unimodal, by Hoggar, 1974.

2. Characteristic polynomial

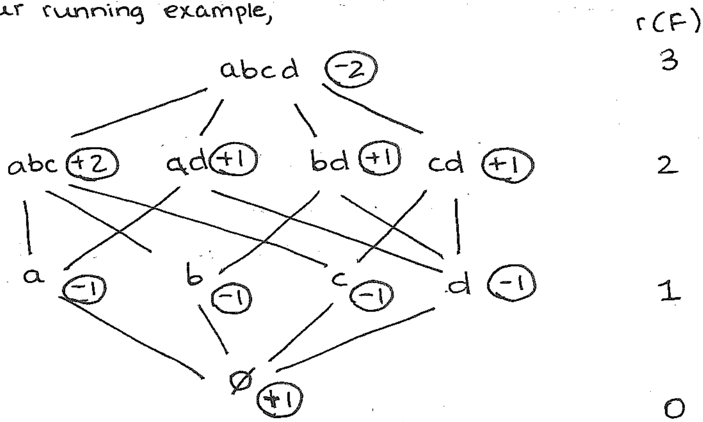
Turns out $p_G(t) = t^{\#\text{connected components}} \underbrace{\chi_{M_G}(t)}_{\text{characteristic polynomial of } M_G}$

where $\chi_{M_G}(t) := \sum_{\substack{\text{flats} \\ F \in L(M)}} \underbrace{\mu(\emptyset, F)}_{\text{mobius inversion function}} t^{r(E) - r(F)}$

$\mu(\emptyset, \emptyset) := +1$

$\mu(\emptyset, F) = -\sum_{G < F} \mu(\emptyset, G)$

EX!! In our running example,



$\mu(\emptyset, F)$ is circled

In fact, $\chi_M(t) = (t-1)\underbrace{\overline{\chi}_M(t)}_{\text{reduced characteristic polynomial of } M}$

$= (t-1)(t^2 - 3t + 2)$

the unsigned coefficient sequence for the reduced polynomial was conjectured unimodal, log concave for all matroids, by Rota - Heron - Welsh - Bydawski.
1971 1972 1976 1977

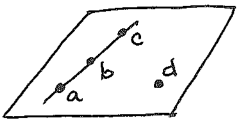
There was almost no progress until 2012...

3. The Chow ring $A(M)$ (Feichtner - Yuzvinsky, 2004)

$$A(M) := \mathbb{Z}[x_F]_{\substack{\text{non-empty,} \\ \text{proper flats} \\ F \in L(M)}} \Big/ \underbrace{(x_F x_G : F \not\subset G, G \not\subset F)}_{\text{quadratic relations}} + \underbrace{(\alpha_i - \alpha_j : i \neq j \text{ in } E)}_{\text{linear relations}}$$

where $\alpha_i = \sum_{\substack{\text{flats } F \\ \text{containing } i}}' x_F$

Ex 11 $M =$



$$A(M) = \mathbb{Z}[x_a, x_b, x_c, x_d, x_{cd}, x_{ad}, x_{bd}, x_{abc}] \Big/ \begin{pmatrix} x_a x_b & x_{ad} x_{bd} & x_d x_{abc} \\ x_a x_c & \vdots & x_{ad} x_b \\ \vdots & \vdots & \vdots \\ x_c x_d & x_{cd} x_{abc} & x_{ad} x_c \end{pmatrix} + (\alpha_a - \alpha_d, \alpha_b - \alpha_d, \alpha_c - \alpha_d)$$

Note $A(M)$ is a graded ring, $A(M) = A^0(M) \oplus A^1(M) \oplus A^2(M) \oplus \cdots$
with $A^i(M) \cdot A^j(M) \subset A^{i+j}(M)$

$\boxed{\text{Exercise 20}}$ In the (running) above example, show

(a) $A(M) = A^0(M) \oplus A^1(M) \oplus A^2(M)$ (i.e., $A^n(M) = 0$ for $n \geq 3$)
$= \mathbb{Z}^1 \oplus \mathbb{Z}^5 \oplus \mathbb{Z}^1$

with an isomorphism $A^2(M) \xrightarrow[\sim]{\deg} \mathbb{Z}$ sending $x_{F_1} x_{F_2} \mapsto +1$
$\forall F_1 \subsetneq F_2$

e.g. $x_a x_{ad} \mapsto +1$
but $x_a^2 \mapsto -1$
$x_d^2 \mapsto -2$
$x_{abc}^2 \mapsto -1$
$x_{ad}^2 \mapsto -1$

(b) These elements in $A(M)$

$$\alpha := \alpha_i \ \forall\, i \in E \qquad \beta := \sum_{\substack{\text{non-empty} \\ \text{flats } F}} x_F - \alpha$$

are linear and satisfy
$$\alpha^2 \xrightarrow{\deg} 1 \ (=\mu_0)$$
$$\alpha\beta \longrightarrow +3 \ (=\mu_1)$$
$$\beta^2 \longrightarrow +2 \ (=\mu_2)$$

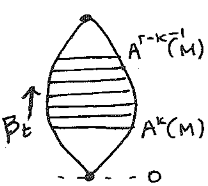
Theorem(s): (A-H-K, 2015) For (simple) matroid M,

(i) One always has $A(M) = \underbrace{A^0(M)}_{\mathbb{Z}} \oplus A^1(M) \oplus \cdots \oplus A^{r-2}(M) \oplus A^{r-1}(M)$, $\quad A^{r-1}(M) \cong \mathbb{Z}$

(and $A^k(M) = 0$ for $k \geq r$)

(ii) $\left.\begin{array}{l} \alpha := \alpha_i \ \forall\, i \in E \\ \beta := \sum_{F \not\ni i} x_F \end{array}\right\} \in A^1(M)$

satisfy $\alpha^{r-1-k}\beta^k \overset{\deg}{\longmapsto} \mu_k$ in $(\mu_0, \mu_1, \ldots, \mu_{r-1})$
$= $ unsigned coefficients of $\overline{\chi}_M(t)$



(iii) One has isomorphisms $A^k(M) \longrightarrow A^{r-1-k}(M)$ $\forall\, k \leq \frac{r-1}{2}$
and working with $\mathbb{R}$ coefficients, $\exists$ (ample) elements $\beta_t \in A^1(M)$ s.t.
it can be realized by $x \longmapsto \beta_t^{r-1-2k} \cdot x$

(iv) the quadratic forms $A^k(M) \xrightarrow{Q_k} \mathbb{R}$
$$x \longmapsto Q_k(x) = \deg\left(x\, \beta^{r-1-2k}\, x\right)$$
have predictable signatures.

(v) β is a limit of ample β_t's and somehow this implies
$\mu_k^2 \geq \mu_{k-1}\mu_{k+1}$ (log-concavity!)

4. $\boxed{\text{REU Problem 9}}$ For various "favorite" matroids, e.g.

- $M = M_G$ where G = complete graph K_n
- uniform (generic) matroids $M_{r,n}$ of rank r, $|E| = n$
- $M = \{$all (non-zero) vectors in $\mathbb{F}_8^n\}$

(a) compute explicit formulas for the Hilbert series of $A(M)$:
$$1 + \dim A^1(M)t^1 + \dim A^2(M) \cdot t^2 + \cdots + \dim A^{r-2} \cdot t^{r-2} + t^{r-1}$$
$$h = (\underset{h_0}{1}, \underset{h_1}{\dim A^1(M)}, \ldots, \underset{h_{r-2}}{\dim A^{r-2}(M)}, \underset{h_{r-1}}{1})$$

(there's a not very pretty formula for M_{K_n} and $M_{r,n}$ in F-Y)

(b) compute γ-vectors and Charney-Davis quantity for h.

(c) compute the eigenvalues of $Q_k = x \cdot \beta^{r-1-2k} \cdot x = x^T A x$, $A^T = A$

(d) When is $A(M)$ a Koszul algebra?

(e) What about the Smith normal forms of these A's?