Correlations in Pattern Avoidance

Marisa Gaetz, Will Hardt, Shruthi Sridhar, and Anh Quoc Tran

UMN Twin Cities Combinatorics REU

Problem 8

August 1, 2017

Overview

1. Preliminaries
2. Correlation Problem
   - Orignial Problem
   - New Problem, $u, v, w \in S_3$
   - New Problem, $v = (k...1)$, $u = (\ell...1)$, $w \in S_3$
3. Characteristic Polynomial Problem
   - Avoiding $k(k-1)...1$
   - Avoiding $t(t-1)..1k(k-1)...(t+1)$
   - Avoiding $12...k$

Preliminaries

- A permutation is a bijection from $\{1, 2, ..., n\}$ to itself

Preliminaries

- A permutation is a bijection from $\{1, 2, ..., n\}$ to itself
- One-line notation

Preliminaries

- A permutation is a bijection from $\{1, 2, ..., n\}$ to itself
- One-line notation

Definition

A permutation $\pi = \pi(1)\pi(2)...\pi(m)$ *contains a pattern* $\sigma = \sigma(1)\sigma(2)...\sigma(k)$ *if there exists a subsequence* $(i_1 < ... < i_k)$ $\pi(i_1)\pi(i_2)...\pi(i_k)$ *of* π *with the same relative ordering as* σ*. Otherwise* π *avoids* σ*.*

Preliminaries

- A permutation is a bijection from $\{1, 2, ..., n\}$ to itself
- One-line notation

Definition

A permutation $\pi = \pi(1)\pi(2)...\pi(m)$ *contains a pattern* $\sigma = \sigma(1)\sigma(2)...\sigma(k)$ *if there exists a subsequence* $(i_1 < ... < i_k)$ $\pi(i_1)\pi(i_2)...\pi(i_k)$ *of* π *with the same relative ordering as* σ*. Otherwise* π *avoids* σ*.*

Example

(**52**341**6**) *contains* (213)*, but avoids* (132)

- $S_n(\sigma_1, ..., \sigma_k) = \{\pi \in S_n \mid \pi \text{ avoids } \sigma_1, ..., \sigma_k\}$

The Original Problem

Problem

How do avoiding u and avoiding w correlate for random permutations in S_n?

The Original Problem

Problem

How do avoiding u and avoiding w correlate for random permutations in S_n?

Solution (Joel Lewis)

If $u = (1, 2, ..., k)$, $w = (\ell, \ell - 1, ..., 1)$, *then negative correlation (Erdős-Szekeres). Otherwise, positive (Marcus-Tardos).*

New Problem

Problem

How do avoiding u and avoiding w correlate for random permutations in $S_n(v)$*?*

New Problem

Problem

How do avoiding u and avoiding w correlate for random permutations in $S_n(v)$?

Criterion

Positive correlation if and only if

$$(\#S_n(v))(\#S_n(v,u,w)) > (\#S_n(v,u))(\#S_n(v,w))$$

Answer for $u, v, w \in S_3$ Case

- Simion and Schmidt (1985) give $\#S_n(\Pi)$ for $\Pi \subset S_3$, $|\Pi| = 1, 2, 3$

Answer for $u, v, w \in S_3$ Case

- Simion and Schmidt (1985) give $\#S_n(\Pi)$ for $\Pi \subset S_3$, $|\Pi| = 1, 2, 3$
- The following (v, u, w) triples negatively correlate:

Answer for $u, v, w \in S_3$ Case

- Simion and Schmidt (1985) give $\#S_n(\Pi)$ for $\Pi \subset S_3$, $|\Pi| = 1, 2, 3$
- The following (v, u, w) triples negatively correlate:

v	(u, w) **unordered pair**
(132)	(123, 231), (123, 312), (213, 231), (213, 312), (231, 312)
(213)	(123, 231), (123, 312), (132, 231), (132, 312), (231, 312)
(231)	(132, 213), (132, 312), (132, 321), (213, 312), (213, 321)
(312)	(132, 213), (132, 231), (132, 321), (213, 231), (213, 321)

Table: Complete list of "interesting" negative correlations for $u, v, w \in S_3$

$v = (k...1)$, $u = (\ell...1)$, $w \in S_3$ Case

Criterion

Positive correlation if and only if

$$(\#S_n(k...1))(\#S_n(w,\ \ell...1)) > (\#S_n(\ell...1))(\#S_n(w,\ k...1))$$

$v = (k\ldots1)$, $u = (\ell\ldots1)$, $w \in S_3$ Case

Criterion

Positive correlation if and only if

$$(\#S_n(k\ldots1))(\#S_n(w,\ \ell\ldots1)) > (\#S_n(\ell\ldots1))(\#S_n(w,\ k\ldots1))$$

Theorem (Reifegerste)

$$\#(S_n(132, m\ldots1)) = \frac{1}{n}\sum_{i=1}^{m-1}\binom{n}{i}\binom{n}{i-1}$$

$v = (k...1)$, $u = (\ell...1)$, $w \in S_3$ cont.

Theorem (Arriata/Regev)

$$\#(S_n(m...1)) \sim \lambda_m \frac{(m-1)^{2n}}{n^{m(m-2)/2}} \textit{ for some constant } \lambda_m$$

$v = (k...1)$, $u = (\ell...1)$, $w \in S_3$ cont.

Theorem (Arriata/Regev)

$$\#(S_n(m...1)) \sim \lambda_m \frac{(m-1)^{2n}}{n^{m(m-2)/2}} \textit{ for some constant } \lambda_m$$

Conclusion

$w = 132$: *positive correlation.*

$v = (k...1)$, $u = (\ell...1)$, $w \in S_3$ cont.

Theorem (Arriata/Regev)

$$\#(S_n(m...1)) \sim \lambda_m \frac{(m-1)^{2n}}{n^{m(m-2)/2}} \textit{ for some constant } \lambda_m$$

Conclusion

$w = 132$: *positive correlation.*

Fact

$$\#S_n(\Pi) = \#S_n(\Pi^R) = \#S_n(\Pi^C)$$

i.e.: $\#S_n(132) = \#S_n(213)$ *and* $\#S_n(132, m...1) = \#S_n(213, m...1)$

Conclusion

$w = 213$: *positive correlation.*

$v = (k...1)$, $u = (\ell...1)$, $w \in S_3$ cont.

Theorem (Arriata/Regev)

$$\#(S_n(m...1)) \sim \lambda_m \frac{(m-1)^{2n}}{n^{m(m-2)/2}} \textit{ for some constant } \lambda_m$$

Conclusion

$w = 132$: *positive correlation.*

Fact

$$\#S_n(\Pi) = \#S_n(\Pi^R) = \#S_n(\Pi^C)$$

i.e.: $\#S_n(132) = \#S_n(213)$ *and* $\#S_n(132, m...1) = \#S_n(213, m...1)$

Conclusion

$w = 213$: *positive correlation.*

What about $w = 231$?

Main conjecture

Albert, Atkinson and Vatter proved that any subclass of 231-avoiding permutations satisfies a linear recurrence.

Conjecture

For any 231-avoiding permutation π, $T(n) = S_n(231, \pi)$ satisfies a linear recurrence, and its characteristic polynomial has all positive real roots.

This implies these coefficients form a *Pólya frequency sequence*.

Avoiding 231 and $k(k-1)...1$

In the rest of the talk, we will denote $D(n,k) = |S_n(231, k(k-1)...1)|$. Note that $D(n,k) = C_n$, the Catalan number, for $n < k$.

Theorem

Let $t = \left\lfloor \frac{k}{2} \right\rfloor$, *then*

$$D(n,k) = \binom{k-1}{1} D(n-1,k) - \binom{k-2}{2} D(n-2,k) + \ldots + (-1)^{t+1} \binom{k-t}{t} D(n-t,k)$$

When $k = 2t$, *this result is true from* $n = t$, *and when* $k = 2t+1$, *this is true from* $n = t+1$.

Proof

We proved the theorem by induction on k with the following recurrence

Lemma

$$D(n,k) = \sum_{0 \le i < n} D(i,k) D(n-i-1, k-1)$$

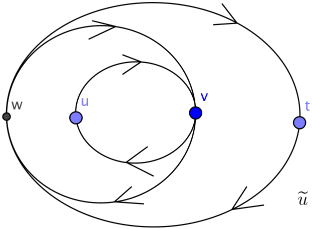
combined with the identity

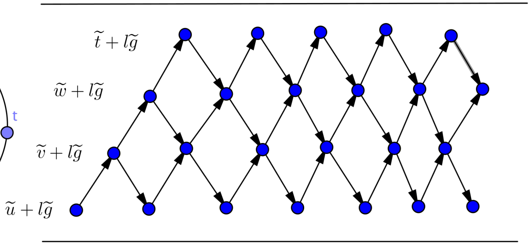
Lemma

$$(-1)^{j+1}\binom{n-j}{j} = (-1)^{j+1}\binom{n-1-j}{j} + C_{j-1} - \sum_{i=1}^{j-1} (-1)^{i+1} C_{j-1-i} \binom{n-1-i}{i}$$

Another proof

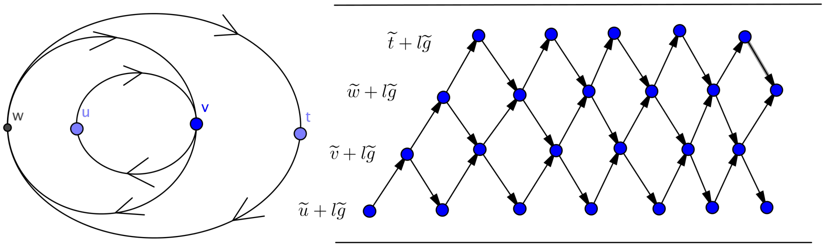
$$S_n(231, k\ldots 1) \leftrightarrow \{\text{Dyck paths of length } 2n \text{ and height} \leq k-1\}$$





$$f(n) = |S_n(231, k\ldots 1)| = \text{number of directed paths connecting } \widetilde{u}, \widetilde{u} + n\widetilde{g}.$$

Another proof



- The RHS: an acyclic weighted (all weighted 1) graph $\widetilde{N}$ in a strip $\mathcal{S}$, invariant under a shift $\widetilde{g}$.
- The LHS is $\widetilde{N}$'s projection N on a cylinder $\mathcal{O} = \mathcal{S}/\mathbb{Z}\widetilde{g}$

Another proof

Galashin and Pylyavskyy proved a general (for any cylindrical network) version of the statement below:

Theorem

Denote $f(n)$ the number of paths connecting $\widetilde{u}, \widetilde{u} + n\widetilde{g}$. Then for all but finitely many n, the sequence f satisfies a linear recurrence with characteristic polynomial

$$Q_N(t) = \sum_{r=0}^{d} (-t)^{d-r} |\mathcal{C}^r(N)|$$

$\mathcal{C}^r(N)$ is the set of $r-$tuples of disjoint simple cycles in N.

More conjecture

Conjecture

For any 231-avoiding pattern π*, we can construct a cylindrical network* $\widetilde{N}$ *such that* $f(n) = |S_n(231, \pi)|$ *has characteristic polynomial* $Q_N(t)$.

Avoiding 231 and $k(k-1)...1$

Proposition

Let $P_k(x)$ denotes the characteristic polynomial for $D(n,k)$. Then P_k has all real roots.

We can prove that the roots of P_k and P_{k+1} are interlaced by the following identities:

$$P_{2k+1}(x) - P_{2k}(x) = -P_{2k-1}(x)$$

$$P_{2k}(x) - xP_{2k-1}(x) = -P_{2k-2}(x)$$

Avoiding 231 and $k(k-1)\ldots 1$

Conjecture

Let $P_k(x)$ denotes the characteristic polynomial for $D(n,k)$. Then $P_k(4(k-1)^2/k^2) < 0$.

This implies that the largest root of P_k is larger then $4(k-1)^2/k^2$, and consequently answer the correlation question earlier.

Avoiding 231 and $t(t-1)...1k(k-1)...(t+1)$

Theorem

$$|S_n(231, t(t-1)...1k(k-1)...(t+1))| = |S_n(231, k(k-1)...1)|$$

.

This is interesting because it isn't known that there is a bijection between permutations that preserves 231-avoiding and maps $k(k-1)...1$ to $t(t-1)...1k...(t+1)$.

Proof

Show that $|S_n(231, t...1k(k-1)...(t+1))|$ satisfies the same linear recurrence as $|S_n(231, k...1)|$.

Proposition

Let $\pi = t..1k...(t+1)$ *and* $T(n,\pi) = |S_n(231,\pi)|$ *and* $D(n,k) = |S_n(231, k..1)|$. *Then,*

$$T(n+1,\pi) = \sum_{0 \le i < n+1} \Big(T(i,\pi)D(n-i, k-t-1) + D(i,t)T(n-i,\pi) - D(i,l)D(n-i,k-t-1) \Big)$$

Let $\rho = \sigma n \tau \in S_n$. Then,

$$\rho \in S_n(231,\pi) \Leftrightarrow \sigma \in S_n(231, t...1) \text{ or } \tau \in S_n(231, k-t-1...1)$$

Avoiding 231 and $12...k$

Conjecture

$I(n,k) = S_n(231, 12...k)$ has characteristic polynomial $(x-1)^{2k-3}$

We know that

$$I(n,k) = \sum_{i=1}^{k-1} \frac{1}{n} \binom{n}{i} \binom{n}{i-1} = \sum_{i=1}^{k-1} \frac{1}{i} \binom{n}{i-1} \binom{n-1}{i-1}$$

so the conjecture above would follow from the identity below, which we believe to be true

Conjecture

$$\sum_{i=0}^{2k+1} (-1)^i \binom{2k+1}{i} \binom{n+i}{k} \binom{n+i-1}{k} = 0$$

References

Astrid Reifesgerste (2003)
On the diagram of 132-avoiding permutations
European Journal of Combinatorics 24(6), 759 – 776.

Richard Arriata (1999)
On the Stanley-Wilf Conjecture for the Number of Permutations Avoiding a Given Pattern
The Electronic Journal of Combinatorics 6.

Amitai Regev (1981)
Asymptotic values for degrees associated with strips of young diagrams
Advances in Mathematics 42(2), 115–136.

Simion and Schmidt (1985)
Restricted Permutations
European Journal of Combinatorics 6(4), 383–406.

Erdős and Szekeres (1935)
A combinatorial problem in geometry
Compositio Mathematica 2.

References

Albert, Atkinson and Vatter (2011)
Subclasses of the separable permutations
Bull. London Math. Soc. 43, 859870.

Galashin and Pylyavskyy (2017)
Linear recurrences for cylindrical networks
arXiv:1704.05160.

Acknowledgements

This research was performed as a part of the 2017 University of Minnesota, Twin Cities Combinatorics REU, and was supported by NSF RTG grant DMS-1148634 and by NSF grant DMS-1351590. We would like to thank Vic Reiner, Pasha Pylyavskyy, and Galen Dorpalen-Barry for their advice, mentorship, and support.