ON THE CRITICAL GROUPS OF CUBES

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1. Introduction

Let G be a graph on n vertices with no self-loops. We can construct its *Laplacian* $L(G)$ to be the $n \times n$ matrix satisfying

$$L(G)_{u,v} = \begin{cases} \deg(u) \text{ if } u = v \\ -m(u,v) \text{ if } u \neq v \end{cases}$$

where $m(u,v)$ is the number of edges between u and v.

By definition, the Laplacian is an integer matrix, so it can be considered as a map of $\mathbf{Z}$-modules $\mathbf{Z}^n \to \mathbf{Z}^n$.

Furthermore, $L(G)$ has kernel equal to the span of the all-ones vector $(1, 1, \ldots, 1)$. It follows that the cokernel can be written as

$$\mathbf{Z}^n / \operatorname{im}(L(G)) \simeq \mathbf{Z} \oplus K(G)$$

where $K(G)$ is a finite abelian group.

The group $K(G)$ is called the *critical group* of G and is an important algebraic invariant in the study of abelian sandpiles. It is also a well-known consequence of Kirchhoff's Matrix-Tree theorem that the order $|K(G)|$ counts the number of spanning trees of G.

In this paper, we investigate the critical group of the *hypercube graph* Q_n, which is equal to the n-fold product of the complete graph on two vertices.

In 2003, H. Bai determined in [1] the p-Sylow subgroups of $K(Q_n)$ for any prime $p > 2$. Last year, Chandler et al [3] calculated the cokernel of the *adjacency matrix* of Q_n and found that it was determined by its eigenvalues.

However, the determination of the 2-Sylow subgroup of the critical group of Q_n remains an open problem.

In Section 1, we make partial progress on this problem by determining an upper bound for the size of the largest cyclic factor of the 2-Sylow subgroup of $K(Q_n)$. We also use the same method to determine similar bounds for related graphs. In Section 2, we outline a possible approach to determination of the critical group via Gröbner basis calculations. In Section 3, we define a cell complex structure on the hypercube and determine the p-Sylow subgroups of its higher critical groups.

2. A Bound On The Top Cyclic Factor

In the previous section, we introduced the hypercube graph Q_n as the n-fold product of the two-cycle C_2. In this section, we prove general results about n-fold products of directed cycles. Toward this end, let C_k denote the directed k-cycle. It will be useful for the rest of the paper to consider C_k^n in the context of *Cayley graphs*.

Definition 2.1. Let G be a group and $E \subset G$ be some subset of elements. The **Cayley graph** $\Gamma(G, E)$ is the (directed) graph with vertex set G and edge set consisting of all pairs of elements (g, h) such that $gh^{-1} \in E$.

If E is a subset such that for every $e \in E$ we have $e^{-1} \in E$, then every pair of adjacent vertices g, h in $\Gamma(G, E)$ will have an edge from g to h and from h to g. In this case, we identify the two edges and consider $\Gamma(G, E)$ as an undirected graph.

Now we can realize C_k^n as $\Gamma((\mathbf{Z}/k\mathbf{Z})^n, E)$ where E is the set of tuples $\{e_i \,|\, 1 \leq i \leq n\}$ which are equal to 1 in the ith index and 0 elsewhere.

Following the approach of [2], we can induce a ring structure on the cokernel of $L(C_k^n)$ from the more general result [2, Proposition 5.20]. We include a proof in our case for the sake of completeness.

Proposition 2.2. $\operatorname{coker}(C_k^n)$ *is isomorphic to* $\mathbf{Z}[x_1, \ldots, x_n]/(x_1^k - 1, x_2^k - 1, \ldots, x_n^k - 1, n - (x_1 + x_2 + \cdots + x_n))$ *as an abelian group.*

Proof. First, note that there is an isomorphism of abelian groups

$$\mathbf{Z}^{k^n} \xrightarrow{\sim} \mathbf{Z}[x_1, \ldots, x_n]/(x_1^k - 1, x_2^k - 1, \ldots, x_n^k - 1),$$

due to the fact that the latter group has a $\mathbf{Z}$-basis consisting of monomials absent k-th powers.

Furthermore, this isomorphism can be realized as the map sending the generator $e_i \in \mathbf{Z}^{k^n}$ equal to the tuple with a 1 in the ith place and a 0 elsewhere to the monomial $\prod_{j=1}^n x_j^{b_j}$, where $b_1 b_2 \ldots b_n$ is the base k expansion of i.

To determine $\operatorname{coker}(L(C_k^n)) = \mathbf{Z}^{2^n} / \operatorname{im}(L(C_k^n))$, it suffices to examine the image of $\operatorname{im}(L(C_k^n))$ under this isomorphism. By definition of C_k^n, the Laplacian maps e_i to ne_i minus the sum of e_j across all j such that the base k expansions of i and j differ in exactly one digit.

Under our isomorphism, this indicates that $L(C_k^n)$ maps the monomial $\prod_{j=1}^n x_j^{b_j}$ to

$$n \prod_{j=1}^{n} x_j^{b_j} - \sum_{k=1}^{n} x_k \prod_{j=1}^{n} x_j^{b_j} = \prod_{j=1}^{n} x_j^{b_j} \left(n - \sum_{k=1}^{n} x_k \right)$$

Therefore, the image of $L(C_k^n)$ in $\mathbf{Z}[x_1, \ldots, x_n]/(x_i^k - 1)$ is the ideal generated by $n - \sum x_i$ and the assertion follows. $\square$

We now use this ring structure to determine an element of maximal order in this abelian group.

Lemma 2.3. *The image of the element* $x_j - 1$ *has maximal additive order in* $\mathbf{Z}[x_1, \ldots, x_n]/(x_1^k - 1, x_2^k - 1, \ldots, x_n^k - 1, n - \sum x_i)$ *for any* j.

Proof. That $x_j - 1$ has finite order in the group follows immediately from [2, Proposition 5.20]

From our Cayley graph interpretation of C_k^n, it is clear that $L(C_k^n)$ is the McKay-Cartan matrix corresponding to the faithful representation γ of $(\mathbf{Z}/k\mathbf{Z})^n$ equal to the sum of irreducible characters $\sum_{e \in E} \chi_e$.

As a result of the proposition, we have that $\mathbf{Z}[x_1, \ldots, x_n]/(x_i^k - 1, n - \sum x_i)$ is isomorphic to the representation ring of $(\mathbf{Z}/k\mathbf{Z})^n$ modulo the ideal generated by $n - \chi_\gamma = n - \sum_{e \in E} \chi_e$.

By the second part of the proposition, an element has finite additive order in this ring iff it lies in the kernel of the map sending all of the χ_{e_i} to 1. The element corresponding to $x_j - 1$ in the representation ring is $\chi_e - 1$ for some irreducible character χ_e under our isomorphism, and it follows that it has finite additive order.

Furthermore, a consequence of this proposition is that *any* polynomial with finite additive order is a linear combination of $x_I - 1$, where x_I denotes a monomial free of k-th powers.

First, we will show for any $j \neq k$ that $x_j - 1$ and $x_k - 1$ have the same additive order. Let C be the additive order of $x_j - 1$. Then there exist polynomials f_i, g such that $C(x_j - 1) = \sum_{i=1}^n f_i(x_1, \ldots, x_n)(x_i^k - 1) + g(x_1, \ldots, x_n)(n - \sum x_i)$. Observe that the generating set of the ideal is invariant under permutation of the variables. Therefore, we can obtain an algebraic combination of the generators that is equal to $C(x_k - 1)$ by interchanging the roles of the variables x_j and x_k. Doing the same in the other direction tells us that $x_j - 1$ and $x_k - 1$ have the same additive order.

Now to show maximality of the additive order of $x_j - 1$, it suffices to show that if $C(x_j - 1) \in (x_i^k - 1, n - \sum x_i)$ for some constant C, then $C(x_I - 1) \in (x_i^k - 1, n - \sum x_i)$ for any monomial x_I. Indeed, we may assume without loss of generality that $x_1 | x_I$. Then writing

$$C(x_I - 1) = C(x_1 - 1) \cdot \frac{x_I}{x_1} + C\left(\frac{x_I}{x_1} - 1\right),$$

we reduce inductively to the case $\deg x_I = 1$, which was done above.

$\square$

Recall by our proposition that the ring $\mathbf{Z}[x_1, \ldots, x_n]/(x_i^k - 1, n - \sum x_i)$ has an underlying abelian group structure, where the torsion part of the group is the critical group $K(C_k^n)$. Therefore, Lemma 2.3 give the following result.

Corollary 2.4. *The size of the largest cyclic factor in $K(C_k^n)$ is equal to the additive order of $x_1 - 1$ in $\mathbf{Z}[x_1, \ldots, x_n]/(x_i^k - 1, n - \sum x_i)$.*

Though this result will suffice in the case $k = 2$, we will need a different term of high order for $k = 3, 4$, and 6.

Lemma 2.5. *For $k > 2$, the image of the element $x_j^{k-1} + x_j^{k-2} + \cdots + x_j - (k-1)$ has order $\frac{\mathrm{ord}(x_j - 1)}{k}$ in $\mathbf{Z}[x_1, \ldots, x_n]/(x_i^k - 1, n - \sum x_i)$.*

Proof. Let C be the least constant such that $x_j - 1$ can be written in the form

$$C(x_j - 1) = \left(x_j^{k-1} + x_j^{k-2} + \cdots + x_j - (k-1)\right) \sum_{\ell=0}^{k-1} \alpha_\ell x_j^\ell \tag{1}$$

for some $\alpha_\ell \in \mathbf{Z}$. We will show that $C = k$ all at once.

Write (1) as the following matrix equation:

$$\begin{bmatrix} -(k-1) & 1 & 1 & \cdots & 1 \\ 1 & -(k-1) & 1 & \cdots & 1 \\ 1 & 1 & -(k-1) & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & -(k-1) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{k-1} \end{bmatrix} = \begin{bmatrix} -C \\ C \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

It suffices to understand the solutions $[\alpha_0\ \alpha_1\ \cdots\ \alpha_{k-1}]^T$ when $C = 1$. Let A denote the $k \times k$ matrix on the left. By observation, it has $[-1\ 1\ 0\ \cdots\ 0]^T$ as eigenvector with eigenvalue $-k$, and kernel $[1\ 1\ \cdots\ 1]^T$. Thus all solutions $[\alpha_0\ \alpha_1\ \cdots\ \alpha_{k-1}]^T$ when $C = 1$ are given by

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{k-1} \end{bmatrix} = \begin{bmatrix} -\frac{1}{k} \\ \frac{1}{k} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

for $c \in \mathbf{Q}$. Thus, when $k \geq 3$, any solution of the above form will have a denominator $\geq k$. So, it is necessary to have $k | C$. Taking $C = k$, we have an explicit solution

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{k-1} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Hence $C = k$, as desired. $\square$

Remark 2.6. Note that when $k = 2$, we can choose $c = \frac{1}{2}$ above and obtain a solution $\alpha_0 = 1$, $\alpha_1 = 0$ with $C = 1$. Indeed, in this case, $x_j^{k-1} + \cdots + x_j - (k-1) = x_j - 1$, so this agrees with Lemma 2.3.

From this, we derive the immediate corollary:

Corollary 2.7. *For $k \geq 3$, the size of the largest cyclic factor in $K(C_k^n)$ is equal to k times the additive order of $x_j^{k-1} + x_j^{k-2} + \cdots + x_j - (k-1)$ in $\mathbf{Z}[x_1, \ldots, x_n]/(x_i^k - 1, n - \sum x_i)$.*

We now turn our attention to finding the additive order of the elements $x_1 - 1$ and $x_1^{k-1} + x_1^{k-2} + \cdots + x_1 - (k-1)$.

By the isomorphism in Proposition 2.2, the polynomial $x_j^{k-1} + x_j^{k-2} + \cdots + x_j - (k-1)$ corresponds to the vector $\mathbf{w} = (-(k-1), 1, \ldots, 1, 0, \ldots, 0) \in \mathbf{Z}^{k^n}$.

In this context, the additive order of $\mathbf{w}$ is the smallest constant C such that there exists an integer vector $\mathbf{v}$ satisfying

$$L(C_k^n)\mathbf{v} = C\mathbf{w}.$$

Before we get to the statement and proof of our bound, we will first consider the Laplacian as a map of vector spaces $\mathbf{Q}^{k^n} \to \mathbf{Q}^{k^n}$.

The vector space $\mathbf{Q}^{k^n}$ has a standard basis consisting of vectors $\{e_i\}_{i=1}^{k^n}$, where e_i is the vector with a 1 in the ith place and a 0 elsewhere. However, observe that we can associate a k-nary string of length n to any integer i between 0 and $k^n - 1$, inclusive. Therefore, we can rewrite these standard basis vectors as the vectors f_u across all k-nary strings of length n, setting e_i to f_u where u is the binary representation of $i - 1$.

We will use these to construct another set of vectors $\{\chi_u\}_{u \in (\mathbf{Z}/k\mathbf{Z})^n}$.

Definition 2.8. The vector χ_u in $\mathbf{Q}^{k^n}$ is defined to be the sum

$$\sum_{v \in (\mathbf{Z}/k\mathbf{Z})^n} \zeta_k^{u \cdot v} f_v,$$

where $u \cdot v$ is the standard dot product and ζ_k is a primitive k-th root of unity.

Remark 2.9. Note that the χ_u are precisely the irreducible complex characters of $(\mathbf{Z}/k\mathbf{Z})^n$.

Proposition 2.10. *The χ_u form an orthogonal basis of $\mathbf{Q}^{k^n}$. Furthermore, each vector χ_u is an eigenvector of $L(C_k^n)$ with eigenvalue equal to $2\omega(u)$, where $\omega(u)$ is the* Hamming weight *of u, equal to the number of non-zero entries of u.*

To see a proof of the above proposition and a more detailed exposition of this material, see [5, Chapter 2].

Now we are ready to state our upper bound theorems. We first consider the cube case, $k = 2$.

Theorem 2.11. *The size of the largest cyclic factor in $K(Q_n)$ is less than or equal to $2^n \cdot \operatorname{lcm}(1, 2, \ldots, n)$.*

Proof. As before, let $\mathbf{w}$ be the vector $(-1, 1, 0, \ldots, 0)$. We wish to find an integer solution to $L(Q_n)\mathbf{v} = C\mathbf{w}$. To do so, we will find a solution to this equation over $\mathbf{Q}$, where we better understand the Laplacian.

We will first show that

$$\mathbf{w} = \frac{1}{2^{n-1}} \sum_{\substack{u \in (\mathbf{Z}/2\mathbf{Z})^n \\ u_n \neq 0}} \chi_u.$$

Let X_n be the character table of $(\mathbf{Z}/2\mathbf{Z}^n)$, or equivalently the change-of-basis matrix for the χ_u basis. From observation, X_n has the following inductive block matrix form:

$$X_n = \begin{bmatrix} X_{n-1} & X_{n-1} \\ X_{n-1} & -X_{n-1} \end{bmatrix}$$

Note that our assertion for the χ_u coordinates of $\mathbf{w}$ for $n = 1$ is trivial. Therefore, it suffices to assume that the statement holds for $n-1$ and prove it inductively.

Let $\mathbf{w}_\chi$ be the vector $(0, \frac{1}{2^{n-1}}, 0, \frac{1}{2^{n-1}}, 0, \ldots)$. Our statement is equivalent to showing $X_n \mathbf{w}_\chi = \mathbf{w}$. By the block matrix definition, we have that the left hand side is the concatenation of the vector $2X_{n-1} \cdot (0, \frac{1}{2^{n-1}}, 0, \frac{1}{2^{n-1}}, \ldots)$ and the zero vector of size 2^{n-1}. By our inductive hypothesis, the upper vector is equal to $(-1, 1, 0, \ldots)$, so the two vectors concatenate together to yield $\mathbf{w}$ as desired.

Now consider the equation $L(Q_n)\mathbf{v} = \mathbf{w}$. Applying a change of basis, we have that this is equivalent to $(X_n^{-1} L(Q_n) X_n)(X_n^{-1}\mathbf{v}) = X_n^{-1}\mathbf{w}$.

The right-hand side is equal to $\mathbf{w}_\chi$ as defined above, while the left-hand side is equal to $D(X\mathbf{v})$ where D is the diagonalization of $L(Q_n)$.

Therefore, we have that $\mathbf{v}$ is equal to $X_n D^{-1} \mathbf{w}_\chi$. The vector $D^{-1}\mathbf{w}_\chi$ expands in the χ_u basis as

$$\frac{1}{2^{n-1}} \sum_{u_n \neq 0} \frac{1}{2\omega(u)} \chi_u,$$

which simplifies to

$$\frac{1}{2^n} \sum_{u_n \neq 0} \frac{\chi_u}{\omega(u)}. \tag{2}$$

The set of $\omega(u)$ is just the integers between 1 and n, so we can cancel out all the denominators by multiplying through by $2^n \operatorname{lcm}(1, 2, \ldots, n)$. Because $k = 2$, the matrix X_n is an integer matrix, and it follows that $2^n \operatorname{lcm}(1, 2, \ldots, n)\mathbf{v} = X_n \cdot (2^n \operatorname{lcm}(1, 2, \ldots, n) D^{-1} \mathbf{w}_\chi)$ is an integer vector.

Therefore, the equation $L(Q_n)\mathbf{v} = 2^n \operatorname{lcm}(1,2,\ldots,n)\mathbf{w}$ has an integer solution, so and the size of the largest cyclic factor must be less than this. $\square$

We deduce a bound on the largest factor in $Syl_2(K(Q_n))$ immediately.

Corollary 2.12. *The size of the largest factor in $Syl_2(K(Q_n))$ is $\leq 2^{n+\lfloor \log_2(n) \rfloor}$.*

Remark 2.13. It is worth noting the difficulties in computing the lowest such C (and thus the size of the highest cyclic factor). First, one must understand additional cancellations that occur due to the parity of the χ_u in (2). Furthermore, as in Lemma 2.5, we can add any rational multiple of the all 1s vector to obtain another solution. So we must also understand divisibility properties of the numerator and denominator of each entry of the vector of (2), namely the greatest common factor of the denominators of every entry modulo which all the numerators agree. This appears to be especially difficult to understand.

We now apply similar methods in the cases $k = 3, 4$, and 6.

Theorem 2.14. *The size of the largest cyclic factor in $K(C_3^n)$ is less than or equal to $2 \cdot 3^{n+1} \cdot \operatorname{lcm}(1,2,\ldots,n)$.*

Proof. This proof will have a very similar flavor to that of Theorem 2.11. Thus, we refer to some of its steps instead of repeating them in several places.

Let $\mathbf{w} = (-2,1,1,0,\ldots,0)$, corresponding to the polynomial $x_1^2 + x_1 - 2$. As before, we wish to find an integer solution $\mathbf{v}$ to $L(C_3^n)\mathbf{v} = C\mathbf{w}$ for $C = 2 \cdot 3^n \cdot \operatorname{lcm}(1,2,\ldots,n)$. Then by Corollary 2.7 the result will follow.

Using the same methods as before, we prove that $\mathbf{w}$ in the χ_u basis is given by

$$\mathbf{w} = \frac{1}{3^{n-1}} \sum_{\substack{u \in (\mathbf{Z}/3\mathbf{Z})^n \\ u_n \neq 0}} \chi_u.$$

(We use the polynomial $x_1^{k-1} + \cdots + x_1 - (k-1)$ precisely to have this nice form in the χ_u basis). In the process of proving this, we set $\mathbf{w}_\chi = (0, \frac{1}{3^{n-1}}, \frac{1}{3^{n-1}}, 0, \frac{1}{3^{n-1}}, \frac{1}{3^{n-1}}, \ldots, 0)$.

Again, we change our matrix equation to the χ_u basis to obtain the equation $(X_n^{-1}L(C_3^n)X_n)(X_n^{-1}\mathbf{v}) = X_n^{-1}\mathbf{w} = \mathbf{w}_\chi$. Manipulating this, we have $\mathbf{v} = X_n D^{-1}\mathbf{w}_\chi$. The vector $D^{-1}w_\chi$ in the χ_u basis is

$$\frac{1}{3^{n-1}} \sum_{u_n \neq 0} \frac{1}{3\omega(u)} \chi_u = \frac{1}{3^n} \sum_{u_n \neq 0} \frac{\chi_u}{\omega(u)}. \tag{3}$$

As before, we need only multiply by $3^n \operatorname{lcm}(1,\ldots,n)$ to clear appearing in (3). However, in this case, the χ_u vectors themselves involve ζ_3, which as a denominator of 2. Hence we must include an extra factor of 2.

$\square$

As before, we deduce a bound on the largest factor of $Syl_3(K(C_3^n))$.

Corollary 2.15. *The size of the largest factor in* $Syl_3(K(Q_n))$ *is* $\leq 3^{n+1+\lfloor \log_3(n) \rfloor}$.

Similarly, we can study the $k = 4$ and $k = 6$ cases. Note that, as we saw above, our ability to do so relies on having a nice enough expression for ζ_k. By a completely analogous means, we arrive at the following theorems and corollaries.

Theorem 2.16. *The size of the largest cyclic factor in* $K(C_4^n)$ *is less than or equal to* $4^{n+1} \cdot \operatorname{lcm}(1, 2, \ldots, n)$.

Corollary 2.17. *The size of the largest factor in* $Syl_2(K(C_4^k))$ *is* $\leq 4^{n+1+\lfloor \log_4(n) \rfloor}$.

Theorem 2.18. *The size of the largest cyclic factor in* $K(C_6^n)$ *is less than or equal to* $2 \cdot 6^{n+1} \cdot \operatorname{lcm}(1, 2, \ldots, n)$.

Corollary 2.19. *The size of the largest factor in* $Syl_2(K(C_6^n))$ *is* $\leq 2^{n+2+\lfloor \log_2(n) \rfloor}$, *and the size of the largest factor in* $Syl_3(K(C_6^n))$ *is* $\leq 3^{n+1+\lfloor \log_3(n) \rfloor}$.

3. Gröbner Basis Calculations

In what follows, R will be a ring and $S = R[x_1, \ldots, x_n]$ will be a polynomial ring over R in n variables.

Definition 3.1. For $I = (i_1, \ldots, i_n) \in (\mathbf{Z}_{\geq 0})^n$, we write

$$x^I = x_1^{i_1} \cdots x_n^{i_n}.$$

For $I = (a_1, \ldots, a_k) \subset \{1, \ldots, n\}$, we write

$$x_I = \prod_{i \in I} x_i.$$

Definition 3.2. A **monomial order** or **term order** on S is a total order of the set of monomial $\{x^I \mid I \in (\mathbf{Z}_{\geq 0})^n\}$.

Example 3.3. Throughout this section, we will use the **grevlex** (or **degrevlex**) term order on S. It is defined as follows: Let $I = (i_1, \ldots, i_n)$ and $J = (j_1, \ldots, j_n)$. Then $x^I < x^J$ if

- $\sum j_k > \sum i_k$; OR
- $\sum j_k = \sum i_k$ and the first non-zero entry of $J - I$ starting from the end is positive.

Throughout the rest of this section, let $<$ be a monomial order.

Definition 3.4. Let $f \in S$. Then the **leading term** of f (with respect to $<$), denoted $\ell\mathrm{t}(f)$, is the term whose monomial is greatest under $<$.

Definition 3.5. Let $f \in S$. Then the **leading coefficient** of f (with respect to $<$), denoted $lc(f)$, is the coefficient of $\ell\mathrm{t}(f)$.

Definition 3.6. Let R a ring and $I \lhd R$ an ideal. Then the **leading term ideal** of I is

$$\mathrm{LT}(I) = \langle \ell\mathrm{t}(f) \mid f \in I \rangle.$$

Definition 3.7. Let $I \lhd S$ an ideal. A **Gröbner basis** of I is a generating set $S = \{g_1, \ldots, g_k\}$ of I satisfying either of the following two properties:

- For every $f \in I$, we can write $\ell\mathrm{t}(f) = c_1\, \ell\mathrm{t}(g_1) + \cdots + c_k\, \ell\mathrm{t}(g_k)$ for some $c_i \in R$.
- $\mathrm{LT}(I) = (\ell\mathrm{t}(g_1), \ldots, \ell\mathrm{t}(g_k))$.

Gröbner bases are of interest, because of the following theorem.

Theorem 3.8. *Let $I \lhd S$ an ideal. Then*

$$S/I \cong S/\,\mathrm{LT}(I)$$

as R-modules.

Theorem 3.9. *When R is a PID, every ideal $I \lhd S$ has a Gröbner basis.*

From now onward, we let $<$ be the degrevlex term order and take $R = \mathbf{Z}$.

Recall that

$$\operatorname{coker} L(Q_n) \cong \mathbf{Z}[x_1, \ldots, x_n]/\langle x_1^2 - 1, \ldots, x_n^2 - 1, n - \sum x_i \rangle$$

from [2, Proposition 5.20]. By Theorem 3.8, we can understand $K(Q_n)$ by understading a Gröbner basis of the ideal $I_n = (x_1^2 - 1, \ldots, x_n^2 - 1, n - \sum x_i)$. In general, however, a Gröbner basis for I_n is difficult to get a grasp on. We therefore restrict our attention factors of $\mathbf{Z}/2^i\mathbf{Z}$ for low i.

First, we show that passing to a Gröbner over $\mathbf{Z}/2^j\mathbf{Z}$ preserves the factors of $\mathbf{Z}/2\mathbf{Z}, \ldots, \mathbf{Z}/2^{j-1}\mathbf{Z}$.

Proposition 3.10. *Fix $j > 1$. Let $S' = \mathbf{Z}/2^j\mathbf{Z}[x_1, \ldots, x_n]$ and $I_n' = \langle x_1^2 - 1, \ldots, x_n^2 - 1, n - \sum x_i \rangle \lhd S'$. Then S'/I_n' and S/I_n are the same in their factors of $\mathbf{Z}/2\mathbf{Z}, \ldots, \mathbf{Z}/2^{j-1}\mathbf{Z}$.*

Proof. Consider $L(Q_n) \otimes \mathbf{Z}/2^j\mathbf{Z} : (\mathbf{Z}/2^j\mathbf{Z})^{2^n} \to (\mathbf{Z}/2^j\mathbf{Z})^{2^n}$. Since tensoring preserves cokernels, we have

$$S'/I_n' \cong \operatorname{coker}(L(Q_n) \otimes \mathbf{Z}/2^j\mathbf{Z}) \cong \operatorname{coker}(L(Q_n)) \otimes \mathbf{Z}/2^j\mathbf{Z} \cong S/I_n \otimes \mathbf{Z}/2^j\mathbf{Z}.$$

The result follows immediately. □

With this in hand, we turn our attention to finding the Gröbner basis for $i = 2$, which will tell us the number of $\mathbf{Z}/2\mathbf{Z}$-factors in $K(Q_n)$. From explicit computations, we arrive at the following conjecture:

Conjecture 3.11. We claim that $LT(I_n)$ has the following form. Define $W_m = \{(2+\epsilon_2, 4+\epsilon_4, \ldots, m-3+\epsilon_{m-3}, m-1, m) \mid \epsilon_i = 0, 1\}$ for m odd. Then $LT(I_n)$ is generated by the set

$$\{x_1\} \cup \{x_2^2, \ldots, x_n^2\} \cup \bigcup_{\substack{m \leq n \\ m \text{ odd}}} \{2x_I \mid I \in W_m\}. \tag{4}$$

This conjecture implies [1, Theorem 1.3] by the following lemma and proposition:

Lemma 3.12. *The sequence*

$$b_n = \begin{cases} 2b_{n-1} & \text{if } n \text{ even} \\ 2b_{n-1} + 2^{\frac{n-3}{2}} & \text{if } n \text{ odd} \end{cases} \tag{5}$$

with initial conditions $b_1 = b_0 = 0$ is given explicitly by

$$2^{n-2} - 2^{\lfloor \frac{n-2}{2} \rfloor}$$

for $n \geq 2$.

Proof. Let $c_n := 2^{n-2} - 2^{\lfloor \frac{n-2}{2} \rfloor}$ for $n \geq 2$. For $n = 2, 3$, we have

$$b_2 = 2b_1 = 0 = 2^0 - 2^0 = c_2$$
$$b_3 = 2b_2 + 2^0 = 1 = 2^1 - 2^0 = c_3.$$

It now remains to show that c_n satisfies the recurrence relation for b_n, by induction on n. For $n = 2k$ even, we have

$$c_n = 2^{2k-2} - 2^{\lfloor \frac{2k-2}{2} \rfloor} = 2^{2k-2} - 2^{k-1} = 2\left(2^{2k-3} - 2^{k-2}\right) = 2c_{n-1}.$$

For $n = 2k+1$ odd, we have

$$\begin{aligned} c_n &= 2^{2k-1} - 2^{\lfloor \frac{2k-1}{2} \rfloor} = 2^{2k-1} - 2^{k-1} = 2\left(2^{2k-2} - 2^{k-1}\right) + \left(2^k - 2^{k-1}\right) \\ &= 2a_{n-1} + 2^{k-1} \\ &= 2a_{n-1} + 2^{\frac{n-3}{2}}. \end{aligned}$$

□

Proposition 3.13. *Suppose Conjecture 3.11 holds, and let a_n be the number of $\mathbf{Z}/2\mathbf{Z}$ factors in $K(Q_n)$. Then*

$$a_n = 2^{n-2} - 2^{\lfloor \frac{n-2}{2} \rfloor}.$$

Proof. The proof is still being worked upon.

□

Remark 3.14. For higher i, the Gröbner basis over $\mathbf{Z}/2^i\mathbf{Z}$ is considerably messier and more difficult to understand. They seem to follow a similar sort of pattern, but we are not sure what it is.

4. Higher Critical Groups

We give a brief exposition of the higher critical groups of Q_n before presenting our result, the calculation of the p-Sylow subgroups of these groups for $p \neq 2$. The material here is presented in more generality in [4].

By definition, the hypercube graph Q_n represents the edges and vertices of the geometric n-dimensional hypercube. From a topological viewpoint, it is thus the 1-skeleton of an n-dimensional cell complex that we will also denote by Q_n and refer to as the "n-cube".

Since Q_n is a cell complex, it comes with an associated chain complex of abelian groups $\{C_k(Q_n)\}_{k\in\mathbb{Z}_{\geq 0}}$ and boundary maps $\partial_k : C_k(Q_n) \to C_{k-1}(Q_n)$. The groups $C_k(Q_n)$ are generated by the k-cells of Q_n. Since Q_n is a finite cell complex, each is a free abelian group of finite rank and therefore it makes sense to define an adjoint *coboundary map* $\partial_k^* : C_{k-1}(Q_n) \to C_k(Q_n)$.

We can then construct the *generalized Laplacians* given by

$$L_k^{ud}(Q_n) = \partial_{k+1}\partial_{k+1}^* \qquad L_k^{du}(Q_n) = \partial_k^*\partial_k \qquad L_k^{tot}(Q_n) = L_k^{ud}(Q_n) + L_k^{du}(Q_n)$$

These are called the *up-down*, *down-up*, and *total* Laplacians. Note that by construction these can be defined for any finite cell complex X and are analogously denoted by $L_k^{ud}(X)$, $L_k^{du}(X)$, and $L_k^{tot}(X)$.

It can be seen by definition that $L_0^{ud}(Q_n)$, $L_0^{tot}(Q_n)$ are both equal to the regular laplacian $L(Q_n)$.

This motivates the definition of the *higher critical groups* of any finite cell complex X:

Definition 4.1.

$$K_i(X) = \frac{\ker\partial_i}{\mathrm{coker} L_i^{ud}(X)}$$

The regular critical group $K(Q_n)$ is equal to $K_0(Q_n)$.

We now present the main result of this section, a calculation of the p-Sylow subgroups of $K_i(Q_n)$ for $p \neq 2$ that generalizes the result in [1].

Theorem 4.2.

$$Syl_p(K_i(Q_n)) \simeq Syl_p(\oplus_{j=i+1}^{n}(\mathbb{Z}/j\mathbb{Z})^{\binom{n}{j}\binom{j-1}{i}})$$

Proof. We first apply a result from section 2 of [4]. Given cell complexes X and Y, this states that $L_i^{tot}(X \times Y)$ is block-diagonal with blocks $L_0^{tot}(X) \otimes \mathrm{id} + \mathrm{id} \otimes L_i^{tot}(Y)$, $L_1^{tot}(X) \otimes \mathrm{id} + \mathrm{id} \otimes L_{i-1}^{tot}(Y)$, ..., $L_i^{tot}(X) \otimes \mathrm{id} + \mathrm{id} \otimes L_0^{tot}(Y)$.

Applying this to $Q_n = Q_1 \times Q_{n-1}$ and using the shorthand $L_{n,k} = L_i^{tot}(Q_n)$, we calculate $L_{n,i}$ as the following block matrix:

$$L_{n,i} = \begin{bmatrix} L_{0,0} \otimes \mathrm{id} + \mathrm{id} \otimes L_{n-1,i} & 0 \\ 0 & L_{0,1} \otimes \mathrm{id} + \mathrm{id} \otimes L_{n-1,i-1} \end{bmatrix}.$$

Then, substituting in

$$L_{0,0} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad L_{0,1} = \begin{bmatrix} 2 \end{bmatrix}$$

we arrive at the following identity:

$$L_{n,i} = \begin{bmatrix} 1 + L_{n-1,i} & -1 & \\ -1 & 1 + L_{n-1,i} & \\ & & 2 + L_{n-1,i-1} \end{bmatrix}$$

Now we will shift to considering $L_{n,i}$ as an operator on the free module of rank $N = 2^{n-i}\binom{n}{i}$ over the ring $\mathbb{Z}_2$ obtained by localization of $\mathbb{Z}$ at the multiplicative subset generated by the element 2. Note the space $(\mathbb{Z}_2)^N$ has a canonical inner product and norm inherited from $\mathbb{R}^N$.

We now make the following two-part assertion. First, we claim that $L_{n,i}$ is orthogonally diagonalizable over $\mathbb{R}^N$, namely it has a set of $2^{n-i}\binom{n}{i}$ eigenvectors that are pairwise orthogonal and linearly independent over $\mathbb{R}$. Second, we claim that these eigenvectors can be chosen such that they lie in $(\mathbb{Z}_2)^N$ and form a basis of this module as well.

For $k = 0$, we have $L_{n,0}$ is just the usual Laplacian of Q_n, and the required eigenvectors are simply the characters χ_u that we constructed earlier.

Thus, it follows that we can make the inductive assumption that $L_{p,q}$ satisfies our assertion for any $0 \leq p < n$, $0 \leq q < i$.

Note that the three blocks $1 + L_{n-1,i}$, $1 + L_{n-1,i}$ and $2 + L_{n-1,i-1}$ on the diagonal of $L_{i,k}$ have size $2^{n-i-1}\binom{n-1}{i}$, $2^{n-i-1}\binom{n-1}{i}$, and $2^{n-i}\binom{n-1}{i-1}$ respectively.

We can therefore write any vector v that $L_{n,i}$ acts on as a tuple (v_1, v_2, v_3), where v_1 is the first $2^{n-i-1}\binom{n-1}{i}$ entries of v and the other two are similarly defined according to the block sizes above.

Then, we can write the action of $L_{n,i}$ on (v_1, v_2, v_3) as

$$L_{n,i} : (v_1, v_2, v_3) \mapsto (v_1 - v_2 + L_{n-1,i}v_1, v_2 - v_1 + L_{n-1,i}v_2, 2v_3 + L_{n-1,i-1}v_3)$$

Finally, we construct our set of eigenvectors in three different classes. The first class of eigenvectors are vectors of the form $(w, w, 0)$ where w is an eigenvector of $L_{n-1,i}$. We see that, if w has corresponding eigenvalue λ, we can calculate $L_{n,i}(w, w, 0) = (\lambda w, \lambda w, 0) = \lambda \cdot (w, w, 0)$.

The second class of eigenvectors are vectors of the form $(w, -w, 0)$, which are mapped to $((2+\lambda)w, -(2+\lambda)w, 0) = (2+\lambda)\cdot(w, -w, 0)$.

The third class of eigenvectors are vectors of the form $(0, 0, u)$ where u is an eigenvector of $L_{n-1,i-1}$. If u has eigenvalue λ, then $L_{n,i}(0,0,u) = (0,0,(2+\lambda)u) = (2+\lambda)\cdot(0,0,u)$.

By our inductive assumption, there are $2^{n-i-1}\binom{n-1}{i}$ eigenvectors of each of the first two classes and $2^{n-i}\binom{n-1}{i-1}$ of the third class. Adding these up and applying Pascal's identity on binomial coefficients gives us a total of $2^{n-i}\binom{n}{i}$. In addition, orthogonality of these vectors follows immediately from the construction above.

Denote the span of the vectors by $S \subseteq (\mathbb{Z}_2)^N$. To show that this inclusion is an equality, observe that for any eigenvector w of $L_{n-1,i}$ the vectors $(w,0,0) = \frac{1}{2}((w,w,0)+(w,-w,0))$ and $(0,w,0) = \frac{1}{2}((w,w,0)-(w,-w,0))$ both lie in S. The vectors of the form $(w,0,0)$ and $(0,w,0)$ both span $(\mathbb{Z}_2)^{2^{n-i-1}\binom{n-1}{i}}$ by our inductive assumption, while the third class of eigenvectors of the form $(0,0,u)$ spans $(\mathbb{Z}_2)^{2^{n-i}\binom{n-1}{i-1}}$. As a result, we find that $S = (\mathbb{Z}_2)^N$ as desired.

To conclude the proof, observe that $L_i^{ud}(Q_n)$ and $L_i^{du}(Q_n)$ are by definition self-adjoint operators that annihilate each other. By simple linear algebra, it follows that these two operators and their sum $L_{n,i}$ have the same eigenvectors.

Denote the multiset of nonzero eigenvalues of $L_i^{ud}(Q_n)$ by $s_i^{ud}(Q_n)$. By the above work, we find that $\ker\partial_i/\mathrm{im}L_i^{ud}(Q_n)$ for $L_i^{ud}(Q_n)$ considered as an operator on $(\mathbb{Z}_2)^N$ is isomorphic to $\oplus_{\lambda\in s_i^{ud}(Q_n)}\mathbb{Z}_2/(\lambda\cdot\mathbb{Z}_2)$.

Considering this as an abelian group, we find that the p-Sylow subgroups of $\ker\partial_i/\mathrm{im}L_i^{ud}(Q_n)$ are isomorphic to those of $K_i(Q_n)$ for $p \neq 2$. Since $s_i^{ud}(Q_n)$ was calculated explicitly in [4], we can conclude

$$Syl_p(K_i(Q_n)) \simeq Syl_p(\oplus_{j=i+1}^{n}(\mathbb{Z}/j\mathbb{Z})^{\binom{n}{j}\binom{j-1}{i}})$$

$\square$

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