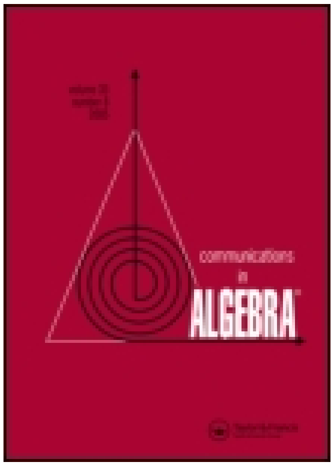

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BETTI NUMBERS OF TORIC VARIETIES AND EULERIAN POLYNOMIALS

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It is well known that the Eulerian polynomials, which count permutations in S_n by their number of descents, give the h-polynomial/h-vector of the simple polytopes known as permutohedra, the convex hull of the S_n-orbit for a generic weight in the weight lattice of S_n. Therefore, the Eulerian polynomials give the Betti numbers for certain smooth toric varieties associated with the permutohedra.

In this article we derive recurrences for the h-vectors of a family of polytopes generalizing this. The simple polytopes we consider arise as the orbit of a nongeneric weight, namely, a weight fixed by only the simple reflections $J = \{s_n, s_{n-1}, s_{n-2}, \ldots, s_{n-k+2}, s_{n-k+1}\}$ for some k with respect to the A_n root lattice. Furthermore, they give rise to certain rationally smooth toric varieties $X(J)$ that come naturally from the theory of algebraic monoids. Using effectively the theory of reductive algebraic monoids and the combinatorics of simple polytopes, we obtain a recurrence formula for the Poincaré polynomial of $X(J)$ in terms of the Eulerian polynomials.

Key Words: Betti numbers; Linear algebraic monaids; Simple polytope; Toric variety.

2010 Mathematics Subject Classification: 14xx; 05xx.

1. INTRODUCTION

Toric varieties and their cohomology have played an increasingly important role in studying the combinatorics of convex polytopes. They started around 1980 with Stanley's spectacular proof of the necessity of McMullen's conditions (characterizing the face numbers of a simple polytope) using the cohomology of rationally smooth projective toric varieties. This connection between the topology of toric varieties and the combinatorial geometry of convex polytopes is of interest to us.

Let (W, S) be a finite Weyl group of type A_n. Let J be any proper subset of S. Associated with J is a certain projective toric variety $X(J)$. We would like to calculate the Betti numbers of $X(J)$ when J is combinatorially smooth, i.e., $X(J)$ is a rationally smooth variety.

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Definition 1 ([5]). Let X be a complex algebraic variety of dimension n. Then X is *rationally smooth at* x if there is a neighbourhood U of x in the complex topology such that, for any $y \in U$,

$$H^m(X, X\backslash\{y\}) = 0, \qquad m \neq 2n$$
$$H^{2n}(X, X\backslash\{y\}) = \mathbb{Q}.$$

Here H^* denotes the cohomology of X with rational coefficients.

The most basic combinatorial data of a d-dimensional convex polytope are the numbers f_i of i-dimensional faces encoded in the face polynomial $f(t) := \sum_{i=0}^{d} f_i t^i$. For simple polytopes, i.e., where each vertex lies on exactly d edges, the possible f-polynomials are expressed in terms of the h-polynomials $h(t) = f(t-1) = \sum_{i=0}^{d} h_i t^i$ where h_i are strictly positive and satisfy the symmetry relation $h_i = h_{d-i}$. When a polytope P is rational, i.e., all its vertices have rational coordinates with respect to some lattice, we associate a toric variety to it X_P using the normal fan construction. It turns out that the Poincaré polynomial of X_P agrees with $h(t^2)$.

Danilov [7, Theorem 10.8, Remark 10.9, Proposition 12.11] proved that X_P is rationally smooth if and only if the polytope P is simple.

Consider a semisimple algebraic group G_0 with maximal torus T_0 and an irreducible representation ρ_λ of G_0 with the highest weight $\lambda \in X(T_0) \otimes \mathbb{Q}$. Consider the action of W on the vector space spanned by the simple roots of G_0 and take the convex hull of the W-orbit of λ, $P_\lambda = \mathrm{Conv}(W.\lambda) \subset X(T_0) \otimes \mathbb{Q}$. Using the inner normal fan construction associated to the polytope P_λ [9], we obtain a projective toric variety $X(J)$. The terminology is justified since $X(J)$ depends only on

$$J = \{s \in S \mid s(\lambda) = \lambda\}.$$

In [21] Renner finds necessary and sufficient conditions for the polytope P_λ to be simple using the theory of algebraic monoids that he developed along with Putcha since 1980. For each Weyl group (W, S), Renner gives a classification of all $J \subseteq S$, such that $X(J)$ is rationally smooth. See Corollary 3.5 in [21].

Definition 2 ([19]). We refer to J *combinatorially smooth* if P_λ is a simple polytope.

According to Renner's classification, the subset J is combinatorially smooth of type A_n if $J \subset \{s_1, s_2, \ldots, s_n\}$ has one of the following forms:

(1) $J_0 = \emptyset$;
(2) $J_0 = \{s_1, \ldots, s_i\}$ where $1 \leq i \leq n$;
(3) $J_0 = \{s_j, \ldots, s_n\}$ where $1 < j \leq n$;
(4) $J_0 = \{s_1, \ldots, s_i, s_j, \ldots, s_n\}$ where $1 \leq i, j \leq n$ and $j - i \leq 3$.

When (W, S) is finite Weyl group of type A_n and $J = \emptyset$, the polytope P_λ is a permutahedron. The Betti numbers of $X(\emptyset)$ are given by the Eulerian numbers. In [3], Brenti studies the descent polynomials (i.e., the Poincaré polynomials of $X(\emptyset)$) as analogues of the Eulerian polynomials.

When $J \neq \emptyset$, the weight λ is allowed to lie on certain reflecting hyperplanes. Of course, the orbit of a point in the complement of the arrangement is just the ordinary permutahedron. Whether the Poincare polynomial of $X(J)$ can be expressed in this case in terms of the Eulerian numbers is an interesting question. In this article, we answer this question by computing the Poincaré polynomial of $X(J)$ when (W, S) is finite Weyl group of type A_n and $J = \{s_{n-k+1}, \dots, s_n\} \subseteq S$ is combinatorially smooth, $1 \leq k \leq n+1$ and $s_k = (k, k+1) \in S_n$.

Theorem. *Let* $J(k, n) = \{s_{n-k+1}, s_{n-k+2}, \dots, s_n\} \subset S$, $1 \leq k \leq n$, *and let* $h_k(t)$ *denote the h-polynomial of the n–dimensional variety* $X(J(k, n))$. *Then* $J(k, n)$ *is combinatorially smooth and the following recurrence relation holds*:

$$h_k(t) = h_{k-1}(t) - \binom{n+1}{k+1}(t^k + t^{k-1} + \cdots + t)E_{n-k}(t).$$

where $J(0, n) = \emptyset$ *and* $h_0 = E_{n+1}$ *the* $(n+1)$*–Eulerian polynomial.*

Finally, the recurrence relation is illustrated for $J(n-1, n) = \{s_2, s_3, s_4, \dots, s_n\}$ and $J(n-2, n) = \{s_3, s_4, \dots, s_n\}$ where the h-polynomial of $X(J(n-2, n))$ and $X(J(n-1, n))$ are computed in [21, Example 4.3] and [22, Example 4.6].

One needs to investigate further to see whether our new technique can provide answer for all types of combinatorially smooth sets J.

This article is structured as follows. In Section 2 we introduce briefly $\mathcal{J}$-irreducible monoids of type J and the projective toric variety associated $X(J)$. Using the notion of cross section lattice associated to a reductive monoid, we have a formula presented in Proposition 1, for the number of i-dimensional faces of the polytope P_λ corresponding to the variety $X(J)$. This gives us a good handle of the h-polynomial of $X(J)$, which can be expressed in terms of all subsets of S. There is an interesting interplay between the geometry of $X(J)$ and the combinatorics of finite sets, illustrated in Corollary 1. In Section 3 we introduce Eulerian polynomials and prove our main results. We conclude this section with an example that illustrates the recurrence formula obtained in Theorem 6.

2. *h*-POLYNOMIAL OF *X*(*J*)

Throughout the article, we work with the field $\mathbb{C}$ of complex numbers. In this section, we establish the definitions and the results needed throughout this text. A very good up-to-date account of the theory of algebraic monoids can be found in [19, 26].

A linear algebraic monoid is an affine variety together with an associative morphism and an identity element. An irreducible monoid M is called *reductive* if its unit group G is a reductive group. Let B be a Borel subgroup of G and $T \subset B$ a maximal torus of G. The set of idempotents in $\overline{T}$ is defined as

$$E(\overline{T}) = \{e \in \overline{T} \mid e^2 = e\},$$

where $\overline{T}$ is the Zariski closure of T in M. The set

$$\Lambda = \{e \in E(\overline{T}) \mid Be = eBe\}$$

is called the cross-section lattice of M relative to B and T.

Definition 3. A reductive monoid M with $0 \in M$ is called $\mathcal{J}$-irreducible if $M - \{0\}$ has exactly one minimal $G \times G$-orbit.

See section 7.3 of [19] for a systematic discussion of the important class of reductive monoids and for a proof of the following theorem.

Theorem 1 ([19, Lemma 7.8]). *Let M be a reductive monoid. The following are equivalent*:

(1) *M is $\mathcal{J}$-irreducible*;
(2) *There is an irreducible rational representation $\rho : M \to End(V)$ which is finite as a morphism of algebraic varieties*;
(3) *If $\overline{T} \subset M$ is the Zariski closure in M of a maximal torus $T \subset G$, then the Weyl group W of T acts transitively on the set of minimal nonzero idempotents of $\overline{T}$.*

Definition 4 ([16, Section 7.3]). If M is $\mathcal{J}$-irreducible, we say that M is $\mathcal{J}$-irreducible of type J if

$$J = \{s \in S \mid se_1 = e_1 s\},$$

where S is the set of reflections relative to T and B and e_1 is the minimal idempotent such that $e_1 B = e_1 B e_1$.

Next, we describe the $G \times G$-orbit structure of a $\mathcal{J}$-irreducible monoid of type $J \subseteq S$. First, recall the partial ordering on the $G \times G$-orbits described as follows:

$$GeG \prec GfG \quad \text{if and only if } GeG \subset \overline{GfG} \quad \text{if and only if } ef = e.$$

The following result was first presented in Theorem 4.16 [16].

Theorem 2 ([21, Theorem 1.2]). *Let M be a $\mathcal{J}$-irreducible monoid of type $J \subseteq S$.*

(1) *There is a canonical one-to-one order-preserving correspondence between the set of $G \times G$-orbits acting on M and the set of W-orbits acting on the set of idempotents of $\overline{T}$. This set is canonically identified with $\Lambda = \{e \in E(\overline{T}) \mid eB = eBe\}$.*
(2) $\Lambda - \{0\} \cong \{I \subseteq S \mid \text{noconnectedcomponentof} I \text{ iscontainedentirelyin} J\}$ *in such a way that e corresponds to $I(e) \subset S$ if $I(e) = \{s \in S \mid se = es \neq e\}$.*
(3) *If $e \in \Lambda - \{0\}$ corresponds to $I(e)$, as in 2 above, then $C_W(e) = W_{I^*(e)}$, where*

$$I^*(e) = I \cup \{s \in J \mid st = ts \text{ for all } t \in I(e)\}.$$

The above theorem, which will be used throughout the article is the bridge between a cross-section lattice and the combinatorics of finite sets.

Next, consider a $\mathcal{J}$-irreducible monoid M of type J and an irreducible representation $\rho : M \to End_{\mathbb{C}}(V)$ which is finite as a morphism.

M has a reductive unit group G. Let $B \subset G$ be a Borel subgroup of G and $T \subset B$ a maximal torus of G. Let $\overline{T}$ denote the Zariski closure of T in M. By Theorem 5.4 of [19], $\overline{T}$ is a normal affine toric variety.

The dimension of the T-orbit corresponding to an idempotent $e \in \overline{T}$ is called the *rank* of e, i.e., $\text{rank}(e) = \dim Te$.

Let G_0 be a semisimple algebraic group defined as the commutator subgroup $G_0 = (G, G)$ of G with maximal torus $T_0 = T \cap G_0$, and let $\rho_\lambda = \rho_{|G_0}$ be the representation of G_0 that corresponds to the highest weight $\lambda \in X(T_0)$. Let the Weyl group W act by reflections on the vector space spanned by the roots of G. Take the W-orbit of λ and consider its convex hull in $X(T_0) \otimes \mathbb{Q}$. We obtain the polytope $P_\lambda = \mathrm{Conv}(W.\lambda) \subset X(T_0) \otimes \mathbb{Q}$.

The next Corollary recorded without a proof in [21] is a key result in our computations of the h–polynomial of the polytope P_λ.

Corollary 1 ([21, Corollary 1.3]). *Let W be a Weyl group, and let $r : W \to GL(V)$ be the usual reflection representation of W. Let $\mathscr{C} \subset V$ be the rational Weyl chamber, and let $\lambda \in \mathscr{C}$. Assume that $J = \{s \in S \mid s(\lambda) = \lambda\}$. Then the set of orbits of W acting on the face lattice $\mathscr{F}_\lambda$ of $\mathscr{P}_\lambda$ is in one-to-one correspondence with the set* $\{I \subseteq S \mid \text{no connected component of } I \text{ is contained entirely in } J\}$.

We conclude the W–orbit contains a representative face whose W–stabilizer/isotropy subgroup is the parabolic subgroup $W_{I_J^*}$, generated by the set

$$I_J^* = \{I\} \cup \{s \in J \mid st = ts \text{ for all } t \in I\}.$$

Next, let $S(J) = \{I \subset S \mid \text{no connected component of } I \text{ is contained entirely in } J\}$, $S(J) \subseteq \mathscr{P}(S)$, the power set of S. For a better understanding we present the following examples of sets of the form $S(J)$ for various types of $J \subset S$ where S is the generating set of the Weyl group $W = S_4$.

Example 1. Let $G = SL_4$ and $W = S_4$ where $S = \{s_1, s_2, s_3\}$.

J	P_λ	h$-$*polynomial*
$\{s_2, s_3\}$	*tetrahedron*	$1 + t + t^2 + t^3$
$\{s_3\}$	*truncated tetrahedron*	$1 + 5t + 5t^2 + t^3$
$\emptyset$	*permutahedron*	$1 + 11t + 11t^2 + t^3$

When $J = \{s_2, s_3\}$ we have $S(J) = \{\emptyset, I_1, I_2, S\}$ where $I_1 = \{s_1\}$, $I_2 = \{s_1, s_2\}$. The subset $I \subseteq S$ corresponds to the unique face F of the polytope P_λ with $I = \{s \in S \mid s(F) = F \text{ and } s_{|F} \neq id\}$ whose relative interior F^0 has nonempty intersection with the Weyl chamber $\mathscr{C}$. To I_1 it corresponds an edge labeled I_1 and to I_2 it corresponds a triangle labeled I_2, both faces drawn in Fig. 1.

When $J = \{s_3\}$ we have $S(J) = \{\emptyset, I_1, I_2, I_3, I_4, S\}$, $I_1 = \{s_1\}$, $I_2 = \{s_2\}$, $I_3 = \{s_1, s_2\}$ and $I_4 = \{s_2, s_3\}$. The corresponding faces to I_i, $1 \leq i \leq 4$ are drawn in Fig. 2.

Example 2. In general, when $J = \{s_2, s_3, \ldots, s_n\}$ the polytope P_λ is a simplex and when $J = \emptyset$ the polytope P_λ corresponds to a permutahedron.

Next, in the cases of interest in this article, we consider the 1-1 correspondence, as posets, between $E(\overline{T})\backslash\{0\}$ and the face lattice of the polytope P_λ, namely

$$e \in E(\overline{T}) \leftrightarrow \mathscr{F}_e, \tag{1}$$

such that $\mathrm{rank}(e) = \dim(\mathscr{F}_e) + 1$. For more details on this correspondence, see [18].

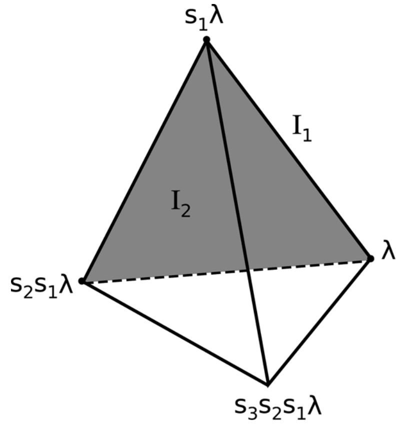

Figure 1 Tetrahedron.

We know that T is a Zariski open subset of $\overline{T}$. Hence the torus

$$\frac{T}{\mathbb{C}^*} \text{ is an open subset of } \frac{\overline{T}-\{0\}}{\mathbb{C}^*}.$$

Our interest is in the projective toric variety

$$X(J) = \frac{\overline{T}-\{0\}}{\mathbb{C}^*} = \mathrm{Proj}[\mathbb{C}[\overline{T}]],$$

which depends only on $J = \{s \in S \mid s(\lambda) = \lambda\}$ and not on λ or M.

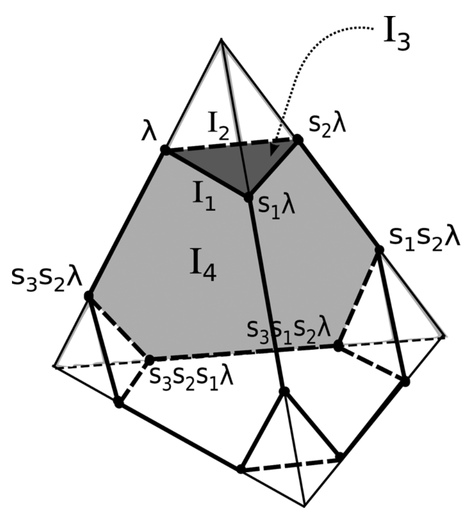

Figure 2 Truncated Tetrahedron.

Next, we mention a formula for calculating the number of i-dimensional faces of the polytope P_λ using the lattice isomorphism (1). The formula can be found in Lemma 4.1 in [11].

Proposition 1 ([11]). *The number of i-dimensional faces of P_λ is*

$$f_i = \sum_{e \in \Lambda_i} \frac{|W|}{|W_{I^*(e)}|},$$

where $\Lambda_i = \{e \in \Lambda \mid \mathrm{rank}(e) = \dim(Te) = i + 1\}$ *and to* $I(e) = \{s \in S \mid se = es \neq e\}$ *it corresponds that* $I^*(e) = I(e) \cup \{s \in J \mid st = ts \text{ for all } t \in I(e)\}$.

This proposition yields an interesting formula of the h-polynomial, which will be used frequently in proving our main results.

Recall that the nodes of a Dynkin diagram corresponds to some special vectors, called *simple roots*, in the vector space $V = X(T) \otimes \mathbb{Q}$. The set of simple roots is denoted by Δ.

A subset $Y \subseteq \Delta$ is called *connected* if it is a connected subset of the underlying graph of the Coxeter diagram.

For (W, S) finite Weyl group, the *graph structure* on S is defined as follows:

$$s \text{ and } t \text{ are joined by an edge if } st \neq ts.$$

Using the previous result we obtain an effective way of combining the geometry of convex polytopes with the cross section lattice associated to reductive monoids. The proof of the following result follows immediately from Corollary 1.

Proposition 2. *The h-polynomial of $X(J)$ is given by*

$$h(t) = \sum_{I \in S(J)} \frac{|W|}{|W_{I_J^*}|}(t-1)^{|I|}.$$

The proof follows immediately from Corollary 1.

3. BETTI NUMBERS OF $X(J)$ IN TERMS OF THE EULERIAN POLYNOMIALS

Definition 5. A permutahedron $P_{n-1} \in \mathbb{R}^n$ is the convex hull in $\mathbb{R}^n$ of the set

$$\{(p_1, p_2, \ldots, p_n) \in \mathbb{R}^n \mid (p_1\ p_2 \cdots p_n) \in S_n\}.$$

Next we introduce Eulerian polynomials associated to a Coxeter system of type A_n. More results on this topic can be found in [2, 10].

Let $\sigma = (p_1, \ldots, p_n) \in S_n$. Define the ascent set of σ

$$A(\sigma) = \{i \mid 1 \leq i \leq n : p_i < p_{i+1}\}.$$

It turns out that

$$i \in A(\sigma) \Longleftrightarrow l(\sigma s_i) = l(\sigma) + 1.$$

Here the notation $l(\sigma)$ stands for the length of a permutation $\sigma \in S_n$.

Let

$$E(n, i) = |\{\sigma \in S_n \mid |A(\sigma) = i\}|.$$

be the Eulerian numbers. When (W, S) is finite Weyl group of type A_{n-1}, we define the n-Eulerian polynomials as follows:

$$E_n(t) = \sum_{i=0}^{n-1} E(n, i)t^i.$$

Theorem 3. *Let h_{n-1} be the h-polynomial of the permutahedron P_{n-1}, and let E_n be the n-Eulerian polynomial. Then*

$$h_{n-1}(t) = E_n(t).$$

Renner gives a proof of this known fact [13] in Theorem 3.1 [25] using algebraic topology and ascent polynomials. As a consequence, we obtain a characterization of the $(n+1)$-Eulerian polynomial in terms of all subsets $I \subseteq S$. Recall that $S = \{s_1, \ldots, s_n\}$ is the minimal set of reflections that generates the permutation group S_{n+1} with $s_i = (i, i+1) \in S_{n+1}$.

Theorem 4. *Let E_{n+1} be the $(n+1)$-Eulerian polynomial. The following identity holds*:

$$E_{n+1}(t) = \sum_{I \subseteq S} \frac{(n+1)!}{|W_I|}(t-1)^{|I|}.$$

Proof. Consider $(W, S) = (S_{n+1}, S)$ finite Weyl group of type A_n, and let $J = \emptyset$. When $J = \emptyset$, the highest weight λ is in the interior of the fundamental Weyl chamber. By applying reflections $s_i = (i,\ i+1) \in S_{n+1}$ about the hyperplanes orthogonal to the simple roots we permute i and $i+1$ coordinates of λ. The polytope P_λ given by the convex hull of the W-orbit of λ turns out to be an n permutahedron since its vertices are obtained by permuting their coordinates. From Theorem 3 and Proposition 2, we have that

$$h(t) = E_{n+1}(t) = \sum_{I \in S(J)} \frac{|W|}{|W_{I_J^*}|}(t-1)^{|I|}.$$

where $I_J^* = I$, $S(J) = \mathcal{P}(S)$ the power set of S, when $J = \emptyset$. □

For the reminder of this section we specialize the discussion to the case of (W, S) Weyl group of type A_n. We are now ready to present the main result of this article.

Theorem 5. *Let* $J(k,n) = \{s_{n-k+1}, s_{n-k+2}, \ldots, s_n\}$ for $1 \leq k \leq n$, *and let* $h_k(t)$ *denote the h-polynomial of the n–dimensional variety* $X(J(k,n))$. *Then* $J(k,n)$ *is combinatorially smooth. The following recurrence relation holds*:

$$h_k(t) = h_{k-1}(t) - \binom{n+1}{k+1}(t^k + t^{k-1} + \cdots + t)E_{n-k}(t), \tag{2}$$

where $J(0,n) = \emptyset$ *and* $h_0 = E_{n+1}$ *the* $(n+1)$*–Eulerian polynomial.*

Proof. From Corollary 3.5 [21], we obtain that $J(k,n)$ is combinatorially smooth. Thus $X(J(k,n))$ is rationally smooth as the corresponding polytopes $P_{\lambda,k}$ are simple polytopes. Let M be a $\mathcal{J}$-irreducible monoid of type $J(k,n)$, where

$$J(k,n) = \{s_{n-k+1}, s_{n-k+2}, \ldots, s_n\} \subseteq S,$$

and let $\Lambda(k)$ denote the cross section lattice associated to M. From Theorem 2 we have that $\Lambda(k)\backslash\{0\} \cong S(J(k,n))$, where $S(J(k,n))$ was previously defined as $S(J(k,n)) = \{I \subset S \mid \text{no connected component of } I \text{ is contained entirely in } J(k,n)\}$. Next, consider

$$M_i = \{A \subseteq S \mid J(k+1,n) - J(i,n) \subseteq A \subseteq S\backslash J(i,n)\},$$

for $0 \leq i \leq k+1$. So, in particular,

$$M_0 = \{A \subseteq S \mid J(k+1,n) \subseteq A\},$$

and

$$M_{k+1} = \{A \subseteq S \mid A \subseteq S\backslash J(k+1,n)\}.$$

Hence, we have

$$S(J(k,n)) = \bigsqcup_{i=0}^{k+1} M_i,$$

for $0 \leq i \leq k+1$.

We associate to each $A \subseteq S(J(k,n))$,

$$A_k^* = A \sqcup \{s \in J(k,n) \mid st = ts \text{ for any } t \in A\}.$$

We compute the h-polynomial of $X(J(k,n)$ using Corollary 2 and obtain

$$h_k(t) = \sum_{i=0}^{k+1} \sum_{A \in M_i} \frac{(n+1)!}{|W_{A_k^*}|}(t-1)^{|A|}. \tag{3}$$

Then for $A \subseteq M_0$ and for $A \subseteq M_1$, we have $A_k^* = A$ and $W_{A_k^*} = W_A$. For $A \subseteq M_i$, for $2 \leq i \leq k+1$, we have $A_k^* = A \sqcup J(i-1)$ and $W_{A_k^*} = W_A \times S_i$. Thus, the h-polynomial of $X(J(k,n))$ is given by

$$h_k(t) = \sum_{i=0}^{k+1} \sum_{A \in M_i} \frac{(n+1)!}{i! \times |W_A|} (t-1)^{|A|}. \tag{4}$$

Consider $J(k-1,n) = \{s_{n-k+2}, \ldots, s_n\}$. Then the cross-section lattice $\Lambda(k-1)\backslash\{0\}$ corresponding to $J(k-1,n)$ has the property that $\Lambda(k-1)\backslash\{0\} \cong S(J(k-1,n))$ according to Theorem 2.

Let

$$S_i = \{A \subseteq S \mid J(k,n)\backslash J(i,n) \subseteq A \subseteq S\backslash J(i,n)\},$$

for $0 \leq i \leq k$. So, in particular,

$$S_0 = \{A \subseteq S \mid J(k,n) \subseteq A\},$$

and

$$S_k = \{A \subseteq S \mid A \subseteq S\backslash J(k,n)\}.$$

Note that for each $i = 0, \ldots, k$, we have

$$S_i \cap \{A \subseteq S \mid s_{n-k} \in A\} = M_i,$$

and

$$S_i \cap \{A \subseteq S \mid s_{n-k} \notin A\} = \{A \subseteq S \mid J(k,n)\backslash J(i,n) \subseteq A \subset S\backslash(J(i,n) \cup \{s_{n-k}\}).$$

Let

$$\begin{aligned} N_i &= \{A \subseteq S \mid J(k,n)\backslash J(i,n) \subseteq A \subseteq S\backslash(J(i,n) \cup \{s_{n-k}\}) \\ &= \{A' \cup (J(k,n)\backslash J(i,n)) \mid A' \subseteq S\backslash J(k+1,n)\} \end{aligned}$$

Note also that $N_k = M_{k+1}$. Then the following relation holds:

$$S(J(k-1,n)) = S(J(k,n)) \sqcup \bigsqcup_{i=0}^{k-1} N_i.$$

Next we compute $A_{k-1}^* = A_{J(k-1,n)}^*$ for all $A \subseteq S(J(k-1,n)$. For $A \subseteq M_0$ and $A \subseteq M_1$, we have

$$A_{k-1}^* = A \qquad \text{and} \qquad W_{A_{k-1}^*} = W_A.$$

For $A \subseteq M_i : i = 2 \cdots k$, we have

$$A_{k-1}^* = A \cup J(k-1,n) \qquad \text{and} \qquad W_{A^*} = W_A \times S_i.$$

We know that

$$M_{k+1} = \mathscr{P}(S \backslash J(k+1, n)).$$

Hence for $A \subseteq M_{k+1}$, we have

$$A^*_{k-1} = A \cup J(k-1, n) \quad \text{and} \quad W_{A^*} = W_A \times S_k.$$

For $A \subseteq N_0$ and $A \subseteq N_1$, we have

$$A^*_{k-1} = A \quad \text{and} \quad W_{A^*_{k-1}} = W_A.$$

For $A \subseteq N_i$ where $i = 1, \ldots, k-1$, we have

$$A^*_{k-1} = A \cup J(i-1, n) \quad \text{and} \quad W_{A^*_{k-1}} = W_A \times S_i.$$

Furthermore, the h-polynomial of $X(J(k-1, n))$ is given by:

$$\begin{aligned} h_{k-1}(t) &= \sum_{i=0}^{k} \sum_{A \in M_i} \frac{(n+1)!}{i! \times |W_A|}(t-1)^{|A|} + \sum_{i=0}^{k-1} \sum_{A \in N_i} \frac{(n+1)!}{|W_A| \times i!}(t-1)^{|A|} \\ &= \sum_{A \in M_{k+1}} \frac{|W|}{k! \times |W_A|}(t-1)^{|A|}. \end{aligned} \tag{5}$$

By (10) and (11), this implies that

$$h_{k-1}(t) - h_k(t) = \sum_{i=0}^{k-1} \sum_{A \in N_i} \frac{(n+1)!}{i! \times |W_A|}(t-1)^{|A|} + \left(\frac{1}{k!} - \frac{1}{(k+1)!}\right) \sum_{A \in M_{k+1}} \frac{(n+1)!}{|W_A|}. \tag{6}$$

The following relations hold: for $A \in N_i$ there exists $A' \in M_{k+1}$ such that

$$A = A' \cup (J(k, n) \backslash J(i, n)) \quad \text{and} \quad W_A = W_{A'} \times S_{k-i+1}.$$

We use these relations in (12) and obtain

$$\begin{aligned} h_{k-1}(t) - h_k(t) &= \sum_{i=0}^{k-1} \sum_{A' \in M_{k+1}} \frac{(n+1)!}{i!(k-i+1)!|W_{A'}|}(t-1)^{|A'|+k-i} \\ &\quad + \frac{(n+1)!}{(n-k)!}\left(\frac{1}{k!} - \frac{1}{(k+1)!}\right) \sum_{A \in M_{k+1}} \frac{(n-k)!}{|W_A|}(t-1)^{|A|} \\ &= \sum_{i=0}^{k-1} \frac{(n+1)!}{(n-k)!(k-i+1)!i!}(t-1)^{k-i} \sum_{A' \in M_{k+1}} \frac{(n-k)!}{|W_{A'}|}(t-1)^{|A'|} \\ &\quad + \frac{(n+1)!}{(n-k)!(k+1)!} k \sum_{A \in M_{k+1}} \frac{(n-k)!}{|W_A|}(t-1)^{A|}. \end{aligned}$$

Theorem 4 allows us to express the $(n-k)$-Eulerian polynomial in terms of the subsets of M_{k+1}:

$$E_{n-k}(t) = \sum_{A \in M_{k+1}} \frac{(n-k)!}{|W_A|}(t-1)^{|A|}.$$

In the next formula, we replace $(k-i+1)! \times i!$ by $\frac{1}{(k+1)!}\binom{k+1}{i}$ and obtain

$$\begin{aligned} h_k(t) - h_{k-1}(t) &= \sum_{i=0}^{k-1} \frac{(n+1)!}{(n-k)!(k-i+1)! \times i!}(t-1)^{k-i}E_{n-k}(t) \\ &\quad + \frac{(n+1)!}{(n-k)!(k+1)!}kE_{n-k}(t) \\ &= \left[\sum_{i=0}^{k-1} \frac{(n+1)!}{(n-k)!(k+1)!}\binom{k+1}{i}(t-1)^{k-i} + \frac{(n+1)!}{(n-k)!(k+1)!}k\right]E_{n-k}(t) \\ &= \binom{n+1}{k+1}\left[\sum_{i=0}^{k-1}\binom{k+1}{i}(t-1)^{k-i} + k\right]E_{n-k}(t). \end{aligned} \tag{7}$$

We need now to show that

$$\sum_{i=0}^{k-1}\left(\binom{k+1}{i}(t-1)^{k-i} + k\right) = \sum_{i=1}^{k} t^i. \tag{8}$$

Let

$$f(t) = \sum_{i=0}^{k-1}\binom{k+1}{i}(t-1)^{k-i}.$$

Observe that

$$\sum_{i=0}^{k+1}\binom{k+1}{i}(t-1)^{k+1-i} = (t-1)f(t) + \binom{k+1}{k}(t-1) + \binom{k+1}{k+1}(t-1)^0.$$

By the binomial theorem, the left-hand side is

$$t^{k+1} = ((t-1)+1)^{k+1}.$$

So

$$f(t) + k = \frac{t^{k+1} - (k+1)(t-1) - 1}{t-1} + k = \frac{t^{k+1} - t}{t-1} = \sum_{i=1}^{k} t^i.$$

This concludes our proof. □

For $n \geq 0$ and for $k = 0, 1, \ldots, n$, let W be the Weyl group of type A_n, and consider $h_{n,k}$ the h–polynomial of the n–dimensional polytope P_λ which is the W–orbit of the highest weight λ fixed by $J = \{s_{n-k+1}, \ldots, s_{n-1}, s_n\}$. Thus

$$\begin{gathered} h_{n,0} \text{ is the usual Eulerian polynomial divided by } t, \\ h_{n,n-1}(t) = 1 + t + \cdots + t^n, \\ h_{n,n}(t) = 1. \end{gathered}$$

For each fixed $k \geq 1$, define the exponential generating function in n

$$H_k := H_k(x, t) = \sum_{n \geq k} h_{n,k}(t) \frac{x^{n+1}}{(n+1)!}$$

along with the convention that $H_0 = 1 + H_1 = \frac{t-1}{1-e^{x(t-1)}}$. The rightmost equality in (2) it is the well-known exponential generating function for the Eulerian polynomials. For more details see [30, Chapter 3, Ex. 80(c), p. 174].

Corollary 2. *For $k \geq 1$ the following recurrence formula holds*:

(1)

$$H_k = H_{k-1} - \frac{x^k}{k!} - H_0 \frac{x^{k+1}}{(k+1)!} \frac{t^{k+1} - t}{t - 1}$$

(2) *Define a bivariate generating function* (*ordinary in k*, *exponential in n*) *by*

$$F := F(x, y, t) := \sum_{k=0}^{\infty} H_k y^k = \sum_{n \geq k \geq 0} h_{n,k}(t) \frac{x^{n+1} y^k}{(n+1)!}.$$

Then

$$F(x, y, t) = \frac{e^{xy}}{y - 1} \left(y + \frac{e^{xy(t-1)-t}}{t - e^{x(t-1)}} \right)$$

Proof. (1) The calculation is rather elementary and uses the fact that

$$\sum_{n=0}^{\infty} E_n(t) \frac{x^n}{n!} = \frac{t-1}{t - e^{(t-1)x}} = H_0.$$

(2) Multiplying the recursion above by y^k and summing on $k \geq 1$ leads to an equation for F that one can solve explicitly. ☐

Example 3. Next, we verify the recurrence formula obtained in Theorem 5.

Let $k = n - 1$. Then $J(n-1, n) = \{s_2, s_3, \ldots, s_n\}$ is combinatorially smooth and the h-polynomial of $X(J(n-1, n))$ computed in Example 4.3 [21], is given by the formula

$$h_{n-1}(t) = 1 + t + t^2 + \cdots + t^n.$$

The recurrence formula obtained in Theorem 6 is equivalent to

$$h_{n-1}(t) = h_{n-2}(t) - \binom{n+1}{n}(t^{n-1} + \cdots + t)E_1(t).$$

We know the h-polynomial of $X(J(n-2, n))$ from Example 4.6 [22]. It is given by the following formula

$$h_{n-2}(t) = 1 + (n+2)t + (n+2)t^2 + \cdots + (n+2)t^{n-1} + t^n.$$

Hence

$$h_{n-2}(t) - h_{n-1}(t) = (n+1)(t^{n-1} + \cdots + t)$$

yields the desired relation.

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