Dihedral Sieving Phenomena

Sujit Rao, Joe Suk

31 July 2017

Outline

Cyclic sieving

Sieving for general groups

Sieving for dihedral groups

Examples of dihedral sieving

Future work

Cyclic Sieving Phenomenon (CSP)

Definition

Let $C_n \circlearrowright X$ be a finite set, $X(q) \in \mathbb{N}[q]$, and $\omega : C_n \to \mathbb{C}^\times$ be an embedding. Then $(X, X(q), C)$ exhibits cyclic sieving if

$$\forall c \in C : |X(q)|_{q=\omega(c)} = |\{x \in X : c(x) = x\}|.$$

Example

- $(\{k\text{-multisubsets of } [n]\}, \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q, C_n)$
- $(\{k\text{-subsets of } [n]\}, \begin{bmatrix} n \\ k \end{bmatrix}_q, C_n)$
- $(\{\text{noncrossing partitions of } n\text{-gon}\}, C_n(q), C_n)$
- $(\{\text{triangulations of a regular } n\text{-gon}\}, C_{n-2}(q), C_n)$
- $\left(\begin{Bmatrix} \text{dissections of } n\text{-gon} \\ \text{with } k \text{ diagonals} \end{Bmatrix}, \frac{1}{[n+k]_q}\begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q \begin{bmatrix} n-3 \\ k \end{bmatrix}_q, C_n\right)$
- $\left(\begin{Bmatrix} \text{noncrossing partitions of} \\ n\text{-gon with } n-k \text{ parts} \end{Bmatrix}, N(n,k;q), C_n\right).$

k-multisubsets

Proposition (Reiner-Stanton-White 2004)

Let V be a f.d. $GL_n(\mathbb{C})$-rep. Assume C permutes a basis $\{v_x\}_{x \in X}$. If

$$X(q) = \chi_\rho(1, q, \ldots, q^{n-1}) = \mathrm{Tr}\left(\mathrm{diag}(1, \ldots, q^{n-1}) : V \to V\right).$$

then $(X, X(q), C)$ has CSP.

Corollary (Reiner-Stanton-White 2004)

Let $V = \mathrm{Sym}^k(\mathbb{C}^N)$. If $\lambda = (k) \vdash k$, then

$$\chi_V(1, q, \ldots, q^{N-1}) = s_\lambda(1, q, \ldots, q^{N-1}) = \begin{bmatrix} N+k-1 \\ k \end{bmatrix}_q$$

and hence $(\{k\text{-multisubsets of } [N]\}, \begin{bmatrix} N+k-1 \\ k \end{bmatrix}_q, C_n)$ has CSP.

Equivalent Definition of Cyclic Sieving

Proposition (Reiner-Stanton-White 2004)

Consider $(X, X(q), C)$. Let A_X be a graded $\mathbb{C}$-vector space

$$A_X = \bigoplus_{i \geqslant 0} A_{X,i}$$

with

$$\sum_{i \geqslant 0} \dim_{\mathbb{C}} A_{X,i} q^i = X(q).$$

Define $C \circlearrowright A_{X,i}$ by $c \cdot v = \omega(c)^i v$. Then $(X, X(q), C)$ has CSP iff $A_X \cong \mathbb{C}^X$.

Representation Ring

Definition

The representation ring of G with coefficients in R is

$$\mathrm{Rep}(G;R) = R[\{\text{iso. classes of f.d. } G\text{-reps}\}]/(I+J)$$

where

$$I = (\{[U \oplus V] - ([U]+[V])\})$$
$$J = (\{[U \otimes V] - [U][V]\}).$$

Fact

An isomorphism $\mathrm{Rep}(G;\mathbb{C}) \to \mathrm{ClFun}(G)$ is given by $[V] \mapsto \chi_V$.

Fact

$\mathrm{Rep}(G;R)$ has an R-basis consisting of irreducible representations.

Cyclic Sieving Rephrased

Example

An isomorphism $\mathbb{Z}[q]/(q^n - 1) \to \mathrm{Rep}(C_n; \mathbb{Z})$ is given by $q \mapsto \omega_n$.

Proposition (Reiner-Stanton-White 2004)

$(X, X(q), C_n)$ has cyclic sieving $\Leftrightarrow \mathbb{C}^X = X(\omega_n)$ in $\mathrm{Rep}(C_n; \mathbb{Z})$.

Definition

Let $\rho_1, \ldots, \rho_k$ be representations of G. Let $G \circlearrowright X$ and $X(q_1, \ldots, q_k) \in \mathbb{C}[q_1, \ldots, q_k]$. Then

$$(X, X(q_1, \ldots, q_k), (\rho_1, \ldots, \rho_k), G)$$

has *G-sieving* if $\mathbb{C}^X = X(\rho_1, \ldots, \rho_k)$ in $\mathrm{Rep}(G; \mathbb{C})$.

Examples

Example (Cyclic Sieving)

$G = C_n$ and $\rho_1 = \omega_n$ for an embedding $\omega_n : C_n \to \mathbb{C}^\times$.

Definition (Barcelo-Reiner-Stanton 2007)

Let $G = C_n \times C_m$ and $\omega_n : C_n \to \mathbb{C}^\times$, $\omega_m : C_m \to \mathbb{C}^\times$ be embeddings. Let $X(t,q) \in \mathbb{Z}[t,q]$. Then $(X, X(t,q), C_n \times C_m)$ has bicyclic sieving if

$$X(\omega_n(c), \omega_m(c')) = |\{x \in X : (c,c')x = x\}|.$$

Example (Bicyclic Sieving)

$G = C_n \times C_m$, $\rho_1 = \omega_n \otimes 1_m$ and $\rho_2 = 1_n \otimes \omega_m$.

Irreducible Representations of $I_2(n)$

Let the dihedral group of order $2n$ be

$$I_2(n) = \left\langle r, s \middle| r^n = s^2 = e, rs = sr^{-1} \right\rangle .$$

n odd $\mathbb{1}$, det, $z_1, \ldots, z_{(n-1)/2}$.

n even $\mathbb{1}$, det, χ_a, $\chi_a \cdot$ det, $z_1, \ldots, z_{(n-2)/2}$.

$$z_i(r) = \begin{bmatrix} \cos(\frac{2\pi k}{n}) & -\sin(\frac{2\pi k}{n}) \\ \sin(\frac{2\pi k}{n}) & \cos(\frac{2\pi k}{n}) \end{bmatrix}$$
$$z_i(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Fact

$z_i = \mathrm{Ind}_{C_n}^{I_2(n)} \omega^i$.

Properties of Fibonomial Coefficients

Definition (Amdeberhan, Chen, Moll, Sagan)

$$\begin{aligned}\{0\}_{s,t} &= 0\\ \{1\}_{s,t} &= 1\\ \{n+2\}_{s,t} &= s\{n+1\}_{s,t} + t\{n\}_{s,t}.\end{aligned}$$

Proposition (Amdeberhan, Chen, Moll, Sagan)

Let $X = \frac{s+\sqrt{s^2+4t}}{2}$ and $Y = \frac{s-\sqrt{s^2+4t}}{2}$. Then

$$\begin{aligned}[n]_q &= \{n\}_{s,t}\big|_{s=q+1,t=-q}\\ \{n\}_{s,t} &= Y^{n-1}\,[n]_q\big|_{q=X/Y}\\ \left\{\begin{matrix} n\\ k\end{matrix}\right\}_{s,t} &= Y^{k(n-k)}\left.\begin{bmatrix} n\\ k\end{bmatrix}_q\right|_{q=X/Y}.\end{aligned}$$

k-multisubsets

Proposition

Let n be odd, V a f.d. $\mathrm{GL}_n(\mathbb{C})$-rep. Assume $I_2(n)$ permutes a basis $\{v_x : x \in X\}$. Let $p \in \mathbb{Z}[x, y]$ be unique such that

$$p(a+b, ab) = \chi_V(a^{n-1}, a^{n-2}b, \ldots, ab^{n-2}, b^{n-1})$$

Then $(X, p, (z_1, -\det), I_2(n))$ exhibits dihedral sieving.

Corollary

Let n be odd and $X = \left(\!\binom{[n]}{k}\!\right)$. Then

$$\left(X, \left\{ {n+k-1 \atop k} \right\}_{s,t}, (z_1, -\det), I_2(n)\right)$$

exhibits dihedral sieving.

Comparison of generating functions

C_n $\rho_1 = \omega_n$

$I_2(n)$ $\rho_1 = z_1,\ \rho_2 = -\det$

n odd	C_n	$I_2(n)$
{k-subsets of $[n]$}	$\left[{n \atop k}\right]$	$\left\{{n \atop k}\right\}$
{k-multisubsets of $[n]$}	$\left[{n+k-1 \atop k}\right]$	$\left\{{n+k-1 \atop k}\right\}$
{NC partitions of $[n]$}	$\frac{1}{[n+1]}\left[{2n \atop n}\right]$	$\frac{1}{\{n+1\}}\left\{{2n \atop n}\right\}$
{NC partitions of $[n]$ with $n-k$ blocks}	$\frac{1}{[n]}\left[{n \atop k}\right]\left[{n \atop k+1}\right]q^{k(k+1)}$	$\frac{1}{\{n\}}\left\{{n \atop k}\right\}\left\{{n \atop k+1}\right\}$

Possibly useful polynomials for even n

$$\{0\}_{s,t,a} = 0$$
$$\{1\}_{s,t,a} = 1$$
$$\{2\}_{s,t,a} = s$$
$$\{n\}_{s,t,a} = \begin{cases} sa\{n-1\} + t\{n-2\} & n \text{ is odd} \\ s\{n-1\} + t\{n-2\} & n \text{ is even.} \end{cases}$$

with substitution $(s, t, a) = (z_1 + b/n, -\det, 1 - b/4)$ gives

$$\left\{ {n \atop k} \right\}_{s,t,a} = \begin{cases} \left[{n \atop k} \right]_{q=\xi_n^{2\ell}} & \{r^\ell, r^{n-\ell}\} \\ \left[{n \atop k} \right]_{q=\xi_2} & \{sr, sr^3, sr^5, \ldots\} \\ \left[{n \atop k} \right]_{q=\xi_2} + 2\left[{n-2 \atop k-1} \right]_{\xi_2} + \left[{n-2 \atop k-2} \right]_{q=\xi_2} & \{sr^2, sr^4, \ldots\} \end{cases}$$

Further Dihedral Actions

- Triangulations and dissections of an n-gon
- Rhoades's promotion-evacuation action on rectangular tableaux
- $I_2(4)$ on alternating sign matrices

Acknowledgments

This research was carried out as part of the 2017 summer REU program at the School of Mathematics, University of Minnesota, Twin Cities, and was supported by NSF RTG grant DMS-1148634. The authors would like to thank Victor Reiner, Pavlo Pylyavskyy, and Benjamin Strasser for their mentorship and support.